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Discontinuous control systems

by
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Abstract. By means of some simple examples from economics, we elucidate certain solution tools for the solution of optimal control problems were the system under study undergoes major changes when certain boundaries are crossed. The "major changes" may be that the state gets a jump discontinuity when crossing a boundary, or that the right hand side of the differential equation changes. Some theoretical result are presented. Among the results presented, at least the sufficient condition related to fields of extremals should be new.

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Introduction In economic control problems, sometimes the underlying system undergoes a major change when the state crosses a boundary. For example, a firm may go bust, when its equity becomes negative. Mathematically speaking, such changes introduce discontinuities in the differential equation that invalidate the assumptions ordinarily required for the standard maximum principle to apply. In other situations, when the state reaches a surface it gets a kick, (a jump discontinuity), a case also necessitating changes in the maximum principle. Below, some simple economic examples are presented in which such features appear. The main purpose of this paper it to show how the solution tools available work in simple situations. These solution tools consist of the standard equations of the maximum principle plus an additional condition involving a jump in the costate variable when boundaries are crossed. There is no reference below for this jump condition, but it must have been stated and used before, directly or indirectly, and more than once, at least in more applied work. (It does appear in Nævdal (2001), see also Nævdal (2003).) The following results pertain to both types of discontinuities described above.

There are in general, of course, a number of results pertaining to jumps in the state variables, including so-called impulse control problems, see e.g. chapter 3 in Seierstad and Sydsæter (1987) and Arutyunov, A. (2005) and papers referred to there. Early economic examples include the works by Arrow and Kurz, and K.Vind, as well as the control theory book by Kamien and Schwartz, all referenced in Seierstad and Sydsæter (1987).

Let us, however, first state a standard "continuous" control problem, together with the maximum principle.

1. Standard Control Problem

Consider the problem

$$\max_{u(.)}\{\int_{t_0}^{t_1} f_0(t,x(t),u(t))dt + h(x(t_1))\}, u(t) \in U \subset \mathbb{R}^r \tag{1.1}$$

subject to the vector differential equation

$$\dot{x}(t) = f(t,x(t),u(t)), x(t_0) = x^0, x^0 \in \mathbb{R}^n, x^0 \text{ fixed,} \tag{1.2}$$

and the terminal conditions

$$x_i(t_1) = x_i^1, i = 1,...,l, \tag{1.3}$$

$$x_i(t_1) \geq x_i^1, i = l+1,...,m, \tag{1.4}$$

$$x_i(t_1) \text{ free, } i = m+1,...,n, \tag{1.5}$$

$(f = (f_1,...,f_n)$, $x = (x_1,...,x_n)$, $u = (u_1,...,u_r)$. Here, h,f_0, f_i are given real–valued functions on $\mathbb{R}^{1+n+r}$, U is a given subset, t_0, t_1, x^0 and x_i^1 are given entities, and $u(t)$, the control function, is subject to choice. The control functions are throughout (except for certain existence results), assumed to be piecewise continuous. Let v.e. t (virtually every t) mean for all t except a finite (or countable number) of points, sometimes v.e. is read as virtually everywhere, having the same meaning. (In fact, (1.2) is required to hold v.e.) Assume that h, h_x,$f_0(t,x,u)$, $f_{0x}(t,x,u)$, $f(t,x,u)$ and $f_x(t,x,u)$ are continuous functions. For this problem the following necessary conditions hold. Let $p = (p_1,...,p_n)$, and define $H = p_0 f_0 + pf = p_0 f_0 + \sum p_i f_i$.

Theorem 1 (Necessary conditions) Let $(x^*(t), u^*(t))$ be an optimal pair. Then there exist a number p_0 and a vector function $p(t) = (p_1(t), ..., p_n(t))$, where $p(t)$ is continuous and piecewise continuously differentiable, such that

$$(p_0, p_1(t), ..., p_n(t)) \neq (0, 0, ..., 0) \text{ for all } t \in [t_0, t_1], \tag{1.6}$$

$$H(t, x^*(t), u^*(t), p(t)) \geq H(t, x^*(t), u, p(t)) \text{ for all } u \in U \text{ for v.e. } t, \tag{1.7}$$

$$\dot{p}_i(t) = -\partial H(t, x^*(t), u^*(t), p(t))/\partial x_i, i = 1, ..., n, \text{ for v.e. } t, \tag{1.8}$$

$$p_0 = 0 \text{ or } p_0 = 1, \tag{1.9}$$

$$p_i(t_1) \text{ no conditions for } i = 1, ..., l, \tag{1.10}$$

$$p_i(t_1) \geq p_0 h_{x_i}(x^*(t_1)),\ p_i(t_1) = p_0 h_{x_i}(x^*(t_1)) \text{ if } x_i^*(t_1) > x_i^1, i = l+1, ..., m, \tag{1.11}$$

$$p_i(t_1) = p_0 h_{x_i}(x^*(t_1)), i = m+1, ..., n. \tag{1.12}$$

□

2. Problems with discontinuities

Let $\phi_1, ..., \phi_k$ be given real-valued C^1-functions on $(t, x)-$space. Let $\Gamma_j := \{(t, x) : \phi_j(t, x) = 0\}$ and $\Gamma = \cup_j \Gamma_j$. Assume that for each $(t, x) \in \Gamma$, at most one function ϕ_i equals zero at (t, x). Assume that the state jumps at points t such that $(t, x(t-)) \in \Gamma_j$ for some j, the jumps being governed by

$$x_i(t+) - x_i(t-) = g_i^j \in \mathbb{R}, \quad g_i^j \text{ fixed}, \tag{2.1}$$

($\pm$ indicates right and left limits).Write $g^j = (g_1^j, ..., g_n^j)$. If $(t_0, x^0) \in \Gamma$, no jump occurs at t_0. Moreover, by assumption, if for any solution $x(.)$ with corresponding control $u(.)$, $(t, x(t-)) \in \Gamma_j$ for some j, so a jump occurs at t, then $x(t+) = x(t-) + g^j$ does not belong to Γ.

By assumption, $(s, x(s-)) \in \Gamma$ for only a finite number of points s. More generally and more precisely, *still by assumption*, for any given control $u(.)$, the following construction of a solution works. Let $x(.)$ be a continuous solution of (1.2) on $[t_0, t_1]$, with $x(t_0) = x^0$. Then $x(s) \notin \Gamma$ for s close to t_0. Let $t = \tau_1$ be the first point for which $(t, x(t)) \in \Gamma$, i.e. $(\tau_1, x(\tau_1)) \in \Gamma_j$ for some $j = j_1$. Change the definition of $x(.)$ on $[\tau_1, t_1]$ by letting $x(.)$ be a continuous solution on $[\tau_1, t_1]$ for which $x(\tau_1) = x(\tau_1-) + g^{j_1}$. Then $x(s) \notin \Gamma$ for $s > \tau_1$, s close to τ_1. Let $t = \tau_2$ be the first point $> \tau_1$ for which $(t, x(t)) \in \Gamma$, i.e. $(\tau_2, x(\tau_2)) \in \Gamma_j$ for some $j = j_2$. Then, change the definition of $x(.)$ on $[\tau_2, t_1]$ by letting $x(.)$ be a continuous solutions on $[\tau_2, t_1]$ for which $x(\tau_2) = x(\tau_2-) + g^{j_2}$. This construction continuous in the same manner, and only a finite number of τ_k 's will be found, say $\tau_1, ..., \tau_{k^*}$. Then the continuous solution $x(t)$, $t \geq \tau_{k^*}$, is used until t_1 is reached. At the end, redefine $x(.)$ to be left continuous. (So, from now on all solutions $x(.)$ are left continuous.) If $x(t_1)$ satisfies the end conditions (1.3)-(1.5), the pair $(x(.), u(.))$ is called admissible.

One can replace the assumption that the above construction works for any $u(.)$ (i.e., the assumptions in the last paragraph), by the convention that we only consider pairs $(x(.), u(.))$, for which the above construction works. Such pairs are called strongly admissible if also (1.3)-(1.5) hold. So in this case we only seek an optimal pair $(x^*(.), u^*(.))$ in the set of strongly admissible pairs.

Theorem 2 (Necessary condition) Assume that the optimal solution $x^*(t)$ crosses or touches each surface $\Gamma_j := \{(t, x) : \phi_j(t, x) = 0\}$ in a nontangent manner (made precise in condition (NT) below), and that the optimal pair $(x^*(.), u^*(.))$ is strongly admissible. Then, provided $(t_1, x^*(t_1-)) \notin \Gamma$ (t_1 the fixed horizon), the standard maximum principle holds even in the present case, except that $p(.)$ has a jump at any "fault point" $\tau \in (t_0, t_1)$, i.e. at any τ for which $(\tau, x^*(\tau-)) \in \Gamma$. The jumps of $p(.)$ are governed by

$$\begin{aligned} p(\tau-) - p(\tau+) &= p_0[f_0(\tau, x^*(\tau-), u^*(\tau-)) - f_0(\tau, x^*(\tau+), u^*(\tau+))]\mu \\ &\quad + [p(\tau+) f(\tau, x^*(\tau-), u^*(\tau-)) - p(\tau+) f(\tau, x^*(\tau+), u^*(\tau+))]\mu. \end{aligned} \tag{2.2}$$

Here, the n-vector $\mu = (\mu_1, ..., \mu_n)$ is determined by the relation

$$[\phi_{jt}(\tau, x^*(\tau-)) + \phi_{jx}(\tau, x^*(\tau-))f(\tau, x^*(\tau-), u^*(\tau-))]\mu_i + \phi_{jx_i}(\tau, x^*(\tau-)) = 0, \tag{2.3}$$

where
$\phi_{jx}(\tau, x^*(\tau-))f(\tau, x^*(\tau-), u^*(\tau-)) = \sum_i(\partial\phi_j(\tau, x^*(\tau-))/\partial x_i)f_i(\tau, x^*(\tau-), u^*(\tau-))$.

The nontangent condition on $x^*(t)$, (NT), is that for any $t \in (t_0, t_1)$,

$$\begin{aligned} \phi_j(t, x^*(t-)) &= 0 \Rightarrow \\ \partial\phi_j(t, x^*(t-))/\partial t &+ \sum_i(\partial\phi_j(t, x^*(t-))/\partial x_i)f_i(t, x^*(t-), u^*(t-)) \neq 0. \end{aligned} \tag{2.4}$$

□

Since all solutions, including $x^*(.)$, are assumed to be left continuous, $x^*(\tau-) = x^*(\tau), x^*(t-) = x^*(t)$ (i.e., above, minus signs inside $x^*(.)$ can be dropped).

Two sketchy proofs of this condition is given in the Appendix.

Remark 1 (Changing dynamics) The above problem is called the jump problem, or the jump case. The present setup can also be used to treat the following type of problem, called the discontinuous (f_0, f)-problem, or simply the discontinuous case. In this case, the right-hand side of the differential equation and/or the integrand in the criterion change abruptly each time the solution crosses one of the surfaces forming Γ, in other word, different differential equations/and or different integrands exist in different parts of the (t, x)-space. On the other hand only continuous solutions $x(.)$ are considered.

Formally, then, for each set of the form $\Phi = \cap_i\Phi_i, \Phi_i = \{(t, x) : \phi_i(t, x) < 0\}$ or $\Phi_i = \{(t, x) : \phi_i(t, x) > 0\}$, (the direction of the inequality sign may depend on i), functions $f_{0\Phi}(t, x, u)$ and $f_\Phi(t, x, u)$ exist, defined on all $\mathbb{R}^{1+n+r}$ and being C^1 here, such that $f_0(t, x, u) = f_{0\Phi}(t, x, u)$ and $f(t, x, u) = f_\Phi(t, x, u)$ for (t, x) in Φ. Define a strongly admissible pair $(x(.), u(.))$ to be a pair such that $x(t)$ belongs to some set Φ except for a finite number of points, such that $f_0(t, x(t), u(t)) = f_{0\Phi}(t, x, u(t))$ and $\dot{x}(t) = f_\Phi(t, x(t), u(t))$ v.e. if $x(t) \in \Phi$, and such that $u(t) \in U$,

$x(t_0) = x^0$, and the terminal conditions (1.3)-(1.5) are satisfied. (For f_0 and f so determined, denote (1.1) and (1.2) by (1.1*) and (1.2*).) Finally, a strengthened version of (NT) is needed, denoted (NT*), namely, for any $t \in (t_0, t_1)$,

$$\begin{aligned} \phi_j(t, x^*(t)) &= 0 \Rightarrow \\ \partial\phi_j(t, x^*(t))/\partial t &+ \sum_i (\partial\phi_j(t, x^*(t))/\partial x_i) f_i(t, x^*(t), u^*(t\pm)) \neq 0, \end{aligned} \quad (2.5)$$

where $f = f_\Phi$ if $(t, x^*(t)) \in \Phi$ for $t < \tau$, t close to τ, and $f = f_{\Phi'}$ if $(t, x^*(t)) \in \Phi'$ for $t > \tau$, t close to τ. Moreover, in (2.3), $f = f_\Phi$ if $(t, x^*(t)) \in \Phi$ for $t < \tau$, t close to τ.

