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Marginal versus average beta of equity under corporate taxation

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Abstract

Even for fully equity-financed firms there may be substantial effects of taxation on the after-tax cost of capital. Among the few studies of these effects, even fewer identify all effects correctly. When marginal investment is taxed together with inframarginal, marginal beta differs from average if there are investment-related deductions like depreciation. To calculate asset betas, one should not only "unlever" observed equity betas, but "untax" and "unaverage" them. Risky tax claims are valued as call options, with closed-form solutions for the exercise probability. Results have practical relevance for multinationals operating under different tax systems.

KEYWORDS: Cost of capital, WACC, loss offset, tax shields, options

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1 Introduction

That the corporate tax rate appears in the cost-of-debt component of the weighted average cost of capital (WACC) is well known. That it affects the cost-of-equity component through investment-related deductions like depreciation is much less known. Even under full equity financing this tax effect can be substantial. It is particularly important for multinationals and other firms operating under different tax systems. The basis for adjusting the WACC for the risk of depreciation tax shields, as suggested here, is that these are proportional to investment. Actually, this proportionality is much more obvious than the idea that debt is a fixed proportion of investment. Thus it is more straightforward to adjust the WACC for this "negative tax leverage" than for debt leverage and its accompanying interest tax shields.

This paper may help convincing practitioners that even if their future tax deductions are risky, taxation in most cases reduces the risk of the net after-tax cash flow substantially. This is true if investment-related deductions are allowed in years after the investment is made. Firms with international operations should observe that different tax systems split the risk differently between the tax cash flow and the cash flow after corporate taxes. Different discount rates may be appropriate in different countries. According to the survey by Graham and Harvey (2001), most firms use a single company-wide discount rate in capital budgeting, even when they evaluate a new project in an overseas market (p. 205). The discount rate is most often based on the Capital Asset Pricing Model (CAPM) (p. 201), which will also be used here for simplicity. As a motivation, the following three paragraphs will highlight the implications of previous studies. Then follows an introduction to the novelties of the present paper.

A simple example of depreciation tax shields (U.S. tax shield numbers from Brealey, Myers, and Allen 2008, p. 561) can be used to show the magnitude of the effects. Consider an asset with an asset beta of 1.00 in the absence of taxation. For an equity financed firm an increase in the tax rate, t, from 35 percent to 70 percent would reduce the beta of equity from 0.694 to 0.388.[1] These are correct betas of equity to be used for finding the required expected rate of return to equity, i.e., the after-tax cost of equity. A theoretical effect of this magnitude should not be neglected.

The example suggests that as a first approximation the beta of equity is proportional to $(1 - t)$. This holds exactly (see Proposition 1 below) under two conditions: The firm will earn the value of future tax deductions with certainty, and deductions are given in years after investment, with interest accumulation so that the present value is equal to investment. However, most of the risk reduction effect of taxes occurs even if there are standard depreciation allowances (as in the example) with no interest accumulation. The present paper extends the analysis to cases with risky future deductions.

In the case with risk free deductions, the intuition behind the result is best seen with a pure cash flow tax (Brown 1948) as a point of reference. This is a proportional tax on non-financial cash flows, with payout of negative taxes in years with negative net cash flows.[2] The systematic risk (the beta) of the after-corporate-tax cash flow is unaffected by a pure cash flow tax. The tax acts cash-flow-wise as just another shareholder. Consider next what happens if negative taxes are not paid out, but postponed and given as tax deductions with interest in later years.[3] This change is like a risk free loan from the firm to the tax authorities (lending by the firm), and acts risk-wise as the opposite of traditional leverage (borrowing by the firm). The systematic risk of the after-tax cash flow is reduced, i.e., multiplied by the factor $(1 - t)$.

The starting point for the present study is Lund (2002a), in which the tax effects on systematic risk of marginal projects were studied. The present paper introduces an analytical production function with decreasing returns to scale. The marginal investment is taxed together with inframarginal investment. Firms choose the scale of investment to maximize after-tax value. This model gives rise to three new insights.

First, the existence of inframarginal profits (rents) leads to a difference between marginal and average expected returns. Most previous studies of these issues have considered projects of various degrees of profitability, and have found tax effects on the systematic risk of the returns of such projects, implicitly or explicitly, numerically or analytically. But this is not the same as finding effects on the *required* expected return. This can only be found by considering the marginal investment, whether it is taxed alone or together with inframarginal investment. The model which follows shows the difference between marginal and average beta of equity when there are investment-related deductions in years after the investment. The model is sufficiently general to invalidate some statements in the earlier literature about tax effects on the required expected return, which were not based on identifying the marginal investment.

The second insight is an analytical description of the risk of tax shields, and thus also of the after-tax cash flow. The risk of, e.g., a future depreciation deduction depends on inframarginal profits. The model gives an analytical solution for the beta of equity based on a combination of the CAPM and an option pricing model.[4] Admittedly, the model is based on a number of simplifying assumptions. It nevertheless gives useful insights into the riskiness of the tax and after-tax cash flows, such as which variables are important and which have no impact.

The third insight is the consequences for how to find asset betas. Starting with observed equity betas, there is a need not only for "unlevering,"[5] but also for "untaxing" (see Proposition 1) and "unaveraging" (see Proposition 2).

In addition to these three main strands of results, there are results on the required expected return before taxes, which is the main focus in related studies in public economics.

These results are less surprising. The most striking feature is their simplicity, i.e., how the tax effects are simply found in a factor which multiplies the required expected return before taxes.

The previous literature on the topic is scattered in public and financial economics, and there are some predecessors which focus on natural resource extraction. In public economics there is a substantial literature on the effect of taxes on the cost of capital before taxes. In King and Fullerton (1984, p. 10) this is formulated as $p = c(r)$, where p is the real cost of capital, r is the real market interest rate, and the function c "depends upon details of the tax code." In most of this literature there is no consideration of uncertainty, and r is taken as given in a partial equilibrium model. Under uncertainty this is inadequate (and easily misleading), even in a similar partial equilibrium framework. There is no single rate of return which can play the role of r. Public economics has the advantage, however, that the studies typically realize the need to identify the marginal project. The relationship to Hall and Jorgenson (1969) is shown in section 5 below.

In financial economics it has long been recognized that the after-tax cost of capital for a partly debt-financed firm depends on taxes. The discount rate is then known as the weighted average cost of capital (WACC). This is typically written[6] as $r_D(1 - t_c)D/V + r_E E/V$, where $r_D(1 - t_c)$ is the after-tax cost of debt, t_c is the corporate tax rate, and r_E is the cost of equity (after tax). This is a value-weighted average, where $V = D + E$ is the market value of the firm, D and E being market values of debt and equity, respectively. In almost all studies and presentations, this formulation neglects the dependence of r_E on corporate taxation of the firm's activities. In contrast, the present study and a few others (see below) imply that even for a fully equity-financed firm, the WACC, in that case r_E, depends on the corporate tax systems in the countries/sectors/jurisdictions where the firm operates.[7]

Since Myers (1974) there have been recommendations in finance textbooks to use valuation by element, known as adjusted present value (APV)[8]. Instead of looking for the correct beta for the net after-tax cash flow, one considers different elements of the cash flow separately, finds their values based on the systematic risk of each, and then finds the sum of the values. The model presented in the present paper is fully consistent with the APV approach. The possibility to use APV and avoid using a discount rate for the net expected cash flow does not mean that the systematic risk of the net cash flow becomes uninteresting. Even though the APV method has been known, Graham and Harvey (2001) show that firms typically rely on a single WACC number. For those who want to advocate the APV method instead, this paper and its predecessors can be used to demonstrate what mistakes will be made with a single WACC. The systematic risk of net cash flows is also needed to find asset betas, cf. section 3 below.

In the sections of Brealey, Myers, and Allen (2008) which do not rely on the APV method, they ignore the possibility to say something systematically about how r_E depends on taxes. They state[9] (p. 561) that "Depreciation tax shields contribute to project cash flow, but they are not valued separately; they are just folded into project cash flows along with dozens, or hundreds, of other specific inflows and outflows. The project's opportunity cost of capital reflects the average risk of the resulting aggregate." This practice is unfortunate if the firm operates under different tax systems.

Eight previous theoretical studies which discuss the effect of taxes on the risk of after-tax rates of return, allowing for tax effects even in the absence of debt, are Levy and Arditti (1973), Galai (1988), Jacoby and Laugton (1992), Derrig (1994), Bradley (1998), Galai (1998), Lund (2002a), and Rao and Stevens (2006). Both Levy and Arditti (1973)[10] and Lund (2002a) determine the marginal investment to find the required expected rate of return, while the others do not. More details are given in section 3 below.

Four of the eight studies assume that firms always pay taxes, so that tax shields are risk free. The last two on the list do not. Earlier, Jacoby and Laughton (1992) and Bradley (1998)[11] use the finance-theoretic approach originally developed for option valuation to study valuation of natural resource projects under taxation. They use Monte-Carlo simulations for specific resource extraction projects, extending the APV method. To find project values after tax there is no need for required expected returns for after-tax net cash flows. But these are found after the net values of cash flows have been calculated, technically like internal rates of return. The present paper highlights, through an analytical model, some typical mechanisms behind these results.

Galai (1998) is, together with Lund (2002a), the paper most closely related to the present one. Galai has a theoretical two-period model with results on the systematic risk of the cash flows to the three claimants, equity, debt, and tax authorities, and on possible conflicts of interest between these. The equity beta is found to be declining in the tax rate, but the required expected return is not determined.

More recently, like the present paper, Rao and Stevens (2006) set up a two-period model of a firm subject to taxation, with investment in the first and a risky outcome in the second period. Like in the present paper, the priced risk is determined by the covariance with an exogenously given process which is unaffected by the tax system to be analyzed. Their model is more general by considering risky debt, and in some other respects. The pricing model is an approximate Arbitrage Pricing Theory (APT), which is robust with respect to different distributional assumptions. The firm in their model can be solvent or insolvent, in tax position or not, and if in tax position, using two different tax shields, debt and non-debt, partly or fully. They improve upon the literature by a simultaneous solution to the cost of debt, the optimal level of debt, and the risks of the tax shields. From this follow also the WACC and the risks and values of the different cash flows, including the tax claim. But they have a different focus for their analysis from that of the present paper.[12] They give no results for effects on their endogenous variables of changes in tax rates or other tax parameters. Their model starts with some exogenous project profitability before tax, and does not determine the risk of the tax shields endogenously based on the taxation of the marginal investment together with inframarginal investment.

