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Returns-to-scale Properties in DEA Models: The Fundamental Role of Interior Points

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RETURNS-TO-SCALE PROPERTIES IN DEA MODELS:

THE FUNDAMENTAL ROLE OF INTERIOR POINTS

by

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Abstract: Attempts can be found in the DEA literature to identify returns to scale at efficient interior points of the production possibility set on the basis of returns to scale at points of the corresponding reference sets. However, an opposite approach is put forward in this paper, advocating that returns-to-scale properties of efficient reference units should be found by identifying first returns to scale of an efficient interior unit that is a radial projection to the frontier of an inefficient unit. Returns-to-scale properties of both the corresponding reference units and units supporting the face in question can then be established.

Keywords: Returns to scale, DEA, Interior points, Vertices

JEL classifications: C61, D20

1 Introduction

Three types of units are mainly considered in DEA models: projections of inefficient points on the frontier, either in input- or output-oriented radial direction (most of these points are interior points of some faces of the frontier), the reference sets for every observed unit, and the set of efficient units spanning the face on which the projection to the frontier of an inefficient unit is located. The latter set may contain several reference sets for every projection.

A question then arises: is there any connection between the returns-to-scale properties of the units of these three sets? In the DEA literature (Tone 1996; Tone 2005; Cooper et al. 2000, 2006), attempts were made to answer partially this question, given that we know the returns-to-scale properties of the reference sets. At first sight, this approach may seem reasonable. Moreover, getting results in this form may be regarded natural because standard optimisation software produces solutions in DEA models in the form of reference sets consisting of efficient units, thus it may seem natural to start with the reference sets.

Our approach involves all the three sets introduced above. However, we consider to take quite an opposite approach than mentioned above: given the returns-to-scale properties of the projected point or interior point of some face, what can then be said of the returns-to-scale properties of not only the reference set, but also of the set of units spanning the face where the projected point is located?

Solving a DEA variable-returns-to-scale problem (Banker et al., 1984) (BCC) using standard software gives us both the solution to the primal problem and the dual problem containing: efficiency score (input- and/or output-oriented) for every unit, list of reference units for every unit, and shadow prices on the constraints in the DEA problem for every unit. From optimisation theory, we know that units that are vertices in the model will not have unique solutions for the shadow prices; this means that the supporting hyperplane is not unique (Banker et al., 1984).

However, on the basis of the primal and dual solutions we can determine the returns-to-scale properties of a projected point on the frontier provided it is an interior point of a face of full dimension (Førsund, 1996; Førsund and Hjalmarsson, 2004; Førsund et al., 2007, Victor et al., 2009). However, the returns-to-scale properties of reference units can only be found by solving two additional LP problems for every reference unit. Moreover, starting with the returns-to-scale properties of a projected point we can also ask: what is the connection between the returns-to-scale properties of this point and the set of units spanning the entire face?

In Section 2, returns to scale is defined, and we consider some examples that show that the returns to scale of the BCC-projected activity cannot be found by observing only the returns-to-scale characteristics of production units in its respective reference set. In Section 3 we prove some theorems which establish that the returns to scale of points on the facet determined by efficient units can be found by observing the returns-to-scale characteristic of some interior point of this facet. Section 4 concludes.

2 Problem statement

Returns to scale in neoclassical production theory

In neoclassical production theory, production possibilities are expressed at the abstract level by a transformation function

$$F(X,Y)=0, \tag{1}$$

where the input vector is $X=(x_1,\dots,x_m)\in E^m$ and the output vector is $Y=(y_1,\dots,y_r)\in E^r$. The transformation function represents efficient input-output combinations in that output are maximised for given inputs; it is assumed to be smooth, i.e., continuously differentiable. The transformation function (1) describes a hypersurface or a frontier in the multidimensional space E^{m+r} of inputs and outputs. Efficient points (X,Y) belong to this hypersurface.

The scale elasticity is a measure of the increase in output relative to a proportional increase in all inputs, evaluated as the marginal change at a point in input-output space. In a multi-output setting, a proportional increase in all outputs is most naturally used instead of an increase in a single output (see Hanoch, 1970; Starrett, 1977; Panzar and Willig, 1977). Expanding inputs proportionally by factor α, the maximal proportional expansion, $\beta = \beta(\alpha, X, Y)$ [with $\beta(1, X, Y) = 1$], of outputs allowed by the transformation function

$$F(\alpha X, \beta(\alpha, X, Y)) = 0 \tag{2}$$

is chosen. Scale elasticity, ε, as a function of inputs and outputs, is defined for a differentiable function as the marginal change in the output-expansion factor caused by a marginal change in the input-expansion factor over the average ratio, or

$$\varepsilon(X, Y) = \frac{\partial \beta(\alpha, X, Y)}{\partial \alpha} \cdot \frac{\alpha}{\beta}. \tag{3}$$