These versions of (2.3) and (NT) are denoted (2.3*) and (NT*), respectively. Similarly, in H and $\partial H/\partial x$ in the necessary conditions (1.6) and (1.7), $(f_0(t, x^*(t), u^*(t)), f(t, x^*(t), u^*(t))) = (f_{0\Phi}(t, x^*(t), u^*(t)), f_\Phi(t, x^*(t), u^*(t)))$ when $(t, x(t))$ belongs to Φ.

Auxiliary state variables can be introduced that keep track of which region is entered, and which makes it possible to reduces the problem to the jump case. This is shown in the particular case considered below.

Let $(x^*(t), u^*(t))$ be a strongly admissible pair satisfying (NT*) and being optimal among all strongly admissible pairs. Then the following theorem holds:

Theorem 3 (Necessary condition). Assume that (NT*) holds. Then the necessary conditions of Theorem 2 hold provided (2.3) is read as just stated (i.e. (2.3*) holds), and (2.2) holds for the following definitions:
$(f_0(\tau, x^*(\tau), u^*(\tau-)), f(\tau, x^*(\tau), u^*(\tau-)) = (f_{0\Phi}(\tau, x^*(\tau), u^*(\tau-)), f_\Phi(\tau, x^*(\tau), u^*(\tau-))$
if $x^*(t) \in \Phi$ for $t < \tau$, t close to t and
$(f_0(\tau, x^*(\tau), u^*(\tau+)), f(\tau, x^*(\tau), u^*(\tau+)) = (f_{0\Phi'}(\tau, x^*(\tau), u^*(\tau+)), f_{\Phi'}(\tau, x^*(\tau), u^*(\tau+))$
if $x^*(t) \in \Phi'$ for $t > \tau$, t close to τ.
(This version of (2.2) is called (2.2*)). □

A sketchy proof is given in the Appendix. Consider the following particular example of the situation in this remark. Assume that $\psi_i(t_0, x_0) < 0$ for all i, and that all admissible solutions first crosses the set $\Gamma^1 = \{(t, x) : \psi_1(t, x) = 0\}$, then $\Gamma^2 = \{(t, x) : \psi_2(t, x) = 0\}$, and so on, such that for any i, before crossing Γ^i, the admissible solution stays in $\{(t,x) : \psi_i(t,x) < 0\}$ and after crossing it stays in $\{(t,x) : \psi_i(t,x) > 0\}$. Let y jump upwards one unit, when $x(.)$ crosses Γ^i, $dy/dt = 0, y(t_0) = 0$. Define $\phi_i(t,x,y) = \psi_i(t,x)(i-y) + (\max\{0, y-(i-1)\})^2$. Then, $\phi_i(t,x(t),y(t)) = \psi_i(t,x(t))$ as long as $y(t) \le i-1$ and $\phi_i(t,x(t),y(t)) \ge \psi_i(t,x(t)) + 1$ when $y(t) \ge i$. Moreover, if $y(\tau-) \le i-1$, then $\psi_i(\tau,x(\tau)) = 0 \Leftrightarrow \phi_i(\tau,x(\tau),y(\tau)) = 0$.

Remark 2 (Does solutions exist?) For the type of discontinuities occurring in Remark 1, it may not always be clear what constitutes a solution of the differential equation. What are the admissible pairs $x(.), u(.)$, do such ones exist at all?. (E.g. has the equation $\dot{x} = -x/|x|$, $x(0) = -1$ a solution on $[0,2]$?) If for some $i, \phi_i(t,x(t)) = 0$ in some interval (if at all this is possible), we have to specify which differential equation $\dot{x} = f_\Phi$ is assumed to holds; we are then at the boundary of two sets Φ. We do not need to remove such ambiguities when stating the necessary conditions, at least when they have the form of Theorem 3 (i.e. when we confine our interest to strongly admissible solutions, in which case problems are assumed away). However, a sufficient condition for such ambiguities not to arise is the following one. For all $(t,x) \in \Gamma$, either (NT*>), i.e. (2.5) holds for all $u \in U$ for the sign $\neq$ replaced by $>$ and $x^*(.)$ replaced by any admissible $x(.)$, or (NT*<) (where $\neq$ is changed to $<$) holds for all $u \in U$ and $x^*(.)$ replaced by any admissible $x(.)$. So in a given problem, one might want to test this last property before venturing further in the solution of the problem. □

Example 1 (**Jumps in a state variable**)
Assume that a person runs a firm that earns no money and goes bust when the equity, x, in the firm is zero. At time 0 the equity is $x_0 > 0$. As long as the firm exists, the person continuously draws an amount $u \in (0,\infty)$ out of the firm. Thus the equity changes according to the equation $dx/dt = -u$. The horizon is T. After going bust, he gets a certain minimum income $\gamma > 0$ from his parents (or from the state). Assume that $x_0/4\gamma < T$. Let $\alpha = \gamma^{1/2}$. The instantaneous utility he gets out of an amount of money w is $w^{1/2}$. Let y be a state equal to 1 before he has gone bust and equal to zero afterwards. (So $\dot{y} = 0$, y has a unit downwards jump at a t such that $x(t) = 0$.) Then what he maximizes is

$$\int_0^T [yu^{1/2} + (1-y)\alpha]dt.$$

What is the optimal policy, i.e. the optimal time profile of u?

Solution

Let τ be the time x reaches 0 and let $\phi_1 := x$ $(k = 1)$. The costate (adjoint variable) p_x corresponding to x is zero for $t > \tau$ (we have a free end). The two component of $\mu = (\mu_x, \mu_y)$ satisfy $\mu_x = 1/u^*(\tau-)$, and $\mu_y = 0$, by (2.3). The costate p_x satisfies

$$p_x(\tau-) := q := (u^*(\tau-)^{1/2} - \alpha)/u^*(\tau-) \qquad (2.6)$$

$(p_0 = 1)$. Write $v := u^*(\tau-) = u^*(t), t < \tau$; because $p_x(.)$ is constant for $t < \tau, u^*(.)$ is constant here. We are not going to need the precise form of $p_y(t)$. The Hamiltonian is $H := u^{1/2} - qu$ for $t < \tau$, and $H_u = 0$ gives $v = (1/2q)^2$. From (2.6) we then get $v^{1/2}/2 = v^{1/2} - \alpha$, or $v = (2\alpha)^2 = 4\gamma$, which yields $x_0 - 4\gamma\tau = 0$, or $\tau = x_0/4\gamma < T$. We have obtained a proposal for the optimal control. The criterion value of $u \equiv 4\gamma$ should be compared with the criterion value for the "candidate" control obtained for the end condition $x(T) = 0$, which is constant equal to x_0/T, (for the latter case Theorem 2 does not work). (The control $u \equiv x_0/T$ gives a lower criterion value.)

Another way to prove optimality of $u \equiv 4\gamma$ is to use the next remark, (Remark 3), and let us do that. Define $u^{**}(t) \equiv x_0/T$. Because $p_x^{**}(t) = 1$, if $u > u^{**}(t) = x_0/T$, then $\alpha(t, u) = -(u - u^{**}(t))/(-u^{**}(t_1-))$ is > 0, and then we shall see that

$$\sqrt{u} - q_T u > \sqrt{u^{**}(t)} - q_T u^{**}(t) \qquad (2.7)$$

for $u > x_0/T$, u close to x_0/T, where $q_T := (u^{**}(T-)^{1/2} - \alpha)/u^{**}(T-) = (T/x_0)^{1/2} - \alpha T/x_0$ $(= p_x^v(t)$ for $t < T)$. The inequality (2.7) should have been the opposite $(\leq)$ if $u^{**}(t)$ was optimal, hence it cannot be optimal. The strict inequality (2.7) follows if we prove that $\beta'(u^{**}(t)) > 0$, where $\beta(u) = \sqrt{u} - q_T u$. So let us do that: $\beta'(u^{**}(t)) = 1/2\sqrt{u^{**}(t)} - q_T = 1/(2\sqrt{x_0/T}) - ((T/x_0)^{1/2} - \alpha T/x_0) = \alpha T/x_0 - (1/2)\sqrt{T/x_0} > (1/2)\sqrt{x_0/T}(T/x_0) - (1/2)\sqrt{T/x_0} = 0$, because $\alpha > (1/2)\sqrt{x_0/T}$ $(\gamma > x_0/4T)$.

Remark 3 (Fault point at t_1) In the situation of Remark 1, it is convenient to have necessary conditions even for the case where a fault point (touch point) occurs at t_1. We restrict attention to the free end case. In this problem, introduce the end condition $\phi_i(t_1, x(t_1)) = 0$ and assume that we have found a candidate $(x^*(t), u^*(t))$ satisfying the necessary conditions with $\phi_i(t_1, x^*(t_1)) = 0$ for some given i for an adjoint function $p^*(t)$ jumping at fault points strictly before t_1 and satisfying the end condition $p_i(t_1) = p_0 h_{x_i}(x^*(t_1)) + \lambda \phi_{ix_i}(t_1, x^*(t_1))$, $i = 1, ..., n$, for some $(p_0, \lambda) \neq 0$. Assume that (NT*) (i.e. (2.5)) is satisfied for the left limit $u^*(t_1-)$. Let $p^{**}(.)$ be the adjoint function arising from (2.1)-(2.3) for the above pair $(x^*(t), u^*(t))$ when $p_0 = 0$, $p^{**}(.)$ jumping at fault points $< t_1$, when we require that $p^{**}(t_1) = \phi_{ix}(t_1, x^*(t_1))$. Assume that $(t, x^*(t)) \in \Phi''$, for $t < t_1$, t close to t_1, so that $(f_0, f) = (f_{0\Phi''}, f_{\Phi''})$ for $(t, x) = (t, x^*(t))$. Let t^* be any given point that is both a non-fault point and a continuity point of $u^*(.)$, and assume that $(t, x^*(t)) \in \Phi$ for all t near t^*. Define

$$\alpha(t^*, u) := \frac{p^{**}(t^*)[f_\Phi(t^*, x^*(t^*), u) - f_\Phi(t^*, x^*(t^*), u^*(t^*))]}{\phi_{it}(t_1, x^*(t_1-)) + \phi_{ix}(t_1, x^*(t_1-)) f_{\Phi''}(t_1, x^*(t_1-), u^*(t_1-))}$$

For any $u \in U$, if $\alpha(t^*, u) > 0$, then $H(t^*, x^*(t^*), u, p^v(t^*)) - H(t^*, x^*(t^*), u^*(t^*), p^v(t^*)) \leq 0$, for $p_0 = 1$, $p^v(t)$ being the adjoint function obtained when $p^v(t_1+) = 0$ and it jumps at fault points before and including t_1, (so $p^v(t)$ satisfied (2.2) in particular for $\tau = t_1$). Here v is any control in U such that (NT*) holds for the right limits at t_1 for $u^*(t_1+) = v$, $(f_0, f) = (f_{0\Phi'}, f_{\Phi'})$ if $(t, x^*(t)) \in \Phi'$, for $t > t_1$, t close to t_1, $(u^*(t) = v$ for such $t)$. For any $u \in U$, if instead $\alpha(t^*, u) < 0$, then $H(t^*, x^*(t^*), u, p(t^*)) - H(t^*, x^*(t^*), u^*(t^*), p(t^*)) \leq 0$ for $p_0 = 1$ and for a function $p(t)$ jumping only at fault points in (t_0, t_1), $p(t_1-) = 0$.

We need to carry out the above procedure for each i. □

A sketchy proof is given in the Appendix.

Remark 4 (State- and time-dependent jumps) In this remark, let g_i^j depend on t and x, $g_i^j(t, x)$ being C^1, so the jump condition is $x_i(t+) - x_i(t-) = g_i^j(t, x(t-))$. Let $g^j(t, x) := (g_1^j(t, x), ..., g_n^j(t, x))$. Strongly admissible solutions are defined as before, the only change is that jumps are determined as just described. In the present case, Theorem 2 must be modified. When $(\tau, x^*(\tau-)) \in \Gamma_j$, the second term on the right hand side of (2.2) must be replaced by
$p(\tau+) g_t^j(\tau, x^*(\tau-))\mu + p(\tau+) g_x^j(\tau, x^*(\tau-)) +$

$p(\tau+)[f(\tau, x^*(\tau-), u^*(\tau-))(1 + g_x^j(\tau, x^*(\tau-))) - p(\tau+)f(\tau, x^*(\tau+), u^*(\tau+))]\mu$. □

We call the just mentioned condition (2.2) modified. A sketchy proof is given in the Appendix.

