MEMORANDUM

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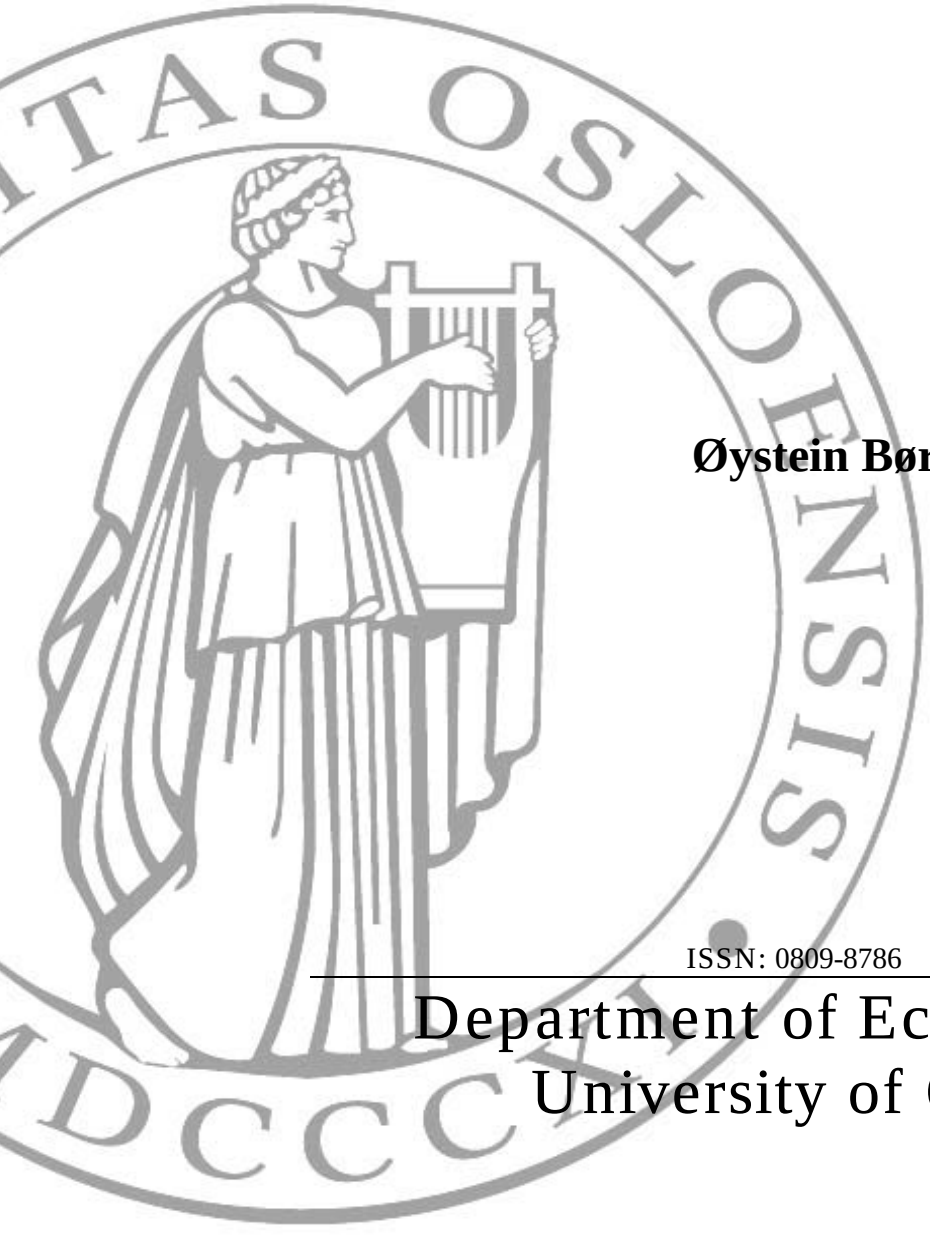
Contagious Mortgage Default



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Last 10 Memoranda

No 09/10	Derek J. Clark and Tore Nilssen *The Number of Organizations in Heterogeneous Societies*
No 08/10	Jo Thori Lind *The Number of Organizations in Heterogeneous Societies*
No 07/10	Olav Bjerkholt *The "Meteorological" and the "Engineering" Type of Econometric Inference: a 1943 Exchange between Trygve Haavelmo and Jakob Marschak*
No 06/10	Dag Kolsrud and Ragnar Nymoen *Macroeconomic Stability or Cycles? The Role of the Wage-price Spiral*
No 05/10	Olav Bjerkholt and Duo Qin *Teaching Economics as a Science: The 1930 Yale Lectures of Ragnar Frisch*
No 04/10	Michael Hoel *Climate Change and Carbon Tax Expectations*
No 03/10	Geir B. Asheim *Comparing the Welfare of Growing Economies*
No 02/10	Rolf Golombek, Mads Greaker and Michael Hoel *Climate Policy without Commitment*
No 01/10	Geir B. Asheim, Max Voorneveld and Jörgen W. Weibull *Epistemically Stable Strategy Sets*
No 27/09	Erik Biørn *Capital Decay and Tax Distortions: How to Abandon Exponential Decay and Benefit from It*

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Contagious Mortgage Default*

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June 29, 2010

Abstract

This paper analyses the default option typical to American mortgages. Households borrow to buy durable housing, but future house prices are uncertain, and households find it advantageous to default on their debt if house prices fall sufficiently. A key assumption of the model is that households are relegated to the rental market upon default, and that there is a small pecuniary inefficiency ("iceberg cost") in renting. This leads defaulters to substitute consumption of other goods for housing; that is, the demand for housing falls upon default. Consequently, when some households default, aggregate demand for housing is reduced, hence house prices fall more, possibly inciting other households to default. This complementarity is a source of multiple equilibria, and a price externality. Using a specific case for which an analytical solution can be derived, I show that contagion is possible: it may be that the default of a minority (interpretable as sub-prime borrowers) spreads to a majority (interpretable as prime borrowers).

JEL classification: E21; G11; R21

Keywords: Housing demand; Mortgage market; Default risk; Multiple equilibria; Contagion

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1 Introduction

Over the last three years, house prices in the US have fallen sharply. According to the S&P/Case-Shiller U.S. National Home Price Index, house prices fell by about one third from the peak in the second quarter of 2006 to the second quarter of 2009.[1] The fall implies that millions of American households have mortgages that exceed the value of their homes, and for many of them, the discrepancy is large. Partly as a result, the rates of delinquent mortgages (one or more payments past due) and mortgages in foreclosure have exploded.[2] According to the Mortgage Bankers Association's National Delinquency Survey (NDS) in the second quarter of 2009, 8.9 percent of all mortgage loans were delinquent, and another 4.3 percent were in the foreclosure process, compared to 4.4 and 1.0 percent, respectively, three years earlier.[3] While mortgage default (understood as the process starting with delinquency and ending in foreclosure) initially sparked within subprime loans, it gradually spread to more conventional mortgages. In the second quarter of 2009, prime fixed-rate loans accounted for one in three foreclosure starts in the NDS. An important worry is that defaults *in themselves* put downward pressure on house prices. A contagious chain may have played out, where default pushed down prices, causing more default and even lower prices, eventually leading to a large scale default crisis.

The purpose of this paper is to demonstrate, by means of a formal model, how the default option in American mortgage finance prepares the ground for contagion. The model is built on three main assumptions: 1) mortgages are non-recourse, non-renegotiable debt contracts, 2) mortgage default is strategic, and 3) renting a house is, all other things equal, more costly than owning the same house. These are interesting and not unreasonable assumptions for the following reasons:

- Only a few states in the U.S. have mandatory non-recourse mortgages.[4] Nevertheless, most American mortgages may be close to non-recourse in practice, because homeowners rarely have significant wealth besides their home, and because U.S. federal bankruptcy law accords the right to a "fresh start." Under Chapter 7 in the Bankruptcy Code, the worst case scenario for a household is to end up with zero net assets and a poor credit rating.[5] Given this limited cost of default, it is important to investigate the economic incentives and consequences of strategic default.

[1] Available on www.standardandpoors.com.

[2] See Sanders (2008) for an early exposition.

[3] The National Delinquency Survey is based on a sample of more than 44 million mortgage loans serviced by mortgage companies, commercial banks, thrifts, credit unions and others. It provides delinquency and foreclosure statistics at the national, regional and state levels. Available on www.mbaa.org.

[4] Capone (1996) includes a thorough discussion on foreclosure and bankruptcy law.

[5] The Bankruptcy Abuse Prevention and Consumer Protection Act of 2005 (Pub.L. 109-8, 119 Stat. 23, enacted April 20, 2005), imposes a means-test to Chapter 7 filings when the debtor's income is above the state median income, in order to prevent e.g. strategic default. However, this means test only applies when debt is primarily consumer debt.

- Most real world loan arrangements are written as non-contingent debt contracts; a recent debate is why lenders do not renegotiate more mortgages in default. Widespread securitisation is the most widely believed reason for the rarity of renegotiation (see, for example, Eggert, 2007). Adelino et al. (2009) offer an alternative view; they argue that lenders expect to recover more from foreclosure than from a modified loan, because many delinquent borrowers eventually resume payments without receiving a modification, and many borrowers receiving a modification eventually re-default, making loan modification a very inefficient strategy.

- It is widely acknowledged that negative home equity (a mortgage that exceeds the value of the home) is the primary driver of mortgage default (Deng et al., 2000). With positive home equity, households would be better off selling the home and prepaying the mortgage, than defaulting. The controversy is to what extent households with negative home equity actually *choose* default (strategic default), as opposed to being forced to default by an exogenously triggered inability to meet scheduled payments. In a study of Massachusetts homeowners in the 1990s, Foote et al. (2008) found that very few people (6%) actually walked away from their mortgages when their home equity was negative, and their study has been taken as evidence against strategic default. Guiso et al. (2009) challenge this view: based on survey data, they estimate that approximately one of four recent defaults are strategic. They argue that the relationship between negative home equity and default is highly non-linear, and that this time home prices have fallen much more than in the episode studied by Foote et al. (2008).

- There are several sources of additional costs involved in renting relative to ownership. One is due to moral hasard, as tenants may have insufficient incentives to take care of the property. Another is the implicit tax advantage of indebted owners, as mortgage interest payments are deductible in the U.S. for federal tax purposes. Moreover, there are overhead costs in the rental business that must be covered.

The key mechanism of the model is as follows. Households borrow to buy durable housing, but future house prices are uncertain, and households find it advantageous to default on their debt if house prices fall sufficiently. Because households cannot obtain new credit after default, they are relegated to the rental market for housing. Because rental housing is more expensive than owned housing per unit of housing, defaulters substitute consumption of other goods for housing; that is, the demand for housing falls upon default. Consequently, aggregate demand for housing is reduced, hence house prices fall more, possibly inciting other households to default. This complementarity is a source of negative externality, multiple equilibria, and possibly, of contagious default.

The theory that I present offers some significant advantages to the bulk of commonly given explanations to the current mortgage default crisis. In my model, both households and banks behave in an optimal manner with symmetric information and rational expectations, showing that there is no need for grand scale fraud, misperceptions or irrationality in order to explain the crisis. Explanations that require key information to be hidden from market participants – information such as the financial position of U.S. households or the possible consequences of a sharp fall in house prices – are quite unsatisfactory for a large and very competitive market such as the U.S. mortgage market. In fact, these issues have since long been hotly debated, as evidenced by e.g. Case and Shiller (2003), and major investment banks explicitly characterised housing "meltdown" scenarios in their research reports on several occasions preceding the crisis (see the evidence presented in Gerardi et al., 2009). Thus, the problem does not seem to be that a housing market collapse was unthinkable or misunderstood, but rather that it was considered too unlikely to be accorded any importance by any individual market participant. Perhaps precisely for this reason did they go on with the lenient lending that made the crisis possible.