The rule for calculating scale elasticity is obtained by differentiating (2) with respect to the input-scaling factor

$$\frac{\partial F(\alpha X, \beta(\alpha)Y)}{\partial \alpha} = \sum_{i=1}^{m} \frac{\partial F(\alpha X, \beta(\alpha)Y)}{\partial(\alpha x_i)} x_i + \sum_{j=1}^{r} \frac{\partial F(\alpha X, \beta(\alpha)Y)}{\partial(\beta y_j)} y_j \frac{\partial \beta(\alpha)}{\partial \alpha} = 0. \tag{4}$$

Evaluating the derivatives at $\alpha = \beta = 1$ without loss of generality, and solving for the scale elasticity as defined by (3) yields:

$$\frac{\partial \beta(\alpha, X, Y)}{\partial \alpha} = \varepsilon(X, Y) = -\frac{\sum_{i=1}^{m} \frac{\partial F(X, Y)}{\partial x_i} x_i}{\sum_{j=1}^{r} \frac{\partial F(X, Y)}{\partial y_j} y_j}. \tag{5}$$

Once a differentiable analytical transformation function is introduced, the scale elasticity value follows from carrying out all the differentiations involved. Returns-to-scale properties of increasing, constant and decreasing returns are then determined by $\varepsilon > 1$, $\varepsilon = 1$, $\varepsilon < 1$, respectively.

Returns to scale in DEA

In the DEA approach the production possibility set is a convex polyhedral. For this reason, function $F(X, Y)$ describing the efficient frontier is not everywhere differentiable (Banker et al., 1984; Banker and Thrall, 1992; Førsund, 1996; Cooper et al., 2000). But it can be shown that this is a continuous convex function that takes a finite value at any finite point (X, Y). However, function $F(X, Y)$ describes the boundary of T implicitly (Krivonozhko et al., 2004; Førsund et al., 2007).

For these very reasons, returns to scale and scale elasticity were calculated in the DEA approach indirectly through the solutions of BCC dual problems (Banker et al., 1984; Banker and Thrall, 1992; Cooper et al., 2000). Let us dwell on this more detail.

The BCC primal input-oriented model is written in the form

$$\min \theta$$

subject to

$$\sum_{j=1}^{n} X_j \lambda_j \le \theta X_0,$$
$$\sum_{j=1}^{n} Y_j \lambda_j \ge Y_0, \tag{6a}$$
$$\sum_{j=1}^{n} \lambda_j = 1,$$
$$\lambda_j \ge 0,$$

where $X_j=(x_{1j},\ldots, x_{mj})$ and $Y_j=(y_{1j},\ldots, y_{rj})$ represent the observed inputs and outputs of production units $j=1,\ldots,n$. In this primal model the efficiency score θ of production unit (X_0, Y_0) is found; (X_0, Y_0) is any unit from the set of production units (X_j, Y_j), $j=1,\ldots,n$.

The dual multiplier form of the BCC model is expressed as

$$\max (u^T Y_0 - u_0)$$

subject to

$$u^T Y_j - v^T X_j - u_0 \le 0, \quad j = 1,\ldots,n \tag{6b}$$
$$v^T X_0 = 1$$
$$v_k \ge 0, \quad k = 1,\ldots,m, \quad u_i \ge 0, \quad i = 1,\ldots,r$$

where (v, u, u_0) is a vector of dual variables, $v \in E^m$, $u \in E^r$, u_0 is a scalar variable associated with the convex constraint.

Banker and Thrall (1992) (see also Cooper et al., 2000) stated the following result:

Assertion 1. Assuming that (X_0, Y_0) is an efficient unit in the BCC model the following conditions identify the situation for returns to scale at this point:

(i) Increasing returns to scale prevails at (X_0, Y_0) if and only if $u_0^* < 0$ for all optimal solutions.

(ii) Decreasing returns to scale prevails at (X_0, Y_0) if and only if $u_0^* > 0$ for all optimal solutions.

(iii) Constant returns to scale prevail at (X_0, Y_0) if and only if $u_0^* = 0$ in some optimal solution.

Hence, to identify returns to scale at point (X_0, Y_0) lying on the efficient frontier it is necessary to solve some additional dual problems of the type (6b).

Banker and Thrall (1992) established that right-hand side and left-hand side elasticities at efficient unit (X_0, Y_0) are determined as

$$\varepsilon^+(X_0, Y_0) = \min_{u_0^*} \left\{1/(1+u_0^*)\right\}$$

$$\varepsilon^-(X_0, Y_0) = \max_{u_0^*} \left\{1/(1+u_0^*)\right\}$$

where u_0^* belongs to the set of optimal dual variables of problem (6b).

Next, in the DEA literature, as mentioned above, attempts were made to identify returns to scale at efficient points of set T on the basis of returns to scale at points of reference set. Let us recall that the reference set of an inefficient unit (X_0, Y_0) is defined as (Cooper et al., 2000):

$$E_0 = \{j \mid \lambda_j^* > 0, j = 1, \ldots, n\},$$

where λ_j^* are optimal variables of the BCC primal optimization model as obtained from solving (6a).