Remark 5 (Existence theorems) Assume that for some $\varepsilon > 0$, for all i, if $\phi_i(t, x) = 0$, then, for any Φ, $\phi_{it}(t, x)+ \phi_{ix}(t, x)f_\Phi(t, x, u) > \varepsilon$ for all $u \in U$ or $\phi_{it}(t, x)+ \phi_{ix}(t, x)f_\Phi(t, x, u) < -\varepsilon$ for all $u \in U$. (In the jump case, $f_\Phi = f$.) Then conditions as in standard existence theorems, namely compactness of U, convexity of $\{f_{0\Phi}(t, x, u) + \gamma, f_\Phi(t, x, u)) : u \in U, \gamma \leq 0\}$ for any Φ, and, for some b, $\sup_t|x(t)| \leq b$ for all admissible $x(.)$, (see e.g. Sydsæter et al. (2005), Theorem 10.4.1) yield existence even for the discontinuities appearing here. More precisely, if a strongly admissible pair exists, the conditions imply the existence of an optimal strongly admissible pair $(x^*(.), u^*(.))$, ($u^*(.)$ measurable). (Note that, by a compactness argument, for any compact set $X \subset \mathbb{R}^n$, for some $\varepsilon' > 0$, $|\phi_i(t, x)| < \varepsilon'$, $(t, x) \in [t_0, t_1] \times X \Rightarrow \phi_{it}+ \phi_{ix}f_\Phi(t, x, u) > \varepsilon'$ for all $u \in U$ or $\phi_{it}+ \phi_{ix}f_\Phi(t, x, u) < -\varepsilon'$ for all $u \in U$). □

The proof is an easy modification of proofs for the standard problem.

3. Sufficient conditions

Theorem 4 (Verification theorem, sufficient condition) Consider problem (1.1)-(1.5), (2.1), and let $(x^*(t), u^*(t))$ be a strongly admissible pair. Assume that (NT) (in the discontinuous case (NT*)) is satisfied for any admissible $x(.)$. Suppose given a subset Q of $(t_0, t_1) \times \mathbb{R}^n$, an open set Q^0 containing Q, and a function $W(s, y)$ on Q^0 such that W is C^1 in $Q^0\backslash\tilde{Z}$, $\tilde{Z} = \{(t, x) \in Q^0, \psi^i(t, x) = 0$ for some $i = 1, ..., k^*\}$, for some given C^1- functions $\psi^i(s, y)$ on $\mathbb{R}^{1+n}$ that includes the ϕ_i -functions, more precisely, $\psi^i(s, y) = \phi_i(s, y)$ for $i = 1, ..., k' \leq k^*$. In the jump case, assume that $W(s, y) = W(s, y+g^j)$ when $(s, y) \in \{(s, y) \in Q : \phi_j(s, y) = 0\} = \Gamma_j$, and that $(s, y) \in \Gamma_j \Rightarrow \psi^i(s, y)$ for all $i \neq j$. Assume also that W satisfies

$$0 = W_s(s, x) + \max_{u\in U}\{f_0(s, x, s) + W_y(s, x)f(s, x, u)\} \tag{3.1}$$

for all (s, y) in $Q^0\backslash\tilde{Z}$, as well as

$$\limsup_{t\to t_1} W(t, x(t)) \geq h(x(t_1)) \tag{3.2}$$

for all strongly admissible solutions $x(.)$ contained in Q, (i.e. $(t, x(t)) \in Q$ for all $t \in (t_0, t_1)$). Assume that $x^*(.)$ is contained in Q, and that

$$h(x^*(t_1)) \geq \limsup_{t \to t_1} W(t, x^*(t)) \tag{3.3}$$

Assume that $W(, x(t))$ is continuous on $[t_0, t_1)$ for all strongly admissible $x(.)$ contained in Q. Moreover, assume that $(t, x^*(t))$ belongs to $Q^0 \backslash \tilde{Z}$, except for a finite number (or countable number) of points t, that $W(s, y)$ is locally Lipschitz continuous on Q^0 (in the jump case only on $Q^0 \backslash \{(s, y) \in Q^0 : \phi_j(s, y) = 0$ for some $j\}$), and that

$$\begin{aligned} & f_0(s, x^*(s), u^*(s)) + W_y(s, x^*(s)) f(s, x^*s), u^*(s)) \\ = \ & \max_{u \in U} \{f_0(s, x^*(s), u) + W_y(s, x^*(s)) f(s, x^*(s), u)\} \text{ v.e.} \end{aligned} \tag{3.4}$$

For each function $\psi^i(s, y)$, $i = 1, \ldots, k^*$, the vector of derivatives (ψ^i_s, ψ^i_x) is nonzero at all (s, y) such that $\psi^i(s, y) = 0$. Then $(x^*(.), u^*(.))$ is optimal in the set of all strongly admissible pairs $(x(.), u(.))$ such that $x(.)$ is contained in Q. □

The words "strongly admissible" are used in two different senses, depending on whether we consider the jump case, or discontinuous (f_0, f) - case. The theorem holds in both cases. (In the discontinuous case, note that $(f_0(t, x, u), f(t, x, u))$ equals $(f_{0\Phi}(t, x, u), f_\Phi(t, x, u))$ when (t, x) belongs to Φ.)

Remark 6 (Continuity on cl$Q$) In Theorem 4, in the discontinuous case, if W is continuous on cl$Q$, and

$$W(t_1, y) \geq h(y) \text{ for all } (t_1, y) \in \ clQ \tag{3.5}$$

and

$$W(t_1, x^*(t_1)) = h(x^*(t_1)) \tag{3.6}$$

then (3.2),and (3.3) can be dropped. □

Note that in many problems with jumps or discontinuities as above the optimal value function is discontinuous. In the above theorem, essentially $W(s, y)$ has to be the optimal value function, so frequently the conditions on $W(s,y)$ cannot be met. Yet, in other problems the optimal value function is after all (at least) continuous, hence a $W(s,y)$ satisfying the conditions in Theorem 4 may exist. In particular this holds frequently when nontangent conditions like those in Remark 5 are satisfied in case of Remark 1.

Proof. For simplicity, assume $f_0 = 0$. Assume first that the set of functions $\{\psi^i\}_i$ coincides with $\{\phi_i\}_i$. Along any strongly admissible trajectory $x(t)$ contained in Q, corresponding to some $u(t)$, the function $s \to W(s, x(s))$ is continuous, and,except perhaps for a finite number of points, locally Lipschitz continuous. It is then also nonincreasing in s, because strong admissibility imply v.e. that $(t, x(t)) \in Q^0 \backslash \{(s,y) \in Q^0 : \phi_j(s,y) = 0 \text{ for some } j\}$ and (3.1) implies

$$(d/ds)W(s,x(s)) = W_s(s,x(s)) + W_y(s,x(s))f(s,x(s),u(s)) \leq 0 \quad \text{v.e.} \qquad (3.7)$$

Moreover, by (3.4) there is equality if $u(s) = u^*(s)$, $x(s) = x^*(s)$, implying constancy of $W(s, x^*(s))$. Thus, $W(T', x^*(T')) = W(0, x^0) \geq W(T', x(T'))$, for any $T' < t_1$, the last inequality by (3.7). By (3.2), (3.3), taking $\limsup$ as $T' \to t_1$, gives $h(x^*(t_1)) \geq h(x(t_1))$, so $(x^*(t), u^*(t))$ is optimal. For the general case ($\{\psi^i\}_i$ larger than $\{\phi_i\}_i$) see Chapter 2 in Seierstad (2008). □

We shall next present a sufficient condition in terms of what is called characteristic solutions. We need the following definitions and preconditions.

Assume that there exists an open set Q^0 in $(t_0, t_1) \times \mathbb{R}^n$, such that for any $(s,y) \in Q^0 \cup \{(t_0, x^0)\}$, for t in $[s, t_1]$, there exist solutions $p(t; s, y)$ and $x(t; s, y)$, piecewise and left continuous in $t \in (s, t_1]$, with $x(s+; s, y) = y$, of the necessary conditions (maximum principle) (1.6)-(1.12), (2.2)-(2.3) for $p_0 = 1$ with corresponding control $u(t; s, y)$, for which the differential equation (1.2), the terminal conditions (1.3)-(1.5) and the following transversality conditions are satisfied.

$$p_i(t_1; s, y) \geq (\partial/\partial x_i)h(x(t_1; s, y)), i = l+1, \ldots, m \qquad (3.8)$$
$$\text{with equality holding if } x_i(t_1; s, y) > x_i^1,$$

$$p_i(t_1; s, y) = (\partial/\partial x_i)h(x(t_1; s, y)),\ i = m+1, \ldots, n\ . \qquad (3.9)$$

In the jump case, for any $(s,y) \in Q^0 \cup \{(t_0, x^0)\}$ it is assumed that $(t, x(t; s, y))$ belongs to $\Gamma := \{(t, y') : \phi_i(t, y') = 0$ for some $i\}$ for only a finite number points $t \in (s, t_1)$, with $(t, x(t+; s, y)) \notin \Gamma$ if $(t, x(t; s, y))$ belongs to Γ. In the discontinuous (f_0, f) - case, for any $(s, y) \in Q^0 \cup \{(t_0, x^0)\}$, it is assume that for all t except a finite number, $(t, x(t; s, y))$ belongs to some set Φ. (As before, $(f_0, f) = f_{0\Phi}, f_\Phi)$ if $(t, x) \in \Phi$.) There is given a subset Q of Q^0, and we shall seek optimality among strongly admissible solutions belonging to Q.Assume that $(t, x(t; t_0, x^0)) \in Q \subset Q^0$ for all $t \in (t_0, t_1)$, (this is the candidate for which optimality will be claimed). The solutions $x(t; s, y)$ are called *characteristic solutions* (sometimes the name *extremals* are used.)

Define $x_0(t; s, y)$ by

$$\dot{x}_0(t; s, y) = f_0(t, x(t; s, y), u(t; s, y)), \; x_0(s; s, y) = 0. \tag{3.10}$$

The function $x_0(t; s, y)$ exists on $[s, t_1]$ for $(s, y) \in \{(0, x^0)\} \cup Q^0$.

Theorem 5. (Sufficient condition involving characteristic solutions) Assume that $W(s, y) := x_0(t_1; s, y) + h(x(t_1; s, y))$ is locally Lipschitz continuous on Q^0 (in the jump case only on $Q^0 \backslash \{(s, y) \in Q^0 : \phi_j(s, y) = 0$ for some $j\}$, in this case, we assume that $W(s, y) = W(s, y + g^j)$ when $(s, y) \in \{(s, y) \in Q : \phi_j(s, y) = 0\}$). Assume that $W(, x(t))$ is continuous on $[t_0, t_1)$ for all strongly admissible $x(.)$ contained in Q.Let Q^* be an open set in $\mathbb{R}^{1+2n}$, and assume that $(t, x(t; s, y), p(t; s, y)) \in Q^*$ for all $t \in (t_0, t_1)$, all $(s, y) \in Q^0$. Assume that there exist C^1 -functions $\phi^k(t, x, p)$, $k = 1, ..., k^*$ on an open set containing clQ^*, such that, for any point (t, x, p) in clQ^*, $\phi^k(t, x, p) = 0$ for at most one k, such that $\hat{H}(t, x, p) := \max_{u \in U}[f_0(t, x, u) + pf(t, x, u)]$, as well as it first and second derivatives with respect to x and p are C^0 in $Q^* \setminus Z, Z := \{(t, x, p) \in Q^* : \phi^k(t, x, p) = 0$ for some $k\}$. (The maximum is assumed to exist for all (t, x, p) in Q^*). We assume that the functions $\phi_i(t, x)$ are included in the set of functions $\phi^k(t, x, p)$ and that, for any $(t, x, p) \in Q^*$, $\phi^k(t, x, p) = 0$ for at most one k. Assume also that $\hat{H}_x(t, x, p)|_A$ and $\hat{H}_p(t, x, p)|_A$, ($|_A$ means restricted to A), have C^1- extensions to an open set containing cl$A$ for any set A of the form $\cap_i \Phi^i$, $\Phi^i = \{(t, x, p) \in Q^* : \phi^i(t, x, p) > 0\}$, or $\Phi^i = \{(t, x, p) \in Q^* : \phi^i(t, x, p) < 0\}$ (the direction of the inequality sign may depend on i).

