MEMORANDUM

No 12/2011

Portfolio Separation with α-symmetric and Pseudo-isotropic Distributions

Nils Chr. Framstad

ISSN: 0809-8786

Department of Economics
University of Oslo

This series is published by the
University of Oslo
Department of Economics

P. O.Box 1095 Blindern
N-0317 OSLO Norway
Telephone: + 47 22855127
Fax: + 47 22855035
Internet: http://www.sv.uio.no/econ
e-mail: econdep@econ.uio.no

In co-operation with
The Frisch Centre for Economic Research

Gaustadalleén 21
N-0371 OSLO Norway
Telephone: +47 22 95 88 20
Fax: +47 22 95 88 25
Internet: http://www.frisch.uio.no
e-mail: frisch@frisch.uio.no

Last 10 Memoranda

No 11/11	Jørgen Modalsli *Inequality and growth in the very long run: inferring inequality from data on social groups*
No 10/11	Eilev S. Jansen, Tord S. H. Krogh *Credit conditions indices: Controlling for regime shifts in the Norwegian credit market*
No 09/11	Karin Jacobsen , Kari H. Eika, Leif Helland, Jo Thori Lind, Karine Nyborg *Are nurses more altruistic than real estate brokers?*
No 08/11	Michael Hoel *The supply side of CO_2 with country heterogeneity*
No 07/11	Anne Line Bretteville-Jensen, Erik Biørn, Randi Selmer *Quitting Behaviour of Cigarette Smokers* *Are there direct effects of a screening program?*
No 06/11	Kari H. Eika *Near and Generous? Gift Propensity and Chosen Emotional Distance*
No 05/11	Finn R. Førsund *Weight Restrictions in DEA: Misplaced Emphasis?*
No 04/11	Vladimir Krivonozhko, Finn R. Førsund, Andrey V. Lychev *Terminal units in DEA: Definition and Determination*
No 03/11	André K. Anundsen, Tord S. H. Krogh, Ragnar Nymoen, Jon Vislie *Overdeterminacy and endogenous cycles: Trygve Haavelmo's business cycle model and its implications for monetary policy*
No 02/11	Nils Chr. Framstad *Portfolio Separation Properties of the Skew-Elliptical Distributions*

Previous issues of the memo-series are available in a PDF® format at:
http://www.sv.uio.no/econ/forskning/publikasjoner/memorandum

Portfolio separation with α-symmetric and pseudo-isotropic distributions*

Memo 12/2011-v2

(First version February 27, 2011, this version February 27, 2013)

Nils Chr. Framstad†‡

Abstract. The shifted pseudo-isotropic multivariate distributions are shown to satisfy Ross' stochastic dominance criterion for two-fund monetary separation in the case with risk-free investment opportunity, and furthermore to admit CAPM under $\mathbb{L}^\alpha$-norm symmetry if $\alpha > 1$. With no risk-free investment opportunity, the $\alpha = 1 + 1/(2d-1)$-norm symmetric cases (any $d \in \mathbb{N}$), are shown to admit $2d$-fund separation, generalizing the well-known elliptical case $d = 1$.

Both discrete-time and continuous-time dynamic models with intermediate consumption are covered.

Key words and phrases: Portfolio separation, mutual fund theorem, CAPM, stochastic dominance, pseudo-isotropic distributions, α-symmetric distributions, stable distributions, stochastic differential equations

MSC (2010): 91G19, 60Exx, 60H05, 49K45.
JEL classification: G11, C61, D81, D53.

---

* version 2 (expanded, revised and corrected), 2013–02–27. Obsoletes version 1 of 2011–02–27.

† University of Oslo, Department of Economics, P.O. Box 1095 Blindern, NO-0317 Oslo, Norway. mailto:ncf+research@econ.uio.no

‡ Also affiliated with the Financial Supervisory Authority of Norway, P.O. Box 1187 Sentrum, NO-0107 Oslo, Norway. The content of this article does not reflect the views of the Norwegian FSA.

1 Introduction

Portfolio separation – i.e. the property of reducing the dimension of a portfolio optimization problem to a low number vectors («funds») without welfare loss to the agents in question – has been treated extensively since Tobin [43]. There are two main directions: the one which is subject of this paper, is the characterization of those returns probability distributions for which those funds will do for all agents. The other is the characterization of preferences which admit the property for all suitable returns distributions (the standard work being Cass and Stiglitz [6], but see even the modern probabilistic approach of Schachermayer et al. [40]); there are also other routes to the separation property, e.g. risk measures, falling somewhat in between beliefs and preferences (contributions include this author [14] and independently, De Giorgi et al. [16]).

This paper is about the distributional side of the theory, where the standard literature reference is Ross [35]. Ross considers preferences compatible with first-order stochastic dominance, and the core of his result is the property that the *returns* distribution (multivariate) be such that the *portfolio* returns distributions (univariate) can be ordered by their mean once a single dispersion parameter is given. Subsequently, Owen and Rabinovitch [33] and Chamberlain [7] establish that the elliptical (also frequently referred to as «elliptically contoured») distributions satisfy Ross' conditions for two-fund separation. Their setting is a mean–variance trade-off, tying the knot back Markowitz [25] approach as employed by Tobin [43]. Over these decades, the development has lent surprises to quite a few of the giants who bear today's theory on their shoulders, and we some: Markowitz turned out predated by more than a decade by De Finetti [8] (see Markowitz account [26] where he also credits Roy [36]). Tobin conjectured that any two-parameter distribution would admit two-fund separation – counterexamples were given by Samuelson [39], Borch [4] and Feldstein [12]. Fama's discovery ([11], can also be read out of Samuelson [38]) that vectors of i.i.d. α-stables admitted two-fund separation, led Cass and Stiglitz to conjecture that α-stability was necessary, until Agnew [1] provided a counterexample. However, the properties that enabled Owen, Rabinovitch and Chamberlain to verify the Ross [35] criterion for the ellipticals, were to be found as far back as Schoenberg [41], [42] in 1938, before modern portfolio theory.

The classical 2-fund separation result, valid for the elliptical returns distributions, is valid both (i) in the presence of a «risk-free» numéraire opportunity (in which case it can be taken as one of the funds, so-called «monetary separation»), and (ii) in the absence of such (in which case, one fund can be chosen as the «minimum variance portfolio»). This paper sets out to generalize, if necessary by admitting a higher number of funds, to the so-called *pseudo-isotropic distributions*, a multivariate class of symmetric random variables such that all linear combinations of the coordinates are of the same type. The pseudo-isotropic distributions admit a dispersion measure which is symmetric and positively homogeneous, and which, together with the excess returns entering via a location shift, characterize the portfolio return distribution completely. To summarize the results briefly:

- The case with risk-free opportunity admits two-fund monetary separation just like the ellipticals, or like the non-elliptical case of i.i.d. symmetric α-stable random variables as established already by Fama [11] (for results in continuous time: this author [13] and Ortobelli et al. [31]). Indeed, pseudo-isotropy generalizes both these classes.