Summers (1987) investigates the riskiness of depreciation deductions, and finds that they have low systematic risk. He recognizes that firms in many cases discount these tax shields too heavily, and states that "patterns of investment may be very substantially distorted in ways not considered in standard analyses of the effects of tax incentives" (p. 302). He goes on to consider consequences for tax reform. Gordon and Wilson (1989) (fn. 10) mention that depreciation deductions are "normally riskfree in nominal terms." One message of the present paper is that before one concludes that these tax shields can be regarded as risk free, one should carefully consider under what circumstances they will be somewhat risky.

This paper can also be seen as a supplement to the empirical work on estimating marginal tax rates of firms taking tax carry-forward and carry-back into consideration. Some central references are Auerbach and Poterba (1987), Shevlin (1990), Graham (1996), and Shanker (2000). While the empirical studies are more realistic by taking multi-period effects into account, the present model gives analytical solutions, identifying which factors are likely to have important effects.

The present paper is organized as follows. Section 2 presents general features of the model. Section 3 deals with the case of risk free tax shields, or full and immediate loss offset. Section 4 introduces imperfect loss offset with uncertainty about whether the firm will be in position to pay taxes. While these two sections focus on the after-tax cost of capital, section 5 gives results on the cost of capital before taxes. Section 6 has a discussion of rents, quasi-rents, and how to reconcile the model with an industry equilibrium. Section 7 contains additional discussion of some aspects of the model. Section 8 concludes. Some proofs and additional details are in the appendices.

2 The model

A firm invests in period 0 and produces in period 1, only. The firm considers an investment project with decreasing returns to scale. It is free to choose the scale of investment, and uses an APV-based method. The optimal choice is endogenous, determined by the tax system and other parameters in each case below. In this way the minimum required expected return to equity in each case is determined. There will also be results on a project with constant returns to scale.

The firm is financed by equity only. This simplifies the analysis and allows a focus on the effects which are the novelties of the paper. For comparison, some simple results with a fixed ratio of riskless debt (Lund 2002a) are stated at the end of section 3. The interaction when both debt, debt tax shields, and investment tax shields are risky, is analyzed by Rao and Stevens (2006). Their model becomes quite complicated and cannot be solved analytically.[13]

The assumption here of full equity financing may be necessary to get an analytical solution, but it also has another justification. It is very different from most of the literature on tax effects on firms' costs of capital, which typically discuss interest tax shields and how the debt capacity of the firm is determined. For many subsidiaries of multinationals that discussion is not directly relevant. Subsidiaries are often financed from the parent company,[14] although the financing may appear as debt, supplied, e.g., by other subsidiaries of the same parent in other jurisdictions. The subsidiary's borrowing and debt service are likely to be determined by minimization of the total taxes on the global operations of the parent, and possibly by limitations on debt ratios set by authorities in host countries.[15] The debt is often formally or de facto guaranteed by the parent. If almost all debt is owed to a related company, bankruptcy and debt capacity cannot be analyzed by the standard methods applicable for stand-alone firms. Instead the assumption of all-equity financing may be a useful simplification, especially since the topic to be analyzed does not depend on debt financing, and the model gets complicated as it is.

The first assumption of the model is:

Assumption 1: *The firm is fully equity financed and maximizes its market value according to the Capital Asset Pricing Model,*

$$E(r_i) = r + \beta_i[E(r_m) - r], \tag{1}$$

where $r > 0$*.*

All variables are nominal. The model is consistent with deterministic inflation, whereas stochastic inflation would require a more complicated model, especially if taxes are not inflation adjusted.

When various tax systems are considered below, these are assumed not to affect the capital market equilibrium. This will be a good approximation if they apply in small sectors of the economy (e.g., natural resource extraction), or abroad in economies ("host countries") which are small in relation to the domestic one (the "home country"). This is thus a partial equilibrium analysis.[16]

The ("home") economy where the firm's shares are traded may have a tax system, which is exogenously given and fixed throughout the analysis, and possibly[17] reflected in r. A consequence of the CAPM is that the claim to any uncertain cash flow X, to be received in period 1, has a period-0 value of

$$\varphi(X) = \frac{1}{1+r}[E(X) - \lambda \operatorname{cov}(X, r_m)], \tag{2}$$

where $\lambda = [E(r_m) - r]/\operatorname{var}(r_m)$. Equation (2) defines a valuation function φ to be applied below.

When betas are found as weighted averages below, all component betas must relate to equilibrium returns. A product price, P, will typically not have an expected rate of price increase which satisfies the CAPM.[18] A claim on one unit of the product will satisfy the CAPM, however, so that the beta of P should be defined in relation to the return $P/\varphi(P)$,

$$\beta_P = \frac{\operatorname{cov}(\frac{P}{\varphi(P)}, r_m)}{\operatorname{var}(r_m)}. \tag{3}$$

Assumption 2: *In period 0 the firm invests an amount $I > 0$ in a project. In period 1 the project produces a quantity Q to be sold at an uncertain price P. The joint probability distribution of (P, r_m) is exogenous to the firm, and $\operatorname{cov}(P, r_m) > 0$. There is no production flexibility; Q is fixed after the project has been initiated. There is no salvage value and no operating cost in period 1.*

The assumption of $\operatorname{cov}(P, r_m) > 0$ can easily be relaxed. It is only a convenience to simplify the verbal discussions below.

3 Case F: Tax deductions are risk free

This section will arrive at two expressions for the beta of equity under the assumption that the firm is certain to be in tax position in the next period, Case F (F for risk Free). The two betas will be referred to as the marginal and average beta. Section 4 will arrive at two other expressions for the beta of equity, under the assumption that the firm is uncertain whether it will pay taxes next period. These two betas will also be referred to as the marginal and average beta, for that case. In addition to these four betas there will be a reference in section 4 to the beta of equity for a marginal project taxed alone, under uncertainty about the tax position, as derived in Lund (2002a). There is also a generalized version of the model in section 6.

Assumption 3: *A tax at rate* $t \in [0, 1)$ *will be paid with certainty in the production period. The tax base is operating revenue less* cI*. There is also a tax relief of* taI *in period 0. The constants* a *and* $c/(1 + r)$ *are in the interval* $[0, 1]$*; moreover,* $t[a + c/(1 + r)] < 1$*.*

This general formulation allows for accelerated depreciation with, e.g., $a > 0$ and $a + c = 1$, or a standard depreciation interpreted (since there is only one period with production) as $a = 0, c = 1$. The requirement $t[a + c/(1 + r)] < 1$ precludes "gold plating incentives," i.e., the tax system carrying more, in present value terms, than one hundred percent of an investment cost.[19]

Assumption 3 implies that a negative tax base gives a negative tax paid out by the authorities. While this is unrealistic for most tax systems when the project stands alone, it is often a good approximation when the marginal project is added to other activity which is more profitable and only weakly correlated with it. An alternative assumption for the second period is considered in section 4. For the first period, however, no alternative is considered.

In the Case FM (M for marginal) of a marginal project alone, the cash flow to equity in period 1 is

$$X_{FM} = PQ(1 - t) + tcI. \tag{4}$$

For each set of tax and other parameters, Q/I is set so that the project is exactly marginal after tax. This does not lead to an optimal scale of investment. The purpose is to characterize marginal investment. Technically this is a project with constant returns to scale (CRS). The market value in period 0 of a claim to this is

$$\varphi(X_{FM}) = \varphi(P)Q(1-t) + \frac{tcI}{1+r}. \tag{5}$$

For a marginal project the expression must be equal to the financing need after taxes, $I(1-ta)$, so that Q/I is determined by

$$I(1-ta) = \varphi(X_{FM}) = \varphi(P)Q(1-t) + \frac{tcI}{1+r}, \tag{6}$$

which implies

$$\varphi(P)Q(1-t) = I\left(1 - ta - t\frac{c}{1+r}\right). \tag{7}$$

The beta of equity is a value-weighted average of the betas of the elements of the cash flow. From (4) this is,

$$\beta_{FM} = \frac{\varphi(P)Q(1-t)}{\varphi(P)Q(1-t) + It\frac{c}{1+r}}\beta_P = \frac{1 - ta - t\frac{c}{1+r}}{1-ta}\beta_P, \tag{8}$$

where the second equality follows from (7) above.[20] This can be summarized as follows:

Proposition 1: *Under Assumptions 1–3, the beta of equity for a marginal investment with constant returns to scale is given by (8). When $tc > 0$, it is strictly decreasing in the tax rate t, in the investment tax credit rate a, and in the present value of the deduction rate c.*

The proof is in Appendix A. The beta of equity is decreasing in the tax rate under any tax system with postponed deductions for investment outlays. Under a pure cash flow tax ($a = 1, c = 0$) there is no such effect of the tax rate. If the investment-related deductions appear in period 1, but have a period-0 present value equal to the investment ($a = 0, c = 1 + r$), equation (7) implies that the marginal investment will be unaffected by the tax rate. But the beta of equity will be $(1 - t)\beta_P$. This is the case mentioned in the introduction with a Brown tax as a point of departure, then a postponement with interest accumulation. The suggestion in the introduction that the beta of equity is approximately equal to $(1-t)\beta_P$ holds when a is small relative to $\frac{c}{1+r}$. In particular, $a=0$ gives $\beta_{FM}=(1-t\frac{c}{1+r})\beta_P$, which is close to $(1-t)\beta_P$ when $\frac{c}{1+r}$ is close to unity.

As mentioned by Lund (2002a), if one wants to calculate asset betas based on observed equity betas, one should "untax" the betas in addition to "unlevering" them.

Consider now the DRS Case, FA (A for average). Instead of technically adjusting Q to find the characteristics of a marginal project, there is now a first-order condition which determines I.

Assumption 4: *Produced quantity is* $Q=f(I)=\omega I^{\alpha}$. *The production function* f *has* $\omega>0,\ \alpha\in(0,1)$.

The cash flow to equity in period 1 is

$$X_{FA}=Pf(I)(1-t)+tcI, \tag{9}$$

The market value of a claim to this is

$$\varphi(X_{FA})=\varphi(P)f(I)(1-t)+\frac{tcI}{1+r}. \tag{10}$$

The firm chooses the optimal scale to maximize

$$\pi_F(I)=\varphi(X_{FA})-I(1-ta). \tag{11}$$

The first-order condition for a maximum is

$$\varphi(P)f'(I)=\frac{1-ta-t\frac{c}{1+r}}{1-t}, \tag{12}$$

which can be rewritten, based on the analytical production function, as

$$\varphi(P)f(I)(1-t)=\frac{I}{\alpha}\left(1-ta-t\frac{c}{1+r}\right). \tag{13}$$

The beta of equity is a value-weighted average of the betas of the elements of the cash flow. From (9) this is

$$\beta_{FA} = \frac{\varphi(P)f(I)(1-t)}{\varphi(P)f(I)(1-t) + It\frac{c}{1+r}}\beta_P. \tag{14}$$

According to (13), the optimal ratio $I/f(I)$ is proportional to α (since the other variables appearing in (13) are exogenous). Consider as a thought experiment what happens when the exogenous α is reduced from unity (which is its implicit value in (7)). The relative weight of the last term in (10), and in the denominator in (14), is reduced, and β_{FA} will get closer to the before-tax β_P.

