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CAPITAL DECAY AND TAX DISTORTIONS: HOW TO ABANDON EXPONENTIAL DECAY AND BENEFIT FROM IT

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Abstract: The appropriate way of quantifying how taxation of a firm's income and capital can distort its optimizing conditions is a recurring issue in the literature on optimal taxation. Exponential decay, although empirically contested, is almost ubiquitous. In the present paper a generalized framework which allows for a general, non-exponential, decay pattern for both true and tax-permitted depreciation, is considered. Both convex and concave survival functions can be accommodated. Three capital concepts are involved, two of which coincide under exponential decay. The trade-off between various departures from neutrality is illustrated. Elements which contribute to non-neutrality are: (i) discrepancy between the definition of the tax-relevant accounting capital and true depreciation, (ii) mis-indexation of depreciation allowances, (iii) incomplete deductibility of interest costs, (iv) asymmetric treatment of interest costs and capital gains, and (v) taxation of the value of the capital stock. Finally, we show that substantial biases can arise in assessing the degree of non-neutrality if non-exponential depreciation schedules are forced, by 'approximation devices', to fit into the exponential decay schedule.

Keywords: Capital taxation. Taxable income. Tax-neutrality. Tax distortion. Survival function. Capital service price. Non-exponential decay. Depreciation. Indexation

JEL classification: D61, E22, H21, H25

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1 Introduction

Income taxation of firms will usually distort the factor input decisions because the optimizing conditions interfere with the way income is defined in the tax code; see *e.g.*, Atkinson and Stiglitz (1980, Lecture 5). The origin of such distortions, notably those relating to capital accumulation, may be, *inter alia*, the tax treatment of financial costs, depreciation allowances and capital gains. Quantifying effects of changes in the tax system on input and output decisions certainly is important in evaluating tax reforms.

In this paper we reconstruct indexes to quantify how properties of the tax system can distort firms' optimizing conditions in a model class which is in several respects substantially more flexible than the standard setup. In describing how investment flows accumulate to capital stocks – the latter playing triple role as an input, a wealth asset and a basis for defining tax-permitted depreciation allowances – *we abandon the ubiquitous exponential decay assumption.* The reason for the popularity of the latter assumption is probably that it implies age-independent retirement and depreciation rates [for recent examples see Devereux (2004) and Gordon, Kalambokidis and Slemrod (2004)]. Yet in several studies it has been empirically contested.[1] Imposing exponential decay, although mathematically convenient, must be considered a strait-jacket from a practical viewpoint. Our framework is based on arbitrary declining functions to describe how interest deductions, depreciation allowances, and capital gains depend on the time path of investment. A core concept is the *capital service price* or the *user cost of capital*, extensively applied in investment and tax research for more than forty years, see *e.g.*, Jorgenson (1963) and Hall and Jorgenson (1967).

The main attention of the paper will be on three issues. First, we show that the following characteristics of the tax system are potentially important: (i) discrepancy between the weight function of the tax-relevant accounting capital and true depreciation, (ii) mis-indexation of depreciation allowances, (iii) incomplete deductibility of interest costs, (iv) asymmetric tax treatment of interest costs and capital gains, and (v) taxation of the value of the capital stock. Second, we demonstrate, by using a parameterization which allows convex as well as concave survival functions for capital, how the applicability of our model for quantitative analysis is considerably extended relative to the exponential decay one. For instance it is perfectly possible that the weight function suitable for constructing capital input by cumulating past gross investment is concave in age – *i.e.*, implies increasing retirement with age – concur with a weight function in constructing capital values which is convex – *i.e.*, implies declining depreciation with age. Exponential decay is useless to describe such a pattern. Possible trade-offs between various sources of non-neutrality are illustrated analytically and by numerical examples. Third, we show by examples how measures of the degree of non-neutrality can be substantially biased if non-

[1] See Biørn (1989, Section 1.1; 1998; 2005, Section 1) for discussion and references.

exponential depreciation schedules are forced – by ‘approximation devices’ – to fit into a system of formulae derived from exponential decay, for example by setting a depreciate rate equal to twice the inverse of the capital’s assumed life time.

The paper proceeds as follows. In Section 2, we develop the basic capital concepts: the capital quantity and the gross capital; the capital value and the net capital; and the capital service price. Next, in Section 3, we describe the tax system, with emphasis on the depreciation allowances, the interest deduction allowances, and the taxable capital gains. This motivates a compact formulation of the tax function, describing how the tax profile depends on the time profile of gross investment. From this we develop, in Section 4, the tax-corrected capital service price. Next, in Section 5, we derive an index of tax distortion, in two alternative forms, and identify five components which specifically contribute to non-neutrality. In Section 6, we give examples of parameter constellations which ensure neutral tax systems. The trade-off between mis-indexation and mis-chosen profile of the depreciation allowances is specifically discussed. In Section 7 pitfalls in using approximative formulas derived from exponential decay in inappropriate situations are discussed. Concluding remarks follow in Section 8.

2 Basic concepts

2.1 Capital quantity and gross capital

Let $J(t)$ be the quantity of capital invested at time t, time being considered as continuous. To characterize the capital’s retirement and the loss of efficiency over time, we introduce the *survival function*, $B(s)$, indicating the share of an investment made s years ago which still exists as productive capital. It satisfies

$$B'(s) \leq 0, \qquad B(0) = 1, \qquad B(\infty) = 0, \tag{2.1}$$

We adopt the common assumption that capital is measured in such a way that one (efficiency) unit of capital produces one unit of capital services per unit of time. Hence, $K(t,s) = B(s)J(t-s)$ represents both the number of capital units of age s at time t, and the instantaneous flow of capital services produced at time t by capital of age s. Making the basic neoclassical assumption that capital units belonging to different vintages are perfect substitutes, we obtain an expression for the total quantity of capital at time t by aggregation across vintages:

$$K(t) = \int_0^\infty K(t,s)\,ds = \int_0^\infty B(s)J(t-s)\,ds. \tag{2.2}$$

It represents the capital’s technical dimension, denoted as the *gross capital*, and could serve to measure capital services in a (neoclassical) model of producer behaviour.

2.2 Capital value and net capital

In order to formalize the tax treatment of income and wealth and the way tax-permitted and true depreciation may differ, we need a capital concept which represents its wealth dimension.

Two auxiliary functions constructed from $B(s)$ will be needed to expose the argument: the *service flow which one new capital unit generates after age* s, discounted at the rate ρ:

$$\omega(s) = \int_s^\infty e^{-\rho(z-s)} B(z) dz, \tag{2.3}$$

and the *service flow per efficiency unit of capital still existing at age* s, likewise discounted at the rate ρ:

$$\phi(s) = \frac{\omega(s)}{B(s)} = \frac{\int_s^\infty e^{-\rho(z-s)} B(z)\, dz}{B(s)}. \tag{2.4}$$

The denominator of the latter expression represents the share of the initial investment, in efficiency units, which attains age s. We interpret ρ as a *real* interest rate, since the variable discounted, $B(s)$, has a quantity dimension.

The value of the capital which is s years old at time t is $V(t,s) = q(t,s)K(t,s)$, where $q(t,s)$ is the price per capital unit of age s at time t, a *vintage price* for short, and $K(t,s)$ is the number of such units. Aggregation across vintages gives the total capital value at time t:

$$V(t) = \int_0^\infty V(t,s)\, ds = \int_0^\infty q(t,s)K(t,s)\, ds. \tag{2.5}$$

How could we construct a capital quantity variable, $H(t)$, such that all of its components can be assumed to have the same price in the value of the aggregate value, $V(t)$?

The answer we will rely on is given by the following line of argument: Assume that the system of vintage prices measured *per efficiency unit* satisfies

$$\frac{q(t,s)}{\phi(s)} = \frac{q(t,0)}{\phi(0)} \iff \frac{q(t,s)B(s)}{\omega(s)} = \frac{q(t,0)}{\omega(0)}, \qquad \forall\, t\, \&\, s \geq 0. \tag{2.6}$$

This equation expresses non-arbitrage between capital of different ages: a firm which at time t buys one *efficiency* unit of capital of age s at the price $q(t,s)$, pays the same price per unit of *prospective* discounted capital services,

$$c(t) = \frac{q(t)}{\phi(0)} = \frac{q(t)}{\omega(0)}, \tag{2.7}$$

as a firm which buys one new unit at the price $q(t) \equiv q(t,0)$.[2]

Condition (2.6) is related to, although not identical with, the condition frequently postulated as an equilibrium condition in the capital market literature, saying that the acquisition price of an asset should equal (the present value of the) its (expected) future rental prices weighted by the relevant remaining efficiencies; confer

[2]See Biørn (1989, Section 4.2) for an extended discussion of this relationship.

Hotelling (1925), Hicks (1973, Chapter II), and Jorgenson (1989, section 1.2). This way of stating the equilibrium condition can, in our notation, be written as

$$q(t) = \int_0^\infty e^{-rz} B(z) c(t+z,t)\, dz, \tag{2.8}$$

where r is the *nominal* interest rate and $c(t+z,t)$ denotes the capital service price at a future time $t+z$, as expected at time t [see, *e.g.*, Jorgenson (1974, p. 205), Takayama (1985, p. 694), and Diewert (2005, Section 12.2)]. Formalizing the arbitrage condition as in (2.6), however, is more convenient from an econometric point of view, since, unlike (2.8), it gives a *closed form* expression for the service prices. Rewriting (2.7) as

$$q(t) = c(t) \int_0^\infty e^{-\rho z} B(z)\, dz, \tag{2.9}$$

we note the close connection between this hypothesis and (2.8). If, starting at time t, the service price $c(t)$ is expected to increase at the (constant) rate γ, so that $c(t+z,t) = c(t)e^{\gamma z}$, then (2.8) can be reexpressed as

$$q(t) = \int_0^\infty e^{-rz} B(z) c(t+z,t) dz = c(t) \int_0^\infty e^{-(r-\gamma)z} B(z) dz.$$

Interpreting ρ as a real interest rate by letting $\rho = r-\gamma$, we see that (2.8) and (2.9) are equivalent.

Substituting (2.6) in (2.5) the capital value can be expressed as

$$V(t) = \int_0^\infty q(t) \tfrac{\phi(s)}{\phi(0)} K(t,s) ds = q(t) \int_0^\infty G(s) J(t-s) ds, \tag{2.10}$$

where

$$G(s) \equiv \frac{\omega(s)}{\omega(0)} \equiv \frac{\int_s^\infty e^{-\rho(z-s)} B(z)\, dz}{\int_0^\infty e^{-\rho z} B(z)\, dz} = \frac{B(s)\phi(s)}{\phi(0)}, \qquad s \geq 0. \tag{2.11}$$

The latter is a weighting function for gross investment in the capital value which may be denoted a *loss of value function*. It describes true depreciation in quantity terms and satisfies

$$G'(s) \leq 0, \qquad G(0) = 1, \qquad G(\infty) = 0, \tag{2.12}$$

To verify this interpretation we decompose the capital value $V(t)$ into a price component equal to the current investment price, $q(t)$, and a quantity component,

$$H(t) = \int_0^\infty G(s) J(t-s)\, ds, \tag{2.13}$$

as follows:

$$V(t) = q(t) H(t). \tag{2.14}$$

We denote $H(t)$ as the *net capital stock*. It is a quantity concept obtained by aggregating past investment, using a weighting function $G(s)$ in which the weight assigned to the quantity invested s years ago is the share of the capital service flow which one capital unit generates *after age* s. The survival function $B(s)$ gives the corresponding weight in $K(t)$. We will in Section 3.8 exemplify cases where a concave $B(s)$ function implies a convex $G(s)$ function and cases where both are convex.