- If no risk-free opportunity exists, separation will only be admitted by a few special cases; most of these will be non-integrable, but there are a few peculiar integrable cases leading to $2d$-fund separation if the index of symmetry is one of the values $\alpha = 1 + 1/(2d-1)$, $d \in \mathbb{N}$, that is, one-and-an-oddth, where $d = 1$ subsumes the elliptical distributions $\alpha =$ one-and-a-whole.
- If the index of symmetry exceeds one, then there is also CAPM for the case *with* risk-free opportunity. Without, there is no securities market line, as there is no two-fund separation.

The exposition will in section 2 establish a *single* period market, review stochastic dominance, and point out the essential property that makes the elliptical distributions admit two-fund monetary separation (Theorem 6). Section 3 will introduce pseudo-isotropic random variables, point out how they fit Theorem 6, then establish a few cases without risk-free opportunity, where agents can do with fewer funds than the entire market and then adapt CAPM to the case with risk-free opportunity. Section 4 then extends to dynamic markets in discrete and continuous time.

1.1 Notation, terminology and standing assumptions

Random variables and stochastic processes are denoted by upright Latin letters (boldfaced if they are vector-valued). Minuscles (Greek/Latin) are non-random constants (vectors if bold), where the (portfolio) choice variable is $\boldsymbol{\xi}$, and $\boldsymbol{1}$ is the vector of ones, and $\boldsymbol{0}$ the null vector. Vectors are columns by default, unless indicated by the transposition superscript «$^\top$». Matrices are Greek uppercase slanted bolds (non-bold if dimension is 1×1 – and consider $\boldsymbol{I}$ a capital Iota. The $\sim$ symbol will denote equality in probability distribution, or for processes: finite-dimensional distributions. The calligraphic fonts will be used for sigma-algebras ($\mathcal{F}$) and filtration ($\boldsymbol{\mathcal{F}}$). As we deal with non-integrable random variables, and the notation $\mathbb{E}[\,\cdot\,|\,\cdot\,]$ may suggest integrability, we shall use notation like $\mathbf{X}|\mathrm{X}_0$ or $\mathbf{X}|\mathcal{F}$ for conditioning and conditional distributions.

A set $H \supsetneq \{\mathbf{0}\}$ is *radial* if it is composed as a union of half-lines from the origin. Constraining the portfolio to the closed first orthant models a «no short sale» constraint, and we shall use that terminology as well. No short sale on some, but not all, investment opportunities, will also correspond to a radial constraint. As commonplace in the literature, we will frequently refer to the numéraire as the «risk-free» investment opportunity, and the other investments as «risky».

Assumption 1. We shall allow for constraints to be specified (in the single-period model, we shall consider either the constraint to a radial set, covering e.g. no short sale conditions, to an affine half-space represening no borrowing or limited degree of leverage, or to the the affine hyperplane of no risk-free opportunity). After having restricted the opportunity set according to these constraints, we shall assume the market to be *free from arbitrage opportunities and from redundant investment opportunities.* (If there is a redundant opportunity, then we can leave it out and rebuild the model without it.) △

In line with the literature on portfolio separation, we make no assumption on limited liability – indeed, all non-elliptical cases will violate that.

2 The single-period market and the preferences

Consider a single period investment in a numéraire (enumerated with as the zeroth coordinate) returning X_0 per monetary unit invested, and another n (finite!) investment opportunities with returns vector assumed to possess the structure $\mathrm{X}_0\boldsymbol{1} + \boldsymbol{\mu}\mathrm{R}_0 + \mathbf{X}\mathrm{R}$ – the decomposition is in part chosen to fit a representation common in the theory of elliptical distributions, see below. This way, the *portfolio return* from investments $\boldsymbol{\xi}$ – any vector, permitting short sale and (for now) unlimited net borrowing without market frictions – in the n opportunities and $w - \boldsymbol{1}^\top\boldsymbol{\xi}$ (where w is initial wealth) in the numéraire, will be

$$w\mathrm{X}_0 + \boldsymbol{\xi}^\top\big(\boldsymbol{\mu}\mathrm{R}_0 + \mathbf{X}\mathrm{R}\big), \tag{1}$$

Here, the probability distributions involved in the $\boldsymbol{\mu}\mathrm{R}_0 + \mathbf{X}\mathrm{R}$ will be specified conditional on X_0, and conditionally, $(\mathrm{R}_0, \mathrm{R})$ will be stochastically independent of $\mathbf{X}$. The location parameter $\boldsymbol{\mu}$ is assumed constant. We will not assume $\mathbf{X}$ to have finite mean, but we will later assume it symmetric about the origin. In view of the above, it represents no loss of generality to interpret – or even formally assume – X_0 as a «risk-free» return. We shall therefore use the term «monetary separation» to refer to separation where the risk-free can be chosen as one of the funds.

2.1 Stochastic dominance and preferences.

Recall that a real-valued random variable X^* first-order (weakly) stochastically dominates another, X, if any of the three following equivalent criteria hold:

(i) (The «mass transfer» property:) there exists some nonpositive random variable X_- such that

$$\mathrm{X}^* + \mathrm{X}_- \sim \mathrm{X} \tag{2}$$

(ii) $\mathrm{CDF}_{\mathrm{X}^*} \leq \mathrm{CDF}_{\mathrm{X}}$

(iii) $\mathbb{E}[u(\mathrm{X}^*)] \geq \mathbb{E}[u(\mathrm{X})]$ for every bounded nondecreasing (utility) function u.

A recent article by Østerdal [32] elaborates on the various definitions, in particular the mass transfer concept, which is the property that will be most useful in this paper.

The following easy observation will sometimes be needed:

Lemma 2. *If neither of* X *and* X^* *dominate the other, then there are two expected utility maximizers who rank the random payoffs different, namely* $u(x) = 1_{x \geq \bar{x}}$ *for two* $\bar{x}$ *values which evaluate* $CDF_{\mathrm{X}}(\bar{x}) - CDF_{\mathrm{X}^*}(\bar{x})$ *to opposite signs.*

For second-order stochastic dominance, assume that $\check{\mathrm{X}}$ is independent of everything else, and consider the formula

$$\mathrm{X}^* + \check{\mathrm{X}} + \mathrm{X}_- \sim \mathrm{X} + \epsilon\check{\mathrm{X}} \tag{3}$$

Analogous to (i) above, second-order stochastic dominance is most frequently taken to be formulated by way of the condition that $\epsilon = \mathbb{E}[\check{\mathrm{X}}] = 0$. However, for our purposes, we cannot assume expectation to exist. A natural extension would be to claim that a risk-averse agent is someone who, when faced with the choice of how much to get of an independent $\check{\mathrm{X}}$ which is *symmetric about zero* (i.e. $-\check{\mathrm{X}} \sim \check{\mathrm{X}}$), then the agent will choose as little as possible. This motivates the following definitions:

Definition 3. In sections 1 through 3, the following definitions will apply.