Let

$$\hat{Z} := \{(s, y) \in Q^0 : \phi^i(s, y, p(s; s, y)) = 0 \text{ for some } i\}. \tag{3.11}$$

Assume that $\hat{Z} \subset \check{Z} := \{(t,x) : \psi^i(t,x) = 0 \text{ for some } i\}$ for some given set of ψ^i -functions with properties as in the preceding theorem that includes the ϕ_j- functions, (in the jump case $\phi_j(s,y) = 0 \Rightarrow \psi^i(s,y) \neq 0$ for all other functions ψ^i). Define $Q_{s,y} := \{t \in (s,t_1) : \phi^k(t,x(t;s,y),p(t;s,y)) = 0$ for some $k\}$. Assume for any $(s,y) \in Q^0 \setminus \check{Z}$, $t \in (s,t_1)$, $t \notin Q_{s,y}$, that $(s',y') \to (x_0(t;s',y'),x(t;s',y'),p(t;s',y'))$ is C^1 in a neighborhood of (s,y). Moreover, for $(s,y) \in Q^0 \setminus \check{Z}, t \in (s,t_1)$, assume that

$$\begin{aligned} &\phi^k(t,x(t;s,y),p(t;s,y)) = 0 \Rightarrow \\ &0 < (d^{\pm}/dt)\phi^k(t,x(t;s,y),p(t;s,y)) \\ \text{or} \quad &0 > (d^{\pm}/dt)\phi^k(t,x(t;s,y),p(t;s,y)) \end{aligned} \tag{3.12}$$

(i.e. both the right derivative and the left derivative $(d^+/dt)\phi^k$ and $(d^-/dt)\phi^k$ are positive or both are negative, $((d^{\pm}/dt)\phi^k = \phi_t^k(t,x(t;s,y),p(t;s,y)) + \phi_x^k(t,x(t;s,y),p(t;s,y))\dot{x}(t^{\pm};s,y) + \phi_p^k(t,x(t;s,y),p(t;s,y))\dot{p}(t^{\pm};s,y))$. In the jump case, when $\phi^k = \phi_k$, (3.12) need only hold for the left limit. We assume also that the nontangent condition (3.12) holds at $t = t_1$, for $t^{\pm}$ replaced by t_1- . Assume finally that $\lim_{t\to t_1}[x_0(t_1;t,\hat{x}(t)) + h(x(t_1;t,\hat{x}(t)))] = h(\hat{x}(t_1))$ for all strongly admissible $\hat{x}(.)$ for which $(t,\hat{x}(t)) \in Q^0$, $t \in (t_0,t_1)$. Then $(x(t;t_0,x^0),u(t;t_0,x^0))$ is optimal among all pairs $(\hat{x}(t),\hat{u}(t))$ for which $\hat{x}(.)$ is strongly admissible and $(t,\hat{x}(t)) \in Q^0$, $t \in (t_0,t_1)$. □

Usually, the triples $x(t;s,y), p(t,s,y), u(t;s,y)$ are found by first finding a control $\hat{u}(t,x,p)$ maximizing $H(t,x,u,p)$, $(p_0 = 1)$. Next, one solves the equations

$$\dot{x} = f(t,x,\hat{u}(t,x,p)), \quad \dot{p} = -H_x(t,x,\hat{u}(t,x,p),p)$$

together with (2.1), (2.2), (2.3), with initial condition $x(s+) = y$, and terminal conditions (1.3)-(1.5), (3.8), (3.9), and we then let $u(t;s,y) = \tilde{u}(t,x(t,s,y),p(t,s,y))$. The two differential equations are one variant of what is called the characteristic equations of the HJB equation, and we call $x(t;s,y), p(t;s,y), u(t;s,y)$ a characteristic triple.

Proof. It suffices to prove the above theorem for the particular case where the criterion is $ax(t_1)$. We shall refer to the proof of Remark 2.23 in Seierstad (2008). Write $v = (s,y)$. We want to prove that $p(\tau+)x_v(\tau+,\hat{v}) = p(\tau-)x_v(\tau-,\hat{v})$ at any point $\tau \in (\hat{s},t_1)$ at which $\phi_j(\tau,x(\tau-;\hat{v})) = 0$ for some j, where $\hat{v} = (\hat{s},\hat{y})$ is any given point in $Q^0 \backslash \check{Z}$. (Of course, $x(\tau-;\hat{v}) = x(\tau;\hat{v})$, still we continue writing

$x(\tau-;\hat{v})$.) Note that $p(.)$ may be discontinuous at τ, so the just mentioned proof does not work for such crossing points. Below, a dot, $(\cdot)$, means scalar product.

For the function $\hat{T}(v)$ that satisfies $\phi_j(\hat{T}(v), x(\hat{T}(v)-, v)) = 0$ for v close to $\hat{v}$, $\hat{T}(\hat{v}) = \tau$, when calculating derivatives with respect to v_i, at $v = \hat{v}$ we get

$$\phi_{jt}(\tau, x(\tau-, \hat{v}))\hat{T}'_{v_i} + \phi_{jx}(\tau, x(\tau-, \hat{v})) \cdot (\dot{x}(\tau-, \hat{v})\hat{T}'_{v_i} + x_{v_i}(\tau-, \hat{v})) = 0. \quad (3.13)$$

Moreover,

$$x_{v_i}(\tau+, \hat{v}) = -\dot{x}(\tau+, \hat{v})\hat{T}'_{v_i} + \dot{x}(\tau-, \hat{v})\hat{T}'_{v_i} + x_{v_i}(\tau-, \hat{v}), \quad (3.14)$$

If $\phi_{jx}(\tau, x(\tau-;\hat{v})) = 0$, then $\mu = 0$ by (2.3), p is continuous at τ by (2.2), and as then $\phi_{jt}(t, x(\tau-;\hat{v})) \neq 0$ by (NT) (or NT*)), (3.13) implies that $\hat{T}'_{v_i} = 0$, and hence by (3.14) that $x_{v_i}(., \hat{v})$ is continuous at τ, so $p(\tau+)x_v(\tau+, \hat{v}) = p(\tau-)x_v(\tau-, \hat{v})$. So assume $\phi_{jx}(\tau, x(\tau-;\hat{v})) \neq 0$ ($\Rightarrow \mu \neq 0$). Combining (2.2) and (2.3), we get

$$p(\tau-) - p(\tau+) = \alpha\phi_{jx}(\tau, x(\tau-, \hat{v})), \text{ for some number } \alpha \quad (3.15)$$

(recall that (NT), respectively (NT*), hold). For simplicity, write $\hat{T}_{v_i} = T'$, $\phi_{jx} = \phi_{jx}(\tau, x(\tau-, \hat{v}))$, $\phi_{jt} = \phi_{jt}(\tau, x(\tau-, \hat{v}))$. Now,

$$\begin{aligned}
&[p(\tau+) \cdot x_{v_i}(\tau+, \hat{v}) - p(\tau-) \cdot x_{v_i}(\tau-, \hat{v})]\mu \\
&= [p(\tau+) \cdot (-\dot{x}(\tau+, \hat{v})T' + \dot{x}(\tau-, \hat{v})T')]\mu + \\
&(p(\tau+) \cdot x_{v_i}(\tau-, \hat{v}))\mu - (p(\tau-) \cdot x_{v_i}(\tau-, \hat{v}))\mu \\
&= (p(\tau-) - p(\tau+))T' + (p(\tau+) \cdot x_{v_i}(\tau-, \hat{v}))\mu - (p(\tau-) \cdot x_{v_i}(\tau-, \hat{v}))\mu \\
&= (p(\tau-) - p(\tau+))(T' - \{(p(\tau-) - p(\tau+)) \cdot x_v(\tau-, \hat{v})\}\mu \\
&= \alpha[\phi_{jx}T' - \{\phi_{jx} \cdot x_v(\tau-, \hat{v})\}\mu] \\
&= \alpha[\phi_{jx} + \{\phi_{jt} + \phi_{jx} \cdot \dot{x}(\tau-, \hat{v})\}\mu]T' \\
&= 0,
\end{aligned}$$

the various equalities by (3.14), (2.2), rearrangement, (3.15), (3.13) and (2.3).

Since $\mu \neq 0$, $[p(\tau+) \cdot x_{v_i}(\tau+, \hat{v}) - p(\tau-) \cdot x_{v_i}(\tau-, \hat{v})] = 0$.

In the proof of Remark 2.23 in Seierstad (2007), this property was show to hold also for points τ at which $\phi^i(\tau, x(\tau, \hat{v}), p(\tau-, \hat{v})) = 0$, for functions $\phi^i \notin \{\phi_i\}_i$. In fact, the remaining proof of Remark 2.23 in Seierstad (2007) can be kept unchanged and yields optimality of $(x^*(t), u^*(t))$. □

Example 2 (Oil production)
An oil field consists of two reservoirs, reservoir I and reservoir II, containing K_1 and K_2 barrels of oil, respectively. The instantaneous profit rate equals $qu - au^2$, where q is the given constant oil price, u is the rate of oil production (the control variable) and au^2 is the cost of production. The parameter a takes two values 1 and 2, 1 for reservoir I, and 2 for reservoir II. The oil in reservoir I will be produced first. The number $x(t)$ denotes the amount of oil remaining in the field at time t, and $x(0) = K = K_1 + K_2$. The rate of change of the oil volume is $\dot{x} = -u$.

The cost function is as follows:

$$C(t, x, u) = au^2 = \begin{cases} u^2 \text{ if } x(t) \geq K_2 \\ 2u^2 \text{ if } x(t) < K_2 \end{cases}$$

There is a given horizon T, and the oil field owner wants to solve the following problem.

$$\max \int_0^T qu - au^2 dt$$

$$\dot{x} = -u, x(0) = K, x(T) \geq 0, u \geq 0, a = \begin{cases} 1, x(t) \geq K_2 \\ 2, x(t) < K_2 \end{cases}, \ \phi(t, x) := x - K_2,$$

where q, K_1, K_2 and T are given constants.

Solution
We will apply Theorem 3. The Hamilton function is $H = qu - au^2 - pu$, and a solution $u(t) = u^*(t) > 0$ must satisfy $H_u = 0$ and

$$H_u = 0 \Rightarrow q - 2au - p = 0 \Rightarrow u = \frac{q - p}{2a}$$

Moreover, $\dot{p} = 0$, and the transversality conditions are $p(T) \geq 0$, $(p(T) = 0$ if $x(T) > 0)$. Finally, $\dot{x} = \frac{p-q}{2a}$.

Four different candidates (i.e. solutions of necessary conditions) will be described. A possible candidate is one where only reservoir I is exploited, (case 1. and 2. below). This is considered first.

1. The case $x(T) > K_2$ gives $p(t) \equiv 0$, $\dot{x} = -q/2$, $x(t) = -\frac{q}{2}t + K, t \in [0, T]$, which works only if $x(T) > K_2$, i.e. in the case $T < 2\frac{K_1}{q}$ this is a candidate for optimality.

2. The case $x(T) = K_2$. For this case Theorem 3 does not give any necessary conditions (and we disregard Remark 3), so we are without necessary conditions. Necessary conditions for continuous problems can, however, be put to work by replacing the free end assumption by the requirement $x(T) \geq K_2$, in which case we know that $x(t) \geq K_2$ for all t, and $p(T) \geq 0$. We now look only at the case $x(T) = K_2$, (the case $x(T) > K_2$ was treated above.) This gives

$$\begin{aligned}
&\dot{p} = 0 \Rightarrow p(t) = C \text{ for some integration constant } C \\
&\Rightarrow u(t) = \tfrac{q-C}{2} \Rightarrow \dot{x} = \tfrac{C-q}{2} \Rightarrow x(t) = \tfrac{C-q}{2}t + D, x(0) = K \Rightarrow x(t) = \tfrac{C-q}{2}t + K \\
&x(T) = K_2 \Rightarrow \tfrac{C-q}{2}T + K = K_2 \Rightarrow C = q - 2\tfrac{K_1}{T} \\
&\Rightarrow x(t) = K - \tfrac{K_1}{T}t,\ u(t) = \tfrac{K_1}{T}, p(t) = q - 2\tfrac{K_1}{T}.
\end{aligned}$$

Now, $p(T) \geq 0 \Rightarrow q - 2\frac{K_1}{T} \geq 0 \Rightarrow T \geq 2\frac{K_1}{q}$. In the case $T \geq 2\frac{K_1}{q}$, where $x(T) = K_2$, we get the candidate

$$x(t) = K - \frac{K_1}{T}t, u(t) = \frac{K_1}{T},$$

with $p(t) = q - 2K_1/T$. (Formally, we should at this point have considered also the case $p_0 = 0$, which however does not given any candidate.)

It may be that the owner exploits both reservoirs. In that case, how do the candidates look like? If $x(.) = x^*(.)$ reaches K_2 at τ before T, then an optimal behavior between 0 and the fixed τ, with $x(0) = K$ and $x(\tau) = K_2$ fixed, implies $\dot{u} =$ constant > 0 in $(0, \tau)$, compare the second case above, hence at least one of the inequalities of (NT*) is satisfied at τ.