Taking time derivatives of (2.13) and (2.14) gives

$$\begin{aligned}
\dot{H}(t) &= \int_0^\infty G(s)\dot{J}(t-s)ds = J(t) - \int_0^\infty G'(s)[-J(t-s)]ds,\\
\dot{V}(t) &\equiv \dot{q}(t)H(t) + q(t)\dot{H}(t),
\end{aligned}$$

in the former using integration by parts. We then obtain

$$\dot{V}(t) = q(t)J(t) + \dot{q}(t)H(t) - q(t)D(t), \tag{2.15}$$

where

$$D(t) = J(t) - \dot{H}(t) = \int_0^\infty g(s)J(t-s)ds, \tag{2.16}$$

$$g(s) = -G'(s) = -\frac{\omega'(s)}{\omega(0)}, \qquad \int_0^\infty g(s)ds = G(0) = 1. \tag{2.17}$$

The variable $D(t)$ will be denoted as *depreciation*. Thus defined it is a *quantity* concept, obtained by assigning to the quantity invested s years ago a weight $g(s)$. When turning to its counterpart in *value* terms it is convenient to distinguish between gross and net concepts, by defining *gross value of depreciation* as $q(t)D(t)$ and *appreciation* as $\dot{q}(t)H(t)$. The *net value of depreciation* is their difference, *i.e.*, $q(t)J(t) - \dot{V}(t) = q(t)D(t) - \dot{q}(t)H(t)$, the latter equality following from (2.15).

2.3 A special case: Exponential decay

It follows from (2.11) that

$$\begin{aligned}
\phi(s) = \phi(0)\,\forall s \iff \quad & \omega(s) = B(s)\omega(0)\,\forall s\\
\implies \quad & G(s) = B(s)\ \forall s \qquad \implies H(t) = K(t)\,\forall t.
\end{aligned} \tag{2.18}$$

Hence, if and only if $\phi(s)$ is age independent, *gross and net capital stock will coincide.* The only per efficiency service flow function $\phi(s)$ which has this property, is the one obtained from the ubiquitous exponential decay assumption:

$$B(s) = e^{-\delta s} \implies \begin{cases} \phi(s) = 1/(\rho+\delta), \\ G(s) = e^{-\delta s}, \quad g(s) = \delta e^{-\delta s} \qquad [\delta > 0,\ s \in [0,\infty)]. \\ D(t) = \delta K(t). \end{cases} \tag{2.19}$$

We will return to this as Example A in Section 3.8. The relationships stated in (2.18) and (2.19) illustrate the particular and restrictive nature of this very common assumption.

2.4 The capital service price in the absence of taxes

The arbitrage condition (2.6) postulates that a firm which invests in an old capital good, regardless of its age, should pay the same price per unit of discounted capital services today and in the future as a firm buying a new good at the current investment price. We interpret the common purchase price per unit of capital services under this perfect market assumption, $c(t)$, defined in (2.7), as the firm's capital service price in the absence of taxes. It follows from (2.19) that under exponential decay, the (tax-free) capital service price gets the very familiar form

$$c(t) = q(t)(\rho+\delta). \tag{2.20}$$

3 The tax system

After these preliminaries we next describe the tax system, including a description of the tax-imposed accounting capital and the definition of the depreciation allowances, the interest deductions, the capital gains, and the taxable capital value.[3]

3.1 The accounting capital

The accounting capital of a firm is the assessment of the value of its production capital which *the firm is obliged to use in defining taxable income* (and possibly also taxable wealth). It is, like the market value $V(t)$, a value concept, but will in general differ from it.

Consider first the *quantity* component of the accounting capital. In analogy with the definition of the survival function, $B(s)$, and the loss of value function, $G(s)$, we let $A(s)$ be the weighting function for gross investment in the accounting capital. It has the properties:

$$A'(s) \leq 0, \qquad A(0) = 1, \qquad A(\infty) = 0. \tag{3.1}$$

Then $A(s)J(t-s)$ is the part of the accounting capital *in quantity terms* at time t which relates to investment made s years ago, and

$$F(t) = \int_0^\infty A(s)J(t-s)\,ds \tag{3.2}$$

is the total for all vintages. This equation is similar to (2.13) for net capital in quantity terms.

The associated *depreciation allowances in quantity terms* can be interpreted as the difference between gross investment and the increase in the accounting capital in quantity terms. From (3.1) and (3.2), using integration by parts, we find that the depreciation allowances in quantity terms at time t, similar to (2.16), is

$$E(t) = J(t) - \dot{F}(t) = \int_0^\infty a(s)J(t-s)\,ds, \tag{3.3}$$

[3] The exposition relies partly on Biørn (1989, Chapter 5).

where

$$\text{(3.4)} \qquad a(s) = -A'(s), \qquad \int_0^\infty a(s)ds = A(0) = 1.$$

The latter expresses the weight given to capital of age s when calculating tax-accounted depreciation in quantity terms.

The correspondence between our descriptions of the tax system and of the true economic evaluation of capital and depreciation is:

SYMBOLS	$A(s)$, $a(s)$, $F(t)$, $E(t)$	*in the description of the tax system*	CORRESPOND TO SYMBOLS	$G(s)$, $g(s)$, $H(t)$, $D(t)$	*for true economic capital valuation and true economic depreciation*
EQUATIONS	(3.1), (3.2), (3.3), (3.4)	*in the description of the tax system*	CORRESPOND TO EQUATIONS	(2.12), (2.13), (2.16), (2.17)	*for true economic capital valuation and true economic depreciation*

3.2 The tax function

Two kinds of taxes are considered: an *income tax* and a *capital value tax*. The former is a tax on net profit, defined as the output value minus the costs of labour, energy, materials, etc., after the tax-permitted net costs related to the purchase, financing, and use of capital have been deducted. The latter may be interpreted either as a tax on capital services, imposed on the user, or – if the firm holds a zero value of its net financial assets – as a net wealth tax.

Let u denote the income tax rate, v the capital value tax rate, $X(t)$ the output value minus total current costs of all inputs other than capital, often denoted as gross operating surplus, $S_D(t)$ the depreciation allowances deductible in the income tax base, $S_I(t)$ the interest costs deductible in income tax base, $S_G(t)$ the capital gains included in income tax base, and $V_A(t)$ the value of accounting capital implicit in the calculation of the depreciation allowances and capital value tax. The tax rates u and v are assumed to be time invariant (or expected so by the firm). All costs of non-capital inputs are immediately deductible in the income tax base. Taxable income is then $X(t) – S_D(t) – S_I(t) + S_G(t)$, so that the tax payment at time t becomes

$$\text{(3.5)} \qquad T(t) = u[X(t) - S_D(t) - S_I(t) + S_G(t)] + vV_A(t).$$

We now proceed by defining $V_A(t)$, $S_D(t)$, $S_I(t)$, and $S_G(t)$.

3.3 Capital value tax and depreciation allowances

We assume that the investment price increases (or is expected to increase) at a constant rate γ, *i.e.*,

$$\text{(3.6)} \qquad q(t) = q(0)e^{\gamma t} \qquad \forall\, t.$$

Let ε be a general *inflation adjustment (indexation) parameter*, defined as the (exponential) rate at which the historic investment cost is allowed to be inflated in the firm's tax accounts when calculating accounting capital and depreciation allowances. The value of the accounting capital at time t, obtained from (3.2), can then be expressed as

$$V_A(t) = \int_0^\infty A(s)e^{\varepsilon s}q(t-s)J(t-s)\,ds = q(t)\int_0^\infty A(s)e^{(\varepsilon-\gamma)s}J(t-s)\,ds. \tag{3.7}$$

If, in particular, $\varepsilon=\gamma$, it follows that

$$V_A(t) = q(t)F(t) \qquad [\varepsilon=\gamma].$$

The expression for the depreciation allowances, obtained from (3.3), is

$$S_D(t) = \int_0^\infty a(s)e^{\varepsilon s}q(t-s)J(t-s)\,ds = q(t)\int_0^\infty a(s)e^{(\varepsilon-\gamma)s}J(t-s)\,ds, \tag{3.8}$$

where $e^{\varepsilon s}q(t-s)J(t-s) \equiv e^{(\varepsilon-\gamma)s}q(t)J(t-s)$ is the inflation adjusted investment cost of vintage $t-s$ at time t. If, in particular, $\varepsilon=\gamma$, it follows that

$$S_D(t) = q(t)E(t) \qquad [\varepsilon=\gamma].$$

Boundary cases are:

(a) $\varepsilon=0$: The accounting capital and the depreciation allowances are calculated from the *historic* investment cost, giving

$$V_A(t) = \int_0^\infty A(s)q(t-s)J(t-s)ds,$$
$$S_D(t) = \int_0^\infty a(s)q(t-s)J(t-s)ds.$$

(b) $\varepsilon=\gamma$: The accounting capital and the depreciation allowances are calculated in terms of the *replacement* cost of the investment, giving

$$V_A(t) = q(t)\int_0^\infty A(s)J(t-s)ds = q(t)F(t),$$
$$S_D(t) = q(t)\int_0^\infty a(s)J(t-s)ds = q(t)E(t).$$

Equations (3.7) and (3.8) can represent any tax code which permits inflation adjustment of the historic investment cost, possibly at a rate different from the rate of price increase of the specific capital category under consideration ($\varepsilon\neq\gamma$).

3.4 Interest deductions

Let $m \in (0,1)$ denote the share of the (imputed) interest cost on the value of the total capital stock which is deductible in the income tax base, and let r be the nominal market interest rate. The imputed interest cost at time t is defined as $(r-\varepsilon)V(t)$, *i.e.*, the difference between the nominal (imputed) interest cost, $rV(t)$, and an (imputed) inflation adjustment of the capital value, $\varepsilon V(t)$. Or stated otherwise, the imputed interest cost is the *tax accounted real interest rate*, $r-\varepsilon$ times the market value of the capital stock.

We then get a *tax-permitted interest cost deduction* at time t equal to

$$\begin{aligned} S_I(t) = m(r-\varepsilon)V(t) &= mq(t)(r-\varepsilon)q(t)H(t) \\ &= mq(t)(r-\varepsilon)\int_0^\infty G(s)J(t-s)\,ds. \end{aligned} \tag{3.9}$$

Boundary cases are:

(a) *A nominal system* ($\varepsilon=0$)*:* If the interest deductions are based on the *nominal interest rate*, which implies that the tax accounted real interest rate coincides with the nominal market interest rate relevant to the capital type under consideration, we have $S_I(t)=mrV(t)$.

(b) *A fully indexed system* ($\varepsilon=\gamma$)*:* If the interest deductions are based on the *true real interest rate*, which implies that the tax accounted real interest rate coincides with the market real interest rate relevant to the equipment type under consideration, we have $S_I(t)=m(r-\gamma)V(t)$.

The indexation parameter for interest deductions may differ from the indexation parameter for the depreciation allowances.

3.5 Capital gains

Let $n\in(0,1)$ be the share of the (imputed) capital gains *on an accrual basis* which is subject to income taxation. The capital gain at time t is defined as $(\gamma-\varepsilon)V(t)$, *i.e.*, the difference between the actual nominal gain, $\gamma V(t)$, and an (imputed) inflation adjustment of the capital value, $\varepsilon V(t)$. Or stated otherwise, the imputed real capital gain is the *tax accounted real rate of inflation*, $\gamma-\varepsilon$, times the market value of the capital stock.

We then get a *taxable capital gain* at time t equal to

$$S_G(t)=n(\gamma-\varepsilon)V(t)=nq(t)(\gamma-\varepsilon)\int_0^\infty G(s)J(t-s)\,ds. \tag{3.10}$$

Boundary cases are:

(a) *A nominal system* ($\varepsilon=0$)*:* If the capital gains are calculated *on a nominal basis*, we have $S_G(t)=n\gamma V(t)$.

(b) *A fully indexed system* ($\varepsilon=\gamma$)*:* If the inflation adjustment of the capital gains is based on the *true inflation rate* relevant to the equipment type under consideration, we have $S_G(t)=n(\gamma-\gamma)V(t)=0$.

The indexation parameter ε used for capital gains may differ from the indexation parameter for depreciation allowances and interest deductions.