Suppose X_- is a nonpositive random variable and that $-\check{\mathrm{X}} \sim \check{\mathrm{X}}$ is independent of everything else. We shall use the term *agent* for a preference ordering which (weakly) prefers X^* to X whenever (2) holds[1]. The agent will be called *risk-averse* if X^* is (weakly) preferred to X whenever (3) holds with $|\epsilon| \leq 1$. △

We shall return to preference assumptions for the purposes of dynamic markets in the respective models in section 4. As we consider separation properties of the distributions, we can define portfolio separation as follows:

Definition 4 ($k+1$-fund (monetary) separation). The returns distribution admits *k-fund monetary separation* if there exist vectors («funds») $\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_k$ such that for any given portfolio $\boldsymbol{\xi}$, there exist $q_1, \ldots, q_k$ so that the return (1) is weakly first-order stochastic dominated by the return obtained using

$$\boldsymbol{\xi}^* = q_1\boldsymbol{\varphi}_1 + \cdots + q_k\boldsymbol{\varphi}_k \tag{4}$$

for $\boldsymbol{\xi}$ in (1). (Fund number $k+1$ is the risk-free.) The distribution admits *k-fund separation* if in addition the position $w - \boldsymbol{1}^\top\boldsymbol{\xi}^*$ in the risk-free opportunity vanishes identically for all agents. △

The latter case will be imposed whenever needed for the model; a model without risk-free opportunity, will have $\boldsymbol{1}^\top\boldsymbol{\xi} = w$ as a constraint.

[1] Of course, as Khanna and Kulldorff [20] point out, an alternative assumption of free disposal would dispose of the need to assume that agents prefer more to less – however our arguments still require preferences to rank only probability distributions.

Remark 5.

- We do not assume that each agent has an optimal (finite) portfolio; rather, the property says that for any given portfolio there is one which is at least as good and which uses only the funds (implying that the restriction to the funds is without welfare loss).
- Risk-aversion is not a main point of this paper, but it is not an uncommon assumption in expositions of portfolio theory. It should be noted that risk-averse agents chould in exceptional cases do with fewer funds. For example, there case of one-fund separation if for e.g. Gaussian $(\boldsymbol{1}, \boldsymbol{I})$ returns without risk-free opportunity not valid among non-riskaverse agents, who will need another to boost variance. Theorem 11 will touch this issue.

△

Observe now that if two portfolios $\boldsymbol{\xi}$ and $\boldsymbol{\xi}^*$ satisfy $\boldsymbol{\xi}^\top \mathbf{X} \sim \boldsymbol{\xi}^{*\top}\mathbf{X}$ and $\boldsymbol{\xi}^\top \boldsymbol{\mu} \leq \boldsymbol{\xi}^{*\top}\boldsymbol{\mu}$, then the return $w\mathrm{X}_0 + \boldsymbol{\xi}^{*\top}\big(\boldsymbol{\mu}\mathrm{R}_0 + \mathbf{X}\mathrm{R}\big)$ using $\boldsymbol{\xi}^*$ will first-order stochastically dominate the return using $\boldsymbol{\xi}$; this follows from the assumed independence, and the nonnegativity of R_0 (the assumed nonnegativity of R is superfluous, but commonplace and without loss of generality). We can therefore work as if R_0 and R were both constants. We shall also say that «$\boldsymbol{\xi}^*$ dominates $\boldsymbol{\xi}$» if the respective portfolio returns returns are ordered that way.

2.2 Portfolio separation: the key argument

Let us first review the elliptical case as an illustration. As is well-known, see e.g. Cambanis et al. [5], any elliptical vector admits the representation $\boldsymbol{\mu} + \mathbf{X}\mathrm{R}$ where the location vector $\boldsymbol{\mu}$ equals the mean iff this exists (it is in fact not uncommon in the theory of elliptical distributions to refer to it as «mean» regardless of integrability), and where $\mathbf{X} = \boldsymbol{\Sigma}\mathbf{U}$ where $\boldsymbol{\Sigma}$ is a positive definite[2] matrix and $\mathbf{U}$ is uniform on the unit sphere. The independent radial variable R can then be used to generate multidimensional versions of any symmetric univariate distribution; for example, we obtain the multinormal by choosing R as a zero-mean normal distribution (or equivalently its absolute value). Analogously, taking R to be a symmetric α-stable, we obtain a symmetric α-stable vector; however, except in the Gaussian case, the marginals will not be stochastically independent.

We can however take $\mathrm{R} = 1$ and simply let $\mathbf{X}|\mathrm{X}_0$ be elliptically distributed and located at zero. Then the (conditional) characteristic function $\mathbb{E}\big[\mathrm{e}^{\mathrm{i}\boldsymbol{\theta}^\top\mathbf{X}}\big|\mathrm{X}_0\big]$ admits the form $h(\sqrt{\boldsymbol{\theta}^\top\boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top\boldsymbol{\theta}})$. If $\boldsymbol{\varphi}$ solves the problem

$$\max_{\boldsymbol{\xi}} \ \boldsymbol{\mu}^\top\boldsymbol{\xi} \quad \text{subject to} \quad \boldsymbol{\xi}^\top\boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top\boldsymbol{\xi} = 1,$$

then the family $\{q\boldsymbol{\varphi}\}_{q\geq 0}$ will yield a portfolio return distribution (from (1)), which first-order stochastically dominate any other possible portfolio returns in the market. This

[2] some authors do include the singular extionsion allowing merely semidefiniteness, which would violate the assumptions of absence of arbitrages and of redundant opportunities.

is two-fund monetary separation, reducing the portfolio optimization problem to the one-dimensional allocation between $\boldsymbol{\varphi}$ and the risk-free.

Realizing that the the key to the result is that h takes as argument a positive homogeneous real-valued functional of $\boldsymbol{\theta}$, we can immediately formulate a much more general result. Notice the q in the exponent on the right-hand side of (5), and that the c function therein is necessarily positively homogeneous.

Proposition 6. *Consider the market (1) with the restriction that the portfolios are restricted to some radial set H (possibly = the entire $\mathbb{R}^n$). Suppose that the conditional characteristic function $\mathbb{E}\big[\mathrm{e}^{\mathrm{i}\boldsymbol{\theta}^\top \mathbf{X}}|\mathrm{X}_0\big]$ admits the representation*

$$\mathbb{E}\big[\mathrm{e}^{\mathrm{i}q\boldsymbol{\theta}^\top \mathbf{X}}|\mathrm{X}_0\big] = h(q\,c(\boldsymbol{\theta})) \quad \forall\ q \geq 0, \qquad \textit{for some } c : \mathbb{R}^n \to [0, \infty). \tag{5}$$

If there is a $\boldsymbol{\varphi}$ that solves the problem

$$\max_{\boldsymbol{\xi}}\ \boldsymbol{\xi}^\top \boldsymbol{\mu} \qquad \textit{subject to} \qquad c(\boldsymbol{\xi}) = 1, \tag{6}$$

then there is two-fund monetary separation over all agents, with $\boldsymbol{\xi}^* = q\boldsymbol{\varphi}$.