In the attempt to determine returns-to-scale properties of a radial projection of an inefficient unit on a face by the returns-to-scale properties of the units belonging to the corresponding reference set E_0, it is stated in Cooper et al. (2006, p.148) and Tone (2005) that if all reference units have constant returns to scale, then the returns to scale of the radial projection cannot be determined. However, firstly the rule developed in Cooper et al. (2006) to determine the scale properties is incomplete, and, secondly, this way of establishing the scale properties between the reference set and the projected point of the inefficient unit actually attempts to establish the connection the other way round determining first the scale property of the efficient projected point, as is the objective of this paper.

Examples of determining returns-to-scale properties

Before going further, let us recall some notions from convex analysis. Faces are formed by an intersection of the supporting hyperplane and the polyhedral set. In the DEA models, the dimension of face may vary from 0 up to $(m+r-1)$, the maximal dimension. Faces of maximal dimension are called facets. Faces of 0-dimension are known as vertices, 1-dimension as edges. In our exposition, $ri\,\Gamma$ stands for relative interior of face Γ.

In order to elucidate the problems with the reference set exhibiting constant returns to scale we will introduce some illustrations. In Figure 1, a two-inputs/one-output BCC model is illustrated. Points $A=(5/4, 5/4, 9/8)$, $B=(1, 3, 3/2)$, $C=(3, 1, 3/2)$, $D=(5, 5, 3)$ and $E=(2, 2/3, 1/2)$ represent the observed production units that determine the production possibility set T. It is easy to see that points A, B, C and D form facet Γ of maximal dimension. This facet belongs to the hyperplane that is described by equation

$$-x_1 - x_2 + 4y - 2 = 0 \,. \tag{7}$$

From Assertion 1 mentioned above and from equation (7), it follows that any unit $(X,Y)\in ri\,\Gamma$ displays decreasing returns to scale since for such units the relation $u_0^* > 0$ holds.

On the other hand, it can be easily checked that units A, B and C display constant returns to scale. Indeed, points *A, B, C* are efficient according to the Charnes et al. (1978) (CCR) model, but point *D* is efficient only on the BCC model. In addition, one can construct a ray from the origin through point *A* (*B*, *C*) in such a way that this ray touches the production possibility set only in point *A* (*B*, *C*). If such ray is constructed through point *D*, then it goes through the interior points of the production possibility set. So, points *A*, *B*, *C* have constant returns to scale and point *D* has decreasing returns to scale. In Figure 1, unit $F\in ri\,\Gamma$ can be represented as a convex combination of units A, B and C of facet Γ. The interior points of facet Γ have decreasing returns to scale, however, units A, B and C have constant returns to scale.

