MEMORANDUM

No 02/2011

Portfolio Separation Properties of the Skew-Elliptical Distributions

Nils Chr. Framstad

ISSN: 0809-8786

Department of Economics
University of Oslo

This series is published by the **University of Oslo Department of Economics**	In co-operation with **The Frisch Centre for Economic Research**
P. O.Box 1095 Blindern N-0317 OSLO Norway Telephone: + 47 22855127 Fax: + 47 22855035 Internet: http://www.sv.uio.no/econ e-mail: econdep@econ.uio.no	Gaustadalleén 21 N-0371 OSLO Norway Telephone: +47 22 95 88 20 Fax: +47 22 95 88 25 Internet: http://www.frisch.uio.no e-mail: frisch@frisch.uio.no

Last 10 Memoranda

Previous issues of the memo-series are available in a PDF® format at:
http://www.sv.uio.no/econ/forskning/publikasjoner/memorandum

Portfolio separation properties of the skew-elliptical distributions

Nils Chr. Framstad*[†]

Version: February 1, 2011.

Abstract. The two fund separation property of the elliptical distributions is extended to the skew-elliptical and by adding a number of funds equalling the rank of the skewness matrix. Some elements of the generalization to *singular* extended skew-elliptical distributions are covered.

Key words and phrases: Portfolio separation, mutual fund theorem, stochastic dominance, singular extended skew-elliptical distributions.

MSC (2000): 91B28, 60E05, 49K45.
JEL classification: G11, C61, D81, D53.

0 Introduction

The concept of portfolio separation, a.k.a. the mutual fund theorem, should be well known. Since Tobin [12], numerous works have generalized the result in terms of the preferences which admit separation (like Cass and Stiglitz [2] or even as recently as Schachermayer et al. [11], using a modern approach), or in terms of distributions (Ross [10]). The concept of risk measures falls somewhat in between, see e.g. this author [6] and independently, De Giorgi et al. [7].

This note extends the results of Owen and Rabinovitch [9] and Chamberlain [3], who point out that the elliptical (also frequently referred to as «elliptically contoured») distributions admit two fund separation. It will turn out that a similar result holds for the skew-elliptical class (Branco and Dey [1] and Díaz-García and González-Farías [4]), at the expense of requiring an additional number of funds corresponding to the rank of the skewness matrix. The latter introduce the wider singular extended skew-elliptical (SESE) class, and one of these generalizations will be covered herein. We shall restrict ourselves to the single-period discrete time case. Using this author's refinement [5] of the approach given by Khanna and Kulldorff [8], there will be a continuous-time analogue if the probability law is infinitely divisible (hence the discrete-time setup is more general in terms of probability distributions).

* University of Oslo, Department of Economics, P.O. Box 1095 Blindern, NO-0317 Oslo, Norway. mailto:ncf+research@econ.uio.no

† Also affiliated with the Financial Supervisory Authority of Norway, P.O. Box 1187 Sentrum, NO-0107 Oslo, Norway. The content of this article does not reflect the views of the Norwegian FSA.

1 The result

Consider a single period investment in a numéraire (enumerated with a zero) returning Y_0 per monetary unit invested, and another p investment opportunities with returns vector $Y_0\mathbf{1} + \boldsymbol{\mu} + \boldsymbol{Y}$, so that the return with investments $\boldsymbol{u}$ in the p opportunities and $w - \boldsymbol{u}^\dagger\mathbf{1}$ (where w is initial wealth) in the numéraire, will be

$$X = wY_0 + \boldsymbol{u}^\dagger\big(\boldsymbol{\mu} + \boldsymbol{Y}\big), \tag{1}$$

(where the «†» superscript denotes transposition). The market will be assumed free of arbitrage opportunities and of redundant investment opportunities (having removed the latter from the market).

The probability distribution of $\boldsymbol{\mu}+\boldsymbol{Y}$ will be considered conditional on Y_0 – therefore, we can (and will) without loss of generality assume $Y_0 = 0$ (or, for that matter, a risk-free return). $\boldsymbol{\mu}$ will be a location parameter, enabling us to assume location at zero in the representation to follow – note however, that we do not assume finite moments of any order.