Proof. If g is the characteristic function of $\mathbf{X}|\mathrm{X}_0$, then $\boldsymbol{\xi}^\top \mathbf{X}|\mathrm{X}_0$ has characteristic function of the form $q \mapsto g(q\boldsymbol{\xi})$, i.e. the conditional distribution is uniquely determined by $q\,c(\boldsymbol{\xi})$; note that if $c(\boldsymbol{\xi}) = 0$, then $\boldsymbol{\xi}^\top \mathbf{X} = 0$, and under the assumption of no arbitrages nor redundant opportunities, this implies $\boldsymbol{\xi} = \boldsymbol{0}$. Assume $\boldsymbol{\varphi}$ to solve the maximization problem as stated (if applicable subject to the additional constraint $\boldsymbol{\xi} \in H$); then for arbitrary $\boldsymbol{\xi} \neq \boldsymbol{0}$, we have $g(\boldsymbol{\xi}/c(\boldsymbol{\xi})) = h(1)$. Let $q = c(\boldsymbol{\xi})/c(\boldsymbol{\varphi}) = c(\boldsymbol{\xi}) > 0$, so that $g(q\boldsymbol{\varphi}) = g(\boldsymbol{\xi})$. In terms of first-order stochastic dominance of the return distribution, we have constructed $\boldsymbol{\varphi}$ so that it dominates $\boldsymbol{\xi}/q$, implying that $\boldsymbol{\xi}^* := q\boldsymbol{\varphi}$ dominates $\boldsymbol{\xi}$. □

For the particular case of the ellipticals, the introduction of another linear constraint requires just another fund, which degenerates if this constraint forbids a risk-free opportunity. This well-known result follows by a Lagrange argument, and is contained as a special case of Theorem 11 below.

3 Pseudo-isotropic distributions

The pseudo-isotropic random vectors form a multivariate distribution class which contains the among others, the ellipticals (and no other square-integrable distributions!) and the vectors of i.i.d. α-stables (α common over the coordinates). The following will give a primer on the defining properties and the development of the theory from the corresponding theory of elliptical distributions, assuming known the basics of the latter.

A univariate symmetric distribution $\tilde{\mathrm{X}}$ is said to admit an n-dimensional version $\mathbf{X}$ if for each non-random n-vector $\boldsymbol{\theta}$, there exists a $c = c(\boldsymbol{\theta})$ (often called the *standard* of $\tilde{\mathrm{X}}$) such that $\boldsymbol{\theta}^\top \mathbf{X} \sim c(\boldsymbol{\theta})\tilde{\mathrm{X}}$. Obviously, this property is preserved under linear transformations. The terminology originated with Eaton, see [10], where the starting point was the univariate $\tilde{\mathrm{X}}$; if one takes as starting point the multivariate $\mathbf{X}$, then $\tilde{\mathrm{X}}$ can be taken as any coordinate of $\mathbf{X}$, except any that is the nonrandom Dirac point mass at zero – we will rule those out in this exposition. The term pseudo-isotropic does refer to the multivariate $\mathbf{X}$, cf. e.g. Jasiulis and Misiewicz [19, Definition 3]:

Definition 7. A *pseudo-isotropic distribution* is one which satisfies (5), extended to arbitrary real q by imposing symmetry: $h(-q) = h(q)$. △

There are some geometric properties to observe: Define an *origin-symmetric star body* K (e.g. Koldobsky [22]) as an origin-symmetric closed set with a continuous boundary crossed precisely twice by each line through the origin, and introducing the K-quasinorm notation $||\boldsymbol{\theta}||_K = \min\{a > 0;\ \boldsymbol{\theta}/a \in K\}$; if the distribution is pseudo-isotropic with $c(\,\cdot\,) = ||\cdot||_K$, then in the terminology of [21, ch. 6], the distribution is called *K-isotropic*. A pseudo-isotropic measure in finite dimension must be of this form ([30], [29, Proposition 4.1.1]). Beware that K is not an isodensity curve for the distribution, except in the elliptical case where rotational invariance of density and of characteristic function coincide.

The only known pseudo-isotropic distributions are those which are – up to a linear transformation – so-called α-symmetric (Cambanis et al. [5]), for which K is an $\mathbb{L}^\alpha$ unit sphere, for $\alpha \in (0, 2]$, the case $\alpha = 2$ being the ellipticals shifted to zero. In order to distinguish out this linear transformation, we make the following choice of terminology:

Definition 8. A pseudo-isotropic random variable $\mathbf{Z}$ is called *standard* α-symmetric, if its standard can be taken to be $||\boldsymbol{\theta}||_\alpha = \left(\sum_i |\theta_i|^\alpha\right)^{1/\alpha}$. (We allow the slight abuse of notation between $||\cdot||_\alpha$ (for a number) and $||\cdot||_K$ (for a set).) A vector $\mathbf{X} = \boldsymbol{\Sigma}\mathbf{Z}$ is then called *$\mathbb{L}^\alpha$-norm symmetric*. We call α the *index of symmetry*. △

Pseudo-isotropic vectors share some properties of the ellipticals – assuming no Dirac coordinate, they are atomless except possibly at the origin; however, unlike the ellipticals, they are absolutely continuous outside $\boldsymbol{0}$. A pseudo-isotropic variable with finite moment of order $\epsilon > 0$, is $\mathbb{L}^\alpha$-norm symmetric, where $\alpha > \epsilon$ or $\alpha = 2$ – again, the elliptical class distinguishes itself. It has been conjectured by Misiewicz (see [28], in particular Theorem II.2.6) that the $\mathbb{L}^\alpha$-symmetrics are the only pseudo-isotropic distributions. Koldobsky ([22], [23]) has further restricted the possible counterexamples to the Misiewicz conjecture by establishing an imbedding of any counterexamples in some limiting space of the $\mathbb{L}^\alpha$ (settling the weaker Lisitskiĭ [24] conjecture).

As mentioned, a vector of i.i.d. symmetric α-stables, will be α-symmetric, the characteristic function for the usual scaling being $\exp\left(-||\boldsymbol{\theta}||_\alpha^\alpha\right)$. However, with dependent components, the index of symmetry could exceed the index of stability (but not the other way around): Precisely whenever $2 \geq \alpha \geq \bar{\alpha} > 0$, $\exp\left(-||\boldsymbol{\theta}||_\alpha^{\bar{\alpha}}\right)$ is the characteristic function of an α-symmetric $\bar{\alpha}$-stable vector. The reader should beware the confusion in the literature, where the notion of symmetry sometimes has the antipodal meaning as in this paper, whilst used for rotational invariance by other authors in the past. This translates to a confusion as to whether the canonical choice for a multidimensional version of a symmetric stable is the one with i.i.d. coordinates, or the elliptical one – indeed, in one of the questions posed by Owen and Rabinovitch [33, footnote 4] they use «symmetric» for rotational invariance. Generally, when it comes to stable laws, the reader should be warned against the literature's inconsistent language and notation, dubbed by Hall [18] as a «comedy of errors». Obviously, an independent radial scaling (the «R» variable) of a pseudo-isotropic vector preserves the star body K (in particular, the sub-Gaussians are elliptical), but there are other known α-symmetrics than the sub-stables. A range of α-symmetric probability distributions is given in terms of valid (i.e. by Bochner's theorem, positive-definite) characteristic functions by Gneiting [17].