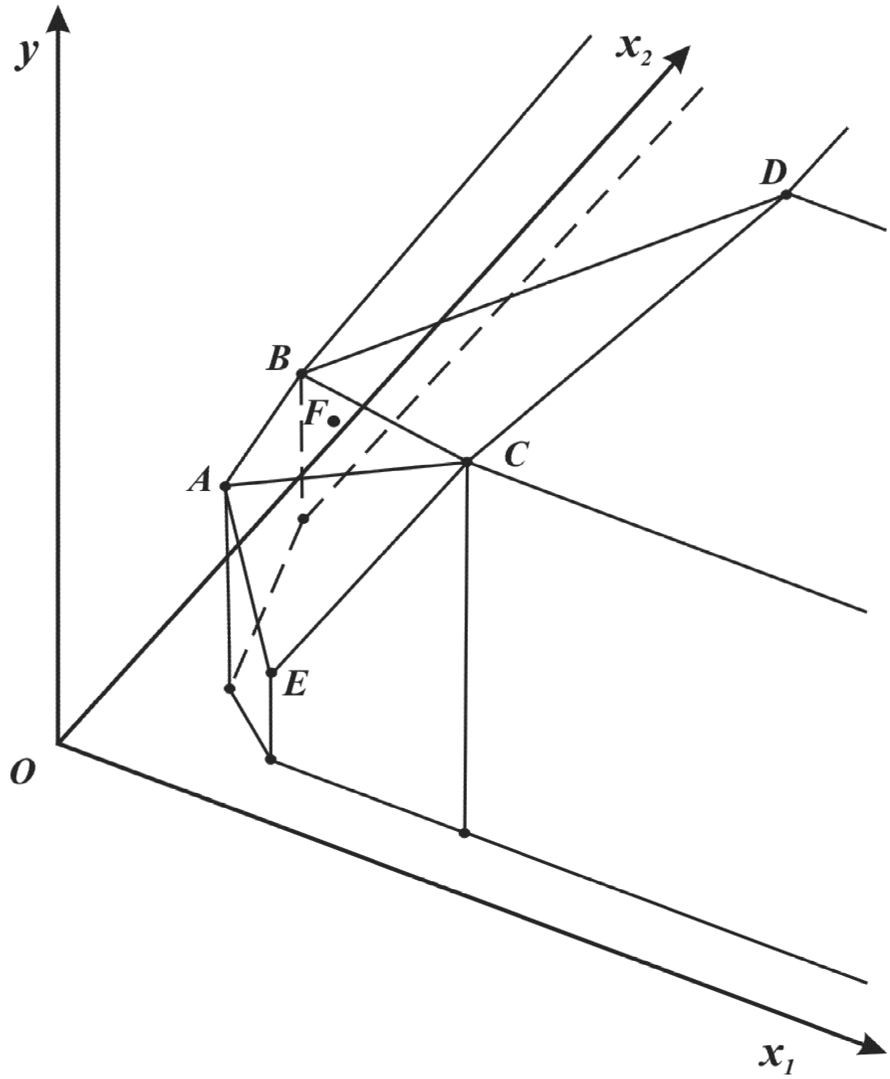

Figure 1. Interior points of facet Γ spanned by points *A,B,D,C* display decreasing returns to scale. However, reference set may contain only units with constant returns to scale

On the other hand, unit F can also be represented as a convex combination of points A, B and D of facet Γ, where point D has decreasing returns to scale. So, we have a situation where the reference set has a combination of constant returns to scale and decreasing returns to scale.

In Figure 2, a three-dimensional BCC model is depicted. Points $A = (5/4, 5/4, 3/4)$, $B = (1, 3, 3/2)$, $C = (3, 1, 3/2)$ and $D = (5, 5, 9/2)$ represent the observed production units that form the production possibility set T. It is easy to prove that points A, B, C and D

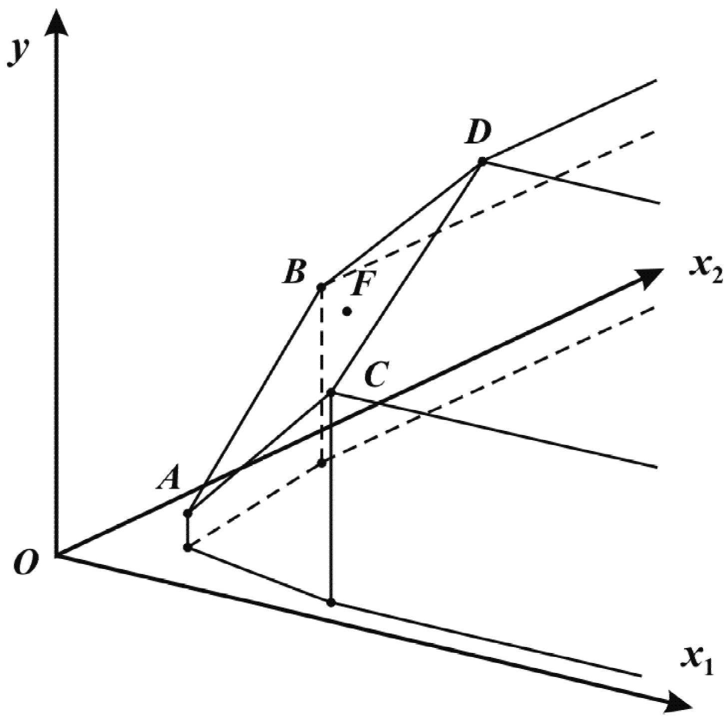

Figure 2. Interior points of facet Γ spanned by points *A,B,D,C* display increasing returns to scale. However, reference set may contain vertices only with constant returns to scale

determine some facet Γ. The supporting hyperplane containing facet Γ is described by the following equation

$$-x_1 - x_2 + 2y + 1 = 0. \tag{8}$$

From (8), it follows that points $(X, Y) \in ri\Gamma$ have increasing returns to scale since $u_0^* < 0$.

It can be easily checked that units B, C and D have constant returns to scale and unit A has increasing returns to scale. Actually, points *B, D* and *C* are efficient according to the CCR model, but point *A* is efficient only on the BCC model.

So, if some point $F \in ri\,\Gamma$ is represented as a convex combination of points B, C and D, then point F has constant returns to scale. On the other hand, if point F is written as a convex combination of points A, B and D, then point F has to display increasing returns to scale.

Thus, these examples show us clearly that the returns to scale of the BCC-projected activity cannot be found only on the basis of the returns-to-scale characteristics of units in its respective reference set that, in fact, contains vertices of some facet.

3. Main results

Now, we will prove some theorems that show that only an interior point of face Γ can identify the returns-to-scale characteristics of points lying on face Γ.

Theorem 1. *For BCC models, let an efficient unit* $(X_0, Y_0) \in ri\Gamma$*, where* $ri\Gamma$ *stands for relative interior of* Γ*, and where* Γ *is a face of maximum dimension of production possibility set* T*. Then any point* $(X,Y) \in ri\Gamma$ *displays the same returns-to- scale property as point* (X_0, Y_0).