3. Again, let us first consider the possibility $x(T) > 0$, $p(T) = 0$. We then have

$$\dot{p} = 0 \Rightarrow \begin{cases} p(t) = C, t \in [0, \tau) \\ p(t) = 0, t \in (\tau, T] \end{cases}$$

$$\Rightarrow u = \begin{cases} \frac{q-C}{2}, t \in [0,\tau) \\ \frac{q}{4}, t \in (\tau, T] \end{cases} \Rightarrow \dot{x} = \begin{cases} \frac{C-q}{2}, t \in [0,\tau) \\ -\frac{q}{4}, t \in (\tau, T] \end{cases} \Rightarrow x(t) = \begin{cases} \frac{C-q}{2}t + D, t \in [0,\tau) \\ -\frac{q}{4}t + E, t \in (\tau, T] \end{cases}$$

Moreover,

$$x(0) = K \Rightarrow D = K \Rightarrow x(t) = \frac{C-q}{2}t + K, t \in [0,\tau)$$

$$\begin{aligned} &x(\tau_-) = K_2 \Rightarrow \tfrac{C-q}{2}\tau + K = K_2 \Rightarrow (C-q)\tau = -2K_1 \\ &\Rightarrow C = q - \tfrac{2K_1}{\tau} \Rightarrow x(t) = K - \tfrac{K_1}{\tau}t, t \in [0,\tau) \end{aligned}$$

This gives

$$\begin{aligned} &a = \begin{cases} 1, t \in [0,\tau) \\ 2, t \in (\tau, T] \end{cases}, \ u = \begin{cases} \frac{K_1}{\tau}, t \in [0,\tau) \\ \frac{q}{4}, t \in (\tau, T] \end{cases} \\ &x(t) = \begin{cases} K - \frac{K_1}{\tau}t, t \in [0,\tau) \\ \frac{q}{4}(\tau - t) + K_2, t \in (\tau, T] \end{cases}, \ p(t) = \begin{cases} q - \frac{2K_1}{\tau}, t \in [0,\tau) \\ 0, t \in (\tau, T] \end{cases} \end{aligned}$$

To find μ, (2.3) is used, ($\phi = \phi_1 = x - K_2$):

$$\left[0 + 1(-\tfrac{K_1}{\tau})\right]\mu + 1 = 0 \Rightarrow \mu = \tfrac{\tau}{K_1}$$

Then the jump condition (2.2) on $p(t)$ yields

$$\begin{aligned} &(q - 2\tfrac{K_1}{\tau}) - 0 = \left[q\tfrac{K_1}{\tau} - \tfrac{K_1^2}{\tau^2} - q\tfrac{q}{4} + 2\tfrac{q^2}{16}\right]\tfrac{\tau}{K_1} + 0 \\ &\Rightarrow q - 2\tfrac{K_1}{\tau} = q - \tfrac{K_1}{\tau} - \tfrac{q^2\tau}{4K_1} + \tfrac{q^2\tau}{8K_1} \Rightarrow -\tfrac{K_1}{\tau} = -\tfrac{q^2\tau}{8K_1} \Rightarrow 8K_1^2 = q^2\tau^2 \\ &\Rightarrow \tau^2 = \tfrac{8K_1^2}{q^2} \Rightarrow \tau = 2\sqrt{2}\tfrac{K_1}{q}. \end{aligned}$$

So

$$\begin{aligned} &a = \begin{cases} 1, t \in [0,\tau) \\ 2, t \in (\tau, T] \end{cases}, \ u = \begin{cases} q/2\sqrt{2}, t \in [0,\tau) \\ \frac{q}{4}, t \in (\tau, T] \end{cases} \\ &x(t) = \begin{cases} K - (q/2\sqrt{2})t, t \in [0,\tau) \\ \frac{q}{4}(\tau - t) + K_2, t \in (\tau, T] \end{cases}, \ p(t) = \begin{cases} q - q/\sqrt{2}, t \in [0,\tau) \\ 0, t \in (\tau, T] \end{cases} \end{aligned}$$

In case T belongs to $(\frac{2\sqrt{2}K_1}{q}, (1/q)[4K_2 + 2\sqrt{2}K_1])$, for $\tau = \frac{2\sqrt{2}K_1}{q}$, the above solution is a candidate, $(x(T) > 0 \Rightarrow 0 < \frac{q}{4}(2\sqrt{2}\frac{K_1}{q} - T) + K_2$, i.e. $T <$

$(1/q)[4K_2 + 2\sqrt{2}K_1])$.

4. Let us now consider the alternative $x(T) = 0$ and $p(T) \geq 0$. We have that

$$\dot{p} = 0 \Rightarrow \begin{cases} p(t) = C_1, t \in [0,\tau] \\ p(t) = C_2, t \in (\tau, T] \end{cases} \Rightarrow u = \begin{cases} \frac{q-C_1}{2}, t \in [0,\tau] \\ \frac{q-C_2}{4}, t \in (\tau, T] \end{cases}$$

$$\Rightarrow \dot{x} = \begin{cases} \frac{C_1-q}{2}, t \in [0,\tau] \\ \frac{C_2-q}{4}, t \in (\tau, T] \end{cases} \Rightarrow x(t) = \begin{cases} \frac{C_1-q}{2}t + D, t \in [0,\tau] \\ \frac{C_2-q}{4}t + E, t \in (\tau, T] \end{cases}$$

Using boundary conditions gives:

$$x(0) = K \Rightarrow D = K \Rightarrow x(t) = \frac{C_1 - q}{2}t + K, t \in [0,\tau)\,,\ x(T) = 0 \Rightarrow$$
$$\frac{C_2 - q}{4}T + E = 0 \Rightarrow E = -\frac{C_2 - q}{4}T \Rightarrow x(t) = \frac{C_2 - q}{4}(t - T), t \in (\tau, T]\,.$$

Furthermore,

$$\begin{aligned} & x(\tau_-) = x(\tau_+) = K_2 \\ & x(\tau_-) = K_2 \Rightarrow \tfrac{C_1-q}{2}\tau + K = K_2 \Rightarrow (C_1 - q)\tau = -2K_1 \\ & \Rightarrow C_1 = q - \tfrac{2K_1}{\tau} \Rightarrow x(t) = K - \tfrac{K_1}{\tau}t, t \in [0,\tau)\,, \end{aligned}$$

and

$$\begin{aligned} & x(\tau_+) = K_2 \Rightarrow \tfrac{C_2-q}{4}(\tau - T) = K_2 \Rightarrow (C_2 - q)(\tau - T) = 4K_2 \Rightarrow C_2 = q + \tfrac{4K_2}{\tau - T} \\ & \Rightarrow x(t) = \tfrac{K_2(t-T)}{\tau - T}, t \in (\tau, T]\,. \end{aligned}$$

Insertion of $C_1$and C_2 in u and p leads to

$$a = \begin{cases} 1, t \in [0,\tau] \\ 2, t \in (\tau, T] \end{cases},\ u = \begin{cases} \frac{K_1}{\tau}, t \in [0,\tau] \\ \frac{K_2}{T-\tau}, t \in (\tau, T] \end{cases}$$
$$x(t) = \begin{cases} K_1 - \frac{K_1}{\tau}t, t \in [0,\tau] \\ \frac{K_2(t-T)}{\tau-T}, t \in (\tau, T] \end{cases},\ p(t) = \begin{cases} q - \frac{2K_1}{\tau}, t \in [0,\tau] \\ q + \frac{4K_2}{\tau-T}, t \in (\tau, T] \end{cases}$$

Again, (2.3) gives $\mu = \tau/K_1$, and the jump condition (2.2) on $p(.)$ then gives

$$\begin{aligned} & = (q - \tfrac{2K_1}{\tau}) - (q + \tfrac{4K_2}{\tau - T}) = \left[q\tfrac{K_1}{\tau} - \tfrac{K_1^2}{\tau^2} - q\tfrac{K_2}{T-\tau} + 2\tfrac{K_2^2}{(T-\tau)^2}\right]\tfrac{\tau}{K_1} + \\ & \left[(q + \tfrac{4K_2}{\tau-T})(-\tfrac{K_1}{\tau}) - (q + \tfrac{4K_2}{\tau-T})(-\tfrac{K_2}{T-\tau})\right]\tfrac{\tau}{K_1} \\ & \Rightarrow -\tfrac{2K_1}{\tau} + \tfrac{4K_2}{T-\tau} = q - \tfrac{K_1}{\tau} - \tfrac{q\tau K_2}{(T-\tau)K_1} + \tfrac{2K_2^2\tau}{(T-\tau)^2K_1} - q + \tfrac{4K_2}{T-\tau} + \tfrac{q\tau K_2}{(T-\tau)K_1} - \tfrac{4K_2^2\tau}{(T-\tau)^2K_1} \\ & \Rightarrow -2K_1(T-\tau)^2 = -K_1(T-\tau)^2 - 2K_2^2\tau^2/K_1 \\ & \Rightarrow -K_1T^2 + 2K_1T\tau - (K_1 - 2K_2^2/K_1)\tau^2 = 0 \\ & \Rightarrow -T^2 + 2T\tau - (1 - 2\alpha^2)\tau^2 = 0, \text{ where } \alpha = K_2/K_1. \end{aligned}$$

Hence, $\tau = T\{-2 + [2^2 - 4(1-2\alpha^2)]^{1/2}\}/2(2\alpha^2-1) = T(\sqrt{2}\alpha - 1))/(2\alpha^2-1) = T/(\sqrt{2}\alpha+1)$.

This works only if $p(T) = q + 4K_2/(\tau - T) \geq 0$, i.e. $T \geq \tau + 4K_2/q = T/(\sqrt{2}\alpha + 1) + 4K_2/q$, or $T \geq (4K_2/q)[1-(\sqrt{2}\alpha+1)^{-1}]^{-1} = (4K_2/q)\{\sqrt{2}\alpha/(\sqrt{2}\alpha+1)\}^{-1} = (4K_2/q)[1+\sqrt{2}/2\alpha] = (1/q)[4K_2+2\sqrt{2}K_1]$. Thus in case $T \geq (1/q)[4K_2+2\sqrt{2}K_1]$, the above solution is a candidate in the problem. Here $u = K_1/\tau = (K_1/T)(\sqrt{2}\alpha+1)$ for $t < \tau = T/(\sqrt{2}\alpha+1)$ and $u = K_2/(T-\tau)$ for $t > \tau$.

The criterion values for the four alternative candidates are as follows:

(1)

$$V^1 = \int_0^T qu - u^2 dt = \int_0^T \frac{q^2}{2} - \frac{q^2}{4} dt = \int_0^T \frac{q^2}{4} dt = \frac{q^2}{4} T$$

(2)

$$V^2 = \int_0^T qu - u^2 dt = \int_0^T q\frac{K_1}{T} - \frac{K_1^2}{T^2} dt = K_1 \left(q - \frac{K_1}{T}\right)$$

(3)

$$\begin{aligned} V^3 &= \int_0^T qu - au^2 dt = \int_0^{\frac{2\sqrt{2}K_1}{q}} q\frac{q}{2\sqrt{2}} - \frac{q^2}{8} dt + \int_{\frac{2\sqrt{2}K_1}{q}}^T q\frac{q}{4} - 2\frac{q^2}{16} dt \\ &= (q\frac{q}{2\sqrt{2}} - \frac{q^2}{8})\frac{2\sqrt{2}K_1}{q} + \frac{q^2}{8}(T - \frac{2\sqrt{2}K_1}{q}) = q^2T/8 + qK_1 - q^2\sqrt{2}K_1/2. \end{aligned}$$

(4)

$$\begin{aligned} V^4 &= \int_0^T qu - au^2 dt = \int_0^\tau qK_1/\tau - K_1^2/\tau^2 dt + \int_\tau^T qK_2/(T-\tau) - 2K_2^2/(T-\tau)^2 dt \\ &= qK_1 - K_1^2/\tau + qK_2 - 2K_2^2/(T-\tau) \\ &= qK_1 - (\sqrt{2}\alpha+1)K_1^2/T + qK_2 - 2K_2^2(1+1/\sqrt{2}\alpha)/T. \end{aligned}$$

For T in $(\frac{2\sqrt{2}K_1}{q}, \frac{4K_2+2\sqrt{2}K_1}{q})$, we have obtained two candidates, no. 2 and no. 3, and for $T = (\frac{4K_2+2\sqrt{2}K_1}{q}, \infty)$, we have obtained two candidates, no. 2 and no. 4. Comparing criterion values, we get that no. 3 and no. 4 are the best ones, respectively. (To compare no. 2 and no. 3, note that $V^2(T) = V^3(T)$ for $T = \frac{2\sqrt{2}K_1}{q}$ as the solutions are the same, and $dV^3(T)/dT > dV^2/dT$ for $T \in [\frac{2\sqrt{2}K_1}{q}, \frac{(4K_2+2\sqrt{2})K_1}{q})$. Similarly, as $\sqrt{2}\alpha + 1 > 1$, then $dV^4(T)/dT > dV^2/dT$ for $T > \frac{(4K_2+2\sqrt{2})K}{q})$ and $V^4(\frac{(4K_2+2\sqrt{2})K_1}{q}) = V^3(\frac{(4K_2+2\sqrt{2})K_1}{q}) > V^2(\frac{(4K_2+2\sqrt{2})K_1}{q})$, the last inequality we have already proved).