3.1 Portfolio separation with risk-free investment opportunity

The symmetry and positive homogeneity of the c functional yields two-fund monetary separation for the pseudoisotropics, much the same way as the elliptical case or the case of i.i.d. α-stable components treated already by Fama [11] (it is already known that the independence of components is not essential, e.g. this author [13]). For $\alpha \leq 1$, the $\mathbb{L}^\alpha$ unit balls are not only non-convex sets – indeed, their complements intersected with any orthant is a convex set (the first-orthant part of the epigraph defining any component as a convex function of the others). This motivates the formulation of the following theorem:

Theorem 9. *Proposition 6 applies to pseudo-isotropic* $\mathbf{X}$*, and yields two-fund monetary separation* $\boldsymbol{\xi}^* = q\boldsymbol{\varphi}$*, where the risky fund* $\boldsymbol{\varphi}$ *can be taken as an extreme point of the convex hull of the unit ball* $\{c \leq 1\}$*. In particular, if there is an extremum on an axis, then one shall only invest in one opportunity, namely the one with highest excess return / dispersion ratio.*

Proof. Since the set K has continuous boundary, then the maximum in Proposition 6 is attained, and this immediately yields two-fund separation. Geometrically, the maximization of $\boldsymbol{\xi}^\top \boldsymbol{\mu}$ has to be obtained on an extreme point.

□

As a consequence of non-diversification, we immediately have that if components are independent and c is the *standard* $\mathbb{L}^\alpha$ unit ball $|| \cdot ||_\alpha$ with $\alpha \leq 1$, then $\mathbb{E}|\mathrm{X}_i| = \infty$, $i \geq 1$. There are particular results available for certain $\mathbb{L}^\alpha$-norm symmetric cases.

3.2 No risk-free investment opportunity: some cases which admit separation or reduced dimensionality

This subsection assumes $\mathbb{L}^\alpha$-norm symmetry – if the Misiewicz conjecture holds, this is the general case, modulo point mass coordinates which we rule out by Assumption 1. It then turns out that we can take the linear transformation $\boldsymbol{\Sigma}$ of Definition 8 to be invertible:

Assumption 10. $\mathbf{X} = \boldsymbol{\Sigma}\mathbf{Z}$, for a standard α-symmetric $\mathbf{Z}$ and an invertible $\boldsymbol{\Sigma}$.

Introducing the notation

$$\boldsymbol{\zeta} = \boldsymbol{\xi}^\top \boldsymbol{\Sigma}, \qquad \boldsymbol{\zeta} = \boldsymbol{\xi}^\top \boldsymbol{\Sigma}, \qquad \boldsymbol{\eta} = \boldsymbol{\Sigma}^{-1}\mathbf{1} \quad (\neq \mathbf{0}) \tag{7}$$

so that the portfolio return is $\boldsymbol{\xi}^\top(\boldsymbol{\mu} + \mathbf{X})$ equals $\boldsymbol{\zeta}^\top\big(\boldsymbol{\rho} + \mathbf{Z}\big)$ and the risky invested amount is $\boldsymbol{\zeta}^\top\boldsymbol{\eta}$, we shall assume one of the following portfolio constraints:

$$\boldsymbol{\zeta}^\top\boldsymbol{\eta} \in L \qquad \text{with the special case } \boldsymbol{\zeta}^\top\boldsymbol{\eta} = w \tag{8}$$

where w is the agent's initial endowment. We shall refer to the $\in L$ constraint as *constrained leverage*, and to the special case $L = \{w\}$ as *no risk-free opportunity*. △

The constraints (8) could in principle remove an arbitrage opportunity even with $\boldsymbol{\Sigma}$ being singular (that is, when $\boldsymbol{\Sigma}^\top\boldsymbol{\theta} = \mathbf{0}$ for some $\boldsymbol{\theta} \perp \boldsymbol{\eta}$); however, that would constitute a riskless investment opportunity, and is therefore ruled out.

The elliptical case $\alpha = 2$ admits 2-fund separation even without risk-free opportunity, and we shall see that this generalizes, at the cost of additional funds, to $\alpha = 1 + 1/\text{odd}$. Note that the further restriction to arbitrary radial H is *not* admitted in Theorem 11. First, introduce the *signed power* notation, which we for notational convenience apply entry-wise to vectors. The following notation will be applied mutandis mutatis to column vectors and scalars as well (we do not need it for matrices):

$$(\boldsymbol{\zeta}^\top)^{<p>} := (|\zeta_1|^p \ \operatorname{sign}\zeta_1 \ , \ \dots \ , \ |\zeta_n|^p \ \operatorname{sign}\zeta_n) \tag{9}$$

For example, $-4^{<1/2>} = -2$ and $\nabla||\boldsymbol{\zeta}||_\alpha^\alpha = \alpha(\boldsymbol{\zeta}^{<\alpha-1>})^\top = \alpha(\boldsymbol{\zeta}^\top)^{<\alpha-1>}$. The signed power is a one-to-one transformation of the base number/vector, for every nonzero exponent.

Theorem 11. *Consider the market under Assumption 10. In parts (a)–(d) assume $\alpha \in (1, 2]$, while in part (e) assume $\alpha \in (0, 1]$. Put*

$$d = \frac{1}{2} \cdot \frac{\alpha}{\alpha - 1} \qquad (\in [1, \infty).) \tag{10}$$

(a) *The* minimum-dispersion portfolio *for no risk-free opportunity – i.e. the one which minimizes the dispersion c subject to the constraint – is*

$$\frac{w}{||\boldsymbol{\eta}||_{2d}^{2d}}\,\boldsymbol{\eta}^{<2d-1>} \tag{11}$$

(b) *Suppose $d \in \mathbb{N}$ (i.e. $\alpha = 1 + 1/\text{odd}$). Then we have $2d + 1$-fund monetary separation under constrained leverage, and $2d$-fund separation if there is no risk-free investment opportunity. In both cases, the $2d$ risky funds are (with the convention $0^0 = 1$)*

$$\boldsymbol{\varphi}_j = \big(\eta_1^{j-1}\rho_1^{2d-j} \ , \ \dots \ , \ \eta_n^{j-1}\rho_n^{2d-j)}\big)^\top \tag{12}$$

(c) *Suppose in addition to part (b) that there is no risk-free opportunity. Then the number of funds needed for all risk-averse agents, is precisely*

$$\min\left\{2d,\ \text{the number of distinct real } \tfrac{\eta_i}{\rho_i} \text{ values plus 1 if } \exists\bar{\imath};\ \eta_{\bar{\imath}} \neq 0 = \rho_{\bar{\imath}}\right\} \tag{13}$$

of linearly independent $\boldsymbol{\varphi}_j$ funds. This number also suffices for all agents with the following single exception: when (13) yields 1, or equivalently when $\boldsymbol{\rho}$ is a multiple of $\boldsymbol{\eta}$, then all risk-averse agents will choose the minimum-dispersion portfolio (11), while other agents require an arbitrary (non-null) free portfolio in addition.

Under constrained leverage, these funds together with the risk-free are sufficient.