The first-order condition and the parameterized production function together give

$$\beta_{FA} = \frac{1 - ta - t\frac{c}{1+r}}{1 - ta - t\frac{c}{1+r}(1-\alpha)}\beta_P, \tag{15}$$

which again is decreasing in the tax rate as long as $c > 0$. As α approaches unity (i.e., CRS), β_{FA} approaches β_{FM}. The result can be summarized as follows:

Proposition 2: *Under Assumptions 1–4, the beta of equity is given by (15). When* $tc > 0$*, it is strictly decreasing in the tax rate* t*, in the investment tax credit rate* a*, in the present value of the deduction rate* c*, and in the scale elasticity* α*.*

The proof is in Appendix A. Observe that $\beta_{FM} < \beta_{FA}$ when $tc(1-\alpha) > 0$. The reader may verify that in this case, the ratio β_{FM}/β_{FA} is decreasing in the tax rate (t), the investment tax credit rate (a), the deduction rate (c), and in $(1-\alpha)$.

The two different expressions for the beta of equity will be called *marginal beta* and *average beta*, respectively. They are both relevant as descriptions of systematic risk for the same project. The average beta will describe the systematic risk of the project as a whole, and in particular, the systematic risk of the shares in a firm with only this project. The marginal beta is still the relevant one for decision making at the margin, which may be decentralized within the firm. The correct beta for calculating the required expected rate of return is the marginal beta. The reason is that at the margin, the ratio Q/I is given by (7), not by (13). The result on the appropriate beta for decision-making is:

Proposition 3: *Under Assumptions 1–4, the firm's optimal investment can be found by maximizing its expected present value with a constant risk-adjusted discount rate based on the beta from (8). The same optimum can be found from maximizing the expected present value with a non-constant risk-adjusted discount rate based on the beta from (14), with beta being a function of the investment level, I.*

The proof is in Appendix A. If the average beta, β_{FA}, is used, it can not be considered a constant. Its value will be endogenously determined as part of the optimization. This restricts its usefulness from a managerial point of view.

Proposition 3 has important implications for all studies which consider the effect of taxation on after-tax returns to equity. Of the eight immediate predecessors of this study, mentioned in the introduction, only Levy and Arditti (1973) and Lund (2002a) identify tax effects on the *required* expected rate of return, i.e., the cost of capital, after tax.[21] Some authors (Galai 1998, Rao and Stevens 2006) have studied tax effects for any exogenously given level of profitability,[22] and some (Jacoby and Laughton 1992, Bradley 1998) have studied the same for specific numerical examples, with various realistic (or "reasonable") profitability levels.[23] This works well when the aim is to find tax effects on the systematic risk of a given project. But if one wants the effect on the required expected rate of return, one needs to consider a project which is exactly marginal after tax.

The reason why the marginal project and the DRS project have two different betas is not that one of them is located outside the security market line (SML), which is a problem sometimes seen in similar analyses. Both are on the SML because of the way their betas are determined in (8) and (14), as value-weighted averages of betas of correctly valued assets. The reason is instead that the period 1 cash flows of the two projects are composed differently. The only reason for this is the tax system, since there are no operating costs or other elements in those cash flows apart from the output values and the tax shields. The phenomenon occurs because the tax system allows investment-related deductions, based on investment which is not equal to the valuation of the subsequent project cash flow. Equation (6), $I(1-ta) = \varphi(X_{FM})$, does not hold for the DRS project.

The term "cost of capital" is generally used for a minimum required expected rate of return. This indicates that the term should not be used for a rate of return of a DRS project which yields supranormal profits (rents). However, the expected return from equations (9) and (10), $E(X_{FA})/\varphi(X_{FA})$, is an equilibrium expected return, and does not in itself exhibit any supranormal profit. This expected return will reflect the systematic risk quantified by β_{FA}. It is (one plus) the correct risk-adjusted discount rate to be used for finding the market value of X_{FA}, but only for some given ratio $tcI/E(P)f(I)(1-t)$. In the model this is optimally chosen and depends on α.

Proposition 3 thus demonstrates the need to "unaverage" observed betas. This is not much different from a correction for unusual operating leverage, except that the correction depends totally on the tax system and goes in the opposite direction: Higher investment cost implies lower beta, while higher operating cost typically would imply higher beta of the net future cash flow. A possibly realistic extension of the model would be to assume that an optimal DRS project would have an (average) operating leverage which is different from that of a marginal project. But this possible complication, which could exist independently of taxes, does not eliminate the tax effect highlighted by the present model.

Lund (2002a) has results on the marginal beta when Case F is extended to allow for a fixed ratio of riskless debt financing. Consider the simplest case with interest payments at the rate r being fully deductible in taxes in period 1.[24] When the equity ratio (market value of equity divided by market value of debt plus equity) is η, the marginal beta is

(Lund 2002a, eqs. (9) and (12)):

$$\beta_{FMB} = \frac{1}{\eta(1-ta)} \left\{ (1-ta) \left[\eta + (1-\eta) \frac{1+r(1-t)}{1+r} \right] - \frac{tc}{1+r} \right\} \beta_P, \tag{16}$$

(Subscript B denotes Borrowing.) The expression in square brackets gives the tax gain from debt through the interest deduction. The equity ratio appears here, but its strongest effect on β_{FMB} comes through its appearance in the denominator of the first fraction. As is well known, a low equity ratio will increase the systematic risk of equity,[25] and it will thus counteract the risk-reducing effect of the investment-related deductions, which are the focus of the present paper, expressed in the tc term. However, for some given η and typical depreciation allowances, the effect of increasing the tax rate will still be a strong reduction in beta.

The WACC is (Lund 2002a, eq. (15)):

$$\eta \left\{ r + \beta_{FMB}[E(r_m) - r] \right\} + (1-\eta) r(1-t). \tag{17}$$

The risk-reducing effects of the investment-related deductions show up via β_{FMB} in the cost-of-equity component of the WACC, which is otherwise a well-known expression.[26]

4 Case R: Uncertain tax position

The results for Case F above are based on the assumption that the firm is certain to be in tax position in period 1. While the tax element tPQ is perfectly correlated with the operating revenue, the depreciation deductions were assumed to be risk free.

To get analytical results the present section assumes that there is no loss offset at all. This means that the two-period model is taken literally and the tax code does not allow carry-backs. One purpose of the present paper is to see how much the results of Case F are modified when deductions are risky. Thus it is relevant to consider this most extreme riskiness.[27] It turns out that even then, the beta of equity is substantially lower than the asset beta before taxes, given reasonable parameter values. The cash flow to equity in period 1 is

$$PQ - t\max(0, PQ - cI). \tag{18}$$

Lund (2002a) arrived at an analytical solution for marginal beta in this case under the assumption that the marginal investment constitutes the whole tax base for the firm.[28]

A marginal beta may now take different meanings. A more realistic marginal beta recognizes that the marginal project is typically part of a larger activity, and that the probability of being in tax position depends on the outcome of that larger activity. This will be analyzed in line with the model of the previous section: The larger activity consists of a DRS investment project, the output of which is being sold at a single stochastic price in the single future period.

Let Case R (for Risky deductions) denote the case with an uncertain tax position. The following assumption replaces Assumption 3 above:

Assumption 5: *The tax base in period 1 is operating revenue less cI. When this is positive, there is a tax paid at a rate t. When it is negative, the tax system gives no loss offset at all. There is also a tax relief of taI in period 0. The constants a and $c/(1+r)$ are in the interval $[0,1]$; moreover, $t[a + c/(1+r)] < 1$.*

The tax cash flow is similar to a cash flow from a European call option. McDonald and Siegel (1984) show how to value this option when the underlying asset has a rate-of-return shortfall. The valuation of the non-linear cash flow is specified as follows:

Assumption 6: *A claim to a period-1 cash flow $\max(0, P - K)$, where K is any positive constant, has a period-0 market value according to the model in McDonald and Siegel (1984).* The value can be written as

$$\varphi(P)N(z_1) - \frac{K}{1+r}N(z_2), \tag{19}$$

where

$$z_1 = \frac{\ln(\varphi(P)) - \ln(K/(1+r))}{\sigma} + \sigma/2, \quad z_2 = z_1 - \sigma, \tag{20}$$

N is the standard normal distribution function, and σ is the instantaneous standard deviation of the price.[29] To apply an absence-of-arbitrage argument for option valuation when there is a rate-of-return shortfall, forward or futures contracts for the output must be traded, or there must exist traded assets which allow the replication of such contracts. The validity of an option valuation formula in an economy with taxation is discussed, e.g., in McDonald (2006), p. 341. He concludes that "When dealers are the effective price-setters in a market, taxes should not affect prices."

The combination of the CAPM and the option pricing model relies on, e.g., the assumptions in Galai and Masulis (1976).[30] The CAPM will now be a single-beta version of the intertemporal CAPM of Merton (1973). Capital markets operate in continuous time, whereas investment, production and taxes happen at discrete points in time.

In what follows it is assumed that the exogenous variables β_P and σ can be seen as unrelated as long as $\sigma > 0$, cf. footnote 14 in McDonald and Siegel (1986). A change in σ could be interpreted as, e.g., additive or multiplicative noise in P, stochastically independent of the previous (P, r_m).[31]

Propositions 4–6 are shown in Appendix B:

Proposition 4: *Under Assumptions 1, 2, 4–6, the beta of equity is given by (21).*

$$\beta_{RA} = \frac{1 - ta - tN(z_{2D})\frac{c}{1+r}}{1 - ta - tN(z_{2D})\frac{c}{1+r}(1-\alpha)}\beta_P, \tag{21}$$

where z_{2D} is given by

$$z_{2D} = \frac{1}{\sigma} \ln \left(\frac{1 - ta - tN(z_{2D}) \frac{c}{1+r}}{\alpha[1 - tN(z_{2D} + \sigma)] \frac{c}{1+r}} \right) - \frac{\sigma}{2}. \tag{22}$$

Although this equation cannot be solved explicitly, it determines z_{2D} implicitly as function of $t, a, c/(1+r), \sigma$, and α. The rate-of-return shortfall (or convenience yield) does not affect the ratio β_{RA}/β_P.