In terms of modelling, I assume that each mortgage is competitively priced to reflect the objective probability of default (individual or risk-based pricing), and that a borrower controls the objective probability of default by his choice of mortgage contract, as in Corbae and Quintin (2010) and Jeske et al. (2010). The housing market is taken to be a market for homogeneous housing space, rather than actual houses, as in Gervais (2002). This conceptualisation is attractive in that there is only one (unit) market price of housing – e.g. the price per square meter of housing space – to be determined in each state.

The paper that comes closest to mine is probably the (parallel and complementary) work by Chatterjee and Eyigungor (2009). They build a large quantitative-theoretic model based on very similar assumptions which they calibrate to the U.S. economy; in simulations, they find that households significantly decrease their demand for housing after default, significantly amplifying price falls. Their work thus suggests that the externality and ensuing contagion mechanism that I study can be economically significant. The simple two-period-two-household-type structure in the present paper permits a detailed, analytical investigation of the mechanisms at hand, which is not possible in a quantitative-theoretic model.[6]

This paper is related to the very diverse literature on financial contagion, and in particular to Allen and Gale (2000). They present a model with complete information in which a small liquidity preference shock in one region can spread contagiously in the economy, through cross holdings of deposits between banks. Their notion of *financial fragility*, where an unforeseen shock (possibly small) can bring down the entire financial system, resembles my case of contagious default equilibrium.

[6] Jeske et al. (2010) derive analytical steady-state solutions in a model without inefficiencies in renting.

2 Model

2.1 Households and housing

Households derive utility from housing services $h > 0$ and the consumption of other goods $c > 0$ according to

$$u(c,h) = \frac{\left[\alpha c^{\rho} + (1-\alpha)h^{\rho}\right]^{\frac{1-\theta}{\rho}} - 1}{1-\theta} \tag{1}$$

for $\rho \leq 1$ and $\theta \geq 0$.[7] This is the form often used in the housing literature (Jeske, 2005). The parameter ρ determines the substitutability between housing and other goods, θ governs risk aversion and intertemporal substitutability, and $\alpha \in (0,1)$ is a weighing parameter for how much the household values the consumption of other goods. It can be verified that the function $u(c,h)$ is two times continuously differentiable and strictly increasing in each of its arguments, and concave. Households seek to maximise their expected discounted utility computed using eq. (1) for each time period, and a finite time discount factor $\beta > 0$.

For tractability and for maximum transparency with respect to the ensuing results, there are only two time periods, *ex ante* and *ex post*, referring to before and after the uncertainty in the model is resolved. Subscript zero denotes the ex ante period; no subscript is used for the ex post period.

Consumption goods are elastically supplied through an international market at the constant price of 1 (they act as numeraire). Housing services can be obtained in two ways: through house ownership or from renting. Housing *stock* is traded in a domestic market. The stock of housing is in fixed aggregate supply H. The relative price of a unit of housing stock will therefore depend on households' aggregate demand. Housing stock is perfectly durable from the first (ex ante) to the last (ex post) period, but after that it no longer has any value, so it is as if it completely depreciated.[8] Owning and occupying one unit of housing stock in any period provides one unit of housing service in that period. Alternatively, households can rent housing service, provided that someone else owns and lets out the housing stock required. For simplicity, I assume that both the stock of housing and the other consumption goods are perfectly divisible, and that there are no transactions cost involved in buying or selling them.

There is a finite number of different *types* of households, indexed $i \in \mathcal{I}$ and distributed according to a fixed, exogenous (discrete) distribution $H(i)$. These types of households differ only in terms of their income. There is no explicit labor market; instead, households receive wage endowments. For simplicity, I fix and normalise the initial wage endowment to $w_{0,i} = \omega$ across all types of households $i \in \mathcal{I}$. In the ex post period, however, wages

[7] Except for $\rho = 0$ and/or $\theta = 1$, in which cases the Cobb-Douglas (logarithmic) form applies.

[8] Other consumption goods perish at the end of each period, unless consumed.

are stochastic and type-dependent, but strictly positive and finite, i.e.[9]

$$w_i = w_i(\sigma) > 0, \tag{2}$$

where $\sigma \sim G(\sigma)$ is a stochastic variable representing the economy's intrinsic uncertainty. Information about household type and wage uncertainty is for simplicity considered to be symmetric, i.e. the type i of any particular household is openly observable, the type-dependent wage uncertainty defined by the correspondence $w_i(\sigma)$ and the distribution $G(\sigma)$ is common knowledge, and the actual wage realisation becomes known to everyone at the same time.

2.2 Mortgage contracts with default

I assume exogenously incomplete financial markets. What I have in mind is a liquidation type bankruptcy institution *à la* chapter 7 in the American bankruptcy code, whereby assets are seized but future wage income is shielded from creditors upon default. Reflecting such an arrangement, I assume that households cannot credibly borrow against future wage income, as they would have no incentive to actually repay their debt. On the other hand, households can use housing stock as collateral since this stock is durable. Consequently, I postulate a market for collateralised non-recourse mortgage contracts.

Mortgage contracts are modelled as straight debt arrangements with a default option. A particular mortgage contract, denoted $m = (b, B, h_0)$, consists of an amount b given to the borrower in the ex ante period in return for a balloon repayment B in the ex post period, which can be thought of as comprising the borrowed amount b plus any compensation for the time value of money and any risk premium.[10] Finally, $h_0 > 0$ is the size of the house bought by means of this contract, which also serves as collateral in that it can be seized by the creditor in case the household defaults on the repayment B.[11]

I assume that the terms of an established mortgage contract cannot be renegotiated. In other words, the contract is *rigid* ex post in that the borrower faces only two alternatives: either pay B as specified by the contract, or default and give up the collateral h_0. However, the mortgage contract is *flexible* ex ante in that the terms of the contract are endogenously determined. The optimal mortgage contract will in general be type dependent, because households with different income uncertainty also will have different default risks.

[9] Assuming that $w_i(\sigma)$ is strictly positive for all $i \in \mathcal{I}$ and all σ ensures that the household's ex post decision problem is always well defined, cf. footnote 19.

[10] It is analytically convenient to separate the amount to be repaid B from the amount received b in this way, rather than specifying an interest rate. The ratio B/b can nevertheless thought of as the contract's implied gross rate of interest.

[11] Liquidation type bankruptcy procedures typically seize all the household's assets, apart from certain exempted ones. Generally, then, there is no reason to posit less collateral than the entire house. Consequently, I assume that the collateral coincides with the amount of housing bought.

Mortgage financing is provided by small, risk neutral banks operating in a competitive financial market, requiring mortgage contracts to break even in expectation. Let $\Delta_i \in \{0,1\}$ denote a function mapping prices of housing stock p and stochastic outcomes σ into a default decision for a household of type i with mortgage contract m, where $\Delta = 0$ means default. Assuming that banks discount time with the same factor β as households, the type-specific set of mortgage contracts for which banks expect to break even is[12]

$$M_i = \big\{(b, B, h_0) : -b + \beta E\big[\Delta_i(p,\sigma,m)B + \big(1 - \Delta_i(p,\sigma,m)\big)ph_0\big] \geq 0\big\}, \tag{3}$$

where E denotes the expectations operator. The set of break even contracts can be thought of as a menu of mortgage contracts available to the household in the credit market, from which he may choose. To keep matters simple, I assume that banks are internationally owned, and funded with abundant resources.[13,14] Competition among banks will imply that mortgage contracts exactly break even in equilibrium, establishing b as a function of (B, h_0). Since only B and h_0 matter for the default decision, it is convenient to use the shorthand notation $m = (B, h_0)$, where it is understood that b is determined from the break even condition in (3) evaluated at equality.

2.3 Cost of default and the rental market

Conform to common perception, I assume that the direct (pecuniary) cost of default is negligible. Instead, I focus on the *indirect* (pecuniary) costs of default. Households will usually find themselves excluded from borrowing in a number of years following default. An important consequence of this is relegation to the house rental market.[15] To the extent that the rental cost exceeds the cost of ownership for an equivalent property, this constitutes an indirect (pecuniary) punishment for default.[16] This is modelled as an "iceberg cost" $1 - \kappa$ in rental: for each dollar rent paid by a renter, the net proceeds to the owner equals $\kappa \in (0,1)$. This cost can be thought to originate from unmodelled moral hasard issues, tax disadvantages, or administrative costs that exist in rental markets but do not apply to home ownership. Because renting is inefficient relative to ownership, then

[12] Eq. (3) assumes no losses in the foreclosure process, though this can easily be added by assuming that the net cash flow to the bank in case of default is $\tau p h_0$, with $\tau \in (0,1)$.

[13] Modelling banks as domestically owned would necessitate handling feed-back effects from banks' cash flows onto households' demand for housing. Besides complicating the model and thus obscuring its main point, this would presumably but aggravate the results.

[14] Abundant funding ensures that banks are able to provide any mortgage contract satisfying eq. (3).

[15] Exclusion from borrowing may also hamper households' ability to self-insure against adverse shocks; see Chatterjee et al. (2007) for an analysis in the context of consumer credit default. With only two periods, however, my model is not well-suited to analyse this issue.

[16] Throughout this paper, I abstract from moral qualms, social stigma, or other non-pecuniary costs that may be attached to default. Though clearly important constraints on the decision to default (see e.g. Guiso et al., 2008, for recent survey evidence), incorporating them is beyond the scope of this paper.

everything else equal, households prefer to own. This motivates a simplifying assumption (maintained throughout the paper) that all households initially start as owners.[17] Households only become renters because they have to, that is, after default.

It follows that there exists a rental market in the ex post period if and only if some households default. To implement this rental market, I let banks trade freely in housing stock and propose rental service. Recall that since the ex post period is the last period, housing stock expires worthless at the end of the period. To preclude arbitrage opportunities the net rental revenues from letting out one unit of housing stock, which equal κ times the unit rental price, must therefore equal the price of that unit of housing stock. Otherwise, the bank could make a riskless profit by either buying or selling housing stock on the margin. Though properly speaking an equilibrium condition, it simplifies the exposition to simply *impose* that the rental price be identically equal to

$$\frac{p}{\kappa}. \tag{4}$$

Evidently, as $\kappa \in (0,1)$, renting is always strictly more expensive than owning, for any size of housing considered.

2.4 Problem definition

Consider a household of any given type $i \in \mathcal{I}$ entering the ex post period with the mortgage contract $m = (B, h_0)$. He obtains a wage realisation $w_i(\sigma)$ and observes prices of housing stock and rental service, p and p/κ, respectively. If he chooses to repay his debt, he owns a house worth ph_0 but repays B, so total available resources are $w_i(\sigma) + ph_0 - B$. He must decide how much of this to spend on housing stock $h > 0$ at price p; the remainder is spent on other consumption goods $c > 0$.[18] That is, the *repayer's problem* is

$$\begin{aligned} v^R\big(p, x_i^R(p,\sigma,m)\big) = \max_{c,h}\ & u(c,h) \\ \text{s.t.}& \\ c + ph &\leq w_i(\sigma) + ph_0 - B \equiv x_i^R(p,\sigma,m) \\ \text{given } m = (B,h_0) &\text{ with } h_0 > 0. \end{aligned} \tag{5}$$

If the household instead chooses to default, his debt B is cleared but he also loses his house h_0, leaving him with total available resources equal to his wage $w_i(\sigma)$. He must decide how much of this to spend on renting housing $h > 0$ at price p/κ; the remainder

[17] In the real world, people presumably stay tenants either because they value the flexibility of tenancy, or because no-one is willing to offer them a mortgage.