Proof. Since point (X_0, Y_0) belongs to the interior of face Γ of maximal dimension, then there exists a unique supporting hyper-plane P that contains face Γ. For points $(X,Y) \in ri\Gamma$ the equation of hyperplane can be written in the form

$$\sum_{i=1}^{m} \frac{\partial F(X,Y)}{\partial x_i} x_i + \sum_{i=1}^{r} \frac{\partial F(X,Y)}{\partial y_i} y_i - u_0 = 0. \tag{9}$$

Using (5) and (9), we obtain

$$\varepsilon(X,Y) = -\frac{u_0 - \sum_{i=1}^{r} \frac{\partial F(X,Y)}{\partial y_i} y_i}{\sum_{i=1}^{r} \frac{\partial F(X,Y)}{\partial y_i} y_i} = 1 - \frac{u_0}{\sum_{i=1}^{r} \frac{\partial F(X,Y)}{\partial y_i} y_i}. \tag{10}$$

It is known (Cooper et al., 2000) that coefficients $-\frac{\partial F(X,Y)}{\partial x_i}$, $\frac{\partial F(X,Y)}{\partial y_i}$, u_0 of hyper-plane P satisfy the constraints of the dual problem (6b), except normalisation constraint $v^T X_0 = 1$ that is associated with point (X_0, Y_0). However, since hyperplane equation (9) is invariant under multiplication by a nonzero number, so the ratio $u_0 \Big/ \sum_{i=1}^{r} \frac{\partial F(X,Y)}{\partial y_i} y_i$ will be the same for all points belonging to the interior of face Γ.

Hence, taking into account (10), we obtain:

$\varepsilon(X,Y) > 1$ if and only if $u_0 < 0$, that is point (X,Y) displays increasing returns to scale;

$\varepsilon(X,Y) < 1$ if and only if $u_0 > 0$, that is point (X,Y) displays decreasing returns to scale;

$\varepsilon(X,Y) = 1$ if and only if $u_0 = 0$, that is point (X,Y) displays constant returns to scale.

The denominator in formulae (10) is always greater than zero, since BCC models consider production units that have $y_i \geq 0$ and there exist nonzero values in every output vector Y_j. In addition dual variables are always greater than zero for interior points of the efficient facet. Thus, in order to identify returns to scale at interior points of face Γ it is sufficient to identify returns to scale at any interior point of face Γ.

This completes the proof.

In the next theorem, we consider the case when the dimension of a face is less than $(r+m-1)$.

Theorem 2. *For BCC models, let an efficient unit* $(X_0, Y_0) \in ri\,\Gamma$*, where* Γ *is a face of* T *and* $\dim\Gamma < (r+m-1)$. *Let there also exist a segment, described by the following equation*

$$F(\alpha X_0, \beta(\alpha) Y_0) = 0, \tag{11}$$

that belongs to the interior of face Γ *under some* $1-\delta \leq \alpha \leq 1+\delta$*, where* δ *is a small parameter. Then any point* $(X,Y) \in ri\,\Gamma$ *displays the same returns to scale as point* (X_0, Y_0).

Proof. Consider some point $(X,Y) \in ri\,\Gamma$. It is easy to prove that the sets of supporting hyperplanes at points (X,Y) and (X_0, Y_0) coincide. Construct a supporting hyper-plane containing face Γ and segment (11). The equation of the hyperplane can be written in the form

$$-v^T X + u^T Y - u_0 = 0. \tag{12}$$

Observe that the part of the frontier corresponding to face Γ is also described by equation (12).

One can show that if there exists segment (11) going through point (X_0, Y_0) and belonging to $ri\Gamma$, then there exists a segment of the form (11) going through point (X, Y) and also belonging to $ri\Gamma$.

The intersection of hyperplane (12) and two-dimensional plane $(\alpha X, \beta Y)$, where α and β are any real numbers, can be written as the following equation

$$-v^T X\alpha + u^T Y\beta - u_0 = 0, \tag{13}$$

that describes a segment of the form (11) in some vicinity of point $(X, Y) \in ri\Gamma$.

From (13), it follows that

$$\frac{\partial \beta}{\partial \alpha} = \frac{v^T X}{u^T Y}.$$

Hence, for any point $(X, Y) \in ri\Gamma$ the following relations hold

$$\varepsilon(X, Y) = \frac{\partial \beta}{\partial \alpha} \cdot \frac{\alpha}{\beta} = \frac{u^T Y\beta - u_0}{\alpha(u^T Y)} \cdot \frac{\alpha}{\beta} = 1 - \frac{u_0}{u^T Y}, \tag{14}$$

since $\alpha = 1$, $\beta = 1$ for point (X, Y).

Observe that there exist a lot of supporting hyper-planes of face Γ, since the dimension of this face is less than $(r + m - 1)$. However the equation of the segment (13) has a unique form for all such hyper-planes, which is invariant under multiplication by a nonzero number. Take any supporting hyper-plane P_1 that contains face Γ, hence this hyper-plane contains a part of the segment (13). Hence for every point $(X, Y) \in ri\Gamma$ the ratio $u_0 / u^T Y$ is determined uniquely.