3.6 The tax function and the age-specific deduction function

Three *indexation parameters* will be needed to establish the tax function: ε_D, the statutory rate of inflation, prescribed in the tax code, used in calculating the value of the accounting capital underlying the tax-permitted depreciation allowances and the capital value tax; ε_I, the statutory rate of inflation implicit in the calculation of the tax-deductible interest cost, and ε_G, the corresponding rate for the capital gains. Inserting (3.7) and (3.8) with $\varepsilon=\varepsilon_D$, (3.9) with $\varepsilon=\varepsilon_I$, and (3.10) with $\varepsilon=\varepsilon_G$ in (3.5), we can express the tax payment at time t compactly as follows:

$$\begin{aligned} T(t) &= u[X(t)-\int_0^\infty \mu(s)q(t)J(t-s)\,ds] \\ &= u[X(t)-\int_0^\infty \mu^*(s)q(t-s)J(t-s)\,ds], \end{aligned} \tag{3.11}$$

where

$$\mu(s)=\left[a(s)-\tfrac{v}{u}A(s)\right]e^{(\varepsilon_D-\gamma)s}+[m(r-\varepsilon_I)-n(\gamma-\varepsilon_G)]G(s), \tag{3.12}$$

$$\mu^*(s)=e^{\gamma s}\mu(s)=\left[a(s)-\tfrac{v}{u}A(s)\right]e^{\varepsilon_D s}+[m(r-\varepsilon_I)-n(\gamma-\varepsilon_G)]G(s)e^{\gamma s}. \tag{3.13}$$

The function $\mu(s)$, or $\mu^*(s)$, to be denoted as *(age specific) tax deduction functions*, summarizes the impact of past investment on the current tax base: (a) Increasing the *replacement value* of an investment made s years ago, $q(t)J(t-s)$, by one unit reduces the current tax base by $\mu(s)$ units. (b) Increasing the *historic value* of an investment made s years ago, $q(t-s)J(t-s)$, by one unit reduces the current tax base by $\mu^*(s)$ units. Their present values, in particular, will be basic characteristics in our description of the distortions that arise via the tax-corrected capital service price. Since $\mu(s)$ and $\mu^*(s)$ operate on replacement values and on historic values [confer (3.11)], we use, respectively, the real and the nominal interest rates, $r-\gamma$ and r, in the discounting. The resulting expression for the present value is

$$\lambda = \int_0^\infty e^{-(r-\gamma)s}\mu(s)\,ds \equiv \int_0^\infty e^{-rs}\mu^*(s)\,ds. \tag{3.14}$$

We want to express λ by means of the tax parameters and the interest and inflation rates. Let, for an arbitrary discounting rate ρ, $Y(\rho)$ and $Z(\rho)$ denote the present value of the weighting function for the net capital $H(t)$ and for the accounting capital $F(t)$, respectively, *i.e.*,[4]

$$Y(\rho) = \int_0^\infty e^{-\rho s}G(s)ds, \tag{3.15}$$

$$Z(\rho) = \int_0^\infty e^{-\rho s}A(s)ds. \tag{3.16}$$

Also, let $y(\rho)$ and $z(\rho)$ be the present value of the weighting functions $g(s)$ and $a(s)$, defining depreciation and depreciation allowances,

$$y(\rho) = \int_0^\infty e^{-\rho s}g(s)ds = 1-\rho Y(\rho), \tag{3.17}$$

$$z(\rho) = \int_0^\infty e^{-\rho s}a(s)ds = 1-\rho Z(\rho), \tag{3.18}$$

where the last equality follows from (3.4), using integration by parts. Inserting (3.12) in (3.14), it follows that the present value of the net deductions in income tax base per unit of investment cost can be written as

$$\begin{aligned}\lambda &= z(r-\varepsilon_D)-\tfrac{v}{u}Z(r-\varepsilon_D) + [m(r-\varepsilon_I)-n(\gamma-\varepsilon_G)]Y(r-\gamma)\\ &= 1-(r-\varepsilon_D+\tfrac{v}{u})Z(r-\varepsilon_D) + [m(r-\varepsilon_I)-n(\gamma-\varepsilon_G)]Y(r-\gamma).\end{aligned} \tag{3.19}$$

3.7 Remark on depreciation rates

Depreciation rates are commonly defined as ratios between depreciation flows and capital stocks, both in quantity terms. They will in general depend on the age distribution of the capital stock, which again reflects the time profile of past gross investment.[5] The only exception is the exponential decay case; confer Section 2.3.

In the special case with constant investment: $J(t) = \bar{J}\,\forall t$, and hence constant (gross and net capital) stock, *i.e.*, stationarity, we get from (2.13), (2.16), (2.17),

[4] $Y(\rho)$ and $Z(\rho)$ have the same relationship to $H(t)$ and $F(t)$ as $\phi(0)=\omega(0)$ has to $K(t)$ [see (2.3)].

[5] For a discussion related to the formally similar *retirement rates*, see Biørn (2005).

(3.15) and the similar equations (3.2), (3.3), (3.4), (3.16) simple expressions for the true depreciation rate and its tax-imposed counterpart:

$$\bar{\delta} = \frac{D(t)}{H(t)} = \frac{\int_0^\infty g(s)\bar{J}}{\int_0^\infty G(s)\bar{J}} = \frac{1}{Y(0)} \qquad [J(t)\ \text{constant}], \tag{3.20}$$

$$\bar{\alpha} = \frac{E(t)}{F(t)} = \frac{\int_0^\infty a(s)\bar{J}}{\int_0^\infty A(s)\bar{J}} = \frac{1}{Z(0)} \qquad [J(t)\ \text{constant}]. \tag{3.21}$$

These are benchmark values only to be used as reference values in the following. If $J(t)$ is non-stationary the magnitude of such rates may be very sensitive to the investment's growth pattern; see Biørn (2005, Section 6) for discussion and examples.

3.8 Examples

Four parametric survival functions, A through D, will be used to illustrate the above results. For B, C, and D we will need the auxiliary function (see Appendix)

$$h(s,\rho,\tau,N) = \left(\tfrac{N-s}{N}\right)^{-\tau} \int_s^N e^{-\rho(z-s)} \left(\tfrac{N-z}{N}\right)^{\tau} dz, \tag{3.22}$$

with $h(s,0,\tau,N) = \frac{N-s}{\tau+1}$. For $\rho \neq 0$ the function satisfies the recursion

$$h(s,\rho,\tau,N) = \tfrac{1}{\rho}\left[1 - \tfrac{\tau}{N-s}\, h(s,\rho,\tau-1,N)\right], \qquad \tau \geq 1,$$
$$h(s,\rho,0,N) = \tfrac{1}{\rho}\left[1 - e^{-\rho(N-s)}\right].$$

Note that $\frac{h(s,0,\tau,N)}{h(0,0,\tau,N)} = 1 - \frac{s}{N}$ for any $\tau \geq 0$, while $\frac{h(s,\rho,\tau,N)}{h(0,\rho,\tau,N)}$ depends on τ when $\rho \neq 0$.

EXAMPLE A: Exponential decay:

$$B(s) = e^{-\delta s}, \quad s \in [0,\infty) \implies G(s) = e^{-\delta s}, \quad s \in [0,\infty).$$

EXAMPLE B: Sudden death:

$$B(s) = 1, \quad s \in [0,N] \implies G_0(s) = 1 - \tfrac{s}{N}, \quad s \in [0,N].$$

EXAMPLE C: Linear retirement:

$$B(s) = 1 - \tfrac{s}{N}, \quad s \in [0,N] \implies G_0(s) = [1 - \tfrac{s}{N}]^2, \quad s \in [0,N].$$

EXAMPLE D: Two-parametric retirement function ($\tau \geq 0$):

$$B(s) = [1 - \tfrac{s}{N}]^{\tau}, \quad s \in [0,N] \implies G_0(s) = [1 - \tfrac{s}{N}]^{\tau+1}, \quad s \in [0,N].$$

For EXAMPLES B, C and D we use, instead of the loss of value function $G(s)$, its special case, $G_0(s)$, for $\rho=0$. The implied $\omega(s), \phi(s), G(s), G_0(s)$ functions are given in Table 1.

TABLE 1: THE FUNCTIONS $\omega(s), \phi(s), G(s), G_0(s)$ FOR EXAMPLES A–D

	EXAMPLE A	EXAMPLE B	EXAMPLE C	EXAMPLE D
$\phi(s)$	$\frac{1}{\rho+\delta}$	$h(s,\rho,0,N)$	$h(s,\rho,1,N)$	$h(s,\rho,\tau,N)$
$\omega(s)$	$\frac{e^{-\delta s}}{\rho+\delta}$	$h(s,\rho,0,N)$	$h(s,\rho,1,N)\left(\frac{N-s}{N}\right)$	$h(s,\rho,\tau,N)\left(\frac{N-s}{N}\right)^{\tau}$
$G(s)$	$e^{-\delta s}$	$\frac{h(s,\rho,0,N)}{h(0,\rho,0,N)}$	$\frac{h(s,\rho,1,N)}{h(0,\rho,1,N)}\left(\frac{N-s}{N}\right)$	$\frac{h(s,\rho,\tau,N)}{h(0,\rho,\tau,N)}\left(\frac{N-s}{N}\right)^{\tau}$
$G_0(s)$	$e^{-\delta s}$	$\frac{N-s}{N}$	$\left(\frac{N-s}{N}\right)^2$	$\left(\frac{N-s}{N}\right)^{\tau+1}$

Example D contains as special cases:

Example B: Sudden death, linear loss-of-value function $G_0(s)$: $\tau=0,\ \tau+1=1$.
Example C: Linear survival function, quadratic loss-of-value function $G_0(s)$: $\tau=1,\ \tau+1=2$.

Example D can also be used to represent:

Concave survival function, convex loss-of-value function $G_0(s)$: $0<\tau<1<\tau+1$.
Convex survival function, convex loss-of-value function $G_0(s)$: $1<\tau<\tau+1$.

Finally, since $\lim_{n\to\infty}(1-\frac{1}{n})^n = \lim_{n\to\infty}(1+\frac{1}{n})^{-n} = e^{-1}$, it follows by substituting $n=\frac{N}{s}$, $c=\frac{\tau+1}{N}$ in $G_0(s)=[1-\frac{s}{N}]^{\tau+1}$ that

$$\lim_{N,\tau\to\infty,\frac{\tau+1}{N}=c} G_0(s) = \lim_{n\to\infty}(1-\tfrac{1}{n})^{ncs} = \Big[\lim_{n\to\infty}(1+\tfrac{1}{n})^n\Big]^{-cs} = e^{-cs}.$$

Thus, Example D contains, for $N\to\infty$, $\tau\to\infty$ and $\frac{\tau+1}{N}$ finite, Example A as a limiting case with $\alpha=\frac{\tau+1}{N}$. So Example D generalizes in a sense not only Examples B and C, but also A.

4 The tax-corrected capital service price

In this section we derive the modified expression for the capital service price, (2.17), when the tax system is taken into account, drawing on Section 3.6 as well as Biørn (1989, Chapter 6),

4.1 The tax-corrected discounted cash-flow function

The firm's *before-tax* and *after-tax cash-flow* are, respectively,

(4.1) $$R(t) = X(t) - q(t)J(t),$$

(4.2) $$R_T(t) = R(t) - T(t) = X(t) - q(t)J(t) - T(t).$$

Inserting for $T(t)$ from (3.11) it follows that their relationship can be written as

(4.3) $$\begin{aligned} R_T(t) &= (1-u)X(t) - q(t)[J(t)-u\textstyle\int_0^\infty \mu(s)J(t-s)\,ds] \\ &\equiv (1-u)R(t) - uq(t)[J(t)-\textstyle\int_0^\infty \mu(s)J(t-s)ds], \end{aligned}$$

or equivalently,

(4.4) $$\begin{aligned} R_T(t) &= (1-u)X(t) - [q(t)J(t)-u\textstyle\int_0^\infty \mu^*(s)q(t-s)J(t-s)\,ds] \\ &\equiv (1-u)R(t) - u[q(t)J(t)-\textstyle\int_0^\infty \mu^*(s)q(t-s)J(t-s)ds]. \end{aligned}$$

Since a cash-flow tax would be characterized by $R_T(t) = (1-u)R(t)$, the term

$$uq(t)[J(t)-\textstyle\int_0^\infty \mu(s)J(t-s)ds] \equiv u[q(t)J(t)-\textstyle\int_0^\infty \mu^*(s)q(t-s)J(t-s)ds]$$

can be interpreted as the adjustment to be made in after-tax cash-flow to account for the firm paying an income tax rather than a cash-flow tax at the rate u.