[18] Since housing stock is perfectly divisible and there are no transaction costs, households fully re-optimise their stock of housing ex post.

is spent on other consumption goods $c > 0$. That is, the *defaulter's problem* is

$$v^D\big(p, x_i^D(\sigma)\big) = \max_{c,h} u(c,h) \tag{6}$$
$$s.t.$$
$$c + \frac{p}{\kappa} h \leq w(\sigma) \equiv x_i^D(\sigma).$$

The *default decision*, denoted $\delta \in \{0,1\}$ where $\delta = 0$ means default, solves[19]

$$v_i(p,\sigma,m) = \max_{\delta\in\{0,1\}} \big\{\delta v^R\big(p, x_i^R(p,\sigma,m)\big) + (1-\delta) v^D\big(\tfrac{p}{\kappa}, x_i^D(\sigma)\big)\big\}, \tag{7}$$

where I shall assume that households do not default when they are in fact indifferent.[20]

Consider the same household in the ex ante period, with initial resources equal to the wage endowment ω. He knows that future wages $w_i(\sigma)$ are stochastic following eq. (2), and that for a particular wage realisation, price and mortgage contract, he will act in a way that solves the period two problem (5)-(7). He must decide how much housing stock $h_0 > 0$ to buy, how much consumption goods $c_0 > 0$ to consume, and choose a mortgage contract (b, B, h_0) from the menu of contracts M_i available to him. Finally, the proceeds b from the contract together with the wage endowment ω must suffice to finance the expenditures on housing stock and other goods. That is, the *contract problem* is

$$V_i = \max_{\{c_0,h_0,b,B\}} \Big\{u(c_0,h_0) + \beta \iint v_i(p,\sigma,m)\, dF(p|\sigma)\, dG(\sigma)\Big\} \tag{8}$$
$$s.t.$$
$$c_0 + p_0 h_0 \leq \omega + b$$
$$m = (b, B, h_0) \in M_i,$$

where $v_i(p,\sigma,m)$ solves the period two problem (5)-(7) for this household, and $F(p|\sigma)$ denotes the distribution of future prices conditional on σ.[21]

[19] The defaulter's problem (6) is well defined for any finite price p because each household's wage $w_i(\sigma)$ is strictly positive and finite, cf. eq. (2). Equilibrium prices will be finite because aggregate resources are finite. Similarly, the repayer's problem (5) is well defined if $w_i(\sigma) + ph_0 - B$ be strictly positive and finite. This may not hold for low values of p if B is large. In this case, the household is in fact not able to repay B, hence I assume he has to default. This assumption is without loss of generality, as households always *choose* to default before they have to, cf. footnote 25.

[20] Without this assumption, the model may feature additional equilibria at indifference points, where a share of identical households defaults and another share does not. I conjecture that such equilibria must be unstable, however, because if any individual household changed action that would move the market price infinitesimally, but sufficiently to induce all other households to strictly prefer the same action.

[21] As the model features multiple equilibria, it is analytically convenient to separate F and G in this way. Whereas $G(\sigma)$ captures the *intrinsic* or *fundamental* uncertainty in the economy, $F(p|\sigma)$ captures the *extrinsic* or *sunspot* uncertainty (see Shell, 2008, for a definition).

2.5 Equilibrium definition

Definition 1 *A rational expectations competitive equilibrium is ex ante (period zero) consumption and housing choices $(c_0, h_0)_i$, mortgage contracts $m_i = (b, B, h_0)_i$, and ex post (period one) contingent default decisions δ_i, consumption and housing choices $(c, h)_i$ for all households $i \in \mathcal{I}$, corresponding default functions $\Delta_i(p, \sigma, m)$ and sets of contracts M_i offered to these households, and a price of housing stock p_0 as well as a contingent distribution of prices of housing stock $F(p|\sigma)$, such that*

1. *Households optimise:*

   (a) *Ex ante (period zero) choices $(c_0, h_0, b, B)_i$ are optimal given the set M_i of contracts offered and prices p_0 and $F(p|\sigma)$, i.e. they solve problem (**??**) for each type of household $i \in \mathcal{I}$.*

   (b) *Ex post (period one) contingent choices $(c, h, \delta)_i$ are optimal given the contract $(B, h_0)_i$ and the price p, i.e. they solve problem (5)-(7) for each type of household $i \in \mathcal{I}$ and each possible value of p in $F(p|\sigma)$.*

2. *All mortgage contracts that would break even in expectation are offered, i.e. for each type of household $i \in \mathcal{I}$,*

$$M_i = \Big\{(b, B, h_0) : -b + \beta \iint \begin{bmatrix} \Delta_i(p, \sigma, m)B \\ +\big(1 - \Delta_i(p, \sigma, m)\big)ph_0 \end{bmatrix} dF(p|\sigma)\, dG(\sigma) \geq 0\Big\}. \quad (9)$$

3. *Projected default behaviour is consistent with households' optimal default decisions for any mortgage contract, i.e. $\Delta_i(p, \sigma, m)$ would solve problem (7) for each type of household $i \in \mathcal{I}$, each possible value of p in $F(p|\sigma)$, and any mortgage contract m, whence*[22]

$$\Delta_i(p, \sigma, m_i) = \delta_i. \quad (10)$$

4. *The market for housing stock clears in each period, i.e.*

$$\int h_{0,i}\, dH(i) = \int h_i\, dH(i) = H. \quad (11)$$

[22] Specifying Δ_i over any mortgage contract (not just equilibrium contracts) ensures that M_i spans out the entire set of potential equilibrium mortgage contracts from which households can then choose.

3 Optimal choices

3.1 Optimal default decision

In order to derive the optimal mortgage contract, we need to know under what conditions households will default. Models of optimal default typically involve a cutoff strategy; that is, default is strictly optimal if and only if the the price of housing stock p falls below some threshold price p^*. As the following proposition shows, a sufficient condition for this to hold in the present model is that other goods and housing are not "too good" substitutes. (This condition, which is convenient but stronger than necessary, will be further discussed below.)

Proposition 1 (Threshold price) *Let $\rho \leq 0$, i.e. let the elasticity of substitution between housing and other consumption be smaller than 1. Then, for any household of type $i \in \mathcal{I}$ with mortgage contract $m = (B, h_0)$, there exists a (type, shock, and contract dependent) threshold price $p_i^*(\sigma, m)$ such that default is strictly optimal if and only if the price of housing stock is strictly below the threshold price, and strictly suboptimal if and only if the price of housing stock is strictly above the threshold price, i.e.*

$$v^R\big(p, x_i^R(p,\sigma,m)\big) < v^D\big(\tfrac{p}{\kappa}, x_i^D(\sigma)\big) \Leftrightarrow p < p_i^*(\sigma, m)$$
$$v^R\big(p, x_i^R(p,\sigma,m)\big) > v^D\big(\tfrac{p}{\kappa}, x_i^D(\sigma)\big) \Leftrightarrow p > p_i^*(\sigma, m).$$

Moreover, whenever $p_i^*(\sigma, m)$ is strictly positive, it is implicitly a continuous and differentiable function of the parameters B and h_0 of the mortgage contract, with

$$\frac{\partial p_i^*(\sigma,m)}{\partial B} > 0 \quad \textit{and} \quad \frac{\partial p_i^*(\sigma,m)}{\partial h_0} < 0, \tag{12}$$

for any given value of σ.

Proof. See appendix. ■

In order to shed light on the underlying mechanism in proposition 1, it is useful to consider a specific case. When preferences are Cobb-Douglas (logarithmic), i.e. in the limit case as ρ tends to zero, a simple analytical solution obtains for the threshold price:[23]

$$p_i^*(\sigma, m) = \frac{B - (1-\kappa^{1-\alpha})w_i(\sigma)}{h_0}. \tag{13}$$

The threshold price in eq. (13) is strictly increasing in leverage B/h_0.[24] That is to say, more levered households default at higher house prices (i.e. more easily). Intuitively,

[23] See appendix.

[24] Whenever $p^* > 0$, which is the interesting case to discuss, as any equilibrium price must be strictly positive.

the more the household is supposed to repay (that is, the higher B is), the higher the temptation to default; and conversely, the more collateral the household has (that is, the higher h_0 is), the more he has to lose from defaulting. The threshold price also depends on the indirect cost of default, which in eq. (13) is represented by the term $(1-\kappa^{1-\alpha})w_i(\sigma)$. First, an increase in the price of renting relative to owning (that is, a lower value of κ) decreases the threshold price. The cost of renting relative to owning is by construction only borne by defaulters, hence it is dissuasive of default. Next, an increase in income $w_i(\sigma)$ decreases the threshold price, suggesting that households default less easily when their income is higher. Importantly, this is not about whether the household is *able* to repay its debt or not.[25] Rather, in this model, income enter into the threshold price as part of the indirect cost of default. This is because with higher income, households would like more housing, thus making it more painful to endure the higher (unit) price of obtaining housing after default.

Since income and the market price of housing are stochastic, default becomes a stochastic event once the mortgage contract is chosen. Given any distribution of market prices, proposition 1 therefore has a meaningful interpretation in terms of likelihood of default. It says that a household is more likely to default when it is more leveraged, when its income is low, and when the price difference between renting and owning is small. Evidently, a household can influence on the likelihood of his own default through his choice of mortgage contract ex ante. In particular, it is always possible for a household to avoid ever defaulting simply by restricting his own borrowing, but this may not be the optimal choice, as will be argued in the following subsection.

Note that the inefficiency in renting implies that households are always strictly "underwater" when they default, in the sense that the value of debt B always strictly exceeds the value of the collateral ph_0. To see this, observe from problems (5) and (6) that if $B = ph_0$, the household's total resources equals $w_i(\sigma)$ whether he defaults or not. But then, since the rental price is always strictly higher than the price of housing stock, the household would strictly prefer repaying debt. Consequently, at any threshold price, we must have

$$B > p_i^* h_0. \tag{14}$$

Finally, I return to discuss the condition that other goods and housing are not "too good" substitutes. Proposition 1 explicitly imposes $\rho \leq 0$, but as its proof shows, $\rho = 0$ is far from a knife-edge case, so the condition is stronger than necessary. This is important to point out, because empirical studies suggest that reasonable parameter values of ρ

[25]To see this, suppose $B > w_i(\sigma)$ so that the household will be unable to repay when the price falls below $p_{\min}(\sigma, m) = \frac{B - w_i(\sigma)}{h_0} > 0$. But from (13) we see that $p_i^*(\sigma, m) > p_{\min}(\sigma, m)$ for all σ and m, which is to say that households always *choose* default before they are *forced* to default.

lie precisely in this range.[26] Moreover, even when other goods and housing are perfect substitutes, it is hard to construct counterexamples where a threshold price does *not* exist, typically requiring that the market price tends to zero.[27] Consequently, it seems likely that a threshold price would exist in any reasonable numerical application. Nevertheless, because the threshold price property greatly simplifies the characterisation of the optimal contract (see proposition 2 below), and since $\rho \leq 0$ is a simple, transparent and not too unreasonable condition, I shall adhere to it for the remainder of the paper.