(d) *Part (b) does not generalize to the case where $d - 1/2 \in \mathbb{N}$ (i.e. $\alpha = 1 + 1/\text{even}$); if we formally consider the funds of (12) with $2d$ being odd, then there are cases where some agents can do with these and other agents can not.*

(e) *Suppose now $\alpha \in (0, 1]$. Then any agent holds the zero position in all but at most two investment opportunities; this is however not a separation result, as different agents may require different pairs. The minimum-dispersion portfolio for the equality constraint $\boldsymbol{\zeta}^\top \boldsymbol{\eta} = w$ can be chosen non-diversified on one axis (possibly non-unique). This portfolio is chosen by all risk-avese agents in the special case where $\boldsymbol{\rho}$ is proportional to $\boldsymbol{\eta}$.*

Before proceeding to the proof, notice that the case where (13) yields 1, is the only where an opportunity with $\eta_i = \rho_i = 0$ is not redundant. Indeed, if $\boldsymbol{\Sigma}$ is the identity, then (13) counts the number of different marginal distributions of nonzero excess returns – then if there are at least two, one with zero excess return (possibly desired by a non-riskaverse agent) can be generated as a linar combination.

Proof. In order not to be first-order dominated, any agent who chooses the level $\hat{w} \in L$ for $\boldsymbol{\zeta}^\top \boldsymbol{\eta}$ and the level $\bar{c}$ for dispersion, must choose a solution of the problem

$$\max_{\boldsymbol{\zeta}} \ \boldsymbol{\zeta}^\top \boldsymbol{\rho} \qquad \text{subject to} \qquad ||\boldsymbol{\zeta}||_\alpha^\alpha = \bar{c}^\alpha, \qquad \boldsymbol{\zeta}^\top \boldsymbol{\eta} = \hat{w} \tag{14}$$

wher $\hat{w} = w$ is mandatory if there is no risk-free opportunity. The Lagrange condition yields

$$\boldsymbol{\rho} - \lambda \boldsymbol{\eta} = \alpha \kappa \boldsymbol{\zeta}^{<\alpha - 1>} \tag{15}$$

(a) The minimum-dispersion portfolio solves the problem

$$\min_{\boldsymbol{\zeta}} \ ||\boldsymbol{\zeta}||_\alpha \qquad \text{subject to} \qquad \boldsymbol{\zeta}^\top \boldsymbol{\eta} = w \tag{16}$$

and in case $\alpha > 1$ this is a concave problem with solution uniquely given by (11) (which is a limiting case of (15)).

(b) Odd signed powers are just ordinary powers, so (15) yields

$$\zeta_i \cdot (\alpha\kappa)^{2d-1} = (\rho_i - \lambda\eta_i)^{2d-1} \tag{17}$$

For $\kappa \neq 0$, expand the power and collect terms to get the $2d$ funds given by (12), and in addition there is the risk-free, unless it vanishes identically. To address degeneracies: the constraint qualification could fail, but only at the minimum-dispersion portfolio, which is the fund $\boldsymbol{\varphi}_{2d}$. And the case $\kappa = 0$ implies $\boldsymbol{\rho} = \lambda\boldsymbol{\eta}$, which will be covered next:

(c) Let us first cover the case when $\boldsymbol{\rho}$ and $\boldsymbol{\eta}$ are proportional. Then the left-hand side of (15) collapse to one vector, a scaling of (11). In addition there is the risk-free, if one such exists, but if it does not, then by proportionality the excess return is uniquely given by w, so that $\kappa = 0$. If $\kappa = 0$, there has to be an additional fund $\perp \boldsymbol{\eta}$ – at zero price, but also not contributing to excess return – to satisfy agents who want a higher dispersion than the minimum. Risk-averse agents will not want to invest in this.

To establish the number of funds needed, i.e. the number of linearly independent vectors in the expansion (12) (cf. (17)), assume $\kappa \neq 0$ (otherwise we have the proportionality just covered) and λ such that

$$(\alpha\kappa)^{2d-1}\boldsymbol{\zeta} = \sum_{j=1}^{2d} \binom{2d-1}{j-1} (-\lambda)^{j-1} \boldsymbol{\varphi}_j \tag{18}$$

We wish to pick $2d$ agents with distinct distinct λ values. That is possible by Lemma 2 as no two distributions with different dispersions can be ordered by first-order dominance; dot each side of equation (15) with $\boldsymbol{\eta}$ to eliminate κ and get, for $w > 0$,

$$\frac{\boldsymbol{\zeta}}{w} = \frac{1}{\boldsymbol{\eta}^\top (\boldsymbol{\rho} - \lambda\boldsymbol{\eta})^{<2d-1>}} \, (\boldsymbol{\rho} - \lambda\boldsymbol{\eta})^{<2d-1>} \tag{19}$$

Scaling the problem by w by replacing $\bar{c}$ by $w\hat{c}$, we have a static maximization problem where different choices of dispersion leads to different λ's, and different.

Gather the $2d$ agents' portfolios in a matrix $\boldsymbol{\Xi}$. Then we can write

$$\alpha^k \boldsymbol{\Xi} \boldsymbol{\Delta} = \boldsymbol{\Gamma} \boldsymbol{\Pi} \boldsymbol{\Lambda}^\top \tag{20}$$

where $\boldsymbol{\Delta}$ is the diagonal $2d \times 2d$ invertible matrix with the agents' κ multipliers on the main diagonal; $\boldsymbol{\Pi}$ is the diagonal $2d \times 2d$ invertible matrix with the binomial coefficients $\binom{2d-1}{j-1}$ on the main diagonal; $\boldsymbol{\Gamma}$ is the matrix of the funds $(\boldsymbol{\varphi}_1, \dots, \boldsymbol{\varphi}_{2d})$; and $\boldsymbol{\Lambda}$ is the Vandermonde matrix of the $(-\lambda)$'s, i.e. with row j being the geometric sequence $\left(1,\, -\lambda_j,\, (-\lambda_j)^2, \dots,\, (-\lambda_j)^{2d-1}\right)^\top$, and invertible as the λ's are distinct.

It remains to find the rank of $\boldsymbol{\Gamma}$, and it follows by properties of Vandermonde determinants and their minors. Pick $\ell \leq 2d$ rows each with ρ_i nonzero; these rows are then ρ_i^{2d-1} times a geometric seqeuence $\left(1,\, \eta_i/\rho_i,\, \dots,\, (\eta_i/\rho_i)^{2d-1}\right)$, and we have full rank whenever these rows have η_i/ρ_i distinct, but not if two such ratios coincide. Let

ℓ be the maximum number of linearly independent $\rho_i \neq 0$ rows, and form a matrix of these rows and an arbitrary non-null row of the form $(0, \ldots, 0, \eta_{\bar{\imath}}^{2d})$ (equivalent to $\rho_{\bar{\imath}} = 0 \neq \eta_{\bar{\imath}}$) – if there is one. If such a non-null row does exist and $\ell < 2d$ (= the number of columns), it is another linearly independent row.

The last statement follows as the unconstrained optimum is spanned by (12), namely the single fund $\boldsymbol{\varphi}_1$.

(d) This part will implicitely use Lemma 2 so that a continuum of multiplier pairs will actually chosen by different agents. Observe that in the even-power case, (15) does not yield (17), but

$$\zeta_i = \big(\frac{\rho_i - \lambda\eta_i}{\alpha\kappa}\big)^k \operatorname{sign}\big(\frac{\rho_i - \lambda\eta_i}{\alpha\kappa}\big) \tag{21}$$

which does not expand to a polynomial. Suppose for a counterexample that $\rho_n/\eta_n > \ldots > \rho_1/\eta_1 > 0$, with all $\eta_i > 0$. Let $\bar{c}$ grow from minimum dispersion (which is of the form of the expansion (12)). At the point where the optimum falls outside the appropriate simplex (e.g. the unit simplex if $\boldsymbol{\eta} = \boldsymbol{1}$ and $w = 1$), opportunity #1 is shorted, requiring one more fund.