To sum up, if $T < 2K_1/q$, candidate no. 1 is optimal, if $T \in (2K_1/q, \frac{2\sqrt{2}K_1}{q}]$ candidate no. 2 is optimal, if T in $(\frac{2\sqrt{2}K_1}{q}, \frac{(4K_2+2\sqrt{2})K_1}{q}]$, candidate no. 3 is optimal, and if $T > \frac{(4K_2+2\sqrt{2})K_1}{q}$ candidate no. 4 is optimal. In fact optimality is only known by carrying out the arguments as below.

Optimality can be proved by means of Theorem 5 and Remark 6. Let us only consider the case of a fixed $T > \frac{(4K_2+2\sqrt{2})K_1}{q}$. Write $x_*(t;T,K,K_2)$, $p(t;T,K,K_2)$, $K = K_1 + K_2$ for the above proposal. Write also $V^i = V^i(T,K,K_2)$, $i = 1, ..., 4$. When $t = 0$ is replaced by s and K is replaced by $y > K_2$, then define $x(t;s,y) = x_*(t-s;T-s,y,K_2)$, $p(t;s,y) = p(t;T-s,y,K_2)$, while for $y \leq K_2$, we define $x(t;s,y) = x_*(t-s;T-s,K_2,y), p(t;s,y) = p(t-s;T-s,K_2,y)$, $(K = K_2 \Rightarrow K_1 = 0, \tau = 0)$. Let us write out in detail the function $x(t;s,y)$.

(a) Consider first the case $y > K_2$. Then if $T-s \leq 2\frac{y-K_2}{q}$, $x(t;s,y) = y - q(t-s)/2$, $t \in [s,T]$. If $2\sqrt{2}(y-K_2)/q \geq T-s > 2\frac{y-K_2}{q}$, $x(t;s,y) = y - \frac{y-K_2}{T-s}(t-s)$, $t \in [s,T]$. If $2\sqrt{2}(y-K_2)/q < T-s \leq (1/q)[4K_2 + 2\sqrt{2}(y-K_2)]$, $x(t;s,y) = y - \frac{q}{2\sqrt{2}}(t-s)$ for $t-s \in [0, 2\sqrt{2}(y-K_2)/q]$ (i.e. for $x(t;s,y) \geq K_2$) and $x(t,s,y) = \frac{q}{4}(2\sqrt{2}(y-K_2)/q - t + s) + K_2$ for $t-s \in \big(2\sqrt{2}(y-K_2)/q, T\big]$ (i.e. for $x(t;s,y) < K_2$). Finally, if $T-s > (1/q)[4K_2 + 2\sqrt{2}(y-K_2)]$, $x(t;s,y) = y - K_2 - \frac{y-K_2}{(T-s)/(\sqrt{2}\alpha+1)}(t-s)$, $t-s \in [0,(T-s)/(\sqrt{2}\alpha+1)]$, $\alpha = K_2/(y-K_2)$ (i.e. for $x(t;s,y) > K_2$), $x(t;s,y) = \frac{K_2(t-T)}{(T-s)/(\sqrt{2}\alpha+1)-(T-s)}$ for $t-s \in ((T-s)/(\sqrt{2}\alpha+1), T]$ (i.e. for $x(t;s,y) < K_2$).

Consider next the case (b) $y \leq K_2$. Then if $T-s \leq 4y/q, x(t;s,y) = y - q(t-s)/4$, and if $T-s > 4y/q$, $x(t;s,y) = y(T-t)/(T-s)$.

We drop writing out the formulas for $p(t;s,y)$ in the corresponding cases. Define $Q^0 = (0,T) \times (0,2K)$, $Q^* = \mathbb{R}^3$, $\phi = K_2 - y = \psi^1(s,y)$, $\psi^2(s,y) = T - s - (1/q)[4K_2 + 2\sqrt{2}(y - K_2)]$, $\psi^3(s,y) = T - s - 2(y - K_2)/q$, $\psi^4(s,y) = T - s - 2\sqrt{2}(y - K_2)/q$, $\psi^5(s,y) = T - s - 4y/q$. Note that the nontangent condition (3.12) holds. Moreover, $x(t;s,y)$ is continuously differentiable at all (s,y), $t \notin \{t : x(t;s,y) = K_2\}$, for which $\psi^i(s,y) \neq 0$, $i > 1$, $y \neq K_2$, and the same holds for $p(t;s,y)$. Moreover, in this example the $x_0(t;s,y)$-function has the same differentiability properties. In case $y \leq K_2$, $x_0(T;s,y) = q^2(T-s)/8$ for $T - s \leq 4y/q$, $x_0(T;s,y) = qy - 2y^2/(T-s)$ for $T - s > 4y/q > 0$. For $y > K_2$, $x_0(T;s,y) = V^1(T-s,y,K_2)$ for $T - s \leq 2(y - K_2)/q$, $x_0(T;s,y) = V^2(T-s,y,K_2)$ for $2\sqrt{2}(y - K_2)/q \geq T - s > 2(y - K_2)/q, x_0(T;s,y) = V^3(T-s,y,K_2)$ for $2\sqrt{2}(y - K_2)/q < T - s \leq (4K_2 + 2\sqrt{2}(y - K_2))/q$, and $x_0(T;s,y) = V^4(T-s,y,K_2)$ for $(4K_2 + 2\sqrt{2}(y - K_2))/q < T - s$.

Formally, Theorem 5, with Remark 6, gives optimality only among strongly admissible pairs, but in this example admissible pairs can be approximated by strongly admissible pairs, so optimality holds among all admissible pairs.

Example 3 (Growth)
A farmer grows only oat and his production of oat is proportionate to his cultivable land. This land he all the time extends by the use of a horse to clear new land. The farmer first clears land on a terrain well suited for cultivation, the size of which is x_1. Afterwords he has to turn to terrain that is more difficult to clear, but which is of very large ("unlimited") size. Per unit of time, on the best terrain, the horse clears two acres per unit of oat given to it as fodder, on the difficult land it only clears 1 acre per unit of oat. The amount of oat given per unit of time to the horse is $ux(t)$, and $x(t)$ is the present production, which equals the present size of the cultivable land, (for simplicity, the proportionality factor has been put equal to 1).

The farmer wants to maximize income from oat production, more precisely he sells $(1-u)x(t)$, for a price equal to 1, so he wants to solve the problem :

$$\max \int_0^T (1-u)x\,dt$$

$$\dot{x} = uax, x(0) = x_0 > 0, x(T) \text{ free}, u \in [0,1], a = \begin{cases} 1 \text{ if } x > x_1 \\ 2 \text{ if } x \leq x_1 \end{cases}, \phi(t,x) = x - x_1$$

The farmer starts with a initial size of land equal to x_0, situated at the best terrain. We assume $x_1/x_0 = e$ and $T > 3/2$.

Solution
The Hamiltonian is

$$H = p_0(1-u)x + puax$$

Since $x(T)$ is free, $p_0 = 1$, and the maximizing value of u satisfies

$$u = \begin{cases} 1 \text{ if } p(t) > 1/a \\ 0 \text{ if } p(t) < 1/a \end{cases}$$

$$\dot{p} = -H'_x = -1 + u - pua = -\max\{1, pa\}$$

Let $(x(t), u(t)) = (x^*(t), u^*(t))$ be an optimal solution and let $\tau = \min\{t : x(t) = x_1\}$ (defined if the last set is nonempty).

A. We first look at the possibility that $x(T) > x_1$, in which case $x(\tau) = x_1$ for some $\tau \in (0, T)$. The condition (NT*) will surely be satisfied if $u(\tau\pm) > 0$, which we here assume. Now, $p(t)$ is strictly decreasing on all intervals at which it is continuous. If $p(t') \leq 1/2$ for a $t' < \tau$, this would lead to $p(t) < 1/2, u(t) = 0$ and $x(t) = x(t') > x_1$ on (t', τ), a contradiction. So $p(t') > 1/2$ for $t' < \tau$, and $\dot{x} = 2x$ here. Hence, $x(t) = x_0e^{2t}$ in $[0, \tau)$. This gives $\tau = 1/2$ by using $x_0e^{2\tau} = x_1$ and $x_1/x_0 = e$. Next, close to T, $p(t)$ is close to zero and $u = 0$ is used, let $[t^*, T]$ be the maximal interval on which $u(t)$ is identically zero. Then $\dot{p} = -1$, so $p(t) = T - t$ here, and $p(t) < 1$, consistent with $u(t) = 0$ on $[t^*, T]$ provided t^* is determined by $T - t^* = 1$, i.e. $t^* = T - 1 > 1/2$. Thus, for $t \in (\tau, t^*)$, $p(t) > 1$ and $u(t) = 1$ here. Furthermore, $\mu = -1/2u(\tau-)x_1 = -1/2x_1$, and $p(\tau-) - p(\tau+) = p(\tau+)[2u(\tau-) - u(\tau+)]\mu = -p(\tau+)[1 - \frac{1}{2}u(\tau+)/u(\tau-)] = -\frac{1}{2}p(\tau+)$. Then as we shall see, for $t^* = T - 1$, $\tau = 1/2$, we have got the following entities satisfying all necessary conditions in Theorem 3:

$$u(t) = \begin{cases} 1, t \in [0, t^*) \\ 0, t \in t^*, T] \end{cases},$$

$$x(t) = \begin{cases} x_0e^{2t}, t \in [0, \tau) \\ x_1e^{t-\tau}, t \in [\tau, t^*) \\ x_1e^{t^*-\tau}, t \in (t^*, T] \end{cases}, \qquad p(t) = \begin{cases} \frac{1}{2}e^{\tau+t^*-2t} > \frac{1}{2}, t \in [0, \tau) \\ e^{-(t-t^*)} > 1, t \in [\tau, t^*) \\ T - t < 1, t \in (t^*, T] \end{cases},$$

To find these solutions, note that $x(\tau) = x_1$ and $x(t^*) = x_1 e^{t^* - \tau}$ were used to determine integration constants for the solution $x(t)$ on $[\tau, t^*)$ and $[t^*, T]$, respectively. Moreover, $\dot{p} = -p$, $\dot{p} = -2p$, together with $p(t^*) = 1$ and $p(\tau-) = (1/2)p(\tau+) = (1/2)e^{-(\tau - t^*)}$ (i.e. the jump condition) were used to determine integration constants for the solution $p(t)$ on $[\tau, t^*)$ and $[0, \tau)$, respectively. (We may check that for this solution $u(\tau\pm) > 0$). Can it be that there exist candidates for which $u(\tau\pm) > 0$ fails?

B. Consider the following fixed end problem:

$$\max \int_0^{T'} (1-u)x dt, T' > 0, T' \text{ fixed},$$
$$\dot{x} = 2ux, x(0) = x_0 > 0, x(T') = x_1 = ex_0, u \in [0,1].$$

In this problem $p(t)$ is continuous and strictly decreasing, and for some t^{**}, $p(t^{**}) = 1$, $p(t) > 1$ and $u = 1$ for $t < t^{**}$, $p(t) < 1$ and $u = 0$ for $t > t^{**}$. Then t^{**} is evidently equal to $\tau = 1/2$, ($x(t) = x_0 e^{2t}$ for $t \leq t^{**}$). If $T' = \tau$, $x(t) < x_1$ for all $t < \tau$, hence $u(t) > 0$ at least at certain points $t < \tau$, t close to τ, hence $p(t) \geq 1/2$ at such points. This in fact implies that $p(t) > 1/2$ for all $t < \tau$, so $u(t) = 1$ here, implying $u^*(\tau-) = 1$. If $T' > \tau = 1/2$, then $p(t) < 1/2$ for $t > \tau$, so $u(t) = 0$, here and $x(t) = x_1$, $t \in [\tau, T]$. Next, let us look at the problem where we start at $\tau = 1/2$, with state x_1, and with free end, with $\dot{x} = ux$, and where, in the criterion we integrate over $[\tau, T]$. For this problem, we have already found the solution; $x(t)$ is as in A. on $[\tau, T]$ and $u(\tau+) = 1$. So $u(\tau\pm)$ *is* > 0.(For the last two problems, actually, the standard maximum principle for "continuous" problems have been used.) Note that $p_0 = 0$ does not give any further candidates.

C. The case $x(T) = x_1$ cannot occur in the original problem, because we saw in B. that x_1 is reached already at time $\tau = 1/2$, but the optimal behavior in the free end problem where we start at (τ, x_1) was seen to be not to have $x(t) = x_1$ for all $t \geq \tau$, but to strictly increase $x(t)$. Let us finally, in the original problem, consider the possibility $x(T) < x_1$. This case is close to the case discussed in A, except that we have $\dot{x} = 2ux$ all the time. Again $\dot{p} = -1$ on $(t^*, T]$, $p(t) = T - t$ here, and now t^* is determined by $p(t^*) = 1/2$, i.e. $1/2 = T - t^*$, so $t^* = T - 1/2$, moreover, before t^*, $p(t) > 1/2$, so $u^* = 1$ here and $x(t) = x_0 e^{2t}$ for such t. But $x_0 e^{2t}$ satisfies $x(T - 1/2) > x_1$, a contradiction. The only candidate remaining in the problem is the one found in A.