Recall that an elliptical (a.k.a. elliptically contoured) random variable $\boldsymbol{Z}$, has characteristic function of the form $e^{-\mathrm{i}\boldsymbol{\theta}^\dagger\boldsymbol{\delta}}\psi(\boldsymbol{\theta}^\dagger\boldsymbol{M}\boldsymbol{\theta})$, where the matrix $\boldsymbol{M}$ is positive definite. The underlying spherical distribution (i.e. $\boldsymbol{M}^{-1/2}(\boldsymbol{Z}-\boldsymbol{\delta})$) can then be written as a mixture $R\boldsymbol{S}$ of a positive radial variable R, and $\boldsymbol{S}$ which is independent and uniform on the sphere. A *singular elliptical distribution* in the sense of [4], is obtained by relaxing the requirement to positive semidefinite $\boldsymbol{M}$. Therein, it is assumed that R is absolutely continuous, but an approximation argument will allow for general R.

This paper does only to a limited extent use singular properties covered by [4], but will utilize their multivariate generalization of the case treated in [1]. Following their notation, one takes as starting point a singular elliptical vector $\boldsymbol{E} = (\boldsymbol{E}_1{}^\dagger, \boldsymbol{E}_2{}^\dagger)^\dagger$ located at $\boldsymbol{\delta} = \mathbf{0}$ and with associated matrix $\boldsymbol{M} = \left(\begin{smallmatrix}\boldsymbol{\Sigma} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\Delta}\end{smallmatrix}\right)$, and where the marginals $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ (p-vector and q-vector, respectively) have associated positive semidefinite matrices $\boldsymbol{\Sigma} \in \mathbf{R}^{p\times p}$ and $\boldsymbol{\Delta} \in \mathbf{R}^{q\times q}$ – observe that each $\boldsymbol{E}_i$ is allowed intra-dependent components. Now for arbitrary non-random $\boldsymbol{\mu} \in \mathbf{R}^p$, $\boldsymbol{\nu} \in \mathbf{R}^q$, $\boldsymbol{D} \in \mathbf{R}^{q\times p}$, then

$$[\boldsymbol{\mu} + \boldsymbol{E}_1 \Big| \boldsymbol{D}\boldsymbol{E}_1 + \boldsymbol{E}_2 - \boldsymbol{\nu} \geq \mathbf{0}] \qquad \text{(component-wise inequality, i.e. positive orthant)}$$

has the *singular vector-variate skew-elliptical distribution*. In [4], this is parametrized as $\mathrm{SESE}_r^{(p)}(q, k_1, \boldsymbol{\mu}, \boldsymbol{\Sigma}, k, \boldsymbol{D}, \boldsymbol{\nu}, \boldsymbol{\Delta}, h_r^{(p)})$ where r, k and k_1 are the ranks of $\boldsymbol{\Sigma}$, $\boldsymbol{\Delta}$ and $\boldsymbol{\Delta}+\boldsymbol{D}\boldsymbol{\Sigma}\boldsymbol{D}^\dagger$, respectively, and $h_r^{(p)}$ denotes the density generating function with respect to some appropriate Hausdorff measure (which is not unique – however, the results won't depend on the choice). We remark that integrability assumptions are not needed, despite the literature's common use of terms like e.g. covariance matrix.

We shall assume $\boldsymbol{\mu}+\boldsymbol{Y}$ to have such a distribution. Then $\boldsymbol{Y}$ belongs to the same class, except with location $\boldsymbol{\mu}$ replaced by null. In order to ensure absence of arbitrage and of redundant investment opportunities, we shall assume $\boldsymbol{\Sigma}$ positive definite (so that in particular, $r = p$); the only «singular» property left then is a possible rank-deficiency of $\boldsymbol{\Delta}$. We can adapt the following special case from [4, Theorem 5.1]:

References

Lemma. *Suppose that $\boldsymbol{Y}$ is absolutely continuous and distributed*

$$\boldsymbol{Y} \sim \mathrm{SESE}_p^{(p)}(q, k_1, \mathbf{0}, \boldsymbol{\Sigma}, k, \boldsymbol{D}, \boldsymbol{\nu}, \boldsymbol{\Delta}, h_p^{(p)}), \tag{2}$$

where $\boldsymbol{\Sigma}$ is positive definite and $h_p^{(p)}$ is the density generating function with respect to p-dimensional Lebesgue measure. Then, for any non-random non-null p-vector $\boldsymbol{u}$:

$$\boldsymbol{u}^\dagger \boldsymbol{Y} \sim \mathrm{SESE}_1^{(1)}(q, k_1, \mathbf{0}, \boldsymbol{u}^\dagger\boldsymbol{\Sigma}\boldsymbol{u}, \mathrm{rank}(\boldsymbol{\Delta}_{\boldsymbol{u}}), \boldsymbol{D}_{\boldsymbol{u}}, \boldsymbol{\nu}, \boldsymbol{\Delta}_{\boldsymbol{u}}, h) \tag{3}$$

where $h = h_1^{(1)}$ is a univariate density-generating function, and

$$\boldsymbol{D}_{\boldsymbol{u}} = \frac{1}{\boldsymbol{u}^\dagger\boldsymbol{\Sigma}\boldsymbol{u}}\boldsymbol{D}\boldsymbol{\Sigma}\boldsymbol{u}, \qquad \boldsymbol{\Delta}_{\boldsymbol{u}} = \boldsymbol{\Delta} + \boldsymbol{D}\boldsymbol{\Sigma}\boldsymbol{D}^\dagger - \boldsymbol{D}_{\boldsymbol{u}}(\boldsymbol{u}^\dagger\boldsymbol{\Sigma}\boldsymbol{u})\boldsymbol{D}_{\boldsymbol{u}}^\dagger. \tag{4}$$

Recalling that non-absolutely continuous components in the underlying radial distribution can be recovered by approximation, we then have the follwing:

Theorem. *Assume the market (1) with the returns distributed according to (2), where $\boldsymbol{\Sigma}$ is positive definite. Suppose the agents rank portfolios according to first-order stochastic dominance of the return. Then we have $2 + \mathrm{rank}(\boldsymbol{D})$ fund separation. Furthermore, under the additional constraint of $\boldsymbol{u}^\dagger\mathbf{1} = w$ (i.e. the absence of opportunity to invest in the («safe») numéraire), we have $1 + \mathrm{rank}(\boldsymbol{D}^\dagger, \mathbf{1}^\dagger\boldsymbol{\Sigma}^{-1})$ fund separation.*

Proof. We observe from (4) that the distribution (3) depends on $\boldsymbol{u}$ only through $\sqrt{\boldsymbol{u}^\dagger\boldsymbol{\Sigma}\boldsymbol{u}} \in \mathbf{R}_+$ and $\boldsymbol{D}\boldsymbol{\Sigma}\boldsymbol{u} \in \mathbf{R}^q$. For given values $Q > 0$ and $Q\boldsymbol{q} \in \mathbf{R}^q$ of these, the agent will

$$\max_{\boldsymbol{u}} \ \boldsymbol{\mu}^\dagger\boldsymbol{u} \qquad \text{subject to} \quad \boldsymbol{u}^\dagger\boldsymbol{\Sigma}\boldsymbol{u} = Q^2, \quad \boldsymbol{D}\boldsymbol{\Sigma}\boldsymbol{u} = Q\boldsymbol{q}$$

or equivalently, putting $\boldsymbol{v} = \boldsymbol{\Sigma}\boldsymbol{u}$, $\boldsymbol{a} = \boldsymbol{\mu}\boldsymbol{\Sigma}^{-1}$

$$\max_{\boldsymbol{v}} \ \boldsymbol{a}^\dagger\boldsymbol{v} \qquad \text{subject to} \quad \boldsymbol{v}^\dagger\boldsymbol{\Sigma}^{-1}\boldsymbol{v} = Q^2, \quad \boldsymbol{D}\boldsymbol{v} = Q\boldsymbol{q},$$

