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On free lunches in random walk markets with short-sale constraints and small transaction costs, and weak convergence to Gaussian continuous-time processes*

Nils Chr. Framstad†‡

June 27, 2011

Abstract. This paper considers a sequence of discrete-time random walk markets with a single risky asset, and gives conditions for the existence of arbitrage opportunities or free lunches with vanishing risk, of the form of waiting to buy and selling the next period, with no shorting, and furthermore for weak convergence of the random walk to a Gaussian continuous-time stochastic process. The conditions are given in terms of the kernel representation with respect to ordinary Brownian motion and the discretisation chosen. Arbitrage examples are established where the continuous analogue is arbitrage-free under small transaction costs – including for the semimartingale modifications of fractional Brownian motion suggested in the seminal Rogers (1997) article proving arbitrage in fBm models.

Key words: Stock price model, random walk, Gaussian processes, weak convergence, free lunch with vanishing risk, arbitrage, transaction costs
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0 Introduction

As well known since Rogers [7], fractional Brownian motions is a troublesome model for uncertainty in price processes, as fBm will introduce arbitrage opportunities to canonical models where the ordinary Brownian motion does not. To remedy this, Rogers proposes a parametrised semimartingale modification, whose moving average kernel converges to the fBm's – in particular, the no-arbitrage property is not preserved under this limit. For the purposes of studying market value gains (or losses), one can however argue that the pointwise

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convergence of the kernel function is less interesting than e.g. weak convergence; heuristically, price process «neighbours» should *in the very least* produce neighbouring profit/loss processes. Weak convergence is however not generally sufficient to preserve arbitrage properties, as upside or downside could converge to 0; In Shiryaev's book [8] (sec. VI.3), more precise conditions for convergence to fair prices are given in terms of weak convergence of the *(driving noise, pricing kernel) pair*. A special case where an arbitrage property seems to – or at least seems to be interpreted as to – carry over a weak limit, emerges from a work by Sottinen [9], who establishes a sequence of discrete-time binary random walk (semimartingale) markets which (a) converges weakly to the geometric fractional Brownian motion Black–Scholes market with Hurst parameter $H > 1/2$, and (b) admits an arbitrage opportunity obtained by waiting for the right moment to buy (nonnegative drift) or short sell (nonpositive drift) the stock, and unwinding the position the very next period; the «right moment» is of course when you might with probability one know that the stock market beats the money market even if tomorrow is a bad day (in which case you buy), or waiting for the conversely «bad» market (in which case you sell).

The purpose of this paper is to generalise this approach to more general Gaussian processes, and give conditions in terms of the moving average kernel and its discretisation. It will also cover cases with sufficiently small transaction costs. The Gaussian law has full support, cf. the no-arbitrage results of Guasoni and coauthours [4], [5], and it is essential for arbitrage in our discrete-time setting – both with and without transaction costs – that the downside be bounded. We give sufficient conditions for the existence of an arbitrage opportunity or a free lunch with vanishing risk (FLVR) of the form wait for a possible time where we buy, and then sell next period; we also give sufficient conditions for weak convergence of the discrete random walks to the Gaussian continuous-time counterparts. Our main contributions in substance are summarised:

- We cover a fairly general class of Gaussian processes, and give examples and counterexamples to the existence of arbitrage opportunities / FLVRs of the abovementioned form.
- Our examples cover both cases where arbitrages are removed, and created, by discretisation. In particular, we argue that for the fBm case treated by [9], the limit transition leads to one arbitrage opportunity vanishing and one emerging.
- We cover any negative drift term (a word which should be interpreted cautiously for non-semimartingales) without shorting, as it turns out that the instantaneous growth from the noise term can tend to infinity.
- For the same reason the arbitrage may also admit sufficiently small *transaction costs*.
- We do not have to assume the discretised market to be binary (hence complete if arbitrage-free) with innovations $\in \{-1, 1\}$. Some regularity on the bound of the support will suffice.
- The weak convergence of the driving noise is likewise shown in this more general setting.

It turns out essential for the free lunch results (given in Section 2) that the innovations of the random walk are bounded, and therefore the weak convergence result of Theorem 1.1 is restricted to this case.

1 The continuous-time and discrete-time market models

Our market has one «safe» asset, taken as numéraire and normalised to price = 1, and one «risky» asset $S^{(n)}$, which for each n is a discretisation of a continuously evolving stochastic process S (referred to as $n = \infty$). S will be constructed from a drift process $A = \int^t a(s)\mathrm{d}s$ and a driving noise Z, assumed to be a Gaussian moving average process with an adapted (hence upper limit of integration is t) kernel representation

$$\begin{aligned} Z(t) &= \int_{-\infty}^{t_0} K(t,s)\mathrm{d}W(s) + \int_{t_0}^{t} K(t,s)\mathrm{d}W(s) \\ &= J(t) + \int_{t_0}^{t} K(t,s)\mathrm{d}W(s), \end{aligned} \tag{1}$$

with respect to standard Brownian motion W, where K is a given deterministic (finite) function which we extend to $s \geq t$ by defining it to be zero there. For example, a fairly large class of moving average processes, including fractional Brownian motion as well as all Gaussian semimartingales, admit a representation of the form $K(t,s) = L(t-s)$ for $t > 0$, and we shall give special attention to these later.