Proposition 5: *Under Assumptions 1, 2, 4–6, the beta for a marginal investment taxed together with the optimally chosen DRS investment is given by (23).*

$$\beta_{RM} = \frac{1 - ta - tN(z_{2D}) \frac{c}{1+r}}{1 - ta} \beta_P. \tag{23}$$

This means that the relationship between marginal and average beta is similar to that of the previous case, which had full certainty about the tax position. There is an extra term containing $tc(1 - \alpha)$ subtracted in the denominator of the average beta.

The two equations (23) and (21) should be compared with (8) and (15). Clearly the effect of the uncertainty in the tax position is similar to a reduced tax rate in period 1, reflecting that the probability of receiving the tax deductions is less than one hundred percent.

For comparison, the marginal beta in the stand-alone CRS case can be found. This is denoted RC because it only applies if the project actually has constant returns to scale. The probability of being in tax position is lower in this case.

Proposition 6: *Under Assumptions 1, 2, 5, and 6, the beta for a marginal investment taxed alone is given by (24).*

$$\beta_{RC} = \frac{1 - ta - tN(z_{2C}) \frac{c}{1+r}}{1 - ta}, \tag{24}$$

where z_{2C} is given by

$$z_{2C} = \frac{1}{\sigma} \ln \left(\frac{1 - ta - tN(z_{2C}) \frac{c}{1+r}}{[1 - tN(z_{2C} + \sigma)] \frac{c}{1+r}} \right) - \frac{\sigma}{2}, \tag{25}$$

which is the limit of (22) as $\alpha \to 1$.

This is the case considered in Lund (2002a), except that equation (25) was not given there. Table I summarizes the five subcases considered. The rightmost column gives the ratio of β_i (the beta of equity) to β_P in each subcase i. TABLE I HERE.

How the marginal and average betas depend on t, σ, and α has been traced through numerical solution to the non-linear equations.[32] All cases considered have $a = 0$ and the ratio $c/(1 + r)$ fixed at $1/1.05$. The central parameter configuration considered is $t = 0.35, \sigma = 0.3$. These are not unreasonable numbers (when the time unit is one year). For simplicity the verbal discussion below will assume $\beta_P = 1$. The five equity betas, divided by β_P, are shown in Figure I as functions of the scale elasticity α. A sixth relevant curve for comparison would be β_P itself, horizontal at 1.0 in the diagram. This would be the beta of equity without taxation or with pure cash flow taxation. FIGURE I HERE.

Figure I shows that the betas have the expected properties. Consider first Case F with riskless tax shields. The sparsely dotted horizonal line gives the marginal β_{FM}, while the heavily/infrequently dashed curve gives the average β_{FA}. The first one is a constant, independent of α. The numerical value, approximately 0.67, is close to $(1 - t)\beta_P$. The average beta declines from β_P to β_{FM} as α goes from zero to unity. The relationship is slightly convex. The upper limit, equal to β_P, comes from the fact that the relative weight on the final term in (10) goes to zero. In the limit as $\alpha \to 0^+$, the future cash flow is proportional to P and has the same systematic risk as P. The ratio β_{FM}/β_{FA} increases towards unity as $\alpha \to 1^-$, as mentioned above, because the whole project approaches a marginal project at this limit.

In Case F the effect on β_{FA} of varying α comes through the changing relative weights of two cash flow elements, one proportional to P, the other risk free. This effect is still present in Case R with risky tax shields. But here there is another, opposing effect: A higher α reduces the probability of being in tax position in period 1. This affects both marginal and average beta in Case R. The densely dashed curve gives the β_{RM} of the marginal investment taxed together with the inframarginal investment. This is increasing and convex as function of α. As $\alpha \to 1^-$, the technology approaches CRS, and the risk of the tax shields increases. The upper limit is thus equal to the β_{RC} of a marginal investment taxed alone. The lower limit, when $\alpha \to 0^+$, is equal to the marginal β_{FM} when the tax shields are risk free. In this limit there is so much income, relative to the investment, that the probability of not paying taxes goes to zero. The solid curve gives the average β_{RA} for the case of risky tax shields. For small α values there is no detectable difference between this and β_{FA}, since the risk is minuscule. As α increases towards unity, β_{RA} approaches β_{RM} from above, since the DRS investment approaches a CRS investment. The feature that β_{RA} is a nonmonotonic function of α is not so easy to explain (and may not be true for all parameter configurations).

The results on β_{RA} can be compared with those of Jacoby and Laughton (1992), although their numerical examples are more complicated, involving also various degrees of operating leverage. In their Figure 5 the systematic risk of the net after-tax cash flow decreases monotonically with increasing rent, which is consistent with the right-hand increasing part of the β_{RA} curve shown in Figure I here. Rents increase to the right in their Figure 5, to the left in Figure I here. The convexity is qualitatively the same in both curves. Their conclusion (p. 44) that "the larger fields will be undervalued relative to the smaller fields if all are discounted with the same discounting structure, as they would be using standard DCF methods" is true within the range they cover, but not in general, due to the non-monotonicity demonstrated here.

Clearly, even the DRS case with risky tax shields can have betas substantially lower than β_P. In this case the marginal beta curve, β_{RM}, satisfies the intuition that it has less risk than the stand-alone marginal beta, β_{RC}, as an effect of being taxed together with an infra-marginal cash flow. But the average beta, β_{RA}, does not exhibit this property uniformly, and in fact, the difference between marginal and average beta is just as large in this case as in the case with risk free deductions. The convexity of the curves strengthens the feature that tax shields, and thus after-tax equity, have relatively low systematic risk when there are moderately decreasing returns to scale (say, $0.6 < \alpha < 0.9$).

Figures II and III show some sensitivities to changes in the tax rate, t, and the volatility, σ. The three non-constant curves from Figure I are reproduced as (similarly) dotted curves, and the corresponding three curves for the new value of t or σ are drawn as dashed or solid. The values of the constant β_{FM} and β_{RC} are now only shown implicitly, as the endpoint values for the curves.

Figure II shows that all betas are decreased if the tax rate is raised (and vice versa), which was also the main point in Lund (2002a) for the cases considered there. The effect on the lowest values (β_{FM}, which is the limit of β_{RM} for low α, and of β_{FA} for high α) seems to be proportional to $(1-t)$, which is almost correct when $c/(1+r)$ is close to unity, see also Corollary 2.2 in Lund (2002a). For a given α, the ratio β_{FM}/β_{FA} is decreasing in t (i.e., the two betas differ more with higher t), as mentioned above.

FIGURE II HERE.

Figure III shows only one β_{FA} curve, as this is unaffected by a change in volatility. The figure shows that except for this, a lower σ works in the same direction as a higher t. But the effects of changes in σ are only discernible for higher values of α, and the magnitudes of the effects are not very large. The effects of σ on the ratio β_{FA}/β_P are robust results in

FIGURE III HERE.

the sense that they do not rely on any assumption about the relationship between σ and β_P. However, the effects of σ on β_{FA} (separately) could also include effects via possible changes in β_P, which have not been analyzed here.[33]

5 Cost of capital before taxes

The cost of capital before corporate taxes is the traditional measure for the effects of the tax system on the acceptance or rejection of real (non-financial) investment projects. This determines the possible distortionary effects of the tax system, although the present paper does not discuss what would be the relevant basis for comparison in various circumstances.

The expected rate of return before corporate taxes, plus 1, is $E(P)Q/I$, which can be rewritten as

$$\frac{E(P)Q}{I} = \frac{E(P)}{\varphi(P)} \cdot \frac{\varphi(P)Q}{I}. \tag{26}$$

Of the two fractions on the right hand side, the first is assumed to be exogenous, and is given by (1) and (3). The second is determined by the requirement that the project should be marginal after tax. For Case F above, this requirement is given by (7), which means that one plus the required expected rate of return before corporate taxes is

$$\frac{E(P)}{\varphi(P)} \cdot \frac{1 - ta - t\frac{c}{1+r}}{1-t}. \tag{27}$$

The distortion in "one plus the expected rate of return" is the second fraction, which appears in Hall and Jorgenson (1969), p. 395. The distortion is independent of (total and systematic) risk, only a function of tax parameters and the risk free interest rate.

For Case R with an uncertain tax position, the relevant $\varphi(P)Q/I$ ratio for a marginal investment taxed together with inframarginal investment is given in equation (B13) in

Appendix B. One plus the required expected rate of return is

$$\frac{E(P)}{\varphi(P)} \cdot \frac{1 - ta - tN(z_{2D})\frac{c}{1+r}}{1 - tN(z_{1D})}. \tag{28}$$

Again the distortion is independent of systematic risk, but now it depends on total risk through the $N(\cdot)$ expressions.

The following proposition summarizes:

Proposition 7: *Under Assumptions 1–3 the required expected return before corporate taxes is given by (27). It is decreasing in a and $c/(1+r)$. It is increasing in the tax rate if $a + c/(1+r) < 1$. The distortion from the tax system does not depend on total or systematic risk. Under Assumptions 1, 2, 4–6, the required expected return before corporate taxes is given by (28). The distortion from the tax system depends on total risk, but not on systematic risk as long as total risk is unchanged.*

The simplicity of the results may be their most surprising feature.

6 Tax deduction for entry costs?

This section investigates whether there are some conditions for industry equilibrium which would undermine the results from the DRS model. The question arises since the existence of rents will attract entry of new firms. The question has not been raised in the studies cited above. It will be shown that the marginal and average betas differ except under the combination of two conditions: Entry costs exactly outweigh quasi-rents for each firm, and the tax treatment of entry costs is equal to the tax treatment of subsequent investment costs.

Under uncertainty it is natural to assume that access to a unique resource or technology to some extent is the result of a random, risky process. An extreme assumption is that firms undertake R&D (or exploration for natural resources) with negligible costs and very low success probability. Call this Case A. Under Case A, a small number of firms will have been lucky, and find themselves in the situation described by the model of the previous sections. These firms have only negligible tax deductions for R&D costs. The opposite extreme is that all firms which have access to the DRS investment opportunity described here, have paid the same entry cost, and that the after-tax entry cost is equal to the after-tax net value of the investment opportunity (after deduction of $I(1-ta)$). Call this Case Z. It will be shown below that the important question for the results of this study is to which extent there may be tax deductions for the entry costs, in particular in period 1.