3.2 Optimal mortgage contract

Proposition 2 (Optimal mortgage contracts) *Let $\rho \leq 0$, so that default behaviour is characterised by a cutoff strategy with respect to a threshold price $p_i^*(\sigma, m)$ as defined in proposition 1. Then, any equilibrium solution $\big(c_0, m = (b, B, h_0)\big)$ to the contract (ex ante) problem (8) for a household of type $i \in \mathcal{I}$ satisfies*

$$\begin{aligned} u_c(c_0, h_0) &\geq \frac{1}{\Pr(p \geq p_i^*(m))} \iint_{p_i^*(\sigma,m)}^{\infty} u_c(c, h)\, dF(p|\sigma)\, dG(\sigma) \\ &= E\big[u_c(c, h) \mid p \geq p_i^*(m)\big] \end{aligned} \tag{15}$$

$$\begin{aligned} p_0 &\geq \frac{u_h(c_0, h_0)}{u_c(c_0, h_0)} + \beta \iint_{p_i^*(\sigma,m)}^{\infty} p\, \frac{u_c(c, h)}{u_c(c_0, h_0)}\, dF(p|\sigma)\, dG(\sigma) + \beta \iint_{0}^{p_i^*(\sigma,m)_-} p\, dF(p|\sigma)\, dG(\sigma) \\ &= \frac{u_h(c_0, h_0)}{u_c(c_0, h_0)} + \beta \left[\begin{array}{c} E\big[p\, \frac{u_c(c,h)}{u_c(c_0,h_0)} \mid p \geq p_i^*(m)\big] \Pr\big(p \geq p_i^*(m)\big) \\ +E\big[p \mid p < p_i^*(m)\big] \Pr\big(p < p_i^*(m)\big) \end{array} \right], \end{aligned} \tag{16}$$

where u_c and u_h denote the partial derivatives of $u(c, h)$, and $E[\cdot|\cdot]$ denotes the conditional expectation operator over distributions $F(p|\sigma)$ and $G(\sigma)$ together. Eqs. (15) and (16) hold with equality when the solution is interior with respect to B and h_0.

Proof. See appendix. ■

Evaluated at equality (that is, for an interior solution), eqs. (15)-(16) are Euler equations. Eq. (15) states that the household seeks to smooth marginal utility of consumption over time. Because the terms B and h_0 of the mortgage contract are not state contingent *per se*, households smooth marginal utility in expected terms. Indeed, if the optimal con-

[26] See Davis and Ortalo-Magné (forthcoming), and the references therein.

[27] The appendix contains an example. What may go wrong is that, when c and h are very close substitutes and the price of housing stock p is low, the household maximises utility by putting all but every penny in housing stock, even after default. At very low p, he may then regret having defaulted, because having to pay p/κ per unit of housing service instead of p makes a big difference in housing consumption when p is sufficiently low.

tract were such that the household would never default, we would recover the result of a standard incomplete markets models such as the permanent income or life cycle hypothesis model. It is clear that the terms B and h_0 cannot impact the household's consumption *in states where he defaults*, cf. problem (6).[28] Therefore, the Euler equation considers marginal utility only across states where the household does not default. This is not to say that the mortgage contract is not state contingent at all. To the contrary, implicit in the optimal choice of contract is the determination of $p_i^*(\sigma, m)$ for every σ; that is, determining in which states there will be default and in which there will not.

In order to interpret eq. (16), start by considering the case when the optimal contract is such that the household would never default; in this case, it becomes

$$p_0 = \frac{u_h(c_0, h_0)}{u_c(c_0, h_0)} + \beta E\Big[p\frac{u_c(c, h)}{u_c(c_0, h_0)}\Big]. \tag{17}$$

The first part of eq. (17) reflects the consumption value of the chosen house ex ante (which is certain), whereas the second part reflects the expected investment value of that house ex post (which is uncertain). It is clear from the repayer's problem (5) that the worth of the house ph_0 impacts the household's resources and consumption. As his marginal utility of consumption is not constant over such realisations, $\frac{u_c(c,h)}{u_c(c_0,h_0)}$ is a stochastic discount factor by which he discounts possible future values of the house. Returning to eq. (16), we recognise one term with the stochastic discount factor, but it applies only to the states where the household repays his debt. The remaining term corresponds to the states when the household defaults, i.e. when the bank recuperates the value of the house. Intuitively, the bank's preferences apply to discount the possible future values of the house in these states, and since the bank is risk neutral, this implies a valuation equal to the expected price.

Competitive pricing of mortgage contracts implies that contracts for which a household would never default have an implied gross rate of interest $B/b = 1/\beta$. But if a contract m is such that the household would default in some states, then since $B > p^*(m)h_0 > ph_0$ from proposition 1 and eq. (14), this contract must have $B/b > 1/\beta$ for the bank to break even. That is, a contract with positive probability of default includes a risk premium.

Collateralised borrowing yields endogenous leverage constraints which can best be described by means of a simple example. Suppose a household faces strong income growth, and for simplicity, suppose that there is no uncertainty, neither about income growth nor about the future house prices p. Strong income growth means that the household

[28] One way to impact consumption in states of default could be to save outside the mortgage contract. To meet this goal, savings would have to be protected from creditors upon default. In the US, when mortgages are not legally non recourse, investing in property exempt from bankruptcy could be a way to achieve it. The present model abstracts from these issues, and can thus be interpreted as the case without exemptions.

would like to borrow heavily to smooth consumption. (As housing and other goods are imperfect substitutes, the household would like to smooth his consumption of both.) Since only housing stock serves a collateral, this implies highly leveraged borrowing. From the properties of the threshold price in (12), however, we see that as leverage grows sufficiently high, the household would eventually default. Absent price uncertainty, this means that the household would default for sure, but default for sure can never be optimal. To see this, start by noting that when the household defaults for sure, the value of B is in itself unimportant: in any case, the bank recuperates the house, which is worth ph_0, and the household obtains the utility of default v^D, which is independent of B. Consider therefore reducing B just enough to avoid default, that is such that $p^*(m) = p$. By construction of p^*, the household is equally well off ex post. Since $B > p^*(m)h_0 = ph_0$, however, the bank receives strictly more than before. Due to competitive pricing, then, it must be that the household obtains strictly more resources ex ante (i.e. b is strictly higher), implying that the household is strictly better off ex ante.

Returning to the general case with income and price uncertainty, the above argument holds for any state of default considered in isolation. The optimal mortgage contract, however, must take all states into account at the same time, as the terms of the contract are not state contingent. While reducing B is desirable in states of default, it is undesirable in states where the household repays debt, as it takes him further away from his desired consumption plan, cf. eq. (15). Summing up, the optimal mortgage contract chooses leverage in the way that best trades off the risk premium from high leverage against its benefits in terms of smoothing consumption.

As demonstrated in the proof of proposition 2, these endogenous leverage constraints are generally not differentiable. If the optimum corresponds to a point where $p^*(\sigma, m) = p$ for some σ, the optimality conditions (15)-(16) may no longer hold with equality; rather, it would be that $u_c(c_0, h_0) > E\big[u_c(c, h) \mid p \geq p^*(m)\big]$. In such a situation, while increased borrowing would in principle smooth consumption and thus marginal utility, the household refrains from it because it would lead to a jump in the risk premium and thus increase the cost of borrowing, giving lower utility overall. That is to say, binding leverage constraints *distort* consumption and investment decisions, leading the household to hold more housing stock and consume less of other goods than he otherwise would wish to. This "excess" demand for housing stock derives from its value as collateral, and pushes the ex ante price p_0 above the value given in the right-hand side of eq. (16).[29]

[29]Distortion of allocations and prices in this manner is a general feature of the collateral equilibrium advanced by Geanakoplos and Zame (2010).

4 Equilibrium

The purpose of this paper is to investigate how mortgage default can spread from some households to others in equilibrium. The model that I have laid out naturally gives rise to multiple equilibria, some of which can be characterised as contagious default equilibria. The mechanisms at work are best illustrated by studying a simple case for which it is possible to derive analytical solutions. While this case study takes up most of the current section, towards the end I return to discuss a generalisation by means of a numerical implementation.

4.1 Limit-symmetric equilibrium

Consider the following assumptions:

1. Preferences are Cobb-Douglas (logarithmic), i.e. $\rho = 0$.

2. There are two types of households, $\mathcal{I} = \{pri, sub\}$, where *pri* households have constant income, and *sub* households risk a $\bar{\sigma} \in (0, 1)$ per cent income shortfall in the ex post period relative to the ex ante period, i.e.[30]

$$w_{pri} = w_0 = \omega \tag{18}$$

$$w_{sub} = \begin{cases} \omega & \text{with prob. } 1 - \nu \\ (1 - \bar{\sigma})\omega & \text{with prob. } \nu. \end{cases} \tag{19}$$

Eq. (19) represents a situation where some households have more uncertain income than other households; here, they are dubbed *sub* with obvious reference to U.S. sub-prime borrowers of late. For example, $\bar{\sigma}$ could be thought of as a business cycle downturn, and *sub* the group of households known to be more sensitive to such shocks, e.g. "fragile" groups that tend to lose their jobs first. To make the contrast as stark as possible, I assume that *pri* households have perfectly stable income, cf. eq. (18). Two types of households is the smallest number required to study contagion, but one could of course consider extensions. Note that although each *individual* household is small, each *type* of household has positive mass. Consequently, an income shortfall to *sub* households reduces aggregate income, thus affecting market prices.

The assumptions laid out above admit analytical solutions in the limit as the probability of income shortfall to *sub* households tends to zero, because both types of households then have the same expected (constant) income profile, which in turn yields ex ante sym-

[30] The notation regarding σ is simplified here for expositional convenience. Eq. (19) is shorthand for $w_{sub}(\sigma) = (1 - \sigma)\omega$, where $\sigma = 0$ with probability $1 - \nu$, and $\sigma = \bar{\sigma}$ with probability ν.

metry.[31] That is, *sub* households are approximately as levered as *pri* households, in spite of the risk of income shortfall. The limit case $\nu \to 0$ is not meant to be taken literally, but rather as a reasonable approximation to cases when ν is strictly positive, but small.[32] Numerical experiments confirm this and show that the goodness of the approximation is a quantitative matter, depending on the parameters of the problem, cf. the discussion in subsection 4.3 below.

Proposition 3 (Limit-symmetric equilibrium) *Let $\rho = 0$, i.e. let preferences be Cobb-Douglas (logarithmic). Let there be two types of households, $i \in \mathcal{I} = \{pri, sub\}$, with ex post income as in eqs. (18)-(19). Let $\gamma \in (0,1)$ denote the share of sub households, and let b, B, c, and h denote limits as $\nu \to 0$. Then*

1. *There is a unique solution to the contract (ex ante) problem (8), and it is identical across household type (ex ante symmetry), with*

$$\begin{aligned} b &= \beta(1-\alpha)\omega \\ B &= (1-\alpha)\omega \\ c_0 &= \alpha\omega \\ h_0 &= H. \end{aligned} \tag{20}$$

2. *When there is no income shortfall to sub households ($w_{sub} = w_0 = \omega$), there is a unique solution to the ex post problem (5)-(7). The equilibrium outcomes are identical across household type (ex post symmetry), without default and with full smoothing of consumption with respect to both housing and other goods, i.e.*

$$\begin{aligned} \delta &= 1 \\ h &= h_0 \\ c &= c_0. \end{aligned} \tag{21}$$

3. *When there is an income shortfall to sub households ($w_{sub} = (1-\bar{\sigma})\omega < w_0$), there may be one, two or three different solutions to the ex post problem (5)-(7), depending on the parameters of the problem. The equilibrium candidates are:*

[31] The key step is obtaining $h_{0,sub} = h_{0,pri}$. Together with the analytical housing demand functions obtained from Cobb-Douglas (logarithmic) preferences, this yields simple expressions for the possible equilibrium prices. Comparing these to the threshold price in eq. (13) yields analytical, manageable existence conditions for different equilibria.

[32] The model is continuous at $\nu = 0$ because utility in the ex post period is always finite; in particular, utility is bounded from below by v^D, which is finite since $x^D = w > 0$. Consequently, whatever the outcome in the event of income shortfall, it can be disregarded in the contract (ex ante) problem (8) in the limit as $\nu \to 0$.