The agent is supposed to enter the market at time t_0, which is the motivation for the splitting there. We might choose to discretise W on the entire time line; however, J will merely enter as a drift term, and we can equally well discretise J directly. We shall choose to do the latter, and results in terms of the former approach will follow analogously. Hence we start by discretising W for $t > t_0$ by replacing its increments $n^{1/2} \cdot (W(\frac{i+1}{n}) - W(\frac{i}{n}))$ by mutually independent random variables $\xi_{i+1} = \xi_{i+1}^{(n)}$, and put

$$Z^{(n)}(t) = J(\tfrac{\lfloor nt \rfloor}{n}) + \sum_{i=\lfloor nt_0 \rfloor}^{\lfloor nt \rfloor - 1} K(\tfrac{\lfloor nt \rfloor}{n}, s_i^{(n)}) \cdot n^{-1/2} \xi_{i+1}^{(n)} \tag{2}$$

$$\text{where} \qquad s_i^{(n)} = t_0 + \tfrac{i - \lfloor nt_0 \rfloor}{n} \tag{3}$$

where $\lfloor \cdot \rfloor$ is the floor function (the integer part). We remark that the choice of $s_i^{(n)}$ is done for simplicity, and could be done more generally, as long as $s_i^{(n)} - t_0 + \frac{\lfloor nt_0 \rfloor}{n} \in [\frac{i}{n}, \frac{i+1}{n})$ – at this stage; we shall however soon use the assumption (3) to simplify notation for the other time arguments in (2) as well. Now define for $n < \infty$

$$A(t) = \int_{t_0}^{t} a(s)\mathrm{d}s, \qquad\qquad A^{(n)}(t) = \frac{1}{n} \sum_{i=\lfloor nt_0 \rfloor}^{\lfloor nt \rfloor - 1} a(s_i^{(n)}) \tag{4}$$

and assume that for all $t \geq t_0$, S and $S^{(n)}$ are given by

$$S = G(A + Z) \qquad\qquad S^{(n)} = G(A^{(n)} + Z^{(n)}) \tag{5}$$

$$\text{for some strictly increasing function } G. \tag{6}$$

The canonical choice is G to be the exponential function, but we shall not need this specific property; for one result we will however use convexity, and for weak convergence we shall need continuity. We shall later see that it is essential for Theorem 2.4 that the ξ_i be bounded, and such an assumption will give rise to a simple proof for the following theorem:

1.1 Theorem: Weak convergence

Let Z be given by (1) with

$$\sup_{t\in[t_0,t_0+T]}\int_{t_0}^{t_0+T}\big((K(t,s))^2+(K_1'(t,s))^2\big)\mathrm{d}s<\infty \tag{7}$$

and $Z^{(n)}$ by (2) with

$$\mathrm{E}[\xi_i]=0=1-\mathrm{E}[\xi_i^2]$$

and such that the ξ_i have support contained in some common interval $[-\check{M},\hat{M}]$. Then on $[t_0,t_0+T]$, $Z^{(n)}$ converges weakly to Z, and for continuous G also $S^{(n)}$ to S.

Proof. See the Appendix. □

For the discrete-time markets, we shall restrict ourselves to the following set of strategies:

1.2 Definition:

Let $n<\infty$ be given, and consider the natural filtration generated by $S^{(n)}$. For any natural m, an *m-period strategy* consists of waiting until some stopping time $t_*=s_{i_*}^{(n)}$, where $i_*\geq j_0:=\lfloor nt_0\rfloor$, buying a predictable positive number b of units, holding these until a stopping time $t^*=s_{i^*}^{(n)}$ where $i^*\in(i_*,i_*+m]$ and then selling all b units. We shall refer to the case $m=1$ as the *single period case*.
The *net return* from this transaction is

$$R=R_{i_*,i^*}:=b\cdot(S^{(n)}(t^*)-S^{(n)}(t_*))-(\lambda\Lambda_*+\lambda\Lambda^*) \tag{8}$$

where $\lambda\Lambda_*$ and $\lambda\Lambda^*$ are the respective transaction costs for buying and selling. △

The reason for the «λ» parameter is that we will consider the properties for small transaction costs, and it will be convenient to scale by a number. The main results will be carried out under the assumption of fixed transaction costs. Proposition 2.3 will show that this is sufficiently general, but preliminarily, the Λ_*, Λ^* will be more general:

1.3 Assumption/notation:

- The family $\{\xi_{j+1}^{(n)}\}_j$ will be assumed independent.
- λ will be a number ≥ 0.
- $\Lambda_*=\Lambda_*(b,S^{(n)}(t_*))$ and $\Lambda^*=\Lambda^*(b,S^{(n)}(t_*),S^{(n)}(t^*))$ will be nonnegative functions, bounded in $(S^{(n)}(t_*),S^{(n)}(t^*))$.
- We shall use the term «transaction cost λ» to imply that $\Lambda_*+\Lambda^*=1$, and we shall refer to «the simple model (9)» to refer to the single period case of transaction cost λ where G is the identity.