The rest of this section considers Case Z. It should be clear that there may be intermediate cases between the extremes, in which those firms that have access to the opportunity, have paid some entry cost, but not as much as the net value they obtain from the opportunity. A detailed model of the entry process, its industry equilibrium, and the tax shield consequences of this is omitted here.[34]

Assumption 7: *An entry cost M is paid for the right to undertake the investment project. This is competitively determined among firms with the same tax position, so that the net value to the firm of paying this entry cost, undertaking the project in optimal scale, and paying taxes, is zero. The sequence of events in period 0 is as follows: (a) The authorities determine the tax system for both periods. (b) The firm pays the entry cost M. (c) The firm determines how much to invest, I.*

In addition to tax deductions defined in Assumption 5, there are tax deductions bM in period 0 and hM in period 1, where b and h are constants in the interval $[0,1]$.

To distinguish the expressions from those above, this situation will be called Case G (for Generalized model). The extension of Case R will be developed, while the similar extension of Case F can be found by setting the probabilities (the $N(\cdot)$ expressions) equal to unity.[35]

Under Assumption 7 there is no economic difference between the two costs, M and I. Only their sum matters to the firm, and the produced quantity might as well have been written as a function of their sum, $M + I$. With no difference between the two, the model would fail to capture the idea of decreasing returns, i.e., the marginal investment project being taxed together with inframarginal investment.

But even under Assumption 7 (i.e., Case Z), the average betas are relevant if there is a difference between the tax treatments of the two costs. The entry cost could be immediately deductible, deductible in the production period, not deductible at all, or some combination of these. The cash flow to equity in period 1 is

$$X_G = Pf(I) - t \cdot \max(Pf(I) - cI - hM, 0). \tag{29}$$

The valuation, as of one period earlier, of a claim to this is

$$\varphi(X_G) = \varphi(P)f(I) - t\left[\varphi(P)f(I)N(z_{1G}) - \frac{cI + hM}{1 + r}N(z_{2G})\right], \tag{30}$$

where

$$z_{1G} = \frac{\ln(\varphi(P)f(I)) - \ln\left(\frac{cI+hM}{1+r}\right)}{\sigma} + \frac{\sigma}{2}, \tag{31}$$

and

$$z_{2G} = z_{1G} - \sigma. \tag{32}$$

Proof of the following proposition is in Appendix C.

Proposition 8: *Under Assumptions 1, 2, 4–7, the beta of equity is given by (33). When there is no deduction for M in period 1 ($h = 0$), then $\beta_{GA} = \beta_{RA}$ (of equation (21)). When the two costs M and I are treated equally by the tax system ($a = b, c = h$), then $\beta_{GA} = \beta_{RC}$ (of equation (24)).*

The average beta in the general case is given by

$$\beta_{GA} = \frac{1 - ta - \frac{tcN(z_{2G})}{1+r}}{1 - ta - \frac{tcN(z_{2G})}{1+r}(1-\alpha) + \frac{thN(z_{2G})}{1+r} \cdot \frac{(1-\alpha)[1-t(a+\frac{cN(z_{2G})}{1+r})]}{[1-t(b+\frac{hN(z_{2G})}{1+r})]}}\beta_P, \tag{33}$$

where $z_{2G} =$

$$\frac{1}{\sigma}\left[\ln\left(\frac{1 - ta - \frac{tN(z_{2G})c}{1+r}}{1 - tN(z_{2G} + \sigma)}(1+r)\right) - \ln\left(c + h\frac{(1-\alpha)[1 - t(a + \frac{cN(z_{2G})}{1+r})]}{\alpha[1 - t(b + \frac{hN(z_{2G})}{1+r})]}\right) - \ln(\alpha)\right] - \frac{\sigma}{2}. \tag{34}$$

Only the two special cases will be discussed. The first case implies that b does not matter for the results when $h = 0$, which is due to the fact that the equilibrium $M(1 - tb)$ is determined endogenously. A higher (lower) b will lead to a higher (lower) M, keeping equilibrium $M(1 - tb)$ unaffected, and when $h = 0$, only $M(1 - tb)$ matters, not M separately. For instance, the two subcases $(b = 0, h = 0)$ and $(b = 1, h = 0)$ give the same beta of equity, β_{RA}, despite the very different tax treatment of M. In these cases with $h = 0$, the difference between marginal and average beta does not go away.

When the two costs are treated equally, the firm's whole activity can be seen as a marginal investment project. In relation to the issues analyzed in this paper, there is nothing which distinguishes this from a case of constant returns to scale, except that the scale of production is determined. The equality of tax treatment, $a = b$ and $c = h$, is an extreme case within the extreme Case Z. Only for this combination of circumstances will the average beta lose its relevance. There are many possible configurations of a, b, c, and h which may be combined with Case Z. Also, Case R above covers at least two interesting possibilities within Case Z, that the entry cost is immediately deductible, and that it is not deductible at all. More generally, outside of Case Z, it also covers the case of negligible entry costs, which was called Case A.

7 Discussion

A seemingly critical assumption in the paper is Assumption 6 on option-like valuation of non-linear cash flows. The underlying assumptions were not detailed, since they are well known. It should be observed that option-like valuation is not limited to the geometric Brownian motion which is most often used. Other processes have been assumed in some studies, and some of them also allow for analytical solutions. Bradley (1998) considers two alternative stochastic processes for the output price. Likewise, Assumption 1 on the CAPM can be relaxed. The crucial assumption is that the risk measure is linear.

There are of course several limitations of the analysis. Leverage effects from debt and operating costs have been left out. If riskless debt is introduced, it is clear (see equation (17)) that the analysis is mainly relevant to characterize the return on equity. The possibility that a multinational may want to change the formal financing of its subsidiaries due to tax changes is neglected here.

Among other simplifications, the production function has a constant elasticity. The uncertainty is multiplicative, which may not be necessary for the model to work (cf. Lund 2003a), but for the simplicity of the results, in particular in the case of risky deductions. The model does not allow for risky inflation, the effect of which would depend on the systematic risk of nominally risk free claims. The source of uncertainty is a single stochastic variable in a single period, and there is no carry-forward or carry-back of losses, all of which exaggerates the risk of the deductions. On the other hand, operating costs in future periods would increase the risk of not being in tax position, thus reducing the risk-adjusted expected values of depreciation tax shields.

Lund (2002a) gives analytical results for a multi-period version of Case F.[36] The results are similar to those of section 3 above. As seen from period 0, the effect of investment-related deductions comes through their present value. But the variation over time in this effect is left for future research. The concept of true economic depreciation may be useful to simplify the picture.

In spite of all this, the model should be a step in the direction of more realism, while retaining the possibility of an analytical solution. Hopefully this can be helpful as a reference for numerical examples and empirical studies, when these include the factors which are left out here.

The results are of particular interest under rent taxation with high rates. From a practical viewpoint, serious mistakes will be made when the same discount rate is applied under high and low tax rates (or variations in a versus c). On the theoretical side, there has been a long discussion in the rent taxation literature on which discount rate to use for expected tax shields, or the firm's after-tax net cash flow, cf. the survey by Lund (2009). Under the Resource Rent Tax proposed by Garnaut and Clunies Ross (1975, 1979) the related question is which rate to use for carrying forward negative cash flows. While many have argued that a general number for the firm's cost of capital should be used, one could hope that newer research (relying on Myers (1974)) will lead to a revision of that view, cf. Lund (2002b).

8 Conclusion

An analysis of the required expected rate of return, i.e., the cost of capital, must identify the marginal project, a project with exactly zero net value. Several studies of tax effects on the after-tax cost of capital have neglected this, and instead analyzed tax effects on the systematic risk of projects with an exogenously given profitability. The present study has shown that when taxes allow investment-related deductions, like depreciation, in years after the investment is made, this gives rise to a difference between the systematic risk of the marginal project and the systematic risk of an inframarginal project. The former is the relevant one for determining the cost of capital.

With imperfect loss offset, the existence of inframarginal profits (rents) are nevertheless important to determine the systematic risk at the margin. This has been shown in an analytical model of a project with decreasing returns to scale, in which the scale of investment is chosen optimally by the firm. The tax cash flow is analogous to a European call option, and its value has been found by option valuation techniques in line with previous literature. A novelty in this connection is the closed-form solution for the probabilities in the option pricing formula.

For practitioners the results are important in showing the need to use either APV or different risk-adjusted discount rates under different tax systems. This is particularly relevant for multinational firms when tax systems differ much. Another result with practical implications concerns the methods for "unlevering" observed equity betas to find asset betas. This study shows how one must also "untax" observed betas if asset betas are to be applicable under different tax systems, and how one must "unaverage" observed betas if the observed companies are taxed and earn inframarginal profits.

The methods and results demonstrated are crucial for discussions on reforms of corporate income taxation and rent taxation. The results on after-tax required returns are at odds with current practices. Only if authorities, firms, and other participants agree on these methods can there be meaningful discussions.

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Appendix A

Proof of Propositions 1 and 2

Define $\hat{c} \equiv \frac{c}{1+r}$ and $\Delta \equiv 1-ta-t\hat{c}(1-\alpha)$. Signs of partial derivatives are needed. These are obtained for β_{FA} to prove Proposition 2. The proof of Proposition 1 follows from setting $\alpha = 1$. The partial derivatives are

$$\frac{\partial \beta_{FA}}{\partial t} = \beta_P \frac{-\hat{c}\alpha}{\Delta^2} < 0, \tag{A1}$$

$$\frac{\partial \beta_{FA}}{\partial a} = \beta_P \frac{-t^2\hat{c}\alpha}{\Delta^2} < 0, \tag{A2}$$

$$\frac{\partial \beta_{FA}}{\partial \hat{c}} = \beta_P \frac{-\alpha t(1-ta)}{\Delta^2} < 0, \tag{A3}$$

$$\frac{\partial \beta_{FA}}{\partial \alpha} = \beta_P \frac{-t\hat{c}(1-ta-t\hat{c})}{\Delta^2} < 0, \tag{A4}$$

q.e.d.

Proof of Proposition 3

Define $\bar{m} \equiv E(r_m) - r$. Observe that the expected return on a claim to one unit of the output price satisfies the CAPM: $E(P)/\varphi(P) = 1 + r + \beta_P \bar{m}$.