The nominal value of the before-tax and the after-tax cash-flow, discounted at the interest rate of the firm, r, are, respectively,

$$(4.5)\qquad W = \int_0^\infty e^{-rt}R(t)dt = \int_0^\infty e^{-rt}[X(t) - q(t)J(t)]\,dt,$$

$$\begin{aligned}(4.6)\qquad W_T &= \int_0^\infty e^{-rt}R_T(t)dt \\ &= \int_0^\infty e^{-rt}[(1-u)X(t)-q(t)\{J(t)-u\int_0^\infty \mu(s)J(t-s)ds\}]dt \\ &\equiv \int_0^\infty e^{-rt}[(1-u)R(t) - u\,q(t)\{J(t)-\int_0^\infty \mu(s)J(t-s)ds\}]dt,\end{aligned}$$

after inserting for $R_T(t)$ from (4.3). It follows that

$$W_T = (1-u)W - u\int_0^\infty e^{-rt}q(t)J(t)dt + u\int_0^\infty e^{-rt}q(t)\int_0^t \mu(s)J(t-s)dsdt + W_{T0},$$

where

$$W_{T0} = u\int_0^\infty e^{-rt}q(t)\int_t^\infty \mu(s)J(t-s)\,ds\,dt$$

is the predetermined part of W_T, *i.e.*, representing investments made before the start of the period over which the cash-flow is cumulated, $t=0$.

It follows, when changing integration variables, that the present value of the after-tax net cash-flow in excess of its predetermined component can be written as

$$\begin{aligned}(4.7)\qquad W_T - W_{T0} = (1-u)W^* &= (1-u)\int_0^\infty e^{-rt}R^*(t)dt \\ &= (1-u)\int_0^\infty e^{-rt}[X(t)-q^*(t)J(t)]dt,\end{aligned}$$

where

$$(4.8)\qquad q^*(t) = q(t)\frac{1-\lambda u}{1-u},$$

$$(4.9)\qquad R^*(t) = X(t) - q^*(t)J(t),$$

$$(4.10)\qquad W^* \equiv \frac{W_T - W_{T0}}{1-u} = \int_0^\infty e^{-rt}R^*(t)dt.$$

We can interpret $q^*(t)$ as a *tax-corrected investment price*, $R^*(t)$ as the firm's net before-tax cash-flow if it had evaluated gross investment at the tax-corrected price, and W^* as the corresponding (hypothetical) before-tax net cash-flow.

Equation (4.7) together with (2.7) motivate the interpretation of the tax-corrected investment price per capital service flow unit over the life cycle as the tax-corrected capital service price. The tax-corrected service price then becomes[6]

$$(4.11)\qquad c^*(t) = \frac{q^*(t)}{\phi(0)} = \frac{q(t)}{\phi(0)}\frac{1-\lambda u}{1-u} = c(t)\frac{1-\lambda u}{1-u}.$$

A non-tax situation, $u=0$, would imply $c^*(t)=c(t)$, $q^*(t)=q(t)$,$R^*(t)=R(t)$, and $W^*=W$. This will also occur, for a non-zero u, if $\lambda=1$, a point to be elaborated below.

[6]It is obvious that the above argument relies on time invariance of the tax rates.

4.2 The tax-corrected capital service price reparameterized

It is convenient to reparameterize the tax-corrected capital service price by expressing λ by means of the real interest rate ρ, the inflation rate γ, the functions Z and Y and the mis-indexation parameters corresponding to $(\varepsilon_D, \varepsilon_I, \varepsilon_G)$. Let

$$\rho = r - \gamma \tag{4.12}$$

be the *market real interest rate*, and define

$$\kappa_D = \gamma - \varepsilon_D, \qquad \kappa_I = \gamma - \varepsilon_I, \qquad \kappa_G = \gamma - \varepsilon_G, \tag{4.13}$$

which can be termed the *mis-indexation* of the depreciation allowances, interest deductions, and capital gains, respectively. Or otherwise stated: the real interest rates which are implicit in the tax system for the three elements are, respectively, $r - \varepsilon_D = \rho + \kappa_D$, $r - \varepsilon_I = \rho + \kappa_I$, and $r - \varepsilon_G = \rho + \kappa_G$.

We can then write λ as follows:

$$\lambda = 1 - \left(\rho + \kappa_D + \tfrac{v}{u}\right) Z(\rho + \kappa_D) + [m\rho + m\kappa_I - n\kappa_G] Y(\rho). \tag{4.14}$$

Inserting this expression in (4.11), we obtain

$$c^*(t) = \frac{q(t)}{\phi(0)}(1 + \xi \tfrac{u}{1-u}) = c(t)(1 + \xi \tfrac{u}{1-u}) = \frac{q(t)}{\phi(0)}(1 + \beta) = c(t)(1 + \beta), \tag{4.15}$$

where β and ξ can be considered *indexes of tax distortion* given by

$$\xi = 1 - \lambda = \left(\rho + \kappa_D + \tfrac{v}{u}\right) Z(\rho + \kappa_D) - (m\rho + m\kappa_I - n\kappa_G) Y(\rho), \tag{4.16}$$

$$\beta = \tfrac{u}{1-u}(1 - \lambda) = \tfrac{u}{1-u}\xi. \tag{4.17}$$

5 Tax distortion: An index and its decomposition

We next decompose the index ξ, defined by (4.16), as

$$\xi = \xi_1 + \xi_2 + \xi_3 + \xi_4 + \xi_5, \tag{5.1}$$

where $(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5)$ are *non-neutrality-inducing components* given by

$$\begin{aligned}
\xi_1 &= \rho[Z(\rho) - Y(\rho)], \\
\xi_2 &= (\rho + \kappa_D) Z(\rho + \kappa_D) - \rho Z(\rho), \\
\xi_3 &= (1 - m)\rho Y(\rho), \\
\xi_4 &= (v/u) Z(\rho + \kappa_D), \\
\xi_5 &= [n\kappa_G - m\kappa_I] Y(\rho).
\end{aligned} \tag{5.2}$$

Then, using (4.15), the tax-corrected capital service price can be expressed as

$$c^*(t) = c(t)[1 + \tfrac{u}{1-u} \textstyle\sum_{i=1}^{5} \xi_i]. \tag{5.3}$$

A non-distortive, *neutral*, tax system is characterized by $\lambda = 1 \iff \xi = \beta = 0 \iff c^*(t) = c(t)$. Hence, if $u \in (0,1)$, a *necessary* condition for neutrality is

$$\xi_1 + \xi_2 + \xi_3 + \xi_4 + \xi_5 = 0. \tag{5.4}$$

The interpretation of the five components in (5.2) is the following:

COMPONENT ONE, when inserting from (3.15) and (3.16), can be written as

$$\xi_1 = \rho \int_0^\infty e^{-\rho s}[A(s) - G(s)]\, ds. \tag{5.5}$$

It represents the effect of *discrepancy between the loss of value function specified in the tax code, $A(s)$, and the true loss of value function, $G(s)$.* It is zero, regardless of ρ, if $A(s) = G(s) \equiv \frac{\omega(s)}{\omega(0)}\ \forall s$. If the weighting function for the accounting capital is below the loss of value function everywhere, *i.e.*, $A(s) < G(s)\ \forall s \iff \int_s^\infty [a(z) - g(z)]dz < 0$, and $\rho > 0$, then this component is *negative* and contributes to the capital service price being below its value under neutrality.

COMPONENT TWO, when inserting from (3.16) and using (3.18), can be written as

$$\xi_2 = z(\rho) - z(\rho + \kappa_D) = \int_0^\infty \left[e^{-\rho s} - e^{-(\rho+\kappa_D)s}\right] a(s) ds \equiv \int_0^\infty e^{-\rho s}[1 - e^{-\kappa_D s}] a(s) ds. \tag{5.6}$$

It represents the effect of the *mis-indexation of the depreciation allowances*. This component is zero for any ρ if $\kappa_D = 0 \iff \varepsilon_D = \gamma$. If insufficient inflation adjustment is permitted, *i.e.*, if $\kappa_D > 0 \iff \varepsilon_D < \gamma$, this component is *positive*, *i.e.*, tends to raise the capital service price. If the depreciation allowances are 'over-indexed', *i.e.*, if $\kappa_D < 0 \iff \varepsilon_D > \gamma$, this component is *negative*.

COMPONENT THREE, when inserting from (3.15), can be written as

$$\xi_3 = (1 - m)\rho \int_0^\infty e^{-\rho s} G(s)\, ds. \tag{5.7}$$

It represents the effect of *incomplete interest deduction* in the tax base. It becomes zero, regardless of ρ, if $m = 1$, *i.e.*, if interest cost is fully deductible. If $m < 1$ and $\rho > 0$, this component is *positive*.

COMPONENT FOUR, when inserting from (3.16), can be written as

$$\xi_4 = \frac{v}{u} \int_0^\infty e^{-\rho s} A(s)\, ds. \tag{5.8}$$

It represents the effect of the *capital value tax* and is zero if $v = 0$. If a capital value tax (subsidy) is imposed, *i.e.*, $v > 0\ (v < 0)$, this component is *positive (negative)*.

COMPONENT FIVE, finally, after inserting from (3.15), can be written as

$$\xi_5 = (n\kappa_G - m\kappa_I) \int_0^\infty e^{-\rho s} G(s)\, ds. \tag{5.9}$$

It represents the *asymmetry in the tax treatment of interest costs and capital gains* and is zero if $n\kappa_G = m\kappa_I$. If capital gains are favoured relative to interest deductions, *i.e.*, if $n\kappa_G < m\kappa_I$, and κ_I and κ_G are positive, this component is *negative*.

The components $\xi_1,\dots,\xi_5$ are *independent of the tax rates u and v*, except that ξ_4 depends on the ratio between the wealth-tax rate and the income-tax rate, v/u.

A VARIANT: A variant of the index ξ is the relative distortion of the capital service price, β, defined in (4.17). It follows that

$$\beta = \beta_1+\beta_2+\beta_3+\beta_4+\beta_5, \quad \text{where } \beta_i = \frac{u}{1-u}\,\xi_i, \qquad i=1,\dots,5, \tag{5.10}$$

i.e., β_i rescales ξ_i by $\frac{u}{1-u}$: $\beta_i \gtreqless \xi_i \iff u \gtreqless \frac{1}{2}$. We can then rewrite (5.3) as

$$c^*(t) = c(t)(1+\textstyle\sum_{i=1}^{5}\beta_i). \tag{5.11}$$

It follows from (5.10) and (5.2) that this variant of the index and its components are – for given $A(s)$, $G(s)$, ρ, ε_D, ε_I, ε_G, $\frac{v}{u}$ – larger the higher is the income tax rate u. Neutrality now requires

$$\beta_1+\beta_2+\beta_3+\beta_4+\beta_5 = 0. \tag{5.12}$$

6 Neutral tax systems

The tax distortion index ξ, with components $\xi_1,\dots,\xi_5$ (or β with components $\beta_1,\dots,\beta_5$), can serve two purposes:

(a) describe formal conditions for neutrality in taxation and
(b) quantify tax distortions through scalar measures.

We now present six configurations for which neutrality, expressed by condition (5.4) [$\iff$ (5.12)], is ensured (Section 6.1) and illustrate from two of them trade-offs between mis-indexation and mis-chosen weights in the accounting capital function (Section 6.2).

6.1 Six cases

CASE 1: $\xi_1=\xi_2=\xi_3=\xi_4=\xi_5=0$.
This constellation will be ensured if

$$A(s)=G(s)\ \ \forall s, \quad m=n=1, \quad \kappa_D=0, \quad \kappa_I=\kappa_G, \quad v=0.$$

In this case, (i) the weight function of the accounting capital implicit in the tax code coincides with true depreciation, (ii) full interest cost deductibility is allowed and full taxation of capital gains prevails, (iii) depreciation allowances are fully indexed, (iv) there is a common indexation of interests and capital gains: $\kappa_I=\kappa_G$, and (v) no tax is imposed on the capital value. *The common degree of indexation of interests and capital gains is immaterial;* the common value of κ_I and κ_G may be non-zero. That (i)–(v) characterize a fully neutral tax system, is well known from the literature for the case $\kappa_D=\kappa_I=\kappa_G=0$. However, we will show that these conditions are stronger than needed. In CASES 2–5 three of the five components are zero, the other two *add to* zero. In the still less restrictive CASE 6, only two components are zero, the remaining three *add to* zero.

CASE 2: $\xi_3=\xi_4=\xi_5=0$, $\xi_1+\xi_2=0$.
This situation will occur if

$$m=n=1, \quad \kappa_I=\kappa_G, \quad v=0, \quad (\rho+\kappa_D)Z(\rho+\kappa_D)=\rho Y(\rho) \iff z(\rho+\kappa_D)=y(\rho),$$

which implies

$$\xi_1 = -\xi_2 = \rho[Z(\rho) - Y(\rho)].$$

This characterizes a situation with (i) full interest deductibility and full taxation of capital, (ii) no tax imposed on the capital value, and (iii) lack of full indexation of the depreciation allowances ($\kappa_D > 0$) compensated by the age profile of the latter.