(a) Repay equilibrium (RR): Both pri and sub households repay their debt, i.e. $\delta_{pri} = \delta_{sub} = 1 \Leftrightarrow p_{RR} \geq p^*_{sub} > p^*_{pri}$*, existing if and only if*

$$\frac{1-\alpha}{\alpha}\gamma\frac{\bar{\sigma}}{1-\bar{\sigma}} \leq 1-\kappa^{1-\alpha}. \tag{22}$$

(b) Partial default equilibrium (RD): pri households repay their debt, while sub households default, i.e. $\delta_{pri} = 1 \Leftrightarrow p_{RD} \geq p^*_{pri}$ *and* $\delta_{sub} = 0 \Leftrightarrow p_{RD} < p^*_{sub}$*, existing if and only if both*

$$(1-\alpha)\gamma(1-\bar{\sigma})\kappa - \big[\alpha+(1-\alpha)\gamma\big]\kappa^{1-\alpha} + \alpha \geq 0 \tag{23}$$

$$(1-\alpha)\gamma(1-\bar{\sigma})\kappa - \big[\alpha+(1-\alpha)\gamma\big]\kappa^{1-\alpha} + \alpha < \bar{\sigma}\big[\alpha+(1-\alpha)\gamma\big]. \tag{24}$$

(c) Contagious default equilibrium (DD): Both pri and sub households default, i.e. $\delta_{pri} = \delta_{sub} = 0 \Leftrightarrow p_{DD} < p^*_{pri} < p^*_{sub}$*, existing if and only if*

$$(1-\alpha)(1-\gamma\bar{\sigma})\kappa - \kappa^{1-\alpha} + \alpha < 0. \tag{25}$$

Proof. See appendix. ■

Proposition 3 shows that when there is no income shortfall to *sub* households, the equilibrium outcomes are "first best": there is no default, hence no inefficiency, and the consumption of both housing service and other goods is perfectly smooth. The more interesting case to investigate is when there *is* an income shortfall to *sub* households.

4.2 Income shortfall

Income shortfall to *sub* households means $w_{sub} < w_{pri}$. Since the threshold price is strictly decreasing in w, and from ex ante symmetry, this implies $p^*_{sub} > p^*_{pri}$. Hence, there are only three possible default configurations, as stated in proposition 3, part 3: either all households repay their debt (no-one defaults), or only *sub* households default, or all households default. Figure 1 plots the existence conditions for these three alternatives, given by eqs. (22)-(25), for the case $\bar{\sigma} = \alpha = \frac{1}{2}$. The horisontal axis measures the share of *sub* households in the economy, γ. The vertical axis measures the square root of κ, which is an inverse measure of the cost of default.[33] (Recall that the larger κ, the smaller the inefficiency in renting; that is, the smaller the gap between the rental price of housing and the price of housing stock, and thus the smaller the indirect cost of default.)

[33] $\kappa^{1-\alpha} = \sqrt{\kappa}$ when $\alpha = \frac{1}{2}$, cf. eqs. (22)-(25). The square root is a monotonic transformation, and $\kappa \in (0,1) \Rightarrow \sqrt{\kappa} \in (0,1)$, so $\sqrt{\kappa}$ is an appropriate (non-linear, inverse) measure of the cost of default.

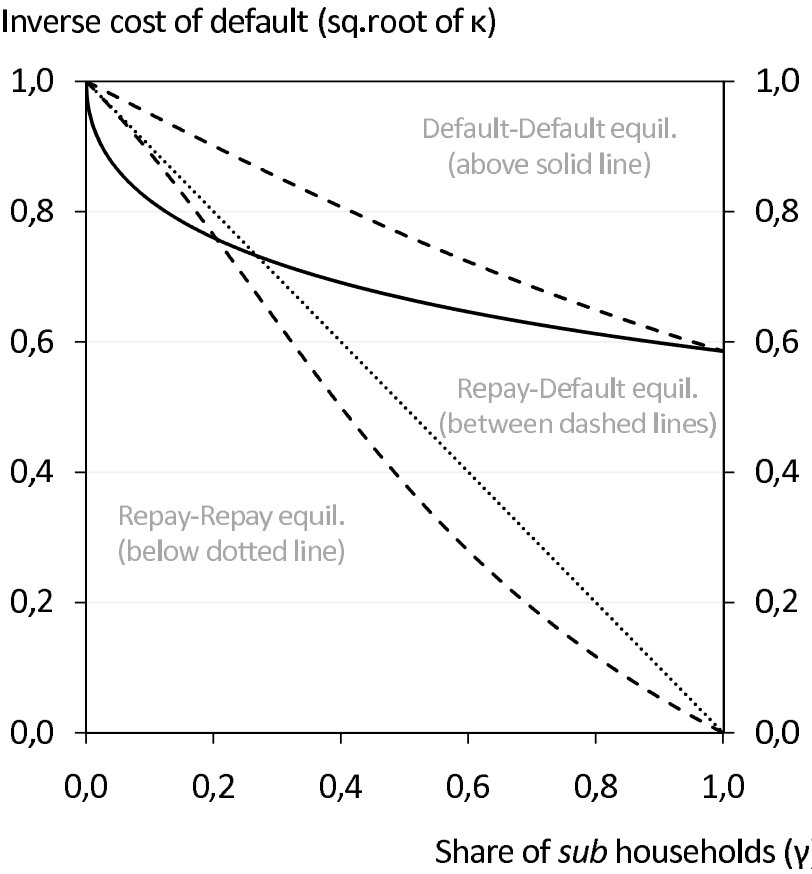

Figure 1: Multiple ex post equilibria with limit-symmetric mortgages, for $\bar{\sigma} = \alpha = \frac{1}{2}$.

4.2.1 Repay equilibrium

The fist possibility is an equilibrium in which both *pri* and *sub* households repay their debt, called the "repay equilibrium" and denoted RR (for "repay-repay"). Its existence condition (22) is satisfied for parameter values in the area below the dotted line separating figure 1 in two equal parts; that is, the repay equilibrium exists when the share of *sub* households is sufficiently small for any given cost of default, or alternatively, when the cost of default is sufficiently high ($\sqrt{\kappa}$ is sufficiently low) for any given share of *sub* households. Since no-one defaults in this equilibrium, it differs from the equilibrium in (21) in a straightforward way: first, the income shortfall has *sub* households reduce their demand for housing; and second, since the supply of housing stock is fixed, the house price must fall for the market to clear; consequently, *pri* households scale up while *sub* households scale down in the housing market.[34] An equilibrium without default may exist for the simple reason that when the aggregate income shortfall, measured by $\gamma\bar{\sigma}$, is sufficiently small, the house price fall is moderate enough for *sub* households to find default unattractive.[35]

[34]Consequently, *pri* households are better off and *sub* households are worse off compared to the equilibrium in (21).

[35]Due to the particularity of Cobb-Douglas (logarithmic) preferences, the effect on house prices from the income shock to *sub* households in this equilibrium is isomorphic to an aggregate income shock $\gamma\bar{\sigma}$.

Conversely, when the house price fall is large enough, the repay equilibrium cannot exist, and so at least *sub* households default. As figure 1 shows, how much is "large enough" depends critically on the cost of default. Note in particular that for $\sqrt{\kappa}$ in the high range, even a small share of *sub* households suffices to bring about default. In other words, when the cost of default is low, mortgages to *sub* households are very sensitive to house prices and may easily end in default.

4.2.2 Partial default equilibrium

An alternative equilibrium to consider is one in which only *sub* households default, while *pri* households repay their debt, called the "partial default equilibrium" and denoted *RD* (for "repay-default"). This is to say that while the house price fall must be "large enough" for *sub* households to choose default, it must not be "too large", in which case *pri* households would also default. In figure 1, these two joint existence conditions are satisfied in the area between the two dashed lines; the lower line corresponds to $p < p^*_{sub}$, and the upper line to $p \geq p^*_{pri}$. As one might expect, this area covers a range of parameter values generally lying to the "north-east" of the repay equilibrium (apart from the overlapping region which is discussed below); that is, it corresponds to larger aggregate income shortfalls (through a larger share of *sub* households) and smaller costs of default (higher $\sqrt{\kappa}$).

Two competing effects determine the housing demand of *sub* households in this equilibrium. Recall from eq. (14) that the a household level is always strictly "underwater" when he defaults, in the sense that the value of debt B always strictly exceeds the value of the collateral ph_0. Default therefore involves a *positive income effect:* the defaulter has *more* resources available after default than he would have had, had he repaid his debt. In isolation, the income effect serves to *increase* the defaulting household's demand for housing. On the other hand, there is a *negative substitution effect:* after default the household faces a strictly higher (unit) price of housing, inducing him to substitute consumption of other goods for housing, that is, reducing the defaulting household's demand for housing. At the indifference point $p = p^*$, the substitution effect dominates the income effect, i.e. default in itself reduces the demand for housing.

A dominating substitution effect has three important implications: First, it implies *amplification*: when house prices fall and some households default, aggregate demand for housing is reduced, hence house prices fall even more. Of course, the default of an individual (atomic) household does not impact market prices much, but for groups of households such as the share of *sub* households considered here, the aggregate effect could be large.[36] Second, a dominating substitution effect implies *complementarity*: the default

[36]See Chatterjee and Eyigungor (2009) for suggestive results in a large, quantitative-theoretic model calibrated to the US economy.

of some, by driving the market price down, may justify the default decision of others.[37] In effect, complementarity is the source of multiple equilibria, evidenced by the overlapping areas in figure 1, e.g. the area below the dotted line but above the lower of the two dashed lines, for which both the repay equilibrium and the partial default equilibrium may exist. While each of these equilibria is locally stable, the model does not provide a guide to which of them will actually be enacted. Third, a dominating substitution effect implies a *price externality*: the individual household fails to take into account the impact of his default decision, through market prices, onto other households' default decisions. In turn, amplification, complementarity and externality may lead to contagion.

4.2.3 Contagious mortgage default

The final equilibrium to consider is one in which both *pri* and *sub* households default, called the "contagious default equilibrium" and denoted DD (for "default-default"). Here, default is "contagious" in the sense that it spreads from *sub* households to *pri* households, even though the income shortfall applies only to *sub* households. For this equilibrium to exist, the house price fall must be so large that it induces not only *sub* but also *pri* households to default, cf. eq. (25). In terms of figure 1, this equilibrium exists for parameter values in the area above the solid line; that is, in the farthest "north-eastern" part where the aggregate income shortfall is the largest (the share of *sub* households is the largest), or the default costs the smallest ($\sqrt{\kappa}$ is the highest).

The model calls for several different interpretations of contagion. The first is based on the aggregate income effect: as house prices depend on aggregate income, an income shortfall to *sub* households will in any case lower the house price. If the aggregate income shock is sufficiently large (through a large share of *sub* households) given the cost of default, it may therefore in itself be enough to induce the default of *pri* households. In this case, default is still contagious in the sense that it spreads to *pri* households even though the income shortfall applies only to *sub* households, but *pri* households' default is due to *sub* households' default *per se*. This contrasts to an interpretation of contagion as due to the *price externality*: when individual *sub* households fail to take into account the impact on the market price from their decisions to default, this may be precisely what leads *pri* households to default. In an aggregate sense at least, this is a negative externality, as default induces an inefficiency in housing that would otherwise not be there, thus reducing the total amount of resources to share.[38]

Evidently from figure 1, there are is a region of overlap between the contagious default

[37] And, of course, vice versa: the repayment of some may justify the repayment decision of others.

[38] The welfare effects at the household level may be either positive or negative. In particular, for a household that would have defaulted anyways, the default of others is beneficial, as it reduces the (unit) price of renting any house.

equilibrium and the partial default equilibrium, and even with the repay equilibrium, in which multiple equilibria are possible.[39] Interestingly, it is when the share of *sub* borrowers is relatively small and the low cost of default relatively low that multiplicity of equilibria is the most frequent. If *sub* households are interpreted as U.S. sub-prime borrowers of late, this is presumably the most plausible region of the parameter space. Consequently, it seems possible that the default of a minority of sub-prime borrowers could spread to a majority of prime borrowers.