The maximand based on a risk-adjusted discount rate using the marginal beta is

$$\frac{E(P)f(I)(1-t)+tcI}{1+r+\beta_{FM}\bar{m}} - I(1-ta).$$

The proposition claims that maximization of this with respect to I gives the same result as (12). The first-order condition is

$$\frac{E(P)f'(I)(1-t)+tc}{1+r+\beta_{FM}\bar{m}} = 1-ta. \tag{A5}$$

Introduce the expression for β_{FM} from (8):

$$E(P)f'(I)(1-t)+tc = (1-ta)\left(1+r+\beta_P\bar{m}\frac{1-ta-t\hat{c}}{1-ta}\right). \tag{A6}$$

For $E(P)$ introduce the expression $\varphi(P)(1+r+\beta_P\bar{m})$, and find

$$\varphi(P)f'(I)(1-t) = \frac{(1+r)(1-ta-t\hat{c})+\beta_P\bar{m}(1-ta-t\hat{c})}{1+r+\beta_P\bar{m}} = 1-ta-t\hat{c}, \tag{A7}$$

which is (12). This proves the first part.

Consider now the other part of the proposition, that the average beta can be used, provided that it is considered as a function of I, i.e., $\beta_{FA} = \beta_{FA}(I)$ as defined by (14). The maximand using the average beta is

$$\frac{E(P)f(I)(1-t)+tcI}{1+r+\beta_{FA}(I)\bar{m}} - I(1-ta).$$

Introduce $E(P) = \varphi(P)(1+r+\beta_P\bar{m})$ and use equation (14) to rewrite the maximand as

$$\frac{\varphi(P)(1+r+\beta_P\bar{m})f(I)(1-t)+tcI}{1+r+\frac{\varphi(P)f(I)(1-t)}{\varphi(P)f(I)(1-t)+It\hat{c}}\beta_P\bar{m}} - I(1-ta) =$$

$$\frac{[\varphi(P)(1+r+\beta_P\bar{m})f(I)(1-t)+tcI][\varphi(P)f(I)(1-t)+It\hat{c}]}{\varphi(P)(1+r+\beta_P\bar{m})f(I)(1-t)+tcI} - I(1-ta),$$

which is the same maximand as in (11), q.e.d.

Appendix B

Proof of Propositions 4–6

This derivation starts with the average beta in Case R. In Case R the cash flow to equity in period 1 is

$$X_R = Pf(I) - t \cdot \max(Pf(I) - cI, 0). \tag{B1}$$

Under Assumption 6 the valuation, as of one period earlier, of a claim to this is

$$\varphi(X_R) = \varphi(P)f(I) - t\left[\varphi(P)f(I)N(z_{1D}) - \frac{cI}{1+r}N(z_{2D})\right], \tag{B2}$$

where

$$z_{1D} = \frac{\ln(\varphi(P)f(I)) - \ln\left(\frac{cI}{1+r}\right)}{\sigma} + \frac{\sigma}{2}, \tag{B3}$$

and

$$z_{2D} = z_{1D} - \sigma. \tag{B4}$$

The expression in square brackets in (B2) can be rewritten in terms of the standard Black and Scholes' formula for option pricing as $C(\varphi(P)f(I), cI, 1, r, \sigma)$, so that

$$\varphi(X_R) = \varphi(P)f(I) - tC(\varphi(P)f(I), cI, 1, r, \sigma). \tag{B5}$$

The firm chooses I to maximize $\pi_R(I) \equiv \varphi(X_R) - I(1 - ta)$. The first-order condition is

$$\varphi(P)f'(I) = \frac{\left(1 - ta - tN(z_{2D})\frac{c}{1+r}\right)}{(1 - tN(z_{1D}))}. \tag{B6}$$

Introducing the constant-elasticity production function gives

$$\varphi(P)f(I)(1 - tN(z_{1D})) = \frac{I}{\alpha}\left(1 - ta - tN(z_{2D})\frac{c}{1+r}\right). \tag{B7}$$

The claim is equivalent to holding a portfolio with $f(I)(1-tN(z_{1D}))$ claims on P, and the rest risk free. The beta is a value-weighted average of the betas of these two elements, i.e.,

$$\beta_{RA} = \frac{\varphi(P)f(I)(1-tN(z_{1D}))}{\varphi(X_R)}\beta_P. \tag{B8}$$

Here, the subscript RA is introduced to show that this is the average beta in Case R. By introducing the expression for $\varphi(X_R)$ from (B2) and the constant-elasticity production function, this can be simplified as

$$\beta_{RA} = \frac{1-ta-tN(z_{2D})\frac{c}{1+r}}{1-ta-tN(z_{2D})\frac{c}{1+r}(1-\alpha)}\beta_P. \tag{B9}$$

It is also possible to express z_{1D} and z_{2D} in terms of exogenous variables, including the elasticity α, avoiding the decision variables of the firm. Plug in from the first-order condition (B7) into (B3)–(B4) to find equation (22) in the main text.

To derive the marginal beta for the same case, consider first the marginal beta derived in Lund (2002a) for the case with an uncertain tax position, equation (24) in that paper. That paper's equation (23) becomes

$$\gamma = \frac{1-ta-tN(z_{2C})\frac{c}{1+r}}{1-tN(z_{1C})}, \tag{B10}$$

and the marginal beta can be written

$$\beta_{RC} = \left(1-ta-tN(z_{2C})\frac{c}{1+r}\right)\beta_P. \tag{B11}$$

The subscript RC (C for CRS) is used here since the case considered in Lund (2002a) did not include the marginal project with some other activity, i.e., as if the case had constant returns to scale.

Again it is possible to express z_{2C} in terms of the exogenous parameters. In this case there is no first-order condition for an interior profit maximum, but the definition of a marginal CRS project, which gives

$$\frac{\varphi(P)Q}{I} = \frac{1 - ta - tN(z_{2C})\frac{c}{1+r}}{1 - tN(z_{1C})}, \tag{B12}$$

cf. equations (5) and (23) in Lund (2002a). This leads to equation (25) in the main text.

What then about the marginal beta for the DRS case? This can be seen as a mixture of the two cases just considered. The marginal beta characterizes a small investment which has a net value of zero. Under imperfect loss offset the value will depend upon the probability of being in tax position. In particular this is crucial in Case R, for which it is assumed that after period one there are no more periods, so that the loss cannot be carried forward (nor backward). The criterion for the project being marginal looks similar to (B12), but in this case the valuation of the option-like cash flow to the marginal project in period 1 is based on the risk-adjusted probabilities $N(z_{1D})$ and $N(z_{2D})$, not $N(z_{1C})$ and $N(z_{2C})$, since they should now reflect the probabilities that the whole DRS project is in tax position at the margin. The project which invests I to yield Q, and which is taxed together with the optimally scaled DRS project, is marginal when

$$\frac{\varphi(P)Q}{I} = \frac{1 - ta - tN(z_{2D})\frac{c}{1+r}}{1 - tN(z_{1D})}. \tag{B13}$$

The marginal beta in the DRS case becomes

$$\beta_{RM} = \frac{1 - ta - tN(z_{2D})\frac{c}{1+r}}{1 - ta}\beta_P, \tag{B14}$$

with z_{2D} given from (22) in the main text.

Appendix C

Proof of Proposition 8

The firm chooses I to maximize $\pi_G(I) \equiv \varphi(X_G) - I(1-ta)$. From the first-order condition follows

$$\varphi(P)f(I)(1-tN(z_{1G})) = \frac{f(I)\left(1-ta-\frac{tc}{1+r}N(z_{2G})\right)}{f'(I)}. \tag{C1}$$

Introducing the constant-elasticity production function gives

$$\varphi(P)f(I)(1-tN(z_{1G})) = \frac{I}{\alpha}\left(1-ta-\frac{tc}{1+r}N(z_{2G})\right). \tag{C2}$$

Equilibrium M is given by

$$M(1-tb) = \varphi(X_G) - I(1-ta)$$

$$= \frac{I}{\alpha}\left(1-ta-\frac{tc}{1+r}N(z_{2G})\right) + \frac{tcIN(z_{2G})}{1+r} - I(1-ta) + \frac{thMN(z_{2G})}{1+r}, \tag{C3}$$

which can be solved for

$$M = I\frac{(1-\alpha)[1-t(a+\frac{cN(z_{2G})}{1+r})]}{\alpha[1-t(b+\frac{hN(z_{2G})}{1+r})]}. \tag{C4}$$

The ratio of the expressions in square brackets in the numerator and the denominator contains the effect of the different tax treatment (if any) of I and M, respectively, in risk-adjusted expected present value terms.

We can now solve for $\varphi(X_G) =$

$$\frac{I}{\alpha}\left[1-ta-\frac{tcN(z_{2G})}{1+r}(1-\alpha) + \frac{thN(z_{2G})}{1+r}\cdot\frac{(1-\alpha)[1-t(a+\frac{cN(z_{2G})}{1+r})]}{[1-t(b+\frac{hN(z_{2G})}{1+r})]}\right]. \tag{C5}$$

This gives the average beta for this case,

$$\beta_{GA} = \frac{1-ta-\frac{tcN(z_{2G})}{1+r}}{1-ta-\frac{tcN(z_{2G})}{1+r}(1-\alpha)+\frac{thN(z_{2G})}{1+r}\cdot\frac{(1-\alpha)[1-t(a+\frac{cN(z_{2G})}{1+r})]}{[1-t(b+\frac{hN(z_{2G})}{1+r})]}}\beta_P, \tag{C6}$$

with $z_{2G} =$

$$\frac{1}{\sigma}\left[\ln\left(\frac{1-ta-\frac{tN(z_{2G})c}{1+r}}{1-tN(z_{2G}+\sigma)}(1+r)\right)-\ln\left(c+h\frac{(1-\alpha)[1-t(a+\frac{cN(z_{2G})}{1+r})]}{\alpha[1-t(b+\frac{hN(z_{2G})}{1+r})]}\right)-\ln(\alpha)\right]-\frac{\sigma}{2}. \tag{C7}$$

In the first special case, when $h = 0$, the fraction

$$\frac{(1-\alpha)[1-t(a+\frac{cN(z_{2G})}{1+r})]}{[1-t(b+\frac{hN(z_{2G})}{1+r})]},$$

which appears in both (C6) and (C7), vanishes, since it is multiplied by h. We find $z_{2G} = z_{2D}$ (of equation (22)), and $\beta_{GA} = \beta_{RA}$ (of equation (21)).

In the second special case, when $a = b$ and $c = h$, α vanishes from both (C6) and (C7), since the last two terms in the large square brackets in (C7) are reduced to

$$-\ln\left(c+h\frac{1-\alpha}{\alpha}\right)-\ln(\alpha) = -\ln\frac{c\alpha+c-c\alpha}{\alpha}-\ln(\alpha) = -\ln(c). \tag{C8}$$

Thus we find $z_{2G} = z_{2C}$ (from (25)), and $\beta_{GA} = \beta_{RC}$ (from (24)), q.e.d.