CASE 3: $\xi_2 = \xi_4 = \xi_5 = 0$, $\xi_1 + \xi_3 = 0$.
This situation will occur if

$$m = n = 0 \;\; \& \; Z(\rho) = Z(\rho+\kappa_D) = 0 \Longleftrightarrow z(\rho) = z(\rho+\kappa_D) = 1,$$

which implies

$$\xi_1 = -\xi_3 = -\rho Y(\rho).$$

The interpretation is that (i) immediate deduction of the full capital expenses is allowed, (ii) no interest deduction is allowed and (iii) capital gain is tax-free. The gain the firm obtains by being allowed instantaneous deduction of the capital expenses is offset by the loss of being allowed no interest deduction.

CASE 4: $\xi_1 = \xi_2 = \xi_4 = 0$, $\xi_3 + \xi_5 = 0$.
This situation will occur if

$$A(s) = G(s) \;\; \forall s, \quad \kappa_D = 0, \quad v = 0, \quad (1-m)\rho = m\kappa_I - n\kappa_G,$$

which implies

$$\xi_3 = -\xi_5 = (1-m)\rho Y(\rho).$$

This characterizes a situation where (i) the depreciation allowances specified in the tax code agree with true depreciation, (ii) the depreciation allowances are fully indexed, (iii) no tax is imposed on the capital value, and (iv) lack of full interest deductions ($m < 1$) is compensated by letting capital gains be favoured relative to interest deductions ($n\kappa_G < m\kappa_I$) (assuming ρ, κ_I, and κ_G positive). The trade-off between the values of m and n which ensure neutrality is then described by

$$(\rho+\kappa_I)m = \rho + n\kappa_G.$$

This implies that if capital gains are tax free ($n=0$), the critical value of m is

$$m = \frac{\rho}{\rho+\kappa_I} = \frac{r-\gamma}{r-\varepsilon_I} \qquad (n=0),$$

which is simply the ratio between the market real interest rate and the real interest rate implied by the indexation of interest deductions. On the other hand it implies that if interest costs are fully deductible ($m=1$), the critical value of n is

$$n = \frac{\kappa_I}{\kappa_G} = \frac{\gamma-\varepsilon_I}{\gamma-\varepsilon_G} \qquad (m=1),$$

which is simply the ratio between the mis-indexations of the interest cost deductions and the capital gains.

CASE 5: $\xi_1 = \xi_2 = \xi_3 = 0$, $\xi_4 + \xi_5 = 0$.
This situation will occur if

$$A(s) = G(s) \;\; \forall s, \quad \kappa_D = 0, \quad m = 1, \quad \frac{v}{u} = \kappa_I - n\kappa_G,$$

which implies

$$\xi_4 = -\xi_5 = (1-n)\kappa_I Y(\rho).$$

This characterizes a situation where (i) the depreciation allowances coincides, for all ages, with true depreciation, (ii) depreciation allowances are fully indexed, (iii) full interest deductions are allowed, and (iv) capital gains are incompletely taxed ($n < 1$), which is (for $\gamma > 0$, *i.e.*, $\varepsilon_I < \gamma$) compensated by taxation of the capital value ($v > 0$).

CASE 6: $\xi_3 = \xi_4 = 0$, $\xi_1 + \xi_2 + \xi_5 = 0$.
This situation will occur if

$$m = n = 1, \qquad v = 0, \qquad (\rho{+}\kappa_D)Z(\rho{+}\kappa_D) = (\rho + \kappa_I - \kappa_G)Y(\rho).$$

This constellation weakens the conditions in CASE 2, by allowing $\kappa_I \neq \kappa_G$. We now have: (i) full interest deductibility, (iii) full taxation of capital gains, (iii) no taxation of the capital value and (iv) the mis-indexation of the depreciation allowances, the interest deductions, and the capital gains ($\kappa_D, \kappa_I, \kappa_G \neq 0$) is neutralized by a departure between the weighting function for the accounting capital and true loss of value function [$A(s) \neq G(s)$ and $Z(\rho{+}\kappa_D) \neq Y(\rho)$].

6.2 Mis-indexation versus mis-chosen depreciation allowances. Illustrations

To further investigate the scope for ensuring tax-neutrality in the presence of mis-indexation of interest costs, depreciation allowances and capital gains we provide *four illustrations*. All of them relate to CASE 2 and CASE 6 above – in which the mis-indexation is neutralized by departures between the weighting function for accounting capital and net capital. The core equation describing neutrality is

$$\text{(6.1)} \qquad \begin{aligned} &(\rho{+}\kappa_D)Z(\rho{+}\kappa_D) = (\rho{+}\kappa_I{-}\kappa_G)Y(\rho) && \text{in CASE 6},\\ &(\rho{+}\kappa_D)Z(\rho{+}\kappa_D) = \rho Y(\rho) && \text{in CASE 2}. \end{aligned}$$

The parametrizations of the weighting functions for capital accumulation to be considered are the $B(s)$ functions described in EXAMPLES A–D in Section 3.8, denoted as ILLUSTRATIONS A–D below. In all examples we let for convenience the implied $G(s)$ [or $G_0(s)$] function and the $A(s)$ function have the same mathematical form.

To elaborate ILLUSTRATIONS B,C and D we will need the functions:[7]

$$\text{(6.2)} \quad \begin{aligned} F(\tau,\theta) &= \int_0^1 e^{\theta s}(1{-}s)^\tau ds = \sum_{j=1}^{\infty} \frac{\tau!\theta^{j-1}}{(\tau{+}j)!} \\ &= \frac{1}{\tau{+}1}\left[1{+}\frac{\theta}{\tau{+}2}\left(1{+}\frac{\theta}{\tau{+}3}\left\{1+\frac{\theta}{\tau{+}4}+\cdots\right\}\right)\right], \end{aligned}$$

$$\text{(6.3)} \quad \begin{aligned} \Lambda(\tau,\theta) &= \theta F(\tau,-\theta) = \theta\int_0^1 e^{-\theta s}(1{-}s)^\tau ds = -\sum_{j=1}^{\infty} \frac{\tau!(-\theta)^j}{(\tau{+}j)!} \\ &= \frac{\theta}{\tau{+}1}\left[1{-}\frac{\theta}{\tau{+}2}\left(1{-}\frac{\theta}{\tau{+}3}\left\{1-\frac{\theta}{\tau{+}4}+\cdots\right\}\right)\right], \qquad \tau{=}0,1,2,\dots\ . \end{aligned}$$

They have a simple relationship to the function (3.22) for $s{=}0$ (see Appendix):

$$\text{(6.4)} \qquad h(0,\rho,\tau,N) = \textstyle\int_0^N e^{-\rho z}\left(1-\frac{z}{N}\right)^\tau dz \equiv NF(\tau,-\rho N) = \frac{1}{\rho}\Lambda(\tau,\rho N),$$

and $\Lambda(\tau,\theta)$ satisfies the recursion

$$\begin{aligned} &\Lambda(\tau,\theta) = \tfrac{1}{\theta}\left[1{-}\tau\,\Lambda(\tau{-}1,\theta)\right], && \tau \geq 1,\\ &\Lambda(0,\theta) = 1{-}e^{-\theta}. \end{aligned}$$

[7] $F(\tau,\theta)$ has the following relationship to $f(b,\theta,\tau)$ defined in (a.3) (see Appendix): $F(\tau,\theta) = f(0,-\theta,\tau)$.

ILLUSTRATION A: First, we assume that the survival function is *exponentially declining*, confer EXAMPLE A in Section 3.8, while the depreciation allowances follow the *declining balance* schedule, which ensures $G(s)$ and A(s) to have the same parametric form, *i.e.*,

$$B(s) = G(s) = e^{-\delta s}, \qquad A(s) = e^{-\alpha s}, \qquad \delta > 0,\, \alpha > 0, \quad s \geq 0,$$
$$\bar{\delta} = \delta, \qquad \bar{\alpha} = \alpha.$$

Since, from (3.15)–(3.16), $Y(\rho)=1/(\rho+\delta)$, $Z(\rho)=1/(\rho+\alpha)$, condition (6.1) implies

$$\text{(6.5)} \quad \begin{array}{llr} \dfrac{\rho+\kappa_D+\alpha}{\rho+\kappa_D} = \dfrac{\rho+\delta}{\rho+\kappa_I-\kappa_G} & \iff \dfrac{\alpha}{\rho+\kappa_D} = \dfrac{\delta-(\kappa_I-\kappa_G)}{\rho+(\kappa_I-\kappa_G)} & \text{in CASE 6} \\ \dfrac{\rho+\kappa_D+\alpha}{\rho+\kappa_D} = \dfrac{\rho+\delta}{\rho} & \iff \dfrac{\alpha}{\rho+\kappa_D} = \dfrac{\delta}{\rho} & \text{in CASE 2} \end{array}$$

This equation describes, for any $\delta\,(>0)$ and $\rho\,(\neq 0)$, the trade-off between the mis-indexation of depreciation allowances, the *gap between* the mis-indexation of interests and capital gains, and the mis-chosen parameter of the declining balance schedule, α.

Consider two special cases.

[1] CASE 2: No gap in interest-gain (mis-)indexation: $\kappa_I=\kappa_G$. The trade-off between α and κ_D can then be described by:

$$\frac{\alpha}{\delta} = 1+\frac{\kappa_D}{\rho}.$$

[2] CASE 6 with no mis-indexation of depreciation: $\kappa_D=0$. Then the trade-off between the gap in the interest-gain (mis-)indexation and the gap between the true and the tax-accounted declining balance parameters, $\delta-\alpha$, is given by

$$\frac{\alpha}{\rho+\kappa_D} = \frac{\delta-(\kappa_I-\kappa_G)}{\rho+(\kappa_I-\kappa_G)} \iff \kappa_I-\kappa_G = \varepsilon_G-\varepsilon_I = \frac{\rho}{\rho+\delta}(\delta-\alpha).$$

The interpretation is that the indexation of interests should be stronger than ($\varepsilon_I>\varepsilon_G$), respectively weaker than ($\varepsilon_I<\varepsilon_G$), the indexation of the capital gains, to neutralize too favourable ($\alpha>\delta$), respectively too unfavourable ($\alpha<\delta$), depreciation allowances.

ILLUSTRATION B: Next, assume that the survival function has the *sudden death* shape with scrapping age N, confer EXAMPLE B in Section 3.8, while the depreciation allowances follow a linear schedule over M years. This again ensures that $G_0(s)$ and $A(s)$ have the same form, *i.e.*,

$$\begin{array}{ll} B(s) = 1, \quad (s\in[0,N]), & \\ G_0(s) = 1-\frac{s}{N} \quad (s\in[0,N]), & A(s) = 1-\frac{s}{M} \quad (s\in[0,M]) \\ \bar{\delta} = \frac{1}{Y(0)} = \frac{2}{N}, & \bar{\alpha} = \frac{1}{Z(0)} = \frac{2}{M}. \end{array}$$

Then (3.15), (3.16) and (6.3) give

$$\begin{array}{l} Y(\rho) = \int_0^N e^{-\rho s}[1-\frac{s}{N}]ds = NF(1,-\rho N) = \frac{1}{\rho}\Lambda(1,N\rho) = \frac{N}{2}[1-\frac{\rho N}{3}(1-\frac{\rho N}{4}\{1-\dots\})], \\ Z(\rho) = \int_0^M e^{-\rho s}[1-\frac{s}{M}]ds = MF(1,-\rho M) = \frac{1}{\rho}\Lambda(1,M\rho) = \frac{M}{2}[1-\frac{\rho M}{3}(1-\frac{\rho M}{4}\{1-\dots\})], \end{array}$$

so that condition (6.1) implies

$$\text{(6.6)} \quad \begin{array}{ll} \Lambda[1,(\rho+\kappa_D)M] = [1+\frac{\kappa_I-\kappa_G}{\rho}]\Lambda[1,\rho N] & \text{in CASE 6,} \\ \Lambda[1,(\rho+\kappa_D)M] = \Lambda[1,\rho N] & \text{in CASE 2.} \end{array}$$

This equation describes, for any maximal service life $N\,(>0)$ and $\rho\,(\neq 0)$, the trade-off between the mis-indexation of depreciation allowances, the *gap between* the mis-indexation of interests and capital gains, and the mis-chosen tax-permitted service life, M.