△

We shall focus on the simple model, where the return (on the event $\{i_* < \infty\}$) will be

$$S^{(n)}(t_0 + \tfrac{i_*+1-\lfloor nt_0 \rfloor}{n}) - S^{(n)}(t_0 + \tfrac{i_*-\lfloor nt_0 \rfloor}{n}) - \lambda$$
$$= y^{(n)}_{i_*+1} + x^{(n)}_{i_*+1} - \lambda, \qquad \text{where} \tag{9a}$$
$$y^{(n)}_{j+1} = \frac{1}{n}\Big\{a(s^{(n)}_j) + J(s^{(n)}_{j+1}) - J(s^{(n)}_j)\Big\} \tag{9b}$$
$$x^{(n)}_{j+1} = \frac{1}{\sqrt{n}}\Big\{K(s^{(n)}_{j+1}, s^{(n)}_j)\xi^{(n)}_{j+1} + \sum_{i=j_0}^{j-1}\big(K(s^{(n)}_{j+1}, s^{(n)}_i) - K(s^{(n)}_j, s^{(n)}_i)\big)\xi^{(n)}_{i+1}\Big\} \tag{9c}$$

adopting the convention that the empty sum, corresponding to $j = j_0$ $(= \lfloor nt_0 \rfloor)$, is zero.

An arbitrage opportunity – to be precisely defined in the next section – occurs if for some bounded $i_* \in [j_0, \infty)$, given the information available then, the transaction costs plus the worst-case possible downside from the innovation $K(s^{(n)}_{i_*+1}, s^{(n)}_{i_*})\cdot\xi^{(n)}_{i_*+1}$ will be fully compensated by the contribution from the dependence of the past (the remaining terms), provided that the event of this happening at j has positive probability at the initial time step j_0. The next section will give conditions for FLVR and arbitrage – obviously there will not be any arbitrage opportunities if $K(s^{(n)}_{j+1}, s^{(n)}_j) \cdot \xi^{(n)}_{j+1}$ has full support for all $j \geq j_0$.

2 Free lunches: sufficient conditions and examples

Starting out with the definitions from the previous section, we now define arbitrage opportunities (as usual) and FLVRs under our admissibility conditions. We shall for simplicity only consider the strictest (L^∞) FLVR definition; in the more general setting, one would replace the ess sup in (10) (and (11)) below, by the L^p norm of the negative part of the return.

2.1 Assumption/notation:

Let us use the term «j-measurable» to mean measurable at step j, i.e. at time $s^{(n)}_j$, and write $\mathrm{P}_j = \mathrm{P}^{(n)}_j$ for the probability measure conditional on the filtration generated up to time $s^{(n)}_j$. In particular, P_0 is conditional on the information available at the time t_0 where the agent enters the market (this information is then most recently updated at time $\lfloor nt_0 \rfloor/n$). We always assume that this information set is given.

Since ξ_j is independent of the past, we shall suppress the dependence of law in terms like e.g. ess $\sup_{\xi_j}$ which is taken to be mean the supremum over the (P_j-)essential support of ξ_j. We shall use the symbol $\gneqq$ to mean «no smaller than and not a.s. equal». △

Informally, we have a free lunch with vanishing risk if we can obtain an arbitrarily small downside to mean return ratio, and an arbitrage opportunity if one can have positive-mean return without downside. The most natural norm to quantify downside is the most restrictive one, namely the essential supremum. The following definition appears notationally a bit cryptical, but will under m-period strategies coincide with the conventional definition of FLVR and arbitrage:

2.2 Definition:

Fix n, m both $< \infty$. Consider the condition

$$\frac{\text{ess sup}^{(\mathrm{P}_j)}\ [-R_{j,i^*}|D_j]}{1 \wedge \mathrm{E}[R_{j,i^*}|D_j]} \le \delta. \tag{10}$$

The market is said to admit an *free lunch with vanishing risk* («FLVR») if for every $\delta > 0$ there exist $i^* \ge j = j_\delta \ge j_0$ (integers, and where $i^* \le j + m$ allowed to be a stopping time) and a j-measurable event D_j such that $\mathrm{P}_0[D_j] > 0$, and such that (10) holds. The market is said to admit an *arbitrage opportunity* if the FLVR definition holds also for $\delta = 0$. △

Notice that in the simple model (9), one can replace the LHS of (10) by

$$\frac{-\operatorname*{ess\ inf}_{\xi_{j+1}} [x_{j+1} + y_{j+1} - \lambda|D_j]}{1 \wedge \mathrm{E}[x_{j+1} + y_{j+1} - \lambda|D_j]}. \tag{11}$$

For the purpose of giving *sufficient* conditions for arbitrage under *small* transaction cost – which is the main object of this section – the simple model turns out fairly close to general:

2.3 Proposition: Free lunches in the simple model (9) vs. in the full model

Fix $b > 0$, $n < \infty$. Assume that for each j, we have

$$\xi_j \operatorname{sign} K(s_j^{(n)}, s_{j+1}^{(n)}) \text{ upper bounded,} \qquad \textit{or} \tag{12a}$$

$$\Lambda^* \text{ of at most linear growth wrt. the last variable (the selling price)} \tag{12b}$$

Then there is arbitrage for sufficiently small λ, provided that so is the case in the simple model (9). If G is convex, then there is FLVR for sufficiently small λ, provided that so is the case in the simple model (9).

Proof. The proof is less interesting, and is relegated to the Appendix. □

From this setup, the strategies described informally at the end of the previous section, immediately give the first part of the following sufficient conditions:

2.4 Theorem: Sufficient conditions for free lunches (I)

Consider the simple model (9).

(a) Fix $n < \infty$. If for some natural $j \ge j_0$ we have

$$\operatorname*{ess\ inf}_{\xi_{j+1}} \ \operatorname*{ess\ sup}_{\{\xi_i\}_{i=j_0+1,\dots,j}} \{x_{j+1} + y_{j+1}\} \ge \bar\lambda \ge 0 \tag{13}$$

we have an arbitrage opportunity for all transaction costs $\lambda \in [0, \bar\lambda)$. Furthermore, we have arbitrage opportunity for transaction cost $\bar\lambda$ if in addition there is a point probability at the ess sup and $K(s_{j+1}^{(n)}, s_j^{(n)})\ \xi_{j+1}$ is non-degenerate.