D. Does an optimal solution exist? Standard existence theorems essentially give existence, because the two right hand sides $2ux$ and ux in the differential equation coincide when $u = 0$. But let us argue a little more carefully. Imagine that $x_n(.), u_n(.)$ is a sequence having criterion values converging to the supremum of the criterion. Let $t_n = \max\{t : x_n(t) \leq x_1\}$, let $t' = \text{limsup}\, t_n$, and, by considering subsequences if necessary, assume $t_n \longrightarrow t'$. By standard existence arguments (see e.g. the arguments in the proof of Theorem 9.2.i, in Cesari (1983)), using compactness of U and convexity of $\{(f_0(t,x,u)+\gamma, f(t,x,u)) : u \in [0,1], \gamma \leq 0\}$ for $x \leq x_1$, there is a pair $x''(.), u''(.)$ such that $\int_0^{t_n} f_0(t, x_n(t), u_n(t))dt \to \int_0^{t'} f_0(x''(t), u''t))dt$, $x''(t) \leq x_1$ for all $t \leq t'$, $x''(t') = x_1$, $x''(0) = x_0$, and satisfying the differential equation $\dot{x} = 2ux$ on $[0, t']$. Using compactness of U and convexity of $\{(f_0(t,x,u) + \gamma, f(t,x,u)) : u \in [0,1], \gamma \leq 0\}$ for $x \geq x_1$, there is also a pair $x'(.), u'(.)$ such that $\int_{t_n}^T f_0(t, x_n(t), u_n(t))dt \to \int_{t'}^T f_0(x'(t), u'(t))dt$, $x'(t') = x_1$, $x'(t) \geq x_1$ for all $t > t'$, $x(t') = x_1$, and satisfying the differential equation $\dot{x} = ux$ on $[t', T]$. If $x'(t) = x_1$ in some interval (which then must be of the form (t', t'')), then $\dot{x}' = ux = 2ux$, since $u = 0$ a.e. here. Thus, $x(.) = x''|_{[0,t']} + x'|_{(t',T]}$, $u = u''|_{[0,t']} + u'|_{(t',T]}$, is an admissible pair in the problem, and it yields maximum of the criterion. (Note that in the original problem, formally, an admissible pair is defined as any pair $(x(t), u(t)), u(t)$ measurable, such that, a.e., $\dot{x}(t) = 2u(t)x(t)$, if $x(t) \leq x_1$, $\dot{x}(t) = u(t)x(t)$ if $x(t) > x_1$, $x(0) = x_0$. In this example an admissible solution $x(t)$ may stay at the level x_1 for some time, there is no problem connected with the differential equation in this case.) It follows that we have got optimality in the set of admissible pairs.

E. The observant reader will have noted that in this problem, a unique candidate can be obtained by applying the standard maximum principle for "continuous" problems on "pieces" of the solution sought for. This is not so in the next problem, which is also of the bang-bang type.

For the two next problems, only the final solutions are presented; the problems were solved in the Master thesis of Stabrun (Stabrun 2007), (as were the problems 2 and 3).

Example 4 (Growth causing pollution with negative effects)
Consider a company producing, say, mineral water. Its production function is x, where x is real capital. A fraction ux is put aside for investment purposes,

while $(1-u)x$ is sold for a price equal to unity. The production causes pollution. The stock of pollution is z, and the increase in pollution $\dot{z}$ equals x. When the accumulated pollution reaches a certain level, the transformation of output into increases in real capital becomes more difficult. (Perhaps a tax on investments sets in, or construction workers must wear burdensome masks or whatever.) The control problem to be solved is then

$$\max \int_0^T (1-u)x dt, \; T \text{ fixed},$$
$$\dot{x} = uax, \; x(0) = x_0 > 0, u \in [0,1], a = \begin{cases} 1 \text{ if } z > z_1 \\ 2 \text{ if } z \leq z_1 \end{cases}, \phi = z - z_1,$$
$$\dot{z} = x, \; z(0) = 0, \; x(T) \text{ and } z(T) \text{ free, } x_0, z_1 \text{ given.}$$

If at all $z(t)$ crosses z_1, define τ as the single point satisfying $z(\tau) = z_1$, and assume that $x_0 = 1, z_1 + x_0/2 = e^4$. We shall here present solutions for different values of T. In all cases, $u = 1$ for $t < t^*, u = 0$ for $t > t^*$. For $T \in (1/2, 5/2)$, $t^* = T - 1/2$ and $z(t) < z_1$ for all t. For $T \in (5/2, 5/2 + \frac{1}{2}\ln 2)$, $T - \frac{1}{2} = t^* < \tau = T - 1 + e^{5-2T}$. For $T \in (5/2 + \frac{1}{2}\ln 2, 3 + \frac{1}{2}\ln 2)$, $t^* = \tau = 2 + \frac{1}{2}\ln 2$. For $T \in (3 + \frac{1}{2}\ln 2, \infty)$, $t^* = T - 1$, $\tau = 2 + \frac{1}{2}\ln 2$. Here, having recourse to the necessary conditions of Theorem 2 reduces the number of candidates needing consideration as compared to the number of candidates arising when using only the standard necessary conditions for continuous problems on "pieces" of the solution sought for, (see the previous example for such an application in discontinuous problems).

Note that here, as $x(t) \geq x_0 > 0$, then the existence result in Remark 5 automatically yields existence of an optimal control.

Example 5 (Capital-dependent income tax)
A firm has production function $2x, x$ being real capital. The owner saves an amount $u2x$, $u \in [0,1]$, so the capital in the firm increases according to $\dot{x} = 2ux - x$, the term $-x$ representing depreciation. If $x > e$, the income taken out of the firm is taxed according to the rate $1/4$, while if $x \leq e$, there is no income tax. Hence, the owner wants to maximize

$$\max \int_0^T 2\alpha(1-u)x dt, \; \alpha = 3/4 \text{ if } x > e, \; \alpha = 1 \text{ if } x \leq e, \; T \text{ fixed} > \ln 2,$$

$\dot{x} = 2ux - x$, $x(0) = 1$, $u \in [0, 1]$, $x(T)$ free, $\phi(t, x) = x - e$.

Solution

Let β be the solution of $(\beta - \ln 2)e = e/2 + 9e^{\beta}/32$, $\beta \simeq 2.63317$, $1 + \ln 2 < \beta < 1 + \ln \frac{16}{3}$. If $T \in (\ln 2, 1 + \ln 2)$, $u = 1$ in $(0, T - \ln 2)$, $u = 0$ in $(T - \ln 2, T)$, $x(t) < e$ for all t in both cases. If $T \in (1 + \ln 2, \beta)$, $x(t) = e$ in $(1, T - \ln 2)$, $u = 1$ in $(0, 1)$, $u = 1/2$ in $(1, T - \ln 2)$, $u = 0$ in $[T - \ln 2, T)$, $x(t) < e$ for $t \notin (1, T - \ln 2)$. If $T \in (\beta, 1 + \ln \frac{16}{3})$ two crossing points exist, $\tau_1 = 1$, and $\tau_2 = 2T + 2\ln \frac{3}{8} - 1$, (i.e. $x(\tau_i) = e, i = 1, 2$), $u = 1$ in $[0, T + \ln \frac{3}{8})$, $u = 0$ in $(T + \ln \frac{3}{8}, T)$, $x(t) < e$ for $t \notin [\tau_1, \tau_2]$, $x(t) > e$ for $t \in (\tau_1, \tau_2)$. If $T \in (1 + \ln \frac{16}{3}, \infty)$, a single crossing point $\tau = 1$ exists (i.e. $x(\tau) = e$ for $\tau = 1$), $u = 1$ in $[0, T - \ln 2), u = 0$ in $(T - \ln 2, T)$, $x(t) < e$ for $t < 1$, $x(t) > e$ for $t > 1$.

Appendix

Sketch of proof of Theorem 2 in the free end case.

To sketchy proofs will be given, the first one (A) assuming continuous differentiability of a certain value function, the second one (B) not relying on such an assumption. (The latter proof can be written out to to yield a rigorous proof.)

(A) For simplicity, assume a single $\phi-$ function and as single fault point τ. Let $S(\tau, x)$ be the value function of the problem P^{τ} : $\max \int_{\tau}^{t_1} f_0(t, x(t), u(t))dt$, with end conditions as before, when solutions start at any given (τ, x). Then the original problem can be written as $\max_{u,\tau}\{\int_{t_0}^{\tau} f_0(t, x(t), u(t))dt + S(\tau, x(\tau))\}$, with end condition $\phi(\tau, x(\tau)) = 0$. The necessary conditions for the last problem is $\dot{p} = -H_x$, the maximum condition, and the transversality condition $H(\tau-) + S_t(\tau, x^*(\tau)) + \gamma\phi_t(\tau, x^*(\tau)) = 0$ (an obvious shorthand notation) with $p(\tau-) = S_x(\tau, x^*(\tau)) + \gamma\phi_x(\tau, x^*(\tau))$, γ some multiplier, see e.g. Theorem 16, p. 398 in Seierstad and Sydsæter, (1987). By standard result concerning derivatives of the value function, see e.g. Theorem 9, p. 219 in Seierstad and Sydsæter (1987), $S_x(\tau, x^*(\tau)) = p(\tau+)$ and $S_t(\tau, x^*(\tau)) = -H(\tau+)$. Now, $H(\tau-) + S_t(\tau, x^*(\tau)) + \gamma\phi_t(\tau, x^*(\tau)) = p_0 f_0(\tau-) - p_0 f_0(\tau+) + (p(\tau+) + \gamma\phi_x) f(\tau-) - p(\tau+) f(\tau+) + \gamma\phi_t = 0$, so $\gamma = -\{p_0 f_0(\tau-) - p_0 f_0(\tau+) + p(\tau+) f(\tau-) - p(\tau+) f(\tau+))\}/(\phi_x f(\tau-) + \phi_t)$, and then $p(\tau-) - p(\tau+) = S_x + \gamma\phi_x - S_x = \gamma\phi_x = \{p_0 f_0(\tau-) - p_0 f_0(\tau+) +$

$p(\tau+)f(\tau-) - p(\tau+)f(\tau+)\}\phi_x/(\phi_x f(\tau-) + \phi_t))$. (Compare (2.2) and 2.3.)

Digression. Presumably, a sufficient condition can be obtained by studying the behavior of $F(\tau) := H(p^\tau(\tau-)) - H(p^\tau(\tau+) + \gamma^\tau\phi_t(\tau, x^\tau(t))$, where x^τ, p^τ, γ^τ are solutions and multipliers of the necessary conditions (for $p_0 = 1$) of the problem $\max_u \int_{t_0}^{\tau} f_0(t, x(t), u(t))dt + S(\tau, x(\tau))$, $\phi(\tau, x(\tau)) = 0$, τ fixed in $[t_0, t_1]$. If the maximized Hamiltonian is concave both in the last problem and in problem P^τ and if the exists a τ^* in $[t_0, t_1]$ such that $F(\tau) \geq 0$ for $\tau \leq \tau^*$ and $F(\tau) \leq 0$ for $\tau \geq \tau^*$, then (presumably) optimality follows.

(B) In the free end case, let us prove the necessary conditions in the case of one function $\phi_1 = \phi$ $(g^1 = g)$,one crossing point $\tau \in (t_0, t_1)$, and a free end, and in the case where $ax(t_1)$ is maximized, a a given vector. (A problem where the criterion is an integral can be rewritten so that it has the just mentioned format, by using an auxiliary state variable, the generalization to a nonlinear scrap value $h(x)$ can be similarly treated). Let $C(t, s)$ be the resolvent of $dq/dt = f_x(t, x^*(t), u^*(t))q$. Let a perturbation u be carried out at t^*, $t^* < \tau$, i.e., let $u^*(.)$ be replaced by the constant $u \in U$ on a short interval $[t^* - \delta, t^*]$. Let $\check{x}(., \delta)$ be the corresponding solution on $[t_0, \tau]$, and let $b = \partial\check{x}(t^*, \delta)/\partial\delta$. When such a perturbation is carried out, it is most convenient for establishing necessary conditions to replace $u^*(.)$ by a function that differs from u^* near the point τ at which $x^*(.)$ enters Γ: Assume first that $\check{x}(t, \delta)$ enters Γ at a time $s(\delta)$ before τ, $(s(\delta) < \tau)$. We then replace $u^*(.)$ on $(s(\delta), \tau)$ by the function $u^{**}(t) = u^*(\tau+)$, $t \in (s(\delta), \tau)$. The solution on $[t_0, t_1]$ corresponding to $u(t) = u1_{[t^*-\delta,t^*]} + u^*(\tau+)1_{(s(\delta),\tau)} + u^*(t)(1 - 1_{[t^*-\delta,t^*]} - 1_{(s(\delta),\tau)})$ is denoted $x(t, \delta)$, $(x(t_0, \delta) = x^0)$. If $\check{x}(t, \delta)$ has not reached Γ even when $t = \tau$, then let us use the function $u^{**}(t) = u^*(\tau-)$ to the right of τ, as long as Γ is not entered. Define now $s(\delta)$ to be the first time Γ is entered, and from then on let us use $u^*(.)$. The solution on $[t_0, t_1]$ corresponding to $u(t) = u1_{[t^*-\delta,t^*]} + u^*(\tau-)1_{(\tau,s(\delta))} + u^*(t)(1 - 1_{[t^*-\delta,t^*]} - 1_{(\tau,s(\delta))})$ is denoted $x(t, \delta)$, $(x(t_0, \delta) = x^0)$.