ILLUSTRATION C: Then we assume that the survival function is *linear*, confer EXAMPLE C in Section 3.8, while the depreciation allowances follow a *quadratic* schedule over M years, so that again $G_0(s)$ and $A(s)$ have the same form, *i.e.*,

$$
\begin{aligned}
&B(s) = [1-\tfrac{s}{N}] \quad (s \in [0,N]), \\
&G_0(s) = [1-\tfrac{s}{N}]^2 \quad (s \in [0,N]), && A(s) = [1-\tfrac{s}{M}]^2 \quad (s \in [0,M]) \\
&\bar{\delta} = \tfrac{1}{Y(0)} = \tfrac{3}{N}, && \bar{\alpha} = \tfrac{1}{Z(0)} = \tfrac{3}{N}.
\end{aligned}
$$

Then (3.15), (3.16) and (6.3) give

$$
\begin{aligned}
Y(\rho) &= \textstyle\int_0^N e^{-\rho s}[1-\tfrac{s}{N}]^2 ds = NF(2,-\rho N) = \tfrac{1}{\rho}\Lambda(2,N\rho) = \tfrac{N}{3}[1-\tfrac{\rho N}{4}(1-\tfrac{\rho N}{5}\{1-\dots\})], \\
Z(\rho) &= \textstyle\int_0^M e^{-\rho s}[1-\tfrac{s}{N}]^2 ds = MF(2,-\rho M) = \tfrac{1}{\rho}\Lambda(2,M\rho) = \tfrac{M}{3}[1-\tfrac{\rho M}{4}(1-\tfrac{\rho M}{5}\{1-\dots\})],
\end{aligned}
$$

so that condition (6.1) implies

$$
\begin{aligned}
&\Lambda[2,(\rho+\kappa_D)M] = [1+\tfrac{\kappa_I-\kappa_G}{\rho}]\Lambda[2,\rho N] && \text{in CASE 6,} \\
&\Lambda[2,(\rho+\kappa_D)M] = \Lambda[2,\rho N] && \text{in CASE 2.}
\end{aligned}
\tag{6.7}
$$

This equation describes, for any maximal service life $N\,(>0)$ and $\rho\,(\neq 0)$, the trade-off between the mis-indexation of depreciation allowances, the *gap between* the mis-indexation of interests and capital gains, and the mis-chosen tax-permitted service life, M.

ILLUSTRATION D: Finally, we generalize the two previous illustrations by assuming that the survival function has the shape assumed in EXAMPLE D in Section 3.8, again letting the implied $G_0(s)$ function and the $A(s)$ function have the same form:

$$
\begin{aligned}
&B(s) = [1-\tfrac{s}{N}]^{\sigma}, \quad (s \in [0,N],\ \sigma \geq 0), \\
&G_0(s) = [1-\tfrac{s}{N}]^{\sigma+1} \quad (s \in [0,N],\ \sigma \geq 0) && A(s) = [1-\tfrac{s}{M}]^{\sigma_*+1} \quad (s \in [0,M],\ \sigma_* \geq 0), \\
&\bar{\delta} = \tfrac{1}{Y(0)} = \tfrac{\sigma+2}{N}, && \bar{\alpha} = \tfrac{1}{Z(0)} = \tfrac{\sigma_*+2}{M}.
\end{aligned}
$$

The shape parameters (N,σ) for $G_0(s)$ correspond to (M,σ_*) for $A(s)$.[8] Then from (3.15), (3.16) and (6.3) it follows that

$$
\begin{aligned}
Y(\rho) &= \textstyle\int_0^N e^{-\rho s}[1-\tfrac{s}{N}]^{\sigma+1} ds = NF(\sigma+1,-\rho N) = \tfrac{1}{\rho}\Lambda(\sigma+1,N\rho) \\
&= \tfrac{N}{\sigma+2}[1-\tfrac{\rho N}{\sigma+3}(1-\tfrac{\rho N}{\sigma+4}\{1-\dots\})], \\
Z(\rho) &= \textstyle\int_0^M e^{-\rho s}[1-\tfrac{s}{M}]^{\sigma_*+1} ds = MF(\sigma_*+1,-\rho N) = \tfrac{1}{\rho}\Lambda(\sigma_*+1,M\rho) \\
&= \tfrac{M}{\sigma_*+2}[1-\tfrac{\rho M}{\sigma_*+3}(1-\tfrac{\rho M}{\sigma_*+4}\{1-\dots\})],
\end{aligned}
$$

so that condition (6.1) implies

$$
\begin{aligned}
&\Lambda[\sigma_*+1,(\rho+\kappa_D)M] = [1+\tfrac{\kappa_I-\kappa_G}{\rho}]\Lambda[\sigma+1,\rho N] && \text{in CASE 6,} \\
&\Lambda[\sigma_*+1,(\rho+\kappa_D)M] = \Lambda[\sigma+1,\rho N] && \text{in CASE 2.}
\end{aligned}
\tag{6.8}
$$

[8] ILLUSTRATIONS B and C follow when $\sigma=\sigma_*=0$ and when $\sigma=\sigma_*=1$, respectively.

This equation describes, for any depreciation profile parameters (σ, N) and $\rho\,(\neq 0)$, the trade-off between the mis-indexation of depreciation allowances, the *gap between* the mis-indexation of interests and capital gains, and the mis-chosen parameters of the tax-permitted depreciation allowances (σ_*, M).

Equation (6.8), of which (6.6) and (6.7) are special cases, prescribes, for any choice of shape parameters (N, σ) and $\rho\,(\neq 0)$, how a gap in the indexation of depreciation allowances, κ_D, and an indexation asymmetry, $\kappa_I - \kappa_G \equiv \varepsilon_G - \varepsilon_I$, (in Case 6) can be neutralized via the shape parameters of the depreciation schedule (M, σ_*). By substituting

$$Q = \rho N, \quad Q_* = (\rho+\kappa_D)M, \quad \Delta = \tfrac{\kappa_I - \kappa_G}{\rho} = \tfrac{\varepsilon_G - \varepsilon_I}{\rho},$$

conditions (6.8) can be written, more conveniently, as

$$(6.9) \qquad \begin{array}{ll} \Lambda[\sigma_*+1, Q_*] = [1+\Delta]\Lambda[\sigma+1, Q] & \text{in Case 6,} \\ \Lambda[\sigma_*+1, Q_*] = \Lambda[\sigma+1, Q] & \text{in Case 2.} \end{array}$$

Table 2: Illustration D, Case 2: Numerical examples.
$(Q_*, \sigma_*) \leftrightarrow (Q, \sigma)$ trade-off $[Q = \rho N, Q_* = (\rho+\kappa_D)M]$ implied by Eq. (6.9)

Q_* as function of (σ, σ_*, Q) for $\Delta = 0$

σ	σ_*	$Q =$ 0.1000	0.5000	1.0000	2.0000
0.0	0.0	0.1000	0.5000	1.0000	2.0000
0.0	0.1	0.1051	0.5264	1.0558	2.1237
0.0	0.5	0.1253	0.6326	1.2809	2.6252
0.0	1.0	0.1506	0.7660	1.5649	3.2624
0.0	2.0	0.2013	1.0341	2.1381	4.5536
0.1	0.0	0.0952	0.4750	0.9474	1.8847
0.1	0.1	0.1000	0.5000	1.0000	2.0000
0.1	0.5	0.1193	0.6005	1.2119	2.4671
0.1	1.0	0.1434	0.7268	1.4793	3.0602
0.1	2.0	0.1916	0.9806	2.0186	4.2619
0.5	0.0	0.0798	0.3962	0.7849	1.5407
0.5	0.1	0.0839	0.4169	0.8277	1.6318
0.5	0.5	0.1000	0.5000	1.0000	2.0000
0.5	1.0	0.1202	0.6043	1.2171	2.4666
0.5	2.0	0.1605	0.8137	1.6544	3.4111
1.0	0.0	0.0665	0.3288	0.6483	1.2638
1.0	0.1	0.0698	0.3458	0.6835	1.3364
1.0	0.5	0.0832	0.4142	0.8237	1.6294
1.0	1.0	0.1000	0.5000	1.0000	2.0000
1.0	2.0	0.1336	0.6721	1.3549	2.7491
2.0	0.0	0.0498	0.2459	0.4840	0.9392
2.0	0.1	0.0523	0.2568	0.5096	0.9913
2.0	0.5	0.0624	0.3192	0.6122	1.2012
2.0	1.0	0.0749	0.3727	0.7410	1.4659
2.0	2.0	0.1000	0.5000	1.0000	2.0000

Case 2: Some examples:

A. Simple analytical example

If $\kappa_I = \kappa_G$ and $\sigma_* = \sigma$ is assumed[9] the following exact condition is implied by (6.9):

$$Q_* = Q \iff (\rho+\kappa_D)M = \rho N \iff \frac{N}{M} = 1 + \frac{\kappa_D}{\rho} \quad \text{for any } \sigma_* = \sigma,\ \kappa_I = \kappa_G.$$

[9]This constellation is exemplified by Illustrations B and C above.

It resembles the condition derived above for the geometric decay case [ILLUSTRATION A, subcase [1]: $\alpha/\delta = 1+(\kappa_D/\rho)$], when we replace (α/δ) by $\bar{\alpha}/\bar{\delta}$, which for $\sigma_* = \sigma$ equals N/M.

B. Numerical examples

Numerical examples are given in Table 2. Four typical cases are commented on below:

[1] *Sudden death and medium value of maximal service life:* Assume that $Q = \rho N = 0.5$, $\sigma = 0$, which implies that capital survival follows the sudden death pattern, with death at age N, so that $G_0(s)$ declines linearly up to this age. Then, to compensate for the *more favourable tax-permitted shape parameter* $\sigma_* = 0.5 \implies A(s) = (1-\frac{s}{M})^{1.5}$, which is convex, neutrality would require $Q_* = 0.6326$. There are numerous parameter constellations which can ensure this, since Q and Q_* can be factorized into $\rho \cdot N$ and $\rho_* \cdot M$, respectively, in numerous ways.

(a) If the constellation giving $Q = 0.5$ is $N = 10$, $\rho = 5\,\%$, then for $M = N = 10$, neutrality would require $\rho_* = 6.3\,\%$, equivalent to a mis-indexation of the depreciation allowances equal to $\kappa_D = 1.3\,\%$. For $\rho = \rho_* = 5\,\%\,(\implies \kappa_D = 0)$ the maximal tax permitted service life should be increased to $M = 12.65$ years.

(b) If the constellation giving $Q = 0.5$ is $N = 25$, $\rho = 2\,\%$, then for $M = N = 25$, neutrality would require $\rho_* = 2.5\,\%$, equivalent to a mis-indexation of the depreciation allowances equal to $\kappa_D = 0.5\,\%$. For $\rho = \rho_* = 2\,\%\,(\implies \kappa_D = 0)$ the maximal tax permitted service life should be increased to $M = 31.6$ years.

[2] *Linear retirement and moderate to long value of maximal service life:* Assume $Q = \rho N = 1$, $\sigma = 1$, which implies that capital survival follows a linearly declining schedule over N years, and convex $G_0(s)$: $G_0(s) = (1-\frac{s}{M})^2$. Then, to compensate for the *less favourable tax-imposed shape parameter* $\sigma_* = 0.5 \implies A(s) = (1-\frac{s}{M})^{1.5}$, which is convex, but less strongly curved than $G_0(s)$, neutrality would require a reduction of Q to $Q_* = 0.8237$. Again, this can be ensured in numerous ways.

(a) If the constellation giving $Q = 1$ is $N = 20$, $\rho = 5\,\%$, then for $M = N = 20$, neutrality would require $\rho_* = 4.1\,\%\,(\implies \kappa_D = -0.9\,\%)$. For $\rho = \rho_* = 5\,\%\,(\implies \kappa_D = 0)$ the maximal tax permitted service life should be reduced to $M = 16.2$ years.