(b) Fix a $T > 0$. Assume that at $t_0 + T$ we have both $K(\,\cdot\,, s)$ differentiable $(s, t_0 + T]$ and J Hölder continuous[1] with exponent $\alpha > 1/2$. Put

$$\eta^{(n)}(s) = \operatorname*{ess\ sup}_{\xi_{\lfloor ns+1 \rfloor}} \{\xi_{\lfloor ns+1 \rfloor} \cdot \operatorname{sign}(K_1'(t_0 + T, s))\}.$$

Then if

$$\liminf_n \operatorname*{ess\ inf}_\omega K(t_0 + T + 1/n, s^{(n)}_{\lfloor j_0 + Tn \rfloor})\xi_{Tn+1} + \int_{t_0}^{t_0+T} |K_1'(t_0 + T, s)|\eta(s)\mathrm{d}s > 0 \quad (14a)$$

– for which it is sufficient that

$$\liminf_n \tfrac{\text{ess inf}\,\xi_n}{\text{ess sup}\,\xi_n} > -\infty, \quad K(t_0 + T^+, t_0 + T) = 0, \quad \text{and } K_1' \text{ not identically zero} \quad (14b)$$

– there is an arbitrage opportunity for all sufficiently small transaction costs within $\lfloor Tn \rfloor$ steps, whenever n is sufficiently large. If (14a) holds with $\liminf_n$ replaced by $\limsup_n$, we have arbitrage opportunity for all sufficiently small transaction costs for *some* sufficiently large n.

(c) Let $\lambda = 0$. Assume that there are sequences $\{j_k\}$, $\{n_k\}$ with $j_k \geq \lfloor n_k t_0 \rfloor$, such that

$$\lim_k \operatorname*{ess\ sup}_{\xi_{j_k+1}} \operatorname*{ess\ inf}_{\xi_{j_0},\ldots,\xi_{j_k}} x^{(n_k)}_{j_k+1} + y^{(n_k)}_{j_k+1} = 0 \quad (15)$$

$$\lim_k \int_{t_0}^{j_k/n_k} |K_1'(j_k/n_k, s)|\eta(s)\mathrm{d}s + \operatorname*{ess\ inf}_\omega K(j_k/n_k, s^{(n_k)}_{j_k})\xi^{(n_k)}_{j_k+1} \geq 0 \quad (16)$$

$$\liminf_k \left| K(s^{(n_k)}_{j_k+1}, s^{(n_k)}_{j_k}) \Big(\mathrm{E}[\xi^{(n_k)}_{j_k+1}] - \operatorname*{ess\ inf}_\omega \xi^{(n_k)}_{j_k+1} \Big) \right| \in (0, \infty) \quad (17)$$

and at each j_k/n_k we have both $K(\,\cdot\,, s)$ differentiable and J Hölder continuous with exponent $\alpha > 1/2$. Then there is a FLVR.

(d) There are infinite-variation semimartingales Z, equalling weak limits of their discretisations $Z^{(n)}$ formed by i.i.d. bounded ξ_i with zero mean and unit variance, for which part (b) applies.

Proof. Part (a) is trivial. For part (b), we first observe that (14a) follows from (14b). The sum part of (9c) will tend to the integral in (14a), the drift term is of order $1/n$ and the Hölder regularity yields $y_j^{(n)} = o(n^{-1/2})$. In the limit we would – on the (possibly null) set of all ξ's attaining values on the boundaries of their supports – get a return of the order $o(n^{-1/2}) - \lambda + n^{-1/2} \cdot$[LHS of (14a)]. With positive probability, one is within a factor $1 - \epsilon$ of the essential extrema, and if (14a) holds, then the return would still exceed any small enough λ for all large enough n. The ess sup case – some large enough n – follows by considering a suitable subsequence.

For part (c), we observe that (15) merely grants that the numerator of (11) tends to 0; we can assume it to converge from below, otherwise there is nothing to prove. Then we rewrite the ratio $-$ ess inf /E into (E $-$ ess inf)/E $- 1$; for each $j = j_k$, $n = n_k$, we then get

$$\frac{K(s^{(n)}_{j+1}, s^{(n)}_j)\Big(\mathrm{E}[\xi^{(n)}_{j+1}] - \text{ess inf}_\omega\, \xi^{(n)}_{j+1}\Big)}{K(s^{(n)}_{j+1}, s^{(n)}_j)\mathrm{E}[\xi^{(n)}_{j+1}] + \text{ess sup}_\omega \sum_{i=j_0}^{j-1} \big(K(s^{(n)}_{j+1}, s^{(n)}_i) - K(s^{(n)}_j, s^{(n)}_i)\big)\xi^{(n)}_{i+1} + n^{1/2} y^{(n)}_{j+1}} - 1.$$

[1]strictly speaking, we need only bound the «downside»; for simplicity, we skip the details

At the limit, the y part will vanish and the sum tend to the integral in (16), by which numerator and denominator tends to the same number Q – which cancels out, as Q is finite and nonzero by (17).