Since for every point $(X, Y) \in ri\Gamma$ we can take the same hyperplane in order to find the scale elasticity by the formulae (14), the sign of the ratio $u_0 / u^T Y$ will be the same for all points $(X, Y) \in ri\Gamma$. Therefore any point $(X, Y) \in ri\Gamma$ displays the same returns to scale as point (X_0, Y_0).

This completes the proof.

The previous two theorems considered the cases when the directional derivative of $F(X,Y)$ at point (X_0, Y_0) of some face along direction αX_0 is continuous. The next theorem deals with the situation when the left-side and the right-side directional derivatives at point (X_0, Y_0) of some face are not equal to each other. In other words, the next theorem takes up the case where face Γ does not contain a segment of the type (11).

Theorem 3. *For BCC models, let efficient production unit* $(X_0, Y_0) \in ri\,\Gamma_0$, *where* Γ_0 *is a face of* T. *Let also the segment described by equation* (11) *belong to face* Γ_1 *under some* $1 \le \alpha \le 1+\delta$, *where* δ *is a small parameter, let another segment described by equation* (11) *belong to face* Γ_2 *under some* $1-\delta \le \alpha \le 1$ *and* $\Gamma_0 = \Gamma_1 \cap \Gamma_2$. *Then any point* $(X,Y) \in ri\,\Gamma_0$ *displays the same returns to scale as point* (X_0, Y_0).

Proof. Consider any point $(X_1, Y_1) \in ri\,\Gamma_0$ that differs from point (X_0, Y_0). One can prove that there exists segment L_1 described by equation $F(\alpha X_1, \beta(\alpha) Y_1) = 0$ and this segment also belongs to face Γ_1 under some $1 \le \alpha \le 1+\delta_1$, where δ_1 is a small parameter. Next, using constructions of Theorem 2 one can show that for any point $(X,Y) \in ri\,\Gamma_0$ the following relation holds

$$\varepsilon^+(X,Y) = 1 - \frac{u_0^+}{u^T Y}, \qquad (15)$$

where supporting hyper-plane P is constructed in such a way that it contains faces Γ_0 and Γ_1, hence value u_0^+ is the same for all points $ri\,\Gamma_0 \subset \Gamma_1$.

By virtue of formulae (15), we obtain that for any point $(X,Y) \in ri\,\Gamma_0$ the following assertions are valid:

$\varepsilon^+(X,Y) > 1$ if and only if $u_0^+ < 0$;

$\varepsilon^+(X,Y) = 1$ if and only if $u_0^+ = 0$;

$\varepsilon^+(X,Y) < 1$ if and only if $u_0^+ > 0$.

For the left part of the segment $F(\alpha X, \beta(\alpha)Y) = 0$ under $1-\delta \le \alpha \le 1$ the proof is conducted similarly. Thus, for any point $(X,Y) \in ri\,\Gamma_0$ the left scale elasticity $\varepsilon^-(X,Y)$ takes the value greater than unity, equal to unity or less than unity depending on the sign of the value u_0^-.

Banker and Thrall (1992) proved that if at some point (X,Y) left-hand side and right-hand side scale elasticities, $\varepsilon^-(X,Y)$ and $\varepsilon^+(X,Y)$, are calculated, then:

a) point (X,Y) displays increasing returns to scale if and only if $\varepsilon^-(X,Y) \ge \varepsilon^+(X,Y) > 1$;
b) point (X,Y) displays decreasing returns to scale if and only if $1 > \varepsilon^-(X,Y) \ge \varepsilon^+(X,Y)$;
c) point (X,Y) displays constant returns to scale if and only if $\varepsilon^-(X,Y) \ge 1 \ge \varepsilon^+(X,Y)$.

As we have already showed, to compare left-hand side elasticity $\varepsilon^-(X,Y)$ and right-hand side elasticity $\varepsilon^+(X,Y)$ with the unity at any point $(X,Y) \in ri\,\Gamma_0$ it is sufficient to conduct such comparison only for point $(X_0,Y_0) \in ri\,\Gamma_0$. Hence, any point $(X,Y) \in ri\,\Gamma_0$ displays the same returns to scale as point $(X_0,Y_0) \in ri\,\Gamma_0$.

This completes the proof.

Theorems 1 - 3 established that in order to identify returns to scale at interior points of a face it is sufficient to identify returns to scale at any interior point of this face. The previous examples showed that vertices of a face cannot in general correctly identify returns to scale at interior points of this face. Thus, the question arises: what is an association between returns to scale at vertices of a face and interior points of this face? The answer is obtained from the following theorem.