4.3 Numerical experiments

The limit case $\nu \to 0$ considered in the preceding subsections was meant as a reasonable approximation to cases with strictly positive, but small probabilities of contagious mortgage default. This subsection discusses the goodness of this approximation based on numerical experiments.[40]

The appropriate numerical implementation of the equilibrium in definition 1 is the sunspot rational expectations equilibrium (see Shell, 2008). Here, a conditional sunspot variable is used to choose between equilibria when there is multiplicity. Its distribution corresponds to $F(p|\sigma)$ in the model. While it remains random and therefore unexplained which equilibrium is actually enacted when there is multiplicity, this approach reveals information about the ranges of probabilities of different outcomes that can be consistent with rational expectations ex ante. The mechanism constraining $F(p|\sigma)$ is endogenous. For instance, if the probability of a contagious default equilibrium is thought to be high, it pays much for the individual *pri* household (through a reduction in his risk premium) to reduce borrowing enough to ensure he would not default at that equilibrium price. If every *pri* household acts in this way, however, the contagious default equilibrium could no longer exist. Consequently, consistent probabilities of contagious default equilibrium tend to be small.[41]

It is, of course, very hard to gauge real-world quantitative importance from a simple two-period-two-type-household model. Nevertheless, the numerical experiments performed suggest an important role for the endogenous leverage constraints discussed in subsection 3.2. Recall from proposition 3 that in the limit symmetric case, the optimal mortgage contract is such that households expect the "first best" solution, i.e. full smoothing of the marginal utility of consumption, with probability 1. Consequently, households are indifferent to a marginal change in borrowing. A strictly positive probability of con-

[39] Adjacent equilibria always overlap; that is, complementarity implies that there is an overlap between the RR and RD equilibria, and between the RD and DD equilibria. Whether the regions for the RR and DD equilibria overlap or not, depends on the parameter values.

[40] Matlab codes are available from the author upon request.

[41] See Peck and Shell (2003) for an analysis of the canonical bank-run model along these lines.

tagious default therefore easily leads a *pri* household to reduce his borrowing to ensure he would not default at that equilibrium price, hence consistent probabilities of contagious default equilibrium are very small. This completely changes when *pri* households expect income growth and therefore are endogenously borrowing constrained. In the numerical experiments performed, I have implemented the income processes

$$w_{pri} = Aw_0 = A\omega$$

$$w_{sub} = \begin{cases} A\omega & \text{with prob. } 1-\nu \\ A(1-\bar{\sigma})\omega & \text{with prob. } \nu, \end{cases}$$

where A is an income growth factor. Setting $A = 1$ recovers the income process studied in the limit-symmetric equilibrium, cf. eqs. (18) and (19), whereas income growth ($A > 1$) naturally gives rise to endogenous borrowing constraints. Under these constraints, the marginal utility of consumption ex ante is strictly higher than the expected marginal utility of consumption ex post, cf. proposition 2, so the household is *not* indifferent to a marginal change in borrowing. Consequently, it becomes much less attractive for a *pri* household to reduce his borrowing to ensure he would not default in the event of a contagious default equilibrium, as this would take him further away from his desired consumption plan. The numerical experiments performed show that the consistent probabilities of contagious default equilibrium rise by several orders of magnitude, even with modest income growth. These results suggest that in an economy with income growth, individual households with no income risk may rationally choose highly leveraged mortgages, accorded by rational banks, even faced with a non-negligible probability of a house price crash so severe it will induce these households' default.

5 Conclusions

This paper analyses the default option typical to American mortgages. A key assumption of the model is that households are relegated to the rental market upon default, and that there is a small pecuniary inefficiency ("iceberg cost") in renting. This leads defaulters to substitute consumption of other goods for housing; that is, the demand for housing falls upon default. Consequently, when some households default, aggregate demand for housing is reduced, hence house prices fall more, possibly inciting other households to default. This complementarity is a source of multiple equilibria, a price externality, and ultimately, contagion.

The complementarity that I analyse is potentially quantitatively important. In parallel and complementary work, Chatterjee and Eyigungor (2009) build a large quantitative-theoretic model based on very similar assumptions to the present model, which they

calibrate to the U.S. economy in order to study the impact of public policy measures to prevent foreclosures. They use a simplified version of the actual U.S. tax regime with respect to deductability of mortgage interest, implying that for most households, the unit cost of renting exceeds the unit cost of ownership. In simulations, they find that households significantly decrease their demand for housing after default, significantly amplifying price falls. Indeed, their work suggests that the externality and ensuing contagion mechanism that I study can be economically significant. Note that Chatterjee and Eyigungor (2009) do not investigate the possibility of multiple equilibria; a warning seems to be in order, for as I have shown, there is every reason to expect multiplicity in this environment.

Many models featuring multiple equilibria use the simplifying but clearly unrealistic assumption that information is complete, and mine is not an exception. In an influential line of research, Morris and Shin (2003) show that multiplicity may be very sensitive to this assumption: for an important class of complete information games, multiplicity disappears once each player is assumed to receive a private, noisy signal about the fundamental state of the economy, no matter how small this noise is. Though not a game but a competitive equilibrium, Morris and Shin's (2003) argument might very well apply to the present model. Note, however, that their argument does not remove the underlying complementarity of the model; indeed, their analysis shows that very similar fundamentals can yield very different outcomes, depending on which way the equilibrium "turns." The important thing about the present paper is not the multiplicity of equilibria as such, but the negative externality and potential contagion resulting from complementarity.

The negative price externality suggests a role for public policy in avoiding mortgage default on the margin. It does not, however, justify removing the default option altogether; defaultable mortgage contracts may be welfare-improving if the alternative is non-defaultable straight debt contracts. Promoting ex post renegotiation for households in a default position, on the other hand, has a clear welfare-enhancing potential: if the outstanding debt B were renegotiated down to $B' = ph_0$, the bank would be equally well off, the household clearly better off, and any negative price externality would be avoided.

The present model assumes that all households interact in a homogeneous housing market, yielding a single unit price of housing. A perhaps more compelling description of the housing market is a series of "overlapping" markets across housing and household type, yielding unit housing prices that are correlated, but imperfectly so. Presumably, the contagion mechanism explored in this article would apply also in such more complex set-ups, so that default in one market could spread to adjacent markets, possibly in an unravelling manner.

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6 Appendix

Proof of proposition 1 (Threshold price). Consider the ex post (period one) problem (5)-(7) for any type of household $i \in \mathcal{I}$. (Subscript i is suppressed below for expositional convenience, but it is understood that since the threshold price depends on σ, it depends on household type inasmuch as wage endowments $w_i(\sigma)$ are type dependent.) For any given finite mortgage contract $m = (B, h_0)$, $ph_0 > B$ must hold for a sufficiently high price p, in which case repaying debt gives strictly more resources than defaulting. Since the rental price is always strictly higher than the price of housing stock, $v^R > v^D$ thus always holds for a sufficiently high price p. Let p^* denote a price for which v^R and v^D cross, i.e. a price such that

$$v^R\big(p^*, x^R(p^*, \sigma, m)\big) = v^D\big(\tfrac{p^*}{\kappa}, x^D\big). \tag{26}$$

Fix any σ and m and consider v^R and v^D as functions of p. From the continuity and differentiability of $u(c, h)$, they are continuous and differentiable wherever well defined (see footnote 19). If we can show that v^R is be steeper than v^D at any crossing point, i.e. if

$$\frac{dv^R\big(p^*, x^R(p^*)\big)}{dp} > \frac{dv^D\big(\frac{p^*}{\kappa}, x^D\big)}{dp} \tag{27}$$

holds for any possible p^*, then continuity implies that v^R and v^D can cross at most once in p-space, which in turn implies the existence of a threshold price for any given σ and m. (If v^R and v^D never cross in p-space, continuity implies that $v^R > v^D$ must hold for all prices, and so zero is trivially a threshold price.) Differentiating $v^R\big(p^*, x^R(p^*)\big)$ and $v^D\big(\frac{p^*}{\kappa}, x^D\big)$, and using Roy's identity, yields

$$\begin{aligned}
\frac{dv^R\big(p^*, x^R(p^*)\big)}{dp} &= \frac{\partial v^R\big(p^*, x^R(p^*)\big)}{\partial p} + \frac{\partial v^R\big(p^*, x^R(p^*)\big)}{\partial x^R}\frac{dx^R(p^*)}{dp} \\
&= -\big[h^R\big(p^*, x^R(p^*)\big) - h_0\big]\frac{\partial v^R\big(p^*, x^R(p^*)\big)}{\partial x^R} \qquad (28) \\
\frac{dv^D\big(\frac{p^*}{\kappa}, x^D\big)}{dp} &= -\frac{h^D\big(\frac{p^*}{\kappa}, x^D\big)}{\kappa}\frac{\partial v^D\big(\frac{p^*}{\kappa}, x^D\big)}{\partial x^D}, \qquad (29)
\end{aligned}$$

where h^R and h^D denote the housing demand functions when the household repays debt and defaults, respectively. Since $u(c, h)$ is homothetic, we have

$$\frac{\partial v^R\big(p^*, x^R(p^*)\big)}{v^R\big(p^*, x^R(p^*)\big)} \Big/ \frac{\partial x^R(p^*)}{x^R(p^*)} = \frac{\partial v^D\big(\frac{p^*}{\kappa}, x^D\big)}{v^D\big(\frac{p^*}{\kappa}, x^D\big)} \Big/ \frac{\partial x^D}{x^D},$$

which from eq. (26) implies

$$\frac{\partial v^D\left(\frac{p^*}{\kappa}, x^D\right)}{\partial x^D} = \frac{x^R(p^*)}{x^D}\frac{\partial v^R\left(p^*, x^R(p^*)\right)}{\partial x^R(p^*)}. \tag{30}$$

Combining eqs. (28)-(30) shows that the "steepness condition" in eq. (27) holds if and only if

$$h^R\left(p^*, x^R(p^*)\right) - h_0 < \frac{1}{\kappa}\frac{x^R(p^*)}{x^D}h^D\left(\tfrac{p^*}{\kappa}, x^D\right),$$

which multiplied by $p^*/x^R(p^*)$ on both sides equals

$$\frac{p^*h^R\left(p^*, x^R(p^*)\right)}{x^R(p^*)} - \frac{p^*h_0}{x^R(p^*)} < \frac{\frac{p^*}{\kappa}h^D\left(\frac{p^*}{\kappa}, x^D\right)}{x^D}. \tag{31}$$

The leftmost and rightmost terms in condition (31) are the expenditure shares on housing when the household repays debt and defaults, respectively. When the elasticity of substitution is smaller than unity (i.e. when $\rho \leq 0$), the household spends a larger share on housing when housing is relatively more expensive, which is the case for renters since the rental price is always higher than the price of housing stock. The middle term, $p^*h_0/x^R(p^*) > 0$, ensures that the inequality holds strictly even when $\rho = 0$. This proves the first part of the proposition. (Moreover, it shows that $\rho = 0$ is not a knife-edge case.) To prove the properties of p^*, start by observing that as $u(c, h)$ is continuous, differentiable and strictly increasing, $v^R\left(p, x^R(p, \sigma, m)\right)$ is continuous, differentiable and strictly increasing in x^R, and thus in $-B$ and h_0 for any given σ and any finite m. Continuity and differentiability of both v^R and v^D with respect to p then means that for any given value of σ, eq. (26) implicitly defines the threshold price p^* as a continuous and differentiable function of the parameters B and h_0 of the mortgage contract whenever $p^* > 0$. Since $dv^R/dp > dv^D/dp$ at any such point, we have

$$\frac{\partial p^*}{\partial B} > 0 \quad \text{and} \quad \frac{\partial p^*}{\partial h_0} < 0.$$