The weak convergence of part (d) is Theorem 1.1. For the rest, we shall for simplicity assume $t_0 = 0$ and note from [3] Theorem 3.9, that Z is a semimartingale if there exists a $\psi \in \mathrm{L}^2$ and a $v \in \mathbf{R}$ such that $K(t,s) = v + \int_0^{t-s} \psi(u)\mathrm{d}u$, and $J(t) = \int_{-\infty}^{0} \int_{-s}^{t-s} \psi(u)\mathrm{d}u\mathrm{d}W(s)$, and that Z has finite variation iff $v = 0$. Then part (b) will apply as long as ψ is sufficiently regular at T, and «large enough»: For the integral in (14a), we note that by the i.i.d. assumption we can choose η constant, equal to some ξ, and $\int_0^T K_1'(T,s)\mathrm{d}s = \int_0^T \psi(T-s)\mathrm{d}s = \int_0^T \psi(s)\mathrm{d}s$, while $K(T+1/n, T) = v + \int_0^{1/n} \psi(s)\mathrm{d}s$. Clearly, if the ξ are bounded and ψ is either a positive function or a negative, then (14a) holds for $v > 0$ small enough. Indeed, if the support of the ξ is symmetric and the worst-case scenario is $\xi_{j+1} = -\xi$, it suffices that

$$0 < v < \int_0^T |\psi(s)|\mathrm{d}s. \tag{18}$$

□

Remark. The conditions in part (a) are also necessary within the simple model (9). △

So the proof of point (b) gives us conditions for when there is a positive set – namely the event that all $\pm\xi_i$ (the sign determined by K_1') up to time T fall within $(1-\epsilon)$ of their respective essential suprema, on which we can buy at time T, sell the next period and even in the (ω-) worst case for ξ_{Tn+1}, still gain nonnegative profit even with transaction costs up to λ. Indeed, the noise term gives rise to an infinite instantaneous growth rate, knocking out the effect of any negative drift, giving rise to arbitrage opportunity even with sufficiently small transaction costs.

The analysis will be simplified if

$$\{\xi_i\} \text{ i.i.d. with} \qquad \check{M} := -\operatorname{ess\,inf} \xi_i \quad \text{and} \quad \hat{M} := \operatorname{ess\,sup} \xi_i \quad \text{both} \quad \in (0,\infty) \tag{19a}$$

$$\begin{gathered} K(t,s) = L(t-s), \quad t > s > t_0, \qquad \text{with } L \text{ not constant} \\ \text{and such that } L(\infty) \text{ and } \ell := L(0^+) \text{ both exist (possibly infinite)} \end{gathered} \tag{19b}$$

This form of K covers a wide class of processes, including semimartingales (cf. the proof of Theorem 2.4 d) and also ones to be covered in the below examples (the assumption that L be non-constant merely rules out the ordinary Brownian motion case, which will not yield free lunches). Under conditions (19), the sum in (9c) can telescope for suitable outcomes $\xi_i = \pm M_0(1-\epsilon)$ or better, $(i \geq j)$, yielding

$$x_{j+1}^{(n)} \geq \frac{1}{\sqrt{n}} L(\tfrac{1}{n})\xi_{j+1}^{(n)} + \frac{1}{\sqrt{n}} \big| L(\tfrac{j+1-\lfloor nt_0 \rfloor}{n}) - L(\tfrac{1}{n})\big| \cdot M_0 \tag{20}$$

where in general, we can choose arbitrary $M_0 \in [0, \min\{\check{M}, \hat{M}\}]$; however we can choose any $M_0 = \hat{M}$ if L is nondecreasing, and $= \check{M}$ if L nondecreasing.

In addition to condition (19a), it would also be natural from an approximation point of view to assume zero mean. However, that is not essential to the following:

2.5 Theorem: Sufficient conditions for free lunches (II)

Consider the simple model (9). Assume eqs. (19) to hold and J Hölder continuous with exponent $\alpha > 1/2$. Put $\ell = L(0^+)$ (finite or infinite) and, whenever well-defined, $c = \big|L(\infty)/\ell - 1\big| \cdot M_0 + \text{ess inf}(\xi \,\text{sign}\, \ell)$.

(a) If $|\ell| \in (0, \infty]$ then

- there is an arbitrage opportunity with small enough $\lambda = \lambda_n > 0$, provided that $|L(\infty)| = +\infty$ or $c > 0$;
- there is a FLVR with $\lambda = 0$, all n large enough, provided that $c = 0$;

(b) If $\ell = 0$ and $\lim_n |n^{\alpha-1/2} L(\frac{1}{n})| \in (0, \infty]$, then there is an arbitrage opportunity with small enough $\lambda = \lambda_n > 0$.

Proof. Obtainable up to j is at least

$$\frac{x_{j+1}\sqrt{n}}{|L(1/n)|}, \text{ which} \geq \Big|\frac{L(\frac{j+1-\lfloor nt_0 \rfloor}{n})}{L(\frac{1}{n})} - 1\Big| \cdot M_0(1-\epsilon) + \xi_{j+1} \,\text{sign}\, L(1/n), \tag{21}$$

any $\epsilon > 0$. Taking $j \to \infty$, the right-hand side of (21) will be > 0 for all large enough j, n if c exists and is > 0; otherwise, it suffices that $L(\infty)$ infinite or $\ell = 0$. In all cases, our conditions grant that the $n^{-1/2}$ term dominates as j, n grow, and yields return large enough to cover some positive (n-dependent) transaction cost.
It remains to prove the FLVR part of (a), assuming $L(\infty)$ finite. The numerator and denominator in (11) both tend to 0, and we can rewrite the reciproke of the ratio as

$$-1 - \frac{\text{E}[\xi_{j+1} \,\text{sign}\, L(1/n)] - \text{ess inf}_\omega(\xi_{j+1} \,\text{sign}\, L(1/n))}{\Big|\frac{L(\frac{j+1-\lfloor nt_0 \rfloor}{n})}{L(\frac{1}{n})} - 1\Big| \cdot M_0(1-\epsilon) + \text{ess inf}_\omega(\xi_{j+1} \,\text{sign}\, L(1/n)) + \frac{\sqrt{n}}{L(\frac{1}{n})} y_{j+1}^{(n)}} \tag{22}$$