Assume first $s(\delta) < \tau$. At time $s(\delta)-$, the change in the state due to the perturbation (i.e. $x(s(\delta)-, \delta) - x^*(s(\delta)-)))$ is approximately $C(s(\delta), t^*)b\delta$, (δ is small.) As δ is small, with $s := s(\delta)$ we have $0 = \phi(s, x(s-, \delta)) \simeq \phi(s, x^*(s) + C(s, t^*)b\delta)) \simeq \phi(s, x^*(\tau-) + f(\tau, x^*(\tau-), u^*(\tau-))(s-\tau) + C(s, t^*)b\delta) \simeq \phi_t(\tau, x^*(\tau-))(s-\tau) + \phi_x(\tau, x^*(\tau-))f(\tau, x^*(\tau-), u^*(\tau-))(s-\tau) + \phi_x(\tau, x^*(\tau-))C(\tau, t^*)b\delta + \phi(\tau, x^*(\tau-)) =: \Psi(s)$, where of course $\phi(\tau, x^*(\tau-)) = 0$. Whether $s - \tau < 0$ or ≥ 0, $\Psi s) = 0$ determines $s = s(\delta)$, because we obtain the same equation also in the case $s - \tau \geq 0$, a case we shall now discuss. In this case we have $0 = \phi(s, x(s-, \delta)) \simeq$

$\phi(s, x(\tau-,\delta) + \dot{x}(\tau-)(s-\tau)) \simeq \phi(s, x(\tau-,\delta) + f(\tau, x^*(\tau-), u^*(\tau-))(s-\tau)) \simeq \phi(s, x^*(\tau-,\delta) + C(\tau, t^*)b\delta) + f(\tau, x^*(\tau-), u^*(\tau-))(s-\tau)) \simeq \Psi(s)$.

Define $t' = \max\{\tau, s(\delta)\}$. At time $t'+$ both $x^*(t'+)$ and $x(t'+,\delta)$ have been affected by a jump of size g, when considering their difference, i.e. the change in the state, g drops out. The effect of using the controls specified instead of $u^*(.)$ near τ adds approximately a change $(-f(\tau, x^*(\tau+), u^*(\tau+)) + f(\tau, x^*(\tau-), u^*(\tau-)))(s-\tau)$ to the change $C(t', t^*)b\delta$ in the state at time $t'+$ caused by the perturbation. It can be checked that this term is the same whether $s-\tau$ is positive or negative. The total change in the state (i.e. $x(t'+,\delta) - x^*(t'+)$, is approximately $h(t') := C(t', t^*)b\delta + (f(\tau-) - f(\tau+))(s-\tau) \simeq C(\tau, t^*)b\delta + (f(\tau-) - f(\tau+))(s-\tau) = h(\tau)$ (a self-explaining shorthand notation). At time t_1, the change $x(t_1, \delta) - x^*(t_1)$ equals approximately $C(t_1, \tau)h(\tau)$, and the change in criterion value is $aC(t_{1,}\tau)h(\tau)$. Note that it follows from the equation $\Psi(s) = 0$ that $s - \tau = \mu C(\tau, t^*)b\delta$, where μ is determined as above, see (2.3). Define $p(t) = aC(t_1, t)$ for $t > \tau$, and define, for $t < \tau$, $p(t) = aC(t_1, \tau)C(\tau, t) + [aC(t_1, \tau)(f(\tau-) - f(\tau+))]\mu C(\tau, t)$. Then $p(\tau-) = p(\tau+) + (p(\tau+)(f(\tau-) - f(\tau+))\mu$, which is (2.2). It is well-known that, for any constants σ and $c \in \mathbb{R}^n$, $(d/dt)cC(\sigma, t) = -cC(\sigma, t)f_x(t, x^*(t), u^*(t))$, from which (1.8) follows. Observe finally that $aC(t_1, \tau)h(\tau) = p(t^*)b\delta$. Now, by optimality, $p(t^*)b\delta \leq 0$. Note that $\partial x(t^*, \delta)/\partial\delta = f(t^*, x^*(t^*), u) - f(t^*, x^*(t^*), u^*(t^*))$, so

$$p(t^*)b\delta = p(t^*)[f(t^*, x^*(t^*), u) - f(t^*, x^*(t^*), u^*(t^*))]\delta \leq 0$$

by optimality, which is the maximum condition for the present type of criterion for $t^* < \tau$. For $t^* > \tau$, the change in the criterion is approximately $aC(t_{1,}t^*)b\delta = p(t^*)b\delta = p(t^*)[f(t^*, x^*(t^*), u) - f(t^*, x^*(t^*), u^*(t^*))]$, which again has to be ≤ 0, by optimality. Hence, the necessary conditions follow in the present case. $\square$

Comment on the proof in the nonfree end case. Consider the proof in (B). In the case of all fault points are crossing points, Ekeland's theorem together with replacing the end constraints by penalty functions can be used to obtain necessary conditions, in a tradition stemming from Clarke (1983). If some touch points also appear, it is better to use the old-fashion methods of Brouwer's fixed point theorem, and a separation argument, (for this tradition see, e.g., Fleming and Rishel (1975)). Let us given some ideas pertaining to the last type of proof, assuming a single fault points and the maximization of $x_1(t_1)$, (i.e. $a = (1, ..., 0)$), and where $x_i(t_i)$ are fixed, $i > 1$. (Again a single function ϕ, and a single fault point τ is considered.) Let K be the set of all convex combinations $c := \sum_m \lambda_m h_m^*$, where $h_m^* =$

$$C(t_1, \tau)\{C(\tau, t_m^*)(f(t_m^*, x^*(t_m^*), u_m) - f(t_m^*, x^*(t_m^*), u^*(t_m^*)) \\ +(f(\tau, -) - f(\tau, +))[\mu C(\tau, t_m^*)(f(t_m^*, x^*(t_m^*), u_m) - f(t_m^*, x^*(t_m^*), u^*(t_m^*)))]\}$$

in case $t_m^* < \tau$, and $h_m^* = C(t_1, t_m^*)[f(t_m^*, x^*(t_m^*), u_m) - f(t_m^*, x^*(t_m^*), u^*(t_m^*))]$ in case $t_m^* > \tau$, t_m^* arbitrary continuity points of $u^*(.)$, u_m arbitrary points in U. If $\text{int}K \neq \emptyset$, then $L = \{\delta'\tilde{1} : \delta' > 0\}$, $\tilde{1} := (1, 0, ..., 0)$ has to be disjoint from $\text{int}K$, otherwise for some $\delta' > 0$, $\delta'\tilde{1}$ is interior in the convex hull $K' := \{\sum_j \lambda_j^* c_j : \sum_j \lambda_j^* = 1, \lambda_j^* \geq 0\}$ of a finite collection of $c's$, say $c_1, ..., c_{j*}$, which leads to a contradiction as we shall see. Let $x(t_1, \lambda^*)$, $\lambda^* = (\lambda_1^*, ..., \lambda_{j^*}^*)$, be the exact solution correspond to perturbations u_m^j used on intervals $[t_{m,j}^* - \lambda_j^* \lambda_m^j, t_{m,j}^*]$, $u_m^j, t_{m,j}^*$ and λ_m^j being the entities occurring in $c = c_j$. (Again near τ, $u^*(.)$ is modified as in the above free end proof on an interval of length $|s - \tau| = |\sum_{j, t_m^* < \tau} \lambda_j^* \lambda_m^i \mu C(\tau, t_m^*)(f(t_m^*, x^*(t_m^*), u_m) - f(t_m^*, x^*(t_m^*), u^*(t_m^*)))|$.) Then $x(t_1, \lambda^*) - x^*(t_1) - \sum_j \lambda_j^* c_j$ is of the first order in $\delta^* := \sum_j \lambda_j^*$, similar to what we have in proofs for the standard control problem. Then, by this first order approximation and a standard use of Brouwer's fixed point theorem, for $\delta > 0$, small enough, $x(t_1, \lambda^*) - x^*(t_1) = \delta\delta'\tilde{1}$, for some λ^*, for which $\sum_i \lambda_j^* < 2\delta$. The last equality contradicts optimality. So L and $\text{int}K$ are disjoint. The separation of $\text{int}K$ and L by a hyperplane implies the maximum condition. If $\text{int}K = \emptyset$, this separation is trivial.

Sketch of proof of Theorem 3 in the free end case. Consider the proof in (B). As noted above, dummy variables can be used to transform the problem in Remark 1 to one with jumps in state variables. However, the above proof sketch also works in the discontinuous (f_0, f) - case, just using that the definition of (f_0, f) varies according to which set Φ the state belongs to, and noting that for the solutions $x(t, \delta)$ the condition (NT*) secures that $x(t, \delta)$ is strongly admissible. The crossing point becomes slightly perturbed, however $x(t, \delta)$ belongs to Φ whenever $x^*(t)$ belongs to Φ, except for t very close to τ.

Sketch of proof of Remark 3 in the free end case. To show the two asserted inequalities related to Hamiltonian, note that $s(\delta) - \tau < 0 \Leftrightarrow \alpha(t^*, u) > 0$.

If these inequalities hold, for any v, letting $u^* (t_1+)$ being equal to v, the arguments in the proof of Theorem 2 can be copied and they yield the condition $H(t^*, x^*(t^*), u, p^v(t^*)) - H(t^*, x^*(t^*), u^*(t^*), p^v(t^*)) \leq 0$, for a function $p^v(t)$ that jumps even at t_1. If $s(\delta) - \tau > 0$, then we are in a case where we can disregard the fault point t_1, and we get the condition $H(t^*, x^*(t^*), u, p(t^*)) - H(t^*, x^*(t^*), u^*(t^*), p(t^*)) \leq 0$ for a $p(t)$-function not jumping at t_1, only at fault points before t_1.

Sketch of proof of Remark 4 in the free end case. To show the result in Remark 4, assume that we have a free end, a single function $\phi = \phi_1$ (with $g = g^1$) is given, and that a single crossing point τ exists. We shall use the arguments of the proof of Theorem 2. Let $x(.)$ be the solution resulting from the perturbation near t^* and let $\hat{g} = I + g$. Note that the jump point s as before is determined approximately by $s - \tau = \mu C(\tau, t) b\delta$, where μ is determined as above, see (2.3). If $s - \tau = s(\delta) - \tau > 0$, then $x(s+) = \hat{g}(s, x(s-)) \simeq \hat{g}(s, x^*(\tau-) + C(\tau, t^*) b\delta + f(\tau, -)(s - \tau)) \simeq \hat{g}(\tau, x^*(\tau-)) + \hat{g}_t(\tau, x^*(\tau-))(s - \tau) + \hat{g}_x(\tau, x^*(\tau-))[C(\tau, t^*) b\delta + f(\tau, -)(s - \tau)]$, and $x^*(s) = \hat{g}(\tau, x^*(\tau-)) + f(\tau, +)(s - \tau)$, so $x(s+) - x^*(s) \simeq \hat{g}_t(\tau, x^*(\tau-))(s - \tau) + \hat{g}_x(\tau, x^*(\tau-))[C(\tau, t^*) b\delta + f(\tau, -)(s - \tau)] - f(\tau, +)(s - \tau) =: \alpha(\delta)$, and $x(t_1) - x^*(t_1) \simeq C(t_1, s)\alpha(\delta) \simeq C(t_1, \tau)\alpha(\delta)$. From this (2.2) modified follows in case $s(\delta) > \tau$. Next, if $s(\delta) < \tau$, $x(\tau) \simeq \hat{g}(s, x(s-)) + f(\tau, +)(s - \tau) \simeq \hat{g}(s, x^*(s) + C(s, t^*) b\delta) + f(\tau, +)(s - \tau) \simeq \hat{g}(s, x^*(\tau-) + f(\tau, -)(s - \tau) + C(s, t^*) b\delta) + f(\tau, +)(s - \tau) \simeq \hat{g}(\tau, x^*(\tau-)) + \alpha(\delta)$, so $x(\tau) - x^*(\tau+) \simeq \alpha(\delta)$ and $x(t_1) - x^*(t_1) \simeq C(t_1, \tau)\alpha(\delta)$. From this (2.2) modified follows in case $s(\delta) < \tau$.

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