Theorem 4. *For the BCC model, let S be a set of vertices (efficient units) that form some face F of the frontier. The following conditions identify the situation for the returns to scale at point $(X,Y) \in S$:*

(i) *if any point* $(X_0, Y_0) \in ri\,\Gamma$ *displays increasing returns to scale then point* (X, Y) *displays increasing returns to scale or constant returns to scale;*

(ii) *if any point* $(X_0, Y_0) \in ri\,\Gamma$ *displays constant returns to scale then point* (X, Y) *displays constant returns to scale;*

(iii) *if any point* $(X_0, Y_0) \in ri\,\Gamma$ *displays decreasing returns to scale then point* (X, Y) *displays decreasing returns to scale or constant returns to scale.*

The proof of Theorem 4 is rather evident. It follows from Theorems 1-3, illustrative examples and Assertion 1.

Proof. Consider every case separately.

(i) If some interior point $(X_0, Y_0) \in ri\,\Gamma$ displays increasing returns-to-scale, then for any supporting hyper-plane P at point (X_0, Y_0) the following relation holds: $u_0^* < 0$ according to Assertion 1. Hyper-plane P contains also point (X, Y). If there exists supporting hyper-plane L at point $(X, Y) \in S$ such that $u_0^* \geq 0$, then point (X, Y) displays constant returns to scale according to Assertion 1. If there does not exist supporting hyper-plane L at point $(X, Y) \in S$ such that $u_0^* \geq 0$, then point (X, Y) displays increasing returns to scale.

(ii) If some point $(X_0, Y_0) \in ri\,\Gamma$ displays constant returns to scale, then there exists supporting hyper-plane P at point (X_0, Y_0) such that $u_0^* = 0$.This hyper-plane contains also point (X, Y). Hence point (X, Y) displays constant returns to scale.

(iii) If point $(X_0, Y_0) \in ri\,\Gamma$ displays decreasing returns to scale, then for any supporting hyper-plane P at point (X_0, Y_0) the following relation holds: $u_0^* > 0$. Hyper-plane P contains also point (X, Y). If there exists a supporting hyper-plane at point $(X, Y) \in S$ such that $u_0^* \leq 0$, then point (X, Y) displays constant returns to scale. If there does not exist a supporting hyper-plane at point $(X,Y) \in S$ such that $u_0^* \le 0$, then point (X,Y) displays decreasing returns to scale according to Assertion 1.

This completes the proof.

Thus, only interior points of a face can serve as indicators of returns to scale at any points of this face.

Further interpretations

In their classic paper Banker et al. (1984) presented an excellent interpretation of term u_0 for the one-input/one-output BCC model. In this case the value u_0 is an intercept of the pertinent linear segment of production function with the vertical axis determined by output variable y.

This interpretation carries over to the multi-dimensional case. In the multi-input/multi-output BCC model, the intersection of the hyperplane containing some face and two-dimensional plane $(\alpha X_1, \beta Y_1)$ is determined by equation (13). Or, in other words, the following equation

$$\beta = \frac{v^T X_1}{u^T Y_1}\alpha + \frac{u_0}{u^T Y_1}$$

describes a pertinent segment of the frontier on the two-dimensional plane spanned by vectors αX_1 and βY_1. Hence, value $\beta_1 = u_0 / u^T Y_1$ is an intercept of the line that contains the pertinent segment with axis βY_1.

In Figure 3, point A on the plane corresponds to point (X_1, Y_1) on the multidimensional frontier. Value $\beta_1 = u_0 / u^T Y_1$ is an intercept of the segment containing point A with axis βY_1. In this case, the positive value β_1 shows that point A displays decreasing returns to scale.

In the above examples, every interior point of face Γ was associated with more than one reference set. In Figure 4, unit F is lying on the facet Γ formed by points A, B, C, D and E. Hence, point F may be represented as a convex combinations of several sets of points: A, B, C or E, B, C or A, B, D or A, D, E and so on. In the multidimensional models the situation may be much more complicated. Thus, one needs to elaborate special algorithms that associate all reference sets with every production unit on the frontier.

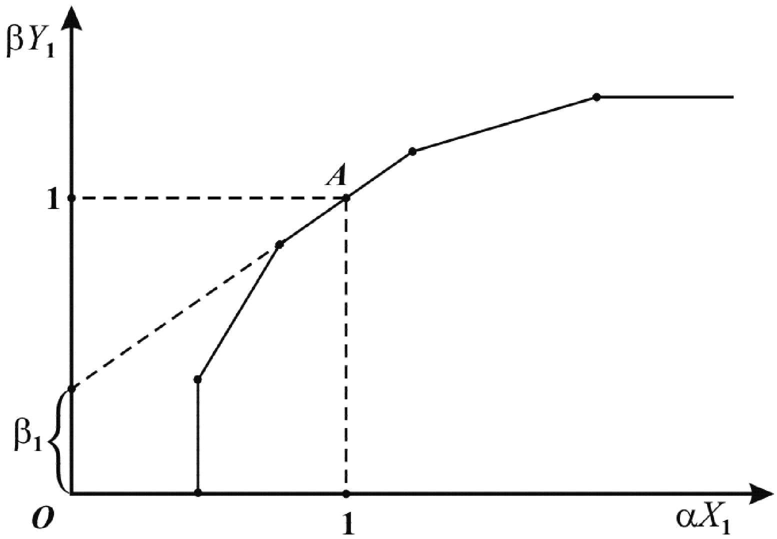

Figure 3. Intersection of the frontier and two-dimensional plane going through point (X_1, Y_1)