Table I: Beta of equity for the five subcases, divided by β_P

	$N(z_2)$	α	marginal vs. average	β_i	β_i/β_P
Case F	1	1	marginal	β_{FM}	$\dfrac{1-ta-t\frac{c}{1+r}}{1-ta}$
		$\in (0,1)$	average	β_{FA}	$\dfrac{1-ta-t\frac{c}{1+r}}{1-ta-t\frac{c}{1+r}(1-\alpha)}$
Case R	$\in (0,1)$	1	marginal	β_{RC}	$\dfrac{1-ta-tN(z_{2C})\frac{c}{1+r}}{1-ta}$
		$\in (0,1)$	marginal	β_{RM}	$\dfrac{1-ta-tN(z_{2D})\frac{c}{1+r}}{1-ta}$
		$\in (0,1)$	average	β_{RA}	$\dfrac{1-ta-tN(z_{2D})\frac{c}{1+r}}{1-ta-tN(z_{2D})\frac{c}{1+r}(1-\alpha)}$
Equations implicitly defining z_{2C} and z_{2D}:					
$z_{2C}=\dfrac{1}{\sigma}\ln\left(\dfrac{1-ta-tN(z_{2C})\frac{c}{1+r}}{[1-tN(z_{2C}+\sigma)]\frac{c}{1+r}}\right)-\dfrac{\sigma}{2}$					
$z_{2D}=\dfrac{1}{\sigma}\ln\left(\dfrac{1-ta-tN(z_{2D})\frac{c}{1+r}}{\alpha[1-tN(z_{2D}+\sigma)]\frac{c}{1+r}}\right)-\dfrac{\sigma}{2}$					

Figure I: β_i/β_P as functions of scale elasticity, α; $t = 0.35, \sigma = 0.3, c/(1+r) = 1/1.05$

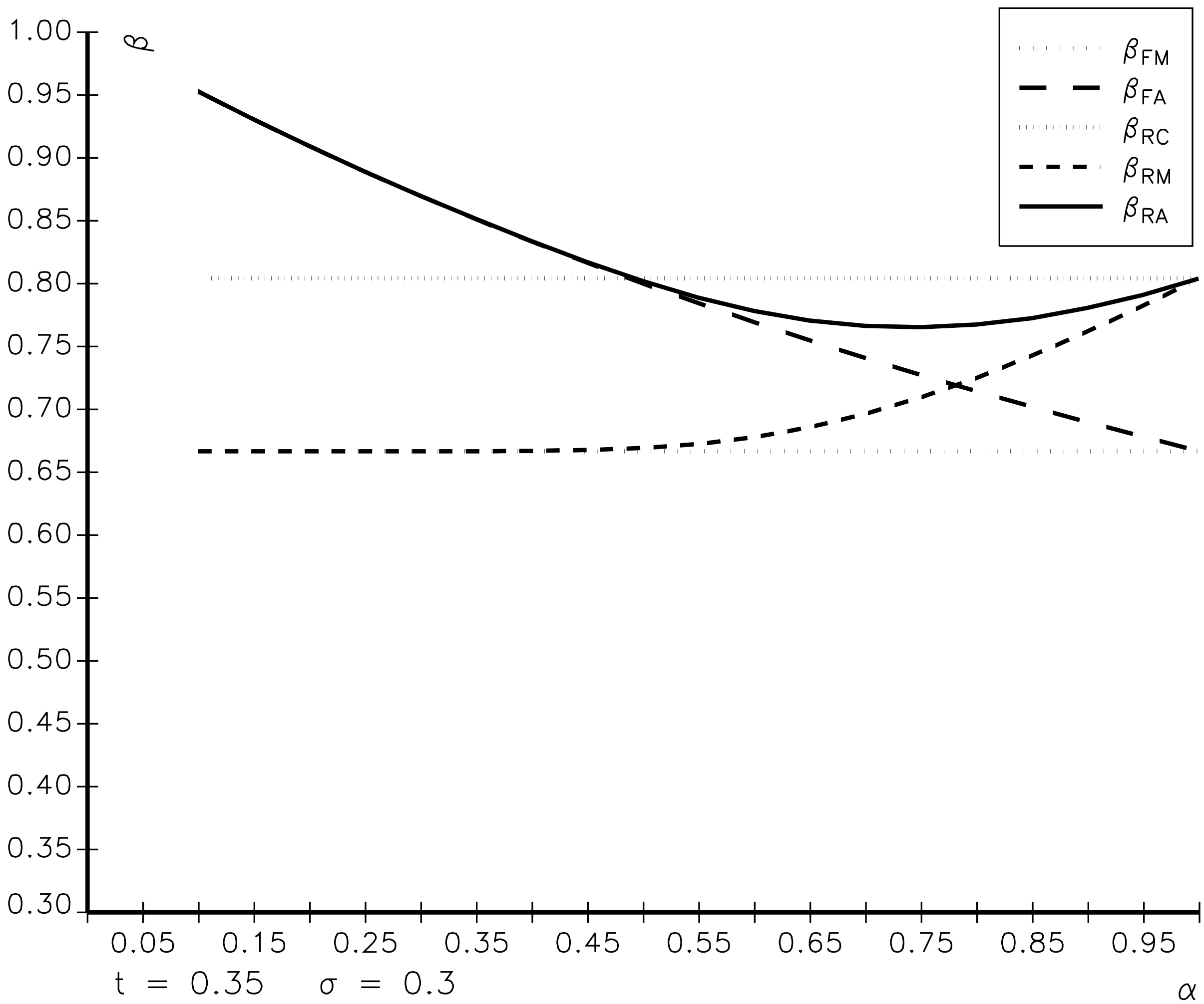


Figure II: β_i/β_P as functions of scale elasticity, α; varying the tax rate

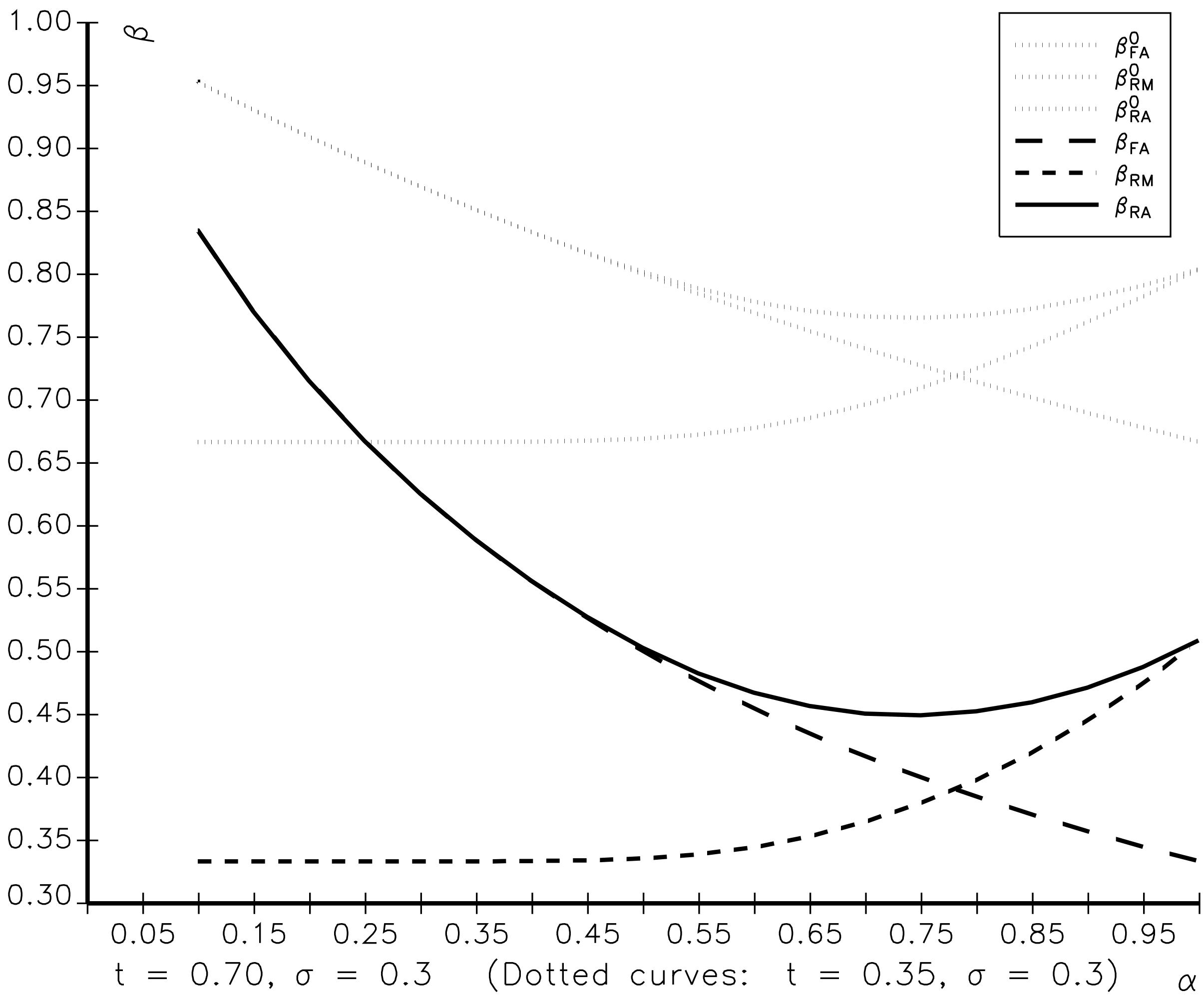
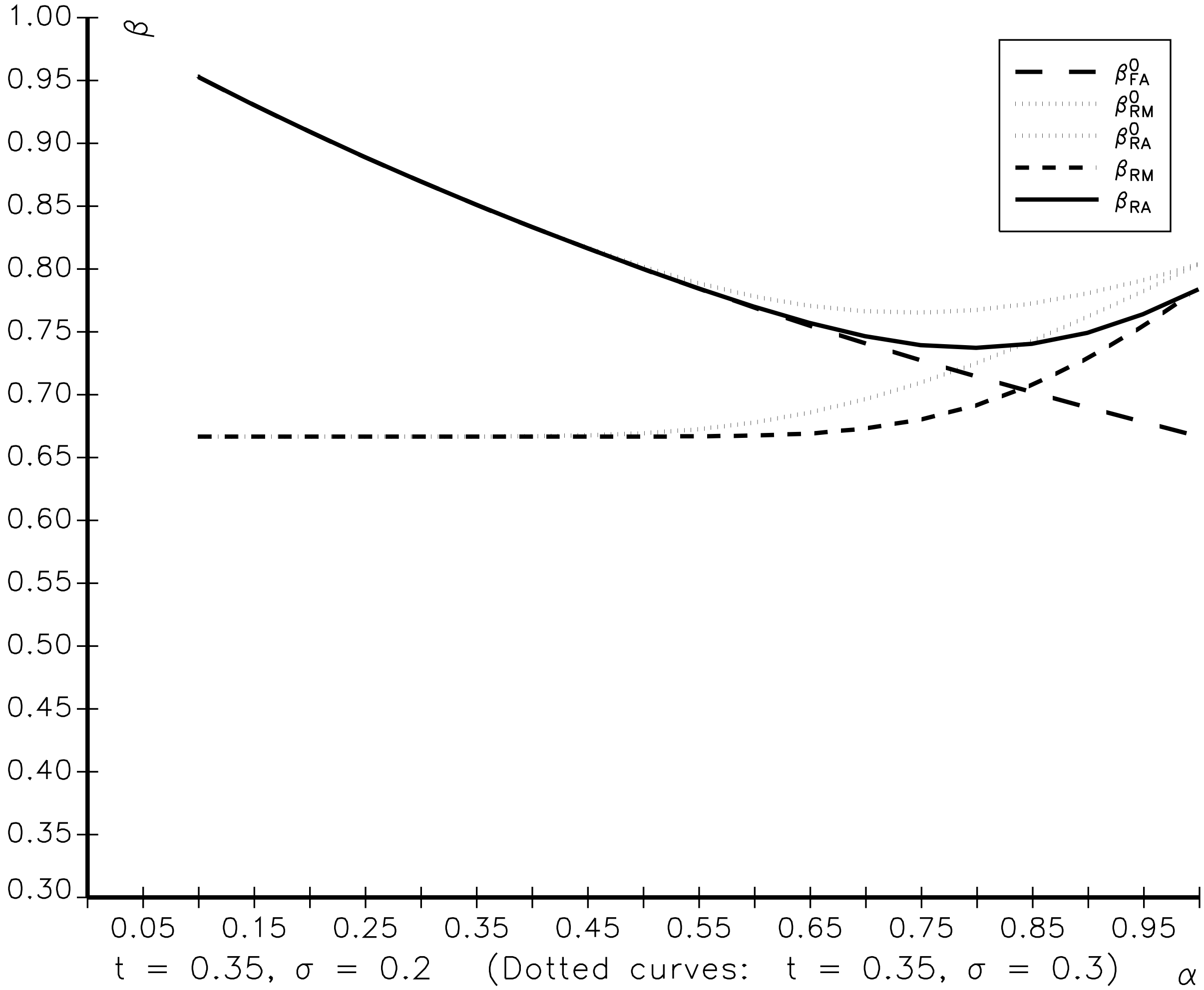