Choose a sequence $\epsilon_{j,k} \searrow 0$. Hölder regularity grants that y term vanishes and thus the denominator tends to c, which is assumed zero. The numerator is nonzero since ξ are nondegenerate and $\ell \neq 0$. □

Remark. Notice that the statement of Theorem 2.5 is not dependent on t_0. From the setup of Theorem 2.4, we have not ruled out that there is an arbitrage opportunity at t_0 when the agent enters the market, but that the event of free lunch fails once and for all – or put otherwise, an agent who enters market one period too late, might learn that one had an arbitrage opportunity which did not materialise «yesterday, but now it is gone forever». On the other hand, when Theorem 2.5 applies, the agent can wait for lunch time, and if the market develops unfavorable there will *always* be a positive probability that the free lunch will materialise at some later j. △

We will in the following discuss a few cases.

2.6 A few cases: examples and non-examples

(a) Sottinen [9] considers fractional Brownian motion with Hurst parameter $H > 1/2$, using the representation $K(t, s) = \int_s^t (u/s)^{H-1/2}(u - s)^{H-3/2}\mathrm{d}u$ (up to an irrelevant positive constant), so that $K(t^+, t) = 0$ and K_1' is positive. In [9] the representation uses only the positive time path of the Wiener process, but our setup covers the case where we enter the market later, yielding $J(t) = \int_{-\infty}^{t_0}\int_s^t (u/s)^{H-1/2}(u - s)^{H-3/2}\mathrm{d}u\mathrm{d}W(s)$. Then J is differentiable at T, so that Theorem 2.4 part (b) applies as long as the ξ_i are bounded and their supports obey some common bound away from zero and infinity. The result holds regardless of the history from t_0 and up to the moment we enter the market, generalising [9].

We can also get an idea of how bad the distributions of the ξ_i must be chosen in order to violate the result. If each ξ_i is a binary variable, then in order to satisfy $\mathrm{E}[\xi_i] = 0 = 1 - \mathrm{E}[\xi_i^2]$, the support must be of the form $\{-\gamma_i^{-1}, \gamma_i\}$. Choosing $\gamma_i = (i - 1)^{-r}$ and hence $\eta(s) = (ns)^{-r}$ and the worst-case value for ξ_{Tn+1} equal to $-(nT)^r$, and observing that $K(T + 1/n, T)$ is $O(n^{-H+1/2})$, we see that although the left hand side of (14a) is zero, it converges in a controlled manner: the return will be

$$o(1/n) + \big[[\text{positive constant}] - O(n^{2r-(H-1/2)})\big] - \lambda,$$

and so it suffices that $2r < H - 1/2$.

(b) Maybe a more common representation for fractional Brownian motion (for any $H \neq 1/2$) is – up to a constant –

$$Z_t = \int_{-\infty}^t \big[(t - s)^{H-1/2} - \max\{0, -s\}^{H-1/2}\big]\mathrm{d}W(s),$$

corresponding to $J(t) = \int_{-\infty}^0 \{(t + |s|)^{H-1/2} - |s|^{H-1/2}\}\mathrm{d}W(s) + \int_0^{t_0} L(t - s)\mathrm{d}W(s)$ with $K(t, s) = L(t - s) = (t - s)^{H-1/2}$. Let us assume that the ξ_i are i.i.d. with bounded support; L is monotoneous, so then conditions (19) hold. In addition, J is Lipschitz. Now the results are different for positively and negatively correlated fBm:

- In the case $H > 1/2$, L positive and decreasing to zero. Theorem 2.5 part (b) applies.
- In the case $H < 1/2$, L is positive and tends to ∞. The parameter c vanishes, so we obtain FLVR. Inspecting the calculations for this particular choice of L, we see that there is no arbitrage opportunity.

(c) In [7], Rogers proposes a modification of fractional Brownian motion, in order to eliminate the arbitrage but preserving the long run memory properties which motivated the use of fBm in finance in the first place. Rogers gives a specific (monotone) example

$$L(t - s) = k((t - s)^2 + \epsilon)^{(H-\frac{1}{2})/2}, \tag{23}$$

but suggests more generally to choose L such that $L(0) = 1$, $L'(0) = 1$, and has the same $\sim (t - s)^{H-1/2}$ behaviour for large t. Assuming i.i.d. innovations with symmetric support, Theorem 2.5 part (a) yields, as for fBM, arbitrage opportunity for $H > 1/2$ (since $L(\infty) = \infty$), and FLVR For $H < 1/2$ (since $c = 0$ by the symmetry of the supports). .