where for the case without safe investment opportunity, augment with the additional constraint $\mathbf{1}^\dagger\boldsymbol{u} = (\mathbf{1}^\dagger\boldsymbol{\Sigma}^{-1})\boldsymbol{v} = w$. Now the constraints $\boldsymbol{D}\boldsymbol{v} = Q\boldsymbol{q}$ form $\mathrm{rank}(\boldsymbol{D})$ linear equations in $\boldsymbol{v}$. Rewriting these constraints – including $\mathbf{1}^\dagger\boldsymbol{\Sigma}^{-1}\boldsymbol{v} = w$ if appropriate – into $\breve{\boldsymbol{D}}\boldsymbol{v} = \breve{\boldsymbol{q}}$ where $\breve{\boldsymbol{D}}$ has full rank, the proof is now a standard procedure: The associated Lagrangian becomes

$$\boldsymbol{a}^\dagger\boldsymbol{v} - \boldsymbol{\lambda}^\dagger\breve{\boldsymbol{D}}\boldsymbol{v} - \Lambda\boldsymbol{v}^\dagger\boldsymbol{\Sigma}^{-1}\boldsymbol{v},$$

which is stationary when $\boldsymbol{a} - \boldsymbol{\lambda}^\dagger\breve{\boldsymbol{D}} = 2\Lambda\boldsymbol{\Sigma}^{-1}\boldsymbol{v} = 2\Lambda\boldsymbol{u}$. To complete the proof, we merely need to address degeneracies: First, if the constraint qualification fails (where the ellipsoid $\boldsymbol{v}^\dagger\boldsymbol{\Sigma}^{-1}\boldsymbol{v} = Q^2$ is tangent to one of the hyperplanes), the solution is obtained as a limiting case, and spanned by the rows of $\breve{\boldsymbol{D}}$. Finally, the case $\Lambda = 0$ is only possible when $\boldsymbol{a}$ is spanned by the rows of $\breve{\boldsymbol{D}}$, and the one fund saved this way will be replaced by an additional orthogonal vector in order to achieve the desired dispersion Q^2 (since no risk aversion is assumed). □

Observe that the result reduces to three-fund separation for the setup of Branco and Dey [1] (who restrict their analysis to $\boldsymbol{D}$ being a vector), and that by putting $\boldsymbol{D} = \mathbf{0}$ we recover the Owen and Rabinovich [9] two-fund separation property as a corollary.

References

[1] M. D. Branco and D. K. Dey, *A general class of multivariate skew-elliptical distributions*, J. Multivariate Anal., 79 (2001), pp. 99–113.

[2] D. Cass and J. E. Stiglitz, *The structure of investor preferences and asset returns, and separability in portfolio allocation: a contribution to the pure theory of mutual funds*, J. Econom. Theory, 2 (1970), pp. 122–160.

[3] G. Chamberlain, *A characterization of the distributions that imply mean–variance utility functions*, J. Econom. Theory, 29 (1983), pp. 185–201.

[4] J. A. Díaz-García and G. González-Farías, *Singular extended skew-elliptical distributions*, J. Korean Statist. Soc., 37 (2008), pp. 385–392.

[5] N. C. Framstad, *Portfolio separation without stochastic calculus (almost)*, University of Oslo: Preprint Pure Mathematics, (2001).

[6] ——, *Coherent portfolio separation—inherent systemic risk?*, Int. J. Theor. Appl. Finance, 7 (2004), pp. 909–917.

[7] E. D. Giorgi, T. Hens, and J. Mayer, *A note on reward-risk portfolio selection and two-fund separation*, Finance Research Letters, In Press, Corrected Proof (2010).

[8] A. Khanna and M. Kulldorff, *A generalization of the mutual fund theorem*, Finance Stoch., 3 (1999), pp. 167–185.

[9] J. Owen and R. Rabinovitch, *On the class of elliptical distributions and their applications to the theory of portfolio choice*, J. Finance, 38 (1983), pp. 745–52.

[10] S. A. Ross, *Mutual fund separation in financial theory—the separating distributions*, J. Econom. Theory, 17 (1978), pp. 254–286.

[11] W. Schachermayer, M. Sîrbu, and E. Taflin, *In which financial markets do mutual fund theorems hold true?*, Finance Stoch., 13 (2009), pp. 49–77.

[12] J. Tobin, *Liquidity preference as behavior toward risk*, Rev. Econom. Stud., 27 (1958), pp. 65–86.