■

Example 1 (Threshold price with Cobb-Douglas preferences) *When preferences are Cobb-Douglas (logarithmic), it is straightforward to show that*

$$\frac{u_h(c, h)}{u_c(c, h)} = \frac{1-\alpha}{\alpha}\frac{c}{h}, \tag{32}$$

where u_c and u_h denote the partial derivatives of $u(c, h)$. The first order condition for the repayer's problem (5) is $u_h/u_c = p$, and for the defaulter's problem (6) it is $u_h/u_c = p/\kappa$. Combining in each case eq. (32) with the first order condition and the budget constraint

yields the familiar demand functions $c^R = \alpha x^R$ *and* $h^R = \frac{1}{p}(1-\alpha)x^R$ *for the repayer, and* $c^D = \alpha x^D$ *and* $h^D = \frac{1}{p/\kappa}(1-\alpha)x^D$ *for the defaulter. The repayer's utility equals the defaulter's utility if and only if*

$$\alpha \log(c^R) + (1-\alpha)\log(h^R) = \alpha \log(c^D) + (1-\alpha)\log(h^D).$$

Inserting for the demand functions, simplifying and solving for the threshold price p^* *yields*

$$p^* = \frac{B - (1-\kappa^{1-\alpha})w(\sigma)}{h_0}. \tag{33}$$

Example 2 (Perfect substitutes, no threshold price) *Suppose other goods and housing are perfect substitute, i.e. suppose* $\rho = 1$*. Households then maximise utility by devoting the entire budget to the cheapest of the two, adjusting for the relative weight* α*. For concreteness, fix* $\alpha = \frac{1}{2}$*. The repayer's problem is then*

$$v^R(p) = \tfrac{1}{2}\max\big\{c^R(p), h^R(p)\big\}$$

where c^R *and* h^R *denote the allocation if the entire budget is spent on consumption goods or housing, respectively, i.e.* $c^R(p) = w + ph_0 - B$ *and* $h^R(p) = h_0 + \frac{w-B}{p}$*. Similarly, the defaulter's problem is*

$$v^D(p) = \tfrac{1}{2}\max\big\{c^D, h^D(p)\big\}$$

for $c_D = w$ *and* $h^D(p) = \frac{\kappa w}{p}$*, and the default decision is*

$$\delta^*(p) = \arg\max_{\delta\in\{0,1\}} \big\{\delta v^R(p) + (1-\delta)v^D(p)\big\}.$$

I claim that there is no threshold price for finite mortgage contracts (B, h_0) *satisfying*

$$B - h_0 > 0 \tag{34}$$

$$w(1-\kappa) - B > 0. \tag{35}$$

Start by observing that repaying debt is optimal when the price is sufficiently high, as

$$\lim_{p\to\infty} v^R(p) = \tfrac{1}{2}\max\big\{\lim_{p\to\infty} c^R(p), \lim_{p\to\infty} h^R(p)\big\} = \tfrac{1}{2}\max\{\infty, h_0\} = \infty$$

$$\lim_{p\to\infty} v^D(p) = \tfrac{1}{2}\max\big\{\lim_{p\to\infty} c^D, \lim_{p\to\infty} h^D(p)\big\} = \tfrac{1}{2}\max\{w, 0\} = \frac{w}{2}.$$

Next, consider the intermediate price $p = 1$*. If the household repays debt, he would then obtain other goods and housing at the same price, and since* $\alpha = \frac{1}{2}$*, he is indifferent between the two, in any case obtaining yielding* $v^R(1) = \frac{1}{2}(w - h_0 - B)$*. However, since*

$B - h_0 > 0$ from cond. (34), the household obtains even higher utility by defaulting and buying other consumption goods, since

$$v^D(1) = \tfrac{1}{2}\max\{w, \kappa w\} = \frac{w}{2}.$$

Roughly speaking, this is an "income" effect: As the household is "underwater", i.e. the value of his debt outweighs the value of his house, he frees up resources by defaulting and is therefore better off. When other consumption goods and housing are very close substitutes, however, the household may eventually regret having defaulted if the price falls sufficiently low, because repaying debt provides housing at an even lower unit price than defaulting. For this to hold, the price difference between renting and owning must be sufficiently large (i.e. κ must be sufficiently small) for this "substitution" effect to more than compensate for the capital loss the household endures by repaying debt when the price is low. To see this, let the price tend to zero. Both the repayer and the defaulter obtain infinite utility in this case, as

$$\begin{aligned}
\lim_{p\to 0} v^R(p) &= \tfrac{1}{2}\max\big\{\lim_{p\to 0} c^R(p), \lim_{p\to 0} h^R(p)\big\} = \tfrac{1}{2}\max\big\{w - B), \lim_{p\to 0}(h_0 + \tfrac{w-B}{p})\big\} = \infty \\
\lim_{p\to 0} v^D(p) &= \tfrac{1}{2}\max\big\{\lim_{p\to 0} c^D, \lim_{p\to 0} h^D(p)\big\} = \tfrac{1}{2}\max\{w, \lim_{p\to 0} \tfrac{\kappa w}{p}\} = \infty,
\end{aligned}$$

but cond. (34) implies that repaying debt is in fact the optimal choice, because

$$\lim_{p\to 0} \frac{h^R(p)}{h^D(p)} = \lim_{p\to 0} \frac{h_0 + \frac{w-B}{p}}{\frac{\kappa w}{p}} = \frac{w - B}{\kappa w} > 1.$$

Proof of Proposition 2 (Optimal mortgage contracts). Consider equilibrium solutions to the contract (ex ante) problem (8) for any type of household $i \in \mathcal{I}$. (Subscript i is suppressed below for expositional convenience, but type dependent threshold prices will in general yield type dependent optimal mortgage contracts.) A few observations help simplify the problem. First, as households are insatiable they fully exhaust their available resources in equilibrium, so the budget constraint must hold with equality. Second, because the set of contracts offered M contains all potential break-even contracts, and because insatiable households always prefer the highest possible b for any given (B, h_0), equilibrium mortgage contracts must exactly break even in expectation. That is, eq. (9) must hold with equality in equilibrium, establishing b as a function of (B, h_0). Moreover, since projected default behaviour is consistent with households' optimal default decisions for any mortgage contract, and these decisions are characterised by a cutoff strategy with respect to a threshold price $p^*(\sigma, m)$, the threshold price can be used to evaluate the function b of (B, h_0). Consequently, (c_0, h_0, B) is an equilibrium solution to the contract

(ex ante) problem (8) if and only if it solves

$$\max_{\{c_0,h_0,B\}} \left\{ u(c_0,h_0) + \beta \int \left[\begin{array}{l} \int_0^{p^*(\sigma,m)-} v^D\big(\frac{p}{\kappa}, x^D(\sigma)\big)\, dF(p|\sigma) \\ + \int_{p^*(\sigma,m)}^{\infty} v^R\big(p, x^R(p,\sigma,m)\big)\, dF(p|\sigma) \end{array} \right] dG(\sigma) \right\} \quad (36)$$

$$s.t.$$

$$c_0 + p_0 h_0 = \bar{\omega} + \beta \int \left[\begin{array}{l} \int_0^{p_i^*(\sigma,m)-} p h_0\, dF(p|\sigma) \\ + \int_{p_i^*(\sigma,m)}^{\infty} B\, dF(p|\sigma) \end{array} \right] dG(\sigma).$$

A natural way to proceed is to employ the Lagrangian method and derive necessary first order conditions for optimality by differentiating conditional on σ. A potential problem, however, is that the constraint in problem (36) is not everywhere differentiable; it is not *à priori* clear whether the optimum corresponds to a point which is differentiable or not. I therefore consider each possibility in turn.

Differentiable case (Interior solution). Consider any particular value of σ and a contract m. The constraint in problem (36) is differentiable with respect to m at σ and m if and only if $F(p|\sigma)$ is such that

$$\Pr(p^*(\sigma,m) = p) = 0. \quad (37)$$

That is, differentiability with respect to m obtains wherever the event that the threshold price exactly equals the market price has probability zero. Clearly, cond. (37) would always hold if $F(p|\sigma)$ were a continuous distribution, but as the proof of proposition 3 shows, the finite number of household types yields a finite number of possible market clearing prices for each value of σ. At any contract m for which cond. (37) holds for every σ, the derivatives of the Lagrangian function $\mathcal{L}$ corresponding to problem (36) are[42]

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_0} &= u_c(c_0,h_0) - \lambda \\
\frac{\partial \mathcal{L}}{\partial B} &= \beta \iint_{p^*(\sigma,m)}^{\infty} \underbrace{\frac{\partial v^R(p,x^R)}{\partial x^R}}_{u_c(c,h)} \underbrace{\frac{\partial x^R}{\partial B}}_{-1} dF(p|\sigma)\, dG(\sigma) + \lambda\beta \iint_{p^*(\sigma,m)}^{\infty} 1\, dF(p|\sigma)\, dG(\sigma) \\
\frac{\partial \mathcal{L}}{\partial h_0} &= u_h(c_0,h_0) + \beta \iint_{p^*(\sigma,m)}^{\infty} \underbrace{\frac{\partial v^R(p,x^R)}{\partial x^R}}_{u_c(c,h)} \underbrace{\frac{\partial x^R}{\partial h_0}}_{p} dF(p|\sigma)\, dG(\sigma) \\
&\quad -\lambda p_0 + \lambda\beta \iint_0^{p^*(\sigma,m)-} p\, dF(p|\sigma)\, dG(\sigma),
\end{aligned}$$

where λ denotes the Lagrangian multiplier on the constraint, and u_c and u_h denote the

[42]Since $p^*(\sigma,m)$ is continuous in B and h_0 and $F(p|\sigma)$ is discrete, cond. (37) also simplified this computation in that there is no need to consider changes in the integration limits resulting from changes in $p^*(\sigma,m)$.

partial derivatives of $u(c,h)$. The equivalences indicated with curly braces follow immediately from applying the envelope theorem to the repayer's problem (5), and the definition of x^R. Equating to zero, substituting for λ, and rearranging, yields

$$u_c(c_0,h_0)\underbrace{\iint_{p^*(\sigma,m)}^{\infty} 1\,dF(p|\sigma)\,dG(\sigma)}_{\Pr(p\geq p^*(m))} = \underbrace{\iint_{p^*(\sigma,m)}^{\infty} u_c(c,h)\,dF(p|\sigma)\,dG(\sigma)}_{E[u_c(c,h)|p\geq p^*(m)]\Pr(p\geq p^*(m))} \tag{38}$$

and

$$p_0 = \frac{u_h(c_0,h_0)}{u_c(c_0,h_0)} + \beta\underbrace{\iint_{p^*(\sigma,m)}^{\infty} p\,\frac{u_c(c,h)}{u_c(c_0,h_0)}\,dF(p|\sigma)\,dG(\sigma)}_{E[p\,\frac{u_c(c,h)}{u_c(c_0,h_0)}|p\geq p^*(m)]\Pr(p\geq p^*(m))} + \beta\underbrace{\iint_{0}^{p^*(\sigma,m)_-} p\,dF(p|\sigma)\,dG(\sigma)}_{E[p|p<p^*(m)]\Pr(p<p^*(m))}, \tag{39}$$

which are the conditions given in proposition 2, holding with equality.