(b) If the constellation giving $Q = 1$ is $N = 50$, $\rho = 2\,\%$, then for $M = N = 50$, neutrality would require $\rho_* = 1.6\,\%\,(\implies \kappa_D = -0.4\,\%)$. For $\rho = \rho_* = 2\,\%\,(\implies \kappa_D = 0)$ the maximal tax permitted service life should be reduced to $M = 41.2$ years.

[3] *Sudden death and very short maximal service life:* Assume that $Q = \rho N = 0.1$, $\sigma = 0$, which implies that capital survival follows the sudden death schedule, with death at age N, so that $G_0(s)$ declines linearly. Then to compensate for the *more favourable tax-permitted shape parameter* $\sigma_* = 1 \implies A(s) = (1-\frac{s}{M})^2$, which is convex, neutrality would require an increase of Q to $Q_* = 0.1506$. Two examples are:

(a) If the constellation giving $Q = 0.1$ is $N = 5$, $\rho = 2\,\%$, then for $M = N = 5$, neutrality would require $\rho_* = 3.0\,\%\,(\implies \kappa_D = 1\,\%)$. For $\rho = \rho_* = 2\,\%\,(\implies \kappa_D = 0)$ the maximal tax permitted service life should be increased to $M = 7.53$ years.

(b) If the constellation giving $Q = 0.1$ is $N = 2$, $\rho = 5\,\%$, then for $M = N = 2$, neutrality would require $\rho_* = 7.5\,\%\,(\implies \kappa_D = 2.5\,\%)$. For $\rho = \rho_* = 5\,(\implies \kappa_D = 0)$ the maximal tax permitted service life should be increased to $M = 3.0$ years.

[4] *Concave retirement and very long maximal service life:* Assume that $Q = \rho N = 2$, $\sigma = 0.1$, which implies that capital survival follows a markedly convex schedule over N years, and weakly concave $G_0(s)$: $G_0(s) = (1-\frac{s}{M})^{1.1}$. Then to compensate for the *more favourable tax-imposed shape parameter* $\sigma_* = 1 \implies A(s) = (1-\frac{s}{M})^2$, which is convex and more strongly curved than $G_0(s)$, neutrality would require an increase of Q to $Q_* = 3.060$. Two examples are:

(a) If the constellation giving $Q=2$ is $N=100$, $\rho=2\,\%$, then for $M=N=100$, neutrality would require $\rho_*=3.1\,\%\,(\Longrightarrow \kappa_D=1.1\,\%)$. For $\rho=\rho_*=2\,\%\,(\Longrightarrow \kappa_D=0)$ the maximal tax permitted service life should be increased to $M=153$ years.

(b) If the constellation giving $Q=2$ is $N=50$, $\rho=4\,\%$, then for $M=N=50$, neutrality would require $\rho_*=6.1\,\%\,(\Longrightarrow \kappa_D=2.1\,\%)$. For $\rho=\rho_*=4\,(\Longrightarrow \kappa_D=0)$ the maximal tax permitted service life should be increased to $M=76$ years.

Remark on quality of approximation: From (6.3) it follows that $\Lambda(\tau,\theta)=\theta/(\tau+1)$ can serve as a first-order approximation. Then (6.9) would imply that Q_*/Q should be approximately equal to $(\sigma_*+2)/(\sigma+2)$. From Table 2 we see that the quality of this approximation may not be good. For example the constellation $(Q,\sigma,\sigma_*)=(2.0,\ 0.5,\ 2.0)$ implies $Q_*/Q=3.4111/2.0000=1.71$ which departs substantially from $(\sigma_*+2)/(\sigma+2)=4.0/2.5=1.6$.

Case 6: Some examples:

Let us again use the approximation defined when truncating the last expression in (6.3) after the first-order term, *i.e.*, $\Lambda(\tau,\theta)\approx\theta/(\tau+1)$. Then the neutrality condition (6.8) gives for Case 6, approximately,

$$\frac{Q_*}{\sigma_*+2}=\frac{Q}{\sigma+2}(1+\Delta) \iff \frac{(\rho+\kappa_D)M}{\sigma_*+2}=\frac{\rho N}{\sigma+2}(1+\Delta) \iff \frac{\rho+\kappa_D}{\bar{\alpha}}=\frac{\rho+\kappa_I-\kappa_G}{\bar{\delta}}.$$

This is equivalent to:

$$(A)\qquad M=N\frac{\sigma_*+2}{\sigma+2}\frac{\rho(1+\Delta)}{\rho+\kappa_D} \iff$$

$$(B)\qquad \rho+\kappa_D=\rho(1+\Delta)\frac{N}{M}\frac{\sigma_*+2}{\sigma+2} \iff$$

$$(C)\qquad 1+\Delta=\frac{\rho+\kappa_D}{\rho}\frac{M}{N}\frac{\sigma+2}{\sigma_*+2} \iff$$

$$(D)\qquad \sigma_*+2=(\sigma+2)\frac{M}{N}\frac{\rho+\kappa_D}{\rho(1+\Delta)}.$$

These expressions, for given (N,σ,ρ), can be used to roughly assess:

(A) How a non-zero κ_D, a non-zero Δ or a $\sigma_*\neq\sigma$ can be 'translated approximately into' a neutrality preserving M.

(B) How a non-zero Δ, an $M\neq N$ or a $\sigma_*\neq\sigma$ can be 'translated approximately into' a neutrality preserving κ_D.

(C) How a non-zero κ_D, an $M\neq N$ or an $\sigma_*\neq\sigma$ can be 'translated approximately into' a neutrality preserving Δ.

(D) How a non-zero κ_D, a non-zero Δ or an $M\neq N$ can be 'translated approximately into' a neutrality preserving σ_*.

It is interesting to contrast the condition under (C) with a corresponding condition derived for the geometric decay case (Illustration A, subcase [2]): The indexation of interests should be stronger than $(\varepsilon_I>\varepsilon_G,\ \Delta<0)$, respectively weaker than $(\varepsilon_I<\varepsilon_G,\ \Delta>0)$, the indexation of the capital gains, to neutralize too favourable ($M<N$ and/or $\sigma_*>\sigma$), respectively too unfavourable ($M>N$ and/or $\sigma_*<\sigma$) depreciation allowances.

More generally, if we truncate the expression for $\Lambda[\cdot,\cdot]$, (6.3), after the quadratic term, the neutrality condition (6.8) gives as an approximation a cubic equation in M:

$$(6.10)\qquad \begin{aligned} &\tfrac{(\rho+\kappa_D)M}{\sigma_*+2}\left[1-\tfrac{(\rho+\kappa_D)M}{(\sigma_*+3)}\left(1-\tfrac{(\rho+\kappa_D)M}{\sigma_*+4}\right)\right]=[1+\tfrac{\kappa_I-\kappa_G}{\rho}]\tfrac{\rho N}{\sigma+2}\left[1-\tfrac{\rho N}{\sigma+3}\left(1-\tfrac{\rho N}{\sigma+4}\right)\right] && \text{in Case 6,}\\ &\tfrac{(\rho+\kappa_D)M}{\sigma_*+2}\left[1-\tfrac{(\rho+\kappa_D)M}{(\sigma_*+3)}\left(1-\tfrac{(\rho+\kappa_D)M}{\sigma_*+4}\right)\right]=\tfrac{\rho N}{\sigma+2}\left[1-\tfrac{\rho N}{\sigma+3}\left(1-\tfrac{\rho N}{\sigma+4}\right)\right] && \text{in Case 2.}\end{aligned}$$

The translations that correspond to (A) through (D) above, will have to be done by numerical methods, but are likely to give more accurate results.

7 Approximation errors and pitfalls

7.1 Approximation errors when exponential decay is invalid

To assess capital service prices and derived indexes of tax (non-)neutrality numerically, we have in this paper recommended the use of expressions of the form (4.15)–(4.16). Assume in particular $m=n=1$, which gives

$$\frac{c^*(t)}{q(t)} = \frac{1+\xi\frac{u}{1-u}}{\phi(0)}, \qquad \xi = \left(\rho+\kappa_D+\tfrac{v}{u}\right) Z(\rho+\kappa_D) - (\rho+\kappa_I-\kappa_G)Y(\rho).$$

Let us from these expressions compare exponential decay with the two-parametric profile (Illustrations A and D in Section 6.2), when using for the latter *different approximations to* $\phi(0)$*,* $Y(\rho)$ *and* $Z(\rho+\kappa_D)$. From the exact definitions we have:

	Exponential decay	Two-parametric
$\phi(0)$	$\frac{1}{\rho+\delta}$	$NF(\sigma,-\rho N) = \frac{1}{\rho}\Lambda(\sigma,\rho N)$
$Y(\rho)$	$\frac{1}{\rho+\delta}$	$NF(\sigma+1,-\rho N) = \frac{1}{\rho}\Lambda(\sigma+1,\rho N)$
$Z(\rho+\kappa_D)$	$\frac{1}{\rho+\kappa_D+\alpha}$	$MF(\sigma_*+1,-(\rho+\kappa_D)M) = \frac{1}{\rho+\kappa_D}\Lambda(\sigma_*+1,(\rho+\kappa_D)M)$

If the valid parametrizations were exponential decay and declining balance for true and tax-permitted depreciation, respectively, we would have

$$\text{(7.1)} \qquad \frac{c^*(t)}{q(t)} = (\rho+\delta)\left[1+\frac{u}{1-u}\left\{\frac{\rho+\kappa_D+\frac{v}{u}}{\rho+\kappa_D+\alpha} - \frac{\rho+\kappa_I-\kappa_G}{\rho+\delta}\right\}\right].$$

If this description is invalid – so that the depreciation rates are not strictly constant and gross and net capital do not coincide – and we still adhere to (7.1) while inserting approximations to δ and α based on the benchmark values under stationary investment, (3.20)–(3.21), *i.e.*, $\delta \approx \bar{\delta} = \frac{\sigma+2}{N}$, $\alpha \approx \bar{\alpha} = \frac{\sigma_*+2}{M}$, we would rely on

$$\text{(7.2)} \qquad \frac{c^*(t)}{q(t)} = (\rho+\bar{\delta})\left[1+\frac{u}{1-u}\left\{\frac{\rho+\kappa_D+\frac{v}{u}}{\rho+\kappa_D+\bar{\alpha}} - \frac{\rho+\kappa_I-\kappa_G}{\rho+\bar{\delta}}\right\}\right].$$

The correct expression would, however, be:

$$\text{(7.3)} \quad \frac{c^*(t)}{q(t)} = \frac{\rho}{\Lambda(\sigma,\rho N)}\left[1+\frac{u}{1-u}\left\{\frac{\rho+\kappa_D+\frac{v}{u}}{\rho+\kappa_D}\Lambda[\sigma_*+1,(\rho+\kappa_D)M] - \frac{\rho+\kappa_I-\kappa_G}{\rho}\Lambda[\sigma+1,\rho N]\right\}\right],$$

where the $\Lambda[\cdot,\cdot]$ function occurs three times *with different arguments*. Values computed from (7.2) and (7.3) may differ substantially. To see this, consider two approximations to an expression of the form $\Lambda = a[1-b(1-c)]$ (a,b,c arbitrary):

$$\Lambda \approx \Lambda_1 = a[1+b]^{-1} \qquad \text{if } b^2, bc \text{ and higher-order terms are small,}$$
$$\Lambda \approx \Lambda_2 = a[1+b(1-c)]^{-1} \qquad \text{if } [b(1-c)]^2 \text{ and higher-order terms are small.}$$

Therefore the truncated expression [confer (6.3)]

$$\Lambda(\tau,\theta) \approx \tfrac{\theta}{\tau+1}\left[1-\tfrac{\theta}{\tau+2}\left(1-\tfrac{\theta}{\tau+3}\right)\right]$$

can be approximated alternatively to

$$\Lambda_1(\tau,\theta) = \frac{\theta}{\tau+1}[1+\frac{\theta}{\tau+2}]^{-1} \quad \text{if } \frac{\theta^2}{(\tau+2)^2}, \frac{\theta^2}{(\tau+2)(\tau+3)} \text{ and higher-order terms are small,}$$

$$\Lambda_2(\tau,\theta) = \frac{\theta}{\tau+1}[1+\frac{\theta}{\tau+2}(1-\frac{\theta}{\tau+3})]^{-1} \quad \text{if } [\frac{\theta}{\tau+2}(1-\frac{\theta}{\tau+3})]^2 \text{ and higher-order terms are small.}$$