(e) Finally, assume $\alpha \in (0, 1]$. Then the intersection of each orthant with the *exterior* of the unit sphere, is convex. Except in the proportional case, and as long as dimension exceeds 2, maximizing $\boldsymbol{\zeta}^\top\boldsymbol{\rho}$ subject to being in the plane $\boldsymbol{\zeta}^\top\boldsymbol{\eta}$ and on the $\mathbb{L}^\alpha$ (quasi!-) sphere, is to move a line in parallel in this plane until the last time intersects the $\mathbb{L}^\alpha$ sphere, which is on a nondifferentable edge of this sphere – that is, when some coordinate is zero. Remove that coordinate from the model and repeat the argument until there are only two left (in which case the constraints form a discrete set and the process cannot be iterated).

Notice that the only way an agent can obtain dispersion as low as $c(\boldsymbol{\zeta}) = |w|/\max_i|\eta_i|$ is to choose all coordinates of $\boldsymbol{\zeta}$ as zero except for a (not necessarily unique) i with highest $|\eta_i|$ (nonzero, as the $\boldsymbol{\Sigma}$ matrix is assumed invertible), in which the position should be w/η_i; note that in case of non-uniqueness, the mimimum dispersion is not attained by mixing two opportunities, except in the case $\alpha = 1$. This resolves the special case. Obviously, a minimum-dispersion portfolio is indespensable, as some agent would choose minimum dispersion. However, an agent choosing higher dispersion could very well choose two different opportunities, as the minimum-dispersion portfolio may not pay off very well in terms of ρ_i (say, it could be zero).

□

Remark 12. The last statement of item (c) does not claim that all funds are needed; although any level of dispersion will be chosen by some agent, it is not necessarily so that any $\hat{w}$ will be chosen. Assume – with no claim to realism – that all $\rho_i < 0$; then the opportunities will be shorted, and any upper bound $\boldsymbol{\zeta}^\top\boldsymbol{\eta}(-\infty, \bar{w}]$ with $w > 0$, would be inactive. △

The leverage constraints makes the separation result admit leverage-dependent interest rates, and also agent-specific. Suppose that agent number a has interest spread of $r_a = r_a(\boldsymbol{\xi}^\top \boldsymbol{1}) = r_a(\boldsymbol{\zeta}^\top \boldsymbol{\eta})$ relative to the risk-free opportunity; intuitively it makes sense that r_a has the same sign as $\boldsymbol{\zeta}^\top \boldsymbol{\eta} - w$ (if it is interest *paid*). Then the agent's excess return at leverage $\hat{w}$ is not anymore $\boldsymbol{\zeta}^\top \boldsymbol{\rho}$, but

$$\boldsymbol{\zeta}^\top \boldsymbol{\rho} - r_a \boldsymbol{\zeta}^\top \boldsymbol{\eta}.$$

The following property then easily carries over from the classical case:

Corollary 13. *Theorem 11 applies to the case of individual leverage-dependent interest rate just as for constrained leverage. Also, it admits $L = L_a$ individual.*

Proof. For whatever choice of $\bar{c}$, $\hat{w}$ agent a considers, the $-r_a \boldsymbol{\zeta}^\top \boldsymbol{\eta} = -r_a(\hat{w})\hat{w}$ term goes outside the maximization, and the problem reduces to the problem for an agent with wealth $w = \hat{w}$, choice $\bar{c}$ and no risk-free opportunity, except that agent a's position in the risk-free opportunity does not vanish.

□

3.3 (When) can we have (what kind of) CAPM?

The answer to the question is of course highly dependent upon what properties one will impose on a model to apply the «CAPM» name. (Elliptical) CAPM has various elements; a bullet-shaped convex risk/return set for the risky opportunities, a pricing formula, and a securities market line where the agents will adapt – applying the elliptical two-fund separation theorem with or without risk-free investment opportunity. However, two-fund separation fails without risk-free opportunity in the $\alpha \in (1, 2)$ norm symmetric case, and although we can have zero-beta portfolios corresponding to efficient ones, we cannot then have a securities market line.

Let us therefore assume that a risk-free opportunity exists, and make the commonplace assumption of risk-aversion. Then there is an established CAPM for shifted (by expectation) symmetric $\bar{\alpha}$-stable returns, see e.g. the derivation by Belkacem et al. [2] for $\bar{\alpha} > 1$. It works as close as we can get it to the elliptical case without having covariance as an inner product: There is a *covariation* measure which, unlike covariance, is not symmetric in the variables (Samorodnitsky and Taqquu's book [37] is a good reference on stable distributions, see in section 2.7 for covariation). Dividing the covariation of a security's return *on* (not «and», and not the other way around!) the market portfolio's return by the dispersion measured by the standard $c(\,\cdot\,)$, we get a non-symmetric correlation coefficient which becomes the security's *beta* (again, see [2]). In fact, it is possible to represent the beta in terms of joint moments of *an arbitrary* order strictly less than $\bar{\alpha}$, see [37, Lemma 2.7.16].

Now consider $\mathbb{L}^\alpha$-norm symmetry; we shall see that CAPM generalizes from the $\bar{\alpha}$-stable or elliptical case, provided that the index of symmetry is > 1. Let us briefly establish the derivation, using general c as far as possible. Unlike the previous section, we shall work with $\boldsymbol{\xi}$ and $\boldsymbol{\Sigma}$ (otherwise, we would have transformed away the betas at the outset). Starting from a position in a location–dispersion efficient portfolio $\boldsymbol{\xi}^*$, the agent can then consider buying a (sufficiently small) portfolio $\boldsymbol{\delta}$, and scale her position in $\boldsymbol{\xi}^*$ by a factor $1-b$ as to maintain the dispersion of the return – by pseudo-isotropy, this is possible. By efficiency, this (marginal) transaction can not increase the excess return. That is, defining b as the $b(\boldsymbol{\delta})$ for which

$$c(\boldsymbol{\delta} + [1-b(\boldsymbol{\delta})]\boldsymbol{\xi}^*) = c(\boldsymbol{\xi}^*) \tag{22}$$

then this perturbation should not yield any higher location parameter («mean», if we have integrability) – nor any lower, as we could choose $-\boldsymbol{\delta}$ in place of $\boldsymbol{\delta}$. Therefore, $\boldsymbol{0}$ maximizes $\boldsymbol{\delta} \mapsto (\boldsymbol{\delta}^\top + [1-b(\boldsymbol{\delta})]\boldsymbol{\xi}^{*\top})\boldsymbol{\mu}$, with first-order condition (necessary[3] if c and hence b is smooth)

$$\boldsymbol{\mu}^\top = (\boldsymbol{\mu}^\top \boldsymbol{\xi}^*)\nabla b(\boldsymbol{0}) \tag{23}$$