Non-differentiable case (Border solution). To see why the constraint in problem (36) is not everywhere differentiable, consider a case in which cond. (37) does not hold. That is, consider some σ and m such that $\Pr\big(p^*(\sigma,m)=p^n\big)=\pi(p^n)>0$, where p^n is shorthand notation for an element in the finite set $\mathcal{P}_\sigma=\big\{p_\sigma^1,p_\sigma^2,...,p_\sigma^{N_\sigma}\big\}$ of possible prices given σ, and $\pi(p^n)$ denotes the corresponding probability. Conditioning on the chosen value of σ, the integral in the constraint in problem (36) can be written

$$\sum_{j=1}^{n-1}\pi(p^j)p^jh_0+\pi(p^n)B+\sum_{j=n+1}^{N_\sigma}\pi(p^j)B. \tag{40}$$

Consider a small increase in B to $B+\varepsilon$. Since $p^*(m)$ is strictly increasing in B, the new value of $p^*(m)$ at $B+\varepsilon$ must be strictly higher than p^n. Consequently, for sufficiently small ε, the corresponding value of eq. (40) becomes

$$\sum_{j=1}^{n-1}\pi(p^j)p^jh_0+\pi(p^n)p^nh_0+\sum_{j=n+1}^{N_\sigma}\pi(p^j)(B+\varepsilon). \tag{41}$$

Eqs. (40) and (41) give the expected value of the mortgage contract, conditional on σ, and thus in the household's available resources ex ante. The *absolute difference* between them as ε tends to zero is

$$\lim_{\varepsilon\to 0}\Big\{-\pi(p^n)[B-p^nh_0]+\varepsilon\sum_{i=n+1}^{N_\sigma}\pi(p^i)\Big\}=-\pi(p^n)[B-p^nh_0], \tag{42}$$

which is finite and strictly negative because $B>p^*(m)h_0$ for any threshold price $p^*(m)$, cf. eq. 14). Since a marginal increase in B implies a discrete reduction in the amount of

resources available, the *derivative* with respect to B does not exist at this point, i.e.

$$\lim_{\varepsilon \to 0} \frac{-\pi(p^n)[B - p^n h_0] + \varepsilon \sum_{i=n+1}^{N_\sigma} \pi(p^i)}{\varepsilon} = -\infty.$$

Moreover, since $p^*(m)$ is strictly decreasing in h_0, differentiability also fails with respect to a small decrease in h_0 at this point. Clearly, a point such as $B + \varepsilon$ above cannot be an optimum, but a point such as B can. The first order conditions for optimality (38)-(39) may no longer hold with equality in such a case. For instance, it could be that $u_c(c_0, h_0) > E\big[u_c(c, h) \mid p \geq p^*(m)\big]$, but the household refrains from borrowing more (which would smooth consumption and thus marginal utility), because doing so would lead to a discrete reduction of the amount of resources available, and thus give lower utility. Note, however, that if $G(\sigma)$ is continuous and cond. (37) only fails to hold for a *finite* number of values σ at the optimum, then the non-differentiable parts of the constraint in problem (36) constitute a zero probability event, and so the constraint is differentiable *almost everywhere*. Since the absolute reduction (42) is finite, the non-differentiable parts can therefore be disregarded, and conds. (38)-(39) will hold. ■

Proof of Proposition 3 (Limit-symmetric equilibrium). Start by observing that ex post utility v is always finite: it is bounded from below by v^D, which is finite since $x^D = w(\sigma) > 0$, and it is bounded from above by some finite value since the equilibrium price must be finite. Therefore, whatever the outcome in the event of income shortfall, it can be disregarded in the contract (ex ante) problem (8) in the limit as $\nu \to 0$. Moreover, in the event of no income shortfall, *pri* and *sub* have the same income. Hence, *pri* and *sub* households solve the same problem in the limit as $\nu \to 0$, yielding *ex ante symmetry*

$$\lim_{\nu \to 0} m_{pri} = \lim_{\nu \to 0} m_{sub}.$$

Let $m = (B, h_0)$ denote this symmetric limit contract, let b denote its expected net present value, and let c_0 denote the limit of the corresponding ex ante (period zero) consumption choice. Ex ante symmetry and housing market clearing impose $h_0 = H$, but solving for B requires a consideration the possible ex post outcomes in the event of no income shortfall. Identical incomes and ex ante symmetry contracts imply *ex post symmetry* in this case, and thus only two possible outcomes: either everyone repays debt, or everyone defaults. When everyone repays debt, the housing demand function $h^R = \frac{1}{p}(1 - \alpha)x^R$ (cf. example 1), where $x^R = \omega + ph_0 - B$, and the housing market clearing condition (11) yields the unique potential market clearing price

$$p_R = \frac{1 - \alpha}{\alpha} \frac{\omega - B}{H},$$

where subscript R indicates that the price pertains to the equilibrium where everyone repays debt. This equilibrium exists whenever $p_R \geq p^*$. Inserting for the threshold price (33) shows that this hold if and only if

$$B \leq \big(1 - \alpha\kappa^{1-\alpha}\big)\omega.$$

Turning to the case when everyone defaults, the housing demand function $h^D = \frac{1}{p/\kappa}(1-\alpha)x^D$, where $x^D = \omega$, yields a unique potential market clearing price

$$p_D = (1-\alpha)\frac{\kappa\omega}{H},$$

where subscript D indicates that it pertains to the equilibrium where everyone defaults. This equilibrium exists whenever $p_D < p^*$, which holds if and only if

$$B > \big((1-\alpha)\kappa - \kappa^{1-\alpha} + 1\big)\omega. \tag{43}$$

Let $\eta \in [0,1]$ denote the probability of the outcome "everyone defaults". In the limit as $\nu \to 0$, the first order conditions (15)-(16) for an interior solution of the contract (ex ante) problem (8) become[43]

$$\begin{aligned} u_c(c_0,h_0) &= \nu \underbrace{E\big[u_c(c,h) \mid p \geq p^*(\sigma,m), \sigma = \bar{\sigma}\big]}_{\text{must be finite}} \\ &\quad + (1-\nu)E\big[u_c(c,h) \mid \underbrace{p \geq p^*(\sigma,m), \sigma = 0}_{\text{case "everyone repays"}}\big] \\ &= u_c(c^R(p_R), h^R(p_R)) \end{aligned} \tag{44}$$

implying $c^R(p_R) = c_0$ since $h^R = h_0 = H$ from symmetry, and

$$\begin{aligned} p_0 &= \frac{u_h(c_0,h_0)}{u_c(c_0,h_0)} + \beta\nu\big[\underbrace{E[...]\Pr(...) + E[...]\Pr(...)}_{\text{must be finite}}\big] \\ &\quad + \beta(1-\nu)\left[\begin{array}{c} (1-\eta)E\big[p\ \underbrace{\frac{u_c(c,h)}{u_c(c_0,h_0)}}_{1} \mid \underbrace{p \geq p^*(\sigma,m), \sigma = 0}_{\text{case "everyone repays"}}\big] \\ +\eta E\big[p \mid \underbrace{p < p^*(\sigma,m), \sigma = 0}_{\text{case "everyone defaults"}}\big] \end{array}\right] \\ &= \frac{u_h(c_0,h_0)}{u_c(c_0,h_0)} + \beta\big[(1-\eta)p_R + \eta p_D\big]. \end{aligned} \tag{45}$$

[43] This is where $\nu \to 0$ and the ensuing symmetry are useful assumptions, as they yield only a single case for which to evaluate $E\big[u(c,h)\big]$. The first condition then explicitly yields $c^R = c_0$, which in turn permits eliminating $u_c(c,h)/u_c(c_0,h_0)$ from the second condition to obtain the form in eq. (45).

Moreover, the budget constraint becomes

$$c_0 + p_0 h_0 = \omega + \beta\big[(1-\eta)B + \eta p_D h_0\big]. \tag{46}$$

Evaluating condition (45) for Cobb-Douglas preferences using eq. (32), and inserting for p_R, p_D, and $h_0 = H$, yields an expression that can be used to substitute for c_0 in the budget constraint (46). The resulting equation can be solved for the ex ante housing price

$$p_0 = (1-\alpha)\big(1 + \beta[1 - \eta + \eta\kappa]\big)\frac{\omega}{H}.$$

Next, evaluating c^R at p_R and $h_0 = H$, and recalling the first order condition (44), shows that $c_0 = c^R = \omega - B$. Using these results to insert for p_0 and c_0 in the budget constraint (46) allows solving for the debt level

$$B = (1-\alpha)\omega.$$

Eq. (6) imposes $\eta = 0$, because for this level of debt, the equilibrium in which everyone defaults cannot exist; that is, B is not large enough to satisfy the existence condition given by eq. (43).[44] Finally, by inserting we obtain

$$\begin{aligned} c_0 = c^R &= \alpha\omega \\ p_R &= (1-\alpha)\frac{\omega}{H} \\ b &= \beta(1-\alpha)\omega, \end{aligned}$$

which concludes the proof of parts 1 and 2 of the proposition. In the event of income shortfall to *sub* households, ex ante symmetry and $w_{sub} = (1-\bar{\sigma})\omega < \omega = w_{pri}$ imply $p^*_{sub} > p^*_{pri}$. Consequently, there are three potential equilibria: either everyone repays debt, or only *sub* households default, or everyone defaults. Solving for these equilibria consists of in each case inserting into the market clearing condition the appropriate housing demand functions for repayers and defaulters, evaluated at w_i and the equilibrium contract $m = (B, h_0) = \big((1-\alpha)\omega, H\big)$, and deriving the market clearing price. The existence conditions are then derived from comparing the potential equilibrium prices with the appropriate threshold prices. When everyone repays debt, the housing demand functions $h_i^R = \frac{1}{p}(1-\alpha)x_i^R$ yield the unique potential market clearing price

$$p_{RR} = (1-\alpha)\Big[1 - \frac{\gamma\bar{\sigma}}{\alpha}\Big]\frac{\omega}{H},$$

[44] $B = (1-\alpha)\omega$ satisfies cond. (43) if and only if $\exists\kappa \in (0,1) : f(\kappa) = (1-\alpha)\kappa - \kappa^{1-\alpha} + \alpha < 0$. But this is not the case since f is strictly decreasing, $f'(\kappa) = (1-\alpha)\big(1 - \frac{1}{\kappa^\alpha}\big) < 0$, and $f(1) = 0$.

where subscript RR indicates that the price pertains to the equilibrium where both types of households repay debt, i.e. the "repay-repay" equilibrium. This equilibrium exists whenever $p_{RR} \geq p^*_{sub}$. Inserting and rearranging shows that this holds if and only if

$$1 - \kappa^{1-\alpha} \geq \frac{1-\alpha}{\alpha}\,\gamma\,\frac{\bar{\sigma}}{1-\bar{\sigma}}.$$

When only *sub* households default, the housing demand functions $h^R_{pri} = \frac{1}{p}(1-\alpha)x^R_{pri}$ and $h^D_{sub} = \frac{1}{p/\kappa}(1-\alpha)x^D_{sub}$ yield the unique potential market clearing price

$$p_{RD} = (1-\alpha)\frac{\alpha(1-\gamma) + \gamma\kappa(1-\bar{\sigma})}{\alpha(1-\gamma)+\gamma}\frac{\omega}{H},$$

where subscript RD indicates "repay-default". This equilibrium exists whenever both $\leq p_{RD} \geq p^*_{pri}$ and $p_{RD} < p^*_{sub}$, which holds if and only if

$$\begin{aligned}
&(1-\alpha)\gamma(1-\bar{\sigma})\kappa - \big[\alpha + (1-\alpha)\gamma\big]\kappa^{1-\alpha} + \alpha \geq 0 \\
&(1-\alpha)\gamma(1-\bar{\sigma})\kappa - \big[\alpha + (1-\alpha)\gamma\big]\kappa^{1-\alpha} + \alpha < \bar{\sigma}\big[\alpha + (1-\alpha)\gamma\big].
\end{aligned}$$

Finally, when everyone defaults, the housing demand functions $h^D_i = \frac{1}{p/\kappa}(1-\alpha)x^D_i$, yield the unique potential market clearing price

$$p_{DD} = (1-\alpha)\big[1-\gamma\bar{\sigma}\big]\frac{\kappa\omega}{H},$$

where subscript RR indicates "default-default". This equilibrium exists whenever $p_{DD} < p^*_{pri}$, which holds if and only if

$$(1-\alpha)(1-\gamma\bar{\sigma})\kappa - \kappa^{1-\alpha} + \alpha < 0.$$

■