Letting, respectively, $\Lambda_1(\tau,\theta)$ and $\Lambda_2(\tau,\theta)$ replace $\Lambda(\tau,\theta)$, while substituting $\theta = \rho N$, $\tau = \sigma+1$, we therefore have

	Exponential decay	Two-parametric, using approx. Λ_1	Two-parametric, using approx. Λ_2
$\frac{1}{\phi(0)}$	$\rho+\delta$	$\frac{\sigma+1}{N}\left[1+\frac{\rho N}{\sigma+2}\right]$	$\frac{\sigma+1}{N}\left[1+\frac{\rho N}{\sigma+2}\left(1-\frac{\rho N}{\sigma+3}\right)\right]$
$\frac{1}{Y(\rho)}$	$\rho+\delta$	$\frac{\sigma+2}{N}\left[1+\frac{\rho N}{\sigma+3}\right]$	$\frac{\sigma+2}{N}\left[1+\frac{\rho N}{\sigma+3}\left(1-\frac{\rho N}{\sigma+4}\right)\right]$
$\frac{1}{Z(\rho+\kappa_D)}$	$\rho+\kappa_D+\alpha$	$\frac{\sigma_*+2}{M}\left[1+\frac{(\rho+\kappa_D)M}{\sigma_*+3}\right]$	$\frac{\sigma_*+2}{M}\left[1+\frac{(\rho+\kappa_D)M}{\sigma_*+3}\left(1-\frac{(\rho+\kappa_D)M}{\sigma_*+4}\right)\right]$

When inserting the approximations Λ_1 and Λ_2 in (7.3) we get two alternative approximations to the relative capital service price

$$(7.4)\quad \frac{c^*(t)}{q(t)} = \left[\rho\frac{\sigma+1}{\sigma+2}+\frac{\sigma+1}{N}\right]\left[1+\frac{u}{1-u}\left\{\frac{\rho+\kappa_D+\frac{v}{u}}{(\rho+\kappa_D)\frac{\sigma_*+2}{\sigma_*+3}+\frac{\sigma_*+2}{M}}-\frac{\rho+\kappa_I-\kappa_G}{\rho\frac{\sigma+2}{\sigma+3}+\frac{\sigma+2}{N}}\right\}\right],$$

$$(7.5)\quad \frac{c^*(t)}{q(t)} = \left[\rho\frac{\sigma+1}{\sigma+2}\left(1-\frac{\rho N}{\sigma+3}\right)+\frac{\sigma+1}{N}\right]$$
$$\times\left[1+\frac{u}{1-u}\left\{\frac{\rho+\kappa_D+\frac{v}{u}}{(\rho+\kappa_D)\frac{\sigma_*+2}{\sigma_*+3}\left(1-\frac{(\rho+\kappa_D)M}{\sigma_*+4}\right)+\frac{\sigma_*+2}{M}}-\frac{\rho+\kappa_I-\kappa_G}{\rho\frac{\sigma+2}{\sigma+3}\left(1-\frac{\rho N}{\sigma+4}\right)+\frac{\sigma+2}{N}}\right\}\right].$$

Some interesting conclusions follow:

[1] If σ, σ_*, N, M are large with σ/N and σ_*/M finite, then (7.4) and (7.5) would be approximately equal and (7.2) would provide a good approximations to both.

[2] If σ is small, say in the range 0–2, it could be very misleading, even approximately, to rely on (7.2) and proceed by letting (a) the value of $\bar{\delta}$ in the first factor after the equality sign and in the curly bracket be equal and (b) ρ in the first bracket in (7.2) have a weight equal to 1. The constellation $\sigma=0$, $\sigma_*=1$ gives for example

$$(7.2)\Longrightarrow \frac{c^*(t)}{q(t)} = \left[\rho+\frac{2}{N}\right]\left[1+\frac{u}{1-u}\left\{\frac{\rho+\kappa_D+\frac{v}{u}}{\rho+\kappa_D+\frac{3}{M}}-\frac{\rho+\kappa_I-\kappa_G}{\rho+\frac{2}{N}}\right\}\right],$$

while

$$(7.4)\Longrightarrow \frac{c^*(t)}{q(t)} = \left[\frac{\rho}{2}+\frac{1}{N}\right]\left[1+\frac{u}{1-u}\left\{\frac{\rho+\kappa_D+\frac{v}{u}}{(\rho+\kappa_D)\frac{3}{4}+\frac{3}{M}}-\frac{\rho+\kappa_I-\kappa_G}{\rho\frac{2}{3}+\frac{2}{N}}\right\}\right],$$

$$(7.5)\Longrightarrow \frac{c^*(t)}{q(t)} = \left[\frac{\rho}{2}-\frac{\rho^2 N}{6}+\frac{1}{N}\right]\left[1+\frac{u}{1-u}\left\{\frac{\rho+\kappa_D+\frac{v}{u}}{(\rho+\kappa_D)\frac{3}{4}(1-\frac{(\rho+\kappa_D)M}{5})+\frac{3}{M}}-\frac{\rho+\kappa_I-\kappa_G}{\rho\frac{2}{3}(1-\frac{\rho N}{4})+\frac{2}{N}}\right\}\right].$$

[3] Approximations relying on (7.2) for observed (discrete time) counterparts to $[D(t), H(t)]$, say $\bar{\delta}\approx\delta(t)=\frac{D(t)}{H(t)}$ for a specific t, or mean values for a sample period $[1,T]$ constructed as $\bar{\delta}\approx\frac{1}{T}\sum_{t=1}^{T}\delta(t)$, may be very misleading.[10]

[10]See Biørn (1989, Section 11.7) for some related evidence.

7.2 Some additional reminders

Assume now that the *mathematical form* of the depreciation profiles is known and, to a good approximation, agrees with the two-parametric Illustration D and that we stick to the correct expression (7.3). However, if our information on the *numerical values* of (N, σ) and how they are related to the tax code parameters (M, σ_*) is ill-founded – owing to lack of data or to inadequate econometric inference – quantitative analysis intending to provide policy recommendations, may give very misleading results, and one may well ask whether such analyses are wasted effort. Unfortunately, it has been difficult to motivate statistical agencies to give priority to collecting information relating to the shape of capital survival functions, say data for capital prices in second-hand markets from which N and σ can be estimated for relevant capital categories and their relation to M and σ_* assessed.[11]

The problems discussed in Section 7.1 exemplify problems in approximating values of non-linear functions by inserting approximate values of the functions' arguments, since the function to be evaluated when assessing tax distortions under non-exponential decay, like ξ and its components. usually is highly non-linear. For instance will $a \approx a^* \,\&\, b \approx b^*$ not in general imply $f(a,b) \approx f(a^*,b^*)$, or the integral $\int_0^n a(s)b(s)ds$ may not be well approximated by $\frac{1}{n}[\int_0^n a(s)ds][\int_0^n b(s)ds]$. If $a(s)$ and $b(s)$ are correlated – as is the case when for instance $a(s) = e^{-\delta s}$ and $b(s) = [1 - \frac{s}{n}]^\tau$ $[s \in (0,n)]$ – the values of the two latter integrals may indeed differ widely.

8 Concluding remarks

In this paper, we have considered ways of describing and measuring departures from neutrality of a corporate tax system. A framework with a general representation of the survival function of the capital has been assumed. Five sources of departure from neutrality have been identified: (i) difference between the depreciation profile of the accounting capital implicit in the tax code and true depreciation, (ii) mis-indexation of the depreciation allowances, (iii) incomplete deduction of interests in the tax base, (iv) taxation of capital values, and (v) asymmetry in the tax treatment of interests and capital gains. The tax system certainly affects the firm neutrally if *all* the five components are zero.

In addition, we have demonstrated that neutrality can be ensured also under the less restrictive requirement that *the sum of* the components be zero, so that for instance a positive contribution from one component is neutralized by a negative contribution from another one. Trade-offs between various departures from neutrality are illustrated – analytically and numerically – for selected parametric survival functions, including the familiar exponential decay and less familiar two-parametric convex and concave functions with a finite maximal capital life-time assumed.

[11] For further discussion, see Biørn (2007).

Within this setting, we have shown that substantial biases can arise when attempting to quantify the degree of non-neutrality if non-exponential two-parametric depreciation schedules are forced, by 'approximation devices', to fit into the formulae derived from exponential decay. On the one hand, exponential decay has been empirically contested by several researchers. On the other hand, in several countries and for an increasing number of capital categories, tax authorities impose depreciation schedules for firms' tax-accounting that deviate substantially from declining balance. Since an analytical framework to handle this exists, it seems to be strong reasons to consider the 'exponential-decay-constant-depreciation-rate' practice in empirical work with great skepticism, or abandon it.

Appendix: An auxiliary function

The function considered is

$$h(a,\rho,\tau,N) = \left(1-\tfrac{a}{N}\right)^{-\tau} \int_a^N e^{-\rho(z-a)} \left(1-\tfrac{z}{N}\right)^{\tau} dz, \tag{a.1}$$

which after substitution of $s=\frac{z}{N}$, $b=\frac{a}{N}$, $\theta=\rho N$ can be written as

$$h(a,\rho,\tau,N) = N(1-b)^{-\tau} \int_b^1 e^{-\theta(s-b)}(1-s)^{\tau} ds. \tag{a.2}$$

Defining

$$f(b,\theta,\tau) = \int_b^1 e^{-\theta(s-b)}(1-s)^{\tau} d\,s, \tag{a.3}$$

we can rewrite (a.2) as

$$h(a,\rho,\tau,N) = N\,(1-b)^{-\tau} f(b,\theta,\tau). \tag{a.4}$$

Now, by using integration by parts, (a.3) can be shown to satisfy the recursion

$$f(b,\theta,\tau) = \tfrac{1}{\theta}[(1-b)^{\tau} - \tau f(b,\theta,\tau-1)], \qquad \theta \neq 0,\ \tau \geq 1. \tag{a.5}$$

To obtain the corresponding recursion for $h(a,\rho,\tau,N)$ we first multiply (a.5) by $N(1-b)^{-\tau}$, to obtain

$$N(1-b)^{-\tau} f(b,\theta,\tau) = \tfrac{N}{\theta}[1 - \tfrac{\tau}{N(1-b)} N(1-b)^{-(\tau-1)} f(b,\theta,\tau-1)],$$

next substitute $s=\frac{z}{N}$, $b=\frac{a}{N}$, $\theta=\rho N$ and finally use (a.4). This yields

$$h(a,\rho,\tau,N) = \tfrac{1}{\rho}\left[1 - \tfrac{\tau}{N-a} h(a,\rho,\tau-1,N)\right], \qquad \rho \neq 0,\ \tau \geq 1. \tag{a.6}$$

The two recursions are not applicable if $\tau=0$ or $\rho=0$. We then obtain directly from (a.1)

$$h(a,\rho,0,N) = \int_a^N e^{-\rho(z-a)} dz = \tfrac{1}{\rho}\left[1 - e^{-\rho(N-a)}\right], \qquad \rho \neq 0,\ \tau = 0, \tag{a.7}$$

$$h(a,0,\tau,N) = \left(1-\tfrac{a}{N}\right)^{-\tau} \int_a^N \left(1-\tfrac{z}{N}\right)^{\tau} d\,z = \tfrac{N-a}{\tau+1}, \qquad \rho = 0,\ \tau \geq 0. \tag{a.8}$$

Equation (a.7) provides the initial value when applying (a.6) recursively for $\rho \neq 0$, $\tau=1,2,\ldots$.

If $a=b=0$, the recursions (a.5)–(a.8) take the simpler forms

$$\begin{aligned}
f(0,\theta,\tau) &= \tfrac{1}{\theta}[1 - \tau f(0,\theta,\tau-1)], && \theta \neq 0,\ \tau \geq 1\\
f(0,\theta,0) &= \tfrac{1}{\theta}[1 - e^{-\theta}], && \theta \neq 0,\ \tau = 0,\\
h(0,\rho,\tau,N) &= \tfrac{1}{\rho}\left[1 - \tfrac{\tau}{N} h(0,\rho,\tau-1,N)\right], && \rho \neq 0,\ \tau \geq 1.\\
h(0,\rho,0,N) &= \tfrac{1}{\rho}\left[1 - e^{-\rho N}\right], && \rho \neq 0,\ \tau = 0,
\end{aligned}$$

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