However, problems with constant returns of scale of units of the reference set and the scale properties of a projected point may also occur when an interior efficient point of some facet is associated with only one reference set. Indeed, consider the first example above (Figure 1). Now, let point D be (5, 5, 2). In this case facet Γ formed by units A, B, C and D is divided into two facets Γ_1 and Γ_2, respectively. Facet Γ_1 is formed by units A, B and C, and these units display constant returns to scale. The reference set of interior point F belonging to Γ_1 consist of points A, B and C. At the same time, unit F has decreasing returns to scale since it is belongs to the unique supporting hyperplane, described by equation (7), that contains facet Γ_1 and point F.

Somebody may object that all cases described above may occur very rarely. Our answer is very simple. Let us recall the situation in linear programming at the beginning of fifties. At that time everybody thought that a degenerate basis that caused cycling may occur very seldom in practice (Dantzig and Thapa, 2003). The degenerate basis means that some basic variables are equal to zero, which may lead to cycling during the solution process. And now, we see that every basis associated with a vertex (efficient unit) in DEA models is degenerate.

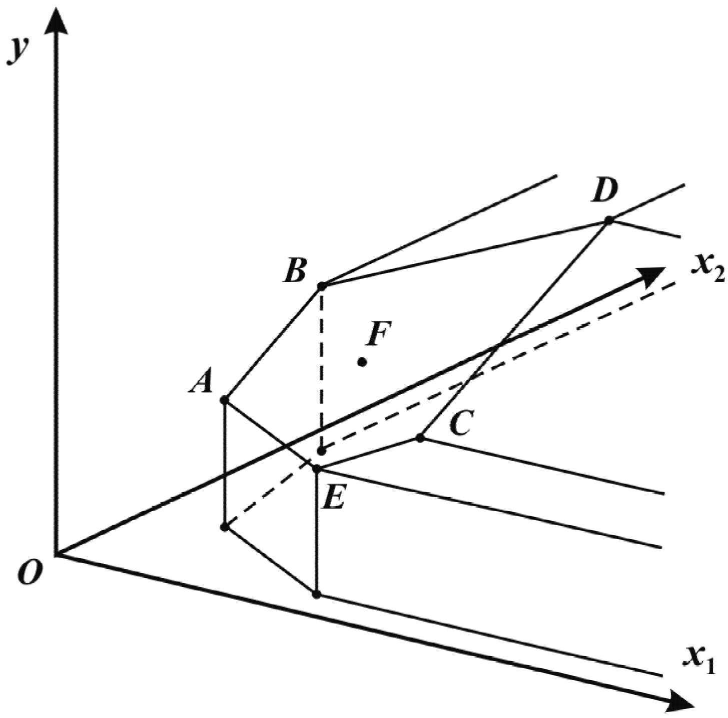

Figure 4. Several reference sets may exist for an interior point F of facet Γ.

This may cause heavy difficulties when constructing visualization methods in DEA models (Krivonozhko et al, 2004).

4. Concluding remarks

In this paper, we establish that the returns to scale of points on the faces of the frontier can be found only by observing the returns-to-scale characteristics of interior points of these faces. This unexpected result has, however, deep methodological reasons. Indeed, in neoclassical production theory the transformation function is assumed to be smooth (continuously differentiable).

In the DEA approach, the frontier is constructed as a convex envelope of actual production units in the multidimensional space, as a result all efficient units lie on the frontier. The "payment" for this is that the frontier function is not differentiable at vertices (efficient units), this means that the gap of derivatives may take place at these points. This may lead to some economic properties (characteristics) being violated.

Interior points of faces keep the properties of smoothness. For this reason, only interior points of faces determine the returns-to-scale characteristics of these faces.

In this paper we have considered all main cases of disposition of faces. In Theorem 1, face Γ has full dimension. In Theorem 2, face Γ has not full dimension, however, it is differentiable along direction αX. In Theorem 3, face Γ_0 has not full dimension, in addition it is not differentiable along direction αX. For this reason, we have to consider two segments, or, in other words, we have to take left-side and right-side derivatives.

Similar theorems can be stated for points $(X,Y) \in ri\,\Gamma \subset WEff_O\,T$, where $WEff_O\,T$ stands for a set of output weakly efficient points with respect to the BCC model. We will not dwell on this, since to reveal the dimension and the type of faces only on the basis of optimal solution of the BCC model is impossible. For this purpose one needs to use significant additional computations.

On the other hand, the direct approach (Krivonozhko et al., 2004; Førsund et al., 2007) for calculations of scale elasticities in DEA models enables us to find scale elasticities at any points of the frontier, and hence to identify returns to scale at any points of the face that contains this point.

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