(d) Ordinary Brownian motion has both upside and downside regardless of history, and the discretised market is easily seen to be arbitrage-free. However, a mix of fractional Brownian motion (where $H > 1/2$) with sufficiently small ordinary Brownian noise, will admit arbitrage under suitably wide conditions: Consider (14a) and assume the ξ_i iid, so that the integral – to which only the fBm part contributes, as the Brownian part has constant kernel k – becomes $\hat{M} \int_0^T K_1'(T,s)\mathrm{d}s$. The ξ_{j+1} coefficient, on the other hand, becomes k since the fBm contribution vanishes in the limit. So there is an arbitrage opportunity as long as the amount k of ordinary Brownian noise is less than $\int_0^T K_1'(T,s)\mathrm{d}s \cdot \hat{M}/\check{M}$. This in contrast with the limiting continuous case, where for $H > 3/4$ the mix is not only arbitrage-free for any positive level of ordinary Brownian volatility, but also the fBm part takes the rôle as drift and does not appear in the Black–Scholes call option price – see [2]. The clear cut at 3/4 does not carry over to the discretised case, while the bordering level of the Brownian volatility plays no part in the continuous case.

(e) The Ornstein–Uhlenbeck process admits the representation $L(t) = e^{-\theta t}$ with $\theta > 0$. It has $L(0^+) = 1$ and $L(\infty) = 0$, and the discretisation with i.i.d.'s with symmetric support does therefore admit FLVR from Theorem 2.5 part (a). Note that $0 < e^{-\theta(t-s)} = 1 + \int_0^{t-s}(-\theta)e^{-\theta r}\mathrm{d}r$ so that $v = 1$ is precisely large enough for (18) to fail; indeed, one might check that it will not admit arbitrage opportunities.

Concluding remarks

We have seen that discrete-time random walk markets may behave fairly different from their weak limits, as former may admit arbitrage opportunities or FLVRs which vanish in the limit even for semimartingales like the Ornstein–Uhlenbeck process. A careful note here is appropriate: while there are arbitrage opportunities for e.g. both fractional Brownian motion with $H > 1/2$ and the discretised counterpart, they are not the same: as pointed out by Rogers [7], a short-memory modification «semimartingalises» fBm, but in the discretised case the arbitrage opportunity is due to the (originally desirable) *long* memory. Hence, in the limit transition, one arbitrage vanishes and another is created. Furthermore, threshold levels seem to lose significance, like the mix between ordinary and fractional Brownian motion, where the arbitrage opportunity vanishes at a certain level of ordinary Brownian noise in the discretised case and for $H > 3/4$ in the continuous case.

At the end of the day, these results underline the need for caution in choice of models of financial markets, choice of continuous vs. discrete time modelling, and in discretisation for numerical analysis. The flaws of an approach where prices alone are discretised this way, are serious enough to be of practical significance. Arguably, a practitioner should be worried even at far less radical modeling issues than distorted *arbitrage* properties – maybe even more so for less radical artifacts, which might not be as easily detected and corrected.

Appendix: two remaining proofs

Proof of Theorem 1.1. The drift and the already occured part will represent no issue, so let us assume $A = A^{(n)} = J = 0$. Also, we can assume without loss of generality, that $t_0 = 0$ and observe that weak limits commute with continuous functions G.

Convergence in finite-dimensional distributions follows like in [9], Theorem 1, by noting that the explicit form of his kernel is not essential to the proof – only the convergence of the variance is, and (7) suffices. To prove tightness, we want to apply [1], Theorem 12.3, for which it suffices to find constants $\gamma \geq 0$ and $\alpha > 1$ and a nondecreasing continuous function F so that for all $t > u$, all positive θ, we have

$$\mathrm{P}[|Z^{(n)}(t) - Z^{(n)}(u)| \geq \theta] \leq \theta^{-\gamma}|F(t) - F(u)|^{\alpha} \tag{24}$$

Now

$$Z^{(n)}(t) - Z^{(n)}(u) = \sum_{i=0}^{\lfloor nt \rfloor - 1} \left(K(\tfrac{\lfloor nt \rfloor}{n}, s_i^{(n)}) - K(\tfrac{\lfloor nu \rfloor}{n}, s_i^{(n)})\right) \cdot n^{-1/2}\xi_{i+1}^{(n)} =: \sum_{i=0}^{\lfloor nt \rfloor} Y_i^{(n)}. \tag{25}$$

Since for each n, the Y_i are bounded, zero-mean, independent variables, we can use [6], Theorem 2, to obtain that for any $\theta > 0$,

$$\mathrm{P}[|\sum_{i=0}^{\lfloor nt \rfloor} Y_i| \geq \theta] \leq 2\exp\{-2\theta/\Gamma_n\} \tag{26}$$

where $\Gamma_n = \sum_{i=0}^{\lfloor nt \rfloor}(\text{ess sup}\, Y_i - \text{ess inf}\, Y_i)^2$. It therefore suffices to show that we can find F, α and γ so that $\theta^{\gamma}\exp\{-2\theta/\Gamma_n\} \leq \frac{1}{2}|F(t) - F(u)|^{\alpha}$ for all $\theta > 0$, all n. By inserting for $\theta = \gamma\Gamma_n/2$ which maximises the left hand side, this is equivalent to

$$\Gamma_n \leq \frac{e}{\gamma}2^{1-1/\gamma}|F(t) - F(u)|^{\alpha/\gamma}, \qquad \text{all } n$$