Figure III: β_i/β_P as functions of scale elasticity, α; varying the volatility



Notes

[1]See Lund (2002a), Proposition 4, which is a multiperiod extension of Proposition 1 of the present paper. Numbers rely on an interest rate of 5 percent, so that the present value of tax shields is 87.5 percent of investment. If the after-tax riskless rate had been 8.45 percent instead, as in Brealey, Myers, and Allen (2008), the tax shield present value had been 80.2 percent, and the tax rate increase would imply a beta reduction from 0.719 to 0.438.

[2]Some have argued that this system is not realistic. In the present paper the system is only used as a pedagogical tool. The relevance of the analysis does not in any way rely on the realism of a Brown tax.

[3]See Garnaut and Clunies Ross (1975, 1979), Fane (1987) and Bond and Devereux (1995). Such deductions are sometimes known as Allowance for Corporate Equity.

[4]Without loss offset the government's tax claim is analogous to a call option on the firm's tax base, cf. Ball and Bowers (1983), Green and Talmor (1985), Majd and Myers (1985, 1987).

[5]See, e.g., Brealey, Myers, and Allen (2008), p. 438.

[6]See, e.g., Brealey, Myers, and Allen (2008), p. 488.

[7]This is different from equilibrium effects of taxes on the shareholders, see footnote 17.

[8]See, e.g., Brealey, Myers, and Allen (2008), pp. 546f. The APV method is seen as "most useful when financing side effects are numerous and important" (p. 549), but there is no mention of tax effects apart from interest tax shields.

[9]Brealey, Myers, and Allen (2008) just describe a practice, and do not endorse it. They have another section called "APV for International Investment" (p. 549). But this section does not focus on taxes.

[10]Lund (2003a) discusses their model and claims that a more realistic alternative turns their results around. An appendix in Lund (2003b) shows the exact relationship between the present paper and Lund (2003a).

[11]Salahor (1998) has results on these effects in the case of linear taxes, assuming that the firm always will be in tax position and that taxes are proportional.

[12]E.g., they state that "Our interest in this paper is not on the tax shield's risk *per se*" (Rao and Stevens 2006, p. 19f). But, "The sensitivity of interest rate and tax policy changes on firms' economic balance sheets, and hence investor's wealth can, in principle, be evaluated in our model" (p. 25).

[13]Pitts (1997) obtains analytical results in a similar model, but does not analyze the cost of capital.

[14]E.g., according to Marin and Schnitzer (2006), 68 percent of foreign direct investment by German firms in Eastern Europe was financed internally.

[15]See Galai (1988), p. 83f., and Pierru and Babusiaux (2008), who find a difference between marginal and average cost of capital, although for reasons different from those of

the present paper. The difference they find originates from differences in (statutory) tax rates.

[16]The tax system of a small host country would hardly affect the capital market in the home country. But even for tax changes in the U.S. (as host) one may assume that the international capital market is unaffected. Bulow and Summers (1984) indicate that this may be a reasonable assumption (their footnote 3).

[17]See Sick (1990), Benninga and Sarig (2003), Cooper and Nyborg (2008). Lund (2002a) includes this possibility, that the CAPM equation has a tax adjusted risk free interest rate, possibly induced by differential taxation at the personal level. Since the model in the present paper has no debt financing, the variable r has only one interpretation, that r which appears as intercept in the CAPM equation, which is also the firm's after-tax discount rate for riskless cash flows.

[18]The product price has what McDonald and Siegel (1984) call an (expected-)rate-of-return shortfall.

[19]In parts of the literature the nominal sum of deductions, here $a+c$, is set to unity (e.g., King 1977, p. 232). In the present paper, a and c are considered as separate, exogenous variables, so that an increase in a is analyzed as if c is kept constant, and vice versa.

[20]This is a special case of Proposition 2 in Lund (2002a).

[21]Lund (2003a) points out that Levy and Arditti (1973) rely on an assumption which may be questionable, that the decision to reinvest indefinitely to compensate for depreciation is made at the beginning of a project.

[22]This level is the unspecified $\frac{V}{S}$ in equation (21S') in Galai (1998), who claims (e.g., bottom of p. 144) to find the cost of capital. Rao and Stevens (2006, p. 11) emphasize that their "analysis accommodates both positive and negative NPV firms." They claim (e.g., middle of p. 2) that the analysis leads to the cost of capital, but the present analysis shows that one must solve for the marginal project. The divergence between the results of Derrig (1994) and the present paper is similar, and is spelt out in Lund (2001).

[23]Jacoby and Laughton (1992, p. 44f) and Bradley (1998, p. 69f) do not claim to identify the required expected return. The discount rates they find are based on average betas and are appropriate for finding values of given projects. They do not explicitly recommend them for decision making.

[24]This is an exception to footnote 17 (see references there), which stated that, in the present paper, the riskless rate r would only be used as the r from the CAPM equation. Using it also as the borrowing rate means that the possible tax adjustments to the CAPM equation, mentioned in that footnote, are ignored. With interest deductibility this implies a preference for debt financing, well known from most of the literature. Alternatively, in a Miller (1977) equilibrium, the fraction within the square brackets in (16) could be equal to unity (Lund 2002a), even with interest deductibility, although hardly across many jurisdictions.

[25]See Modigliani and Miller (1963), p. 439.

[26]Under some circumstances, all elements of the WACC will be reduced by a factor of approximately $(1 - t)$: In the cost-of-equity component, the riskless rate possibly by one minus the home-country tax rate (see footnote 17), and β_{FMB} due to investment-related deductions.

[27]Shevlin (1990) describes intermediate cases.

[28]The present paper improves upon the solution for the case considered in Lund (2002a), by pointing out that the variables z_1 and z_2, called x_1 and x_2 in equation (19) in that paper, can be rewritten in terms of the exogenous parameters, given that the production function has a constant elasticity. Observe in particular that whereas the option value in general depends on a rate-of-return shortfall, this dependency disappears here, given that the first-order condition of the firm is satisfied.

[29]As shown in any textbook in finance, $N(z_2)$ is a risk-adjusted probability for the option to be exercised, and $N(z_1)$ multiplies this with a conditional expectation. See, e.g., McDonald (2006).

[30]An alternative would be to rely on an approximate Arbitrage Pricing Theory. Rao and Stevens (2006) rely on this for a related analysis, assuming ad hoc that the approximate valuation equation holds with equality. For the purpose of the present paper, to use the option pricing formula applied here, one must in addition assume that the output price follows a geometric Brownian motion. Leland (1999) points out weaknesses in combining option pricing models with the CAPM. The differences between standard betas and the Bs suggested by Leland are small in relation to the effects pointed out in the present paper.

[31]Davis (2002) argues that covariances are likely to change when volatilities of commodity prices change, but does not give any arguments for his assumption that correlations are unchanged. See also the discussion in section 4 of Lund (2005b). Of course, the method used here does not mean that σ could be zero while β is different from zero.

[32]An attempt at finding the signs of analytical partial derivatives is included in Lund (2005a), pp. 23–28. Ad hoc assumptions were needed, so the attempts are not included here.

[33]See footnote 31.

[34]The equilibrium will depend on, e.g., whether there is diversification of the entry costs or they are paid by risk averse entrepreneurs, moreover, whether entry is like a lottery with no systematic risk, or a process which in itself has systematic risk, and perhaps decreasing returns to scale.

[35]The term Case G denotes a tax system, which might have been combined with either Case R or Case F, which distinguish whether tax shields are risky or not. Each of these combinations, GR and GF, might have been considered in combinations with either Case A, Case Z, or an intermediate case. These refer to the amount of entry cost in relation to the subsequent net value for those firms which obtain access to the investment opportunity. Since Case A has negligible entry costs, there is no need to consider it in conjunction with tax system Case G. But intermediate cases might have been considered.

[36]Jacoby and Laughton (1992) and Bradley (1998) give numerical results in multi-period models with imperfect loss offset.

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