To establish such a bound, we use the boundedness of supports to obtain

$$\begin{aligned}
\Gamma_n &\leq Cn^{-1}\sum_{i=0}^{\lfloor nT \rfloor}\left(K(\tfrac{\lfloor nt \rfloor}{n}, s_i^{(n)}) - K(\tfrac{\lfloor nu \rfloor}{n}, s_i^{(n)})\right)^2 \\
&\to C\int_0^T (K(t,s) - K(u,s))^2 \mathrm{d}s \\
&= C\int_0^T (\int_u^t K_1'(r,s)\mathrm{d}r)^2 \mathrm{d}s \\
&\leq C \sup_{r \in [0,T]} \int_0^T (K_1'(r,s))^2 \mathrm{d}s \cdot (t-u)^2.
\end{aligned}$$

where C is a suitable constant. By (7), the $(t-u)^2$ coefficient is finite, and we are done. □

Proof of Proposition 2.3. Put $b = m = 1$. Denote the buying and selling prices in the simple model (9) by $\acute{\zeta}$ and $\grave{\zeta}$, and in the full model by $\acute{S} = G(\acute{\zeta})$ and $\grave{S} = G(\grave{\zeta})$. Observe first that the we may take $\acute{\zeta}$ bounded by restricting D_j without voiding the property $\mathrm{P}_0[D_j] > 0$, and we will do so in the following. Assume that the simple model (9) has arbitrage for transaction cost $c > 0$; then

$$\grave{\zeta} \geqslant \acute{\zeta} + c$$

which by applying G and rearranging, is equivalent to

$$\grave{S} - \acute{S} - \lambda(\Lambda^*(1, \acute{S}, \grave{S}) + \Lambda_*(1, \acute{S})) \geqslant G(\acute{\zeta} + c) - G(\acute{\zeta}) - \lambda(\Lambda^*(1, G(\acute{\zeta}), G(\grave{\zeta})) + \Lambda_*(1, G(\acute{\zeta})))$$

so we have arbitrage if the right hand side is nonnegative, so it suffices that

$$0 < \lambda \leq \frac{G(\acute{\zeta} + c) - G(\acute{\zeta})}{\Lambda^*(1, G(\acute{\zeta}), G(\grave{\zeta})) + \Lambda_*(1, G(\acute{\zeta}))} \tag{27}$$

As already remarked, we can assume $\acute{\zeta}$ bounded, so it suffices to bound Λ^* for given $\acute{\zeta}$. If (12a) holds, then also $\grave{\zeta}$ is bounded. Assume therefore (12b), i.e. $\Lambda^*(1, \acute{S}, \grave{S}) \leq \lambda_0(\acute{S}) + \lambda_1(\acute{S}) \cdot \grave{S}$, where λ_0, λ_1 are locally bounded functions of $\acute{S}$. Then the return is

$$\begin{aligned} Y &= \grave{S} - \acute{S} - \lambda(\Lambda^*(1, \acute{S}, \grave{S}) + \Lambda_*(1, \acute{S})) \\ &\geq \grave{S}(1 - \lambda\lambda_1(\acute{S})) - \acute{S}(1 - \lambda\lambda_1(\acute{S})) - \lambda\lambda_1(\acute{S})\acute{S} - \lambda(\Lambda_* + \lambda_0) \end{aligned}$$

By boundedness of $\acute{S}$ we can take $\lambda\lambda_1(\acute{S}) < 1$, in which case the return will be $\geqslant 0$ if

$$\grave{S} - \acute{S} \geqslant \lambda \frac{\lambda_0 + \lambda_1 \acute{S} + \Lambda_*}{1 - \lambda\lambda_1(\acute{S})}$$

which at least equals $\lambda \cdot 2(\lambda_0 + \lambda_1 \acute{S} + \Lambda_*) =: \lambda\tilde{\Lambda}$, where $\tilde{\Lambda}$ is a locally bounded function of $\acute{S}$ alone. Hence we can consider the problem with $\tilde{\Lambda}$ in place of Λ_* and assuming $\Lambda^* = 0$, and then the right-hand side of (27) will not depend on the selling price $\grave{\zeta}$. We are done with the arbitrage part of the proposition.

For FLVR, it suffices to point out that Jensen's inequality improves both the upside and downside for convex G, compared to for linear ones. □

References

[1] Patrick Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons Inc., New York, first edition, 1968.

[2] Patrick Cheridito. Mixed fractional Brownian motion. *Bernoulli*, 7(6):913–934, 2001.

[3] Patrick Cheridito. Gaussian moving averages, semimartingales and option pricing. *Stochastic Process. Appl.*, 109(1):47–68, 2004.

[4] Paolo Guasoni. No arbitrage under transaction costs, with fractional Brownian motion and beyond. *Math. Finance*, 16(3):569–582, 2006.

[5] Paolo Guasoni, Miklós Rásonyi, and Walter Schachermayer. Consistent price systems and face-lifting pricing under transaction costs. *Ann. Appl. Probab.*, 18(2):491–520, 2008.

[6] Wassily Hoeffding. Probability inequalities for sums of bounded random variables. *J. Amer. Statist. Assoc.*, 58:13–30, 1963.

[7] L. C. G. Rogers. Arbitrage with fractional Brownian motion. *Math. Finance*, 7(1):95–105, 1997.

[8] Albert N. Shiryaev. *Essentials of stochastic finance*, volume 3 of *Advanced Series on Statistical Science & Applied Probability*. World Scientific Publishing Co. Inc., River Edge, NJ, 1999. Facts, models, theory, Translated from the Russian manuscript by N. Kruzhilin.

[9] Tommi Sottinen. Fractional Brownian motion, random walks and binary market models. *Finance Stoch.*, 5(3):343–355, 2001.