Notice that the set of attainable excess return/volatility pairs for risky portfolios,

$$\{(\boldsymbol{\xi}^\top \boldsymbol{\mu},\, c(\boldsymbol{\xi}))\,;\, \boldsymbol{\xi}^\top \boldsymbol{1} = w\} \tag{24}$$

is convex whenever $\boldsymbol{\xi} \mapsto c$ is a convex function, and then stationarity is sufficient. By (23), the betas are $\nabla b(\boldsymbol{0})$, which we find by differentiating (22) wrt. $\boldsymbol{\delta}$:

$$\boldsymbol{\mu} = (\boldsymbol{\xi}^{*\top}\boldsymbol{\mu}) \cdot \boldsymbol{\beta} \qquad \text{where the betas are } \boldsymbol{\beta}^\top = \frac{\nabla c(\boldsymbol{\xi}^*)}{\nabla c(\boldsymbol{\xi}^*)\boldsymbol{\xi}^*} \tag{25}$$

– for excess returns (add the risk-free return times $\boldsymbol{1}$ on both sides if desired). For symmetric $\bar{\alpha}$-stables treated by [2], this reduces to their formula; indeed, the gradient of c is one of the possible ways to define covariation ([37, Definition 2.7.3]).

We have a securities market line for $\alpha \in (1, 2]$: The efficient frontier is a concave function in the (c, excess return) plane, namely a shifted $\sqrt[\alpha]{\ }$ function, whose derivative tends to zero at infinity. Just like in elliptical CAPM, there is a unique increasing line from the risk-free opportunity at the origin, touching the efficient frontier at precisely one tangency point; this point is the equilibrium market portfolio (by scaling $\boldsymbol{\mu}$ until markets clear; as in the elliptical case, monetary two-fund separation ensures that the sum of all agents' portfolios is indeed on the line). To summarize what is shown:

Proposition 14. *If a risk-free opportunity exists and the excess returns are of the form $\boldsymbol{\mu}$ plus an $\mathbb{L}^\alpha$-norm symmetric returns distribution with $\alpha > 1$, we have a CAPM that characterize excess returns and a securities market line in terms of the betas obtained by taking $\boldsymbol{\xi}^*$ to be the market portfolio in (25). The betas can be rewritten as*

$$\boldsymbol{\beta}^\top = \frac{\nabla c(\boldsymbol{\xi}^*)}{\nabla c(\boldsymbol{\xi}^*)\boldsymbol{\xi}^*} = \frac{\left(\boldsymbol{\xi}^{*\top}\boldsymbol{\Sigma}^{-1}\right)^{<\alpha-1>}(\boldsymbol{\Sigma}^{-1})^\top}{||(\boldsymbol{\Sigma}^{-1})^\top\boldsymbol{\xi}^*||_\alpha^\alpha} \tag{26}$$

where $\boldsymbol{\Sigma}$ being such that $\boldsymbol{\Sigma}^{-1}\mathbf{X}$ is standard α-symmetric.

[3] As we assume that a risk-free opportunity, $\boldsymbol{\delta}$ could be arbitrary, and therefore and therefore the directional derivative must vanish in all directions; this in contrast to the case without risk-free opportunity, where only the zero-market value $\boldsymbol{\delta}$ are possible.

The argument breaks down when $\boldsymbol{\xi} \mapsto c(\boldsymbol{\xi})$ is not convex, i.e. when $\alpha < 1$. And, even for $\alpha = 1$ the shifted $\sqrt[\alpha]{}$ curve fails to be strictly convex and fails to have a tangent line from the origin.

4 Dynamic market models in discrete and continuous time

The results generalize to dynamic models where the price processes have the appropriately distributed increments – in continuous time, under the addtional assumption of the semimartingale property. For a motivating example, consider the single-period market treated this far as a model with decision at the beginning of the period, and consumption before investment and at the end, where the terminal wealth is consumed. Suppose now that an agent considers to consume γ and invest $\boldsymbol{\xi}$. Under the hypothesis of Proposition 6, the agent can instead invest $\boldsymbol{\xi}^* = c(\boldsymbol{\xi})\boldsymbol{\varphi}$ (which yields the same dispersion) and consume $\gamma + (\boldsymbol{\xi}^* - \boldsymbol{\xi})^\top \boldsymbol{\mu}\mathrm{R}_0$; then terminal wealth (= terminal consumption) has the same distribution, but initial consumption is higher. This should be preferable to an agent who prefers more to less, and we will adjust the preference assumption accordingly by modifying the mass transfer criterion. The case of a bounded number of periods could then be covered by backwards induction, but we shall treat a case with unbounded horizon. The approach is based on ([13], which in turn is based on an approach of Khanna and Kulldorff [20] for the geometric Brownian case), though somewhat more rigorous; we shall first define the problem on continuous time.

Assumption 15. Assume given the usual stochastic basis, i.e. a filtered probability space where the filtration $\boldsymbol{\mathcal{F}} = \{\mathcal{F}_t\}_{t\geq 0}$ is right-continuous with $\mathcal{F}_0$ complete. We shall consider the controlled (through *cumulative consumption* $\gamma(t)$ and the $\mathbf{R}^n$-valued *portfolio* $\boldsymbol{\xi} = \boldsymbol{\xi}(t)$, both assumed predictable) Itô stochastic process

$$\mathrm{dV}(t) = \boldsymbol{\xi}(t)^\top \left[\boldsymbol{\mu}(t)\,\mathrm{d}t + \boldsymbol{\Sigma}(t)\,\mathrm{d}\mathbf{Z}(t)\right] - \mathrm{d}\gamma(t) \qquad \text{(with } \mathrm{V}(t_0) \text{ given)} \tag{27}$$

driven by a right-continuous adapted process $\mathbf{Z}$, where $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are given predictable processes. We shall consider the following two types of portfolio constraints:

$$\boldsymbol{\xi} \in \text{ some radial set } H, \qquad\qquad \text{OR} \qquad\qquad \boldsymbol{\xi}^\top \boldsymbol{1} \in \text{ some set } L \tag{28}$$

where both H and L could be time-dependent, even stochastic if (i) adapted and (ii) given the past path of V, conditionally independent of everything else. The case without risk-free opportunity is $L = \{V(t)\}$. △

The dynamics (27) can be given the standard justification as the market value of a portfolio, self-financing except for consumption, in a continuously traded market with one «risk-free» opportunity with price $\mathrm{S}_0(t) = 1\ \forall t$ and n risky investment opportunities $\{\mathrm{S}_i\}$ each satisfying an Itô stochastic differential equation with driving noise $\mathbf{Z}$:

$$\mathrm{dS}_i(t) = \mathrm{S}_i(t)\left[\mu_i(t)\,\mathrm{d}t + \boldsymbol{\sigma}_i(t)^\top \mathrm{d}\mathbf{Z}(t)\right]. \tag{29}$$

However, it can also be justified as the development of the value of an insurance portfolio, without trading. It must be remarked that the distributions might (and usually will) violate limited liability; notice also that we are not logging returns (the Gaussian is exceptional in that the driving noise and the log of the geometric SDE, are both of the same type, modulo different drift). It should also be noted that redundance of an investment opportunity may change over time, and hence the number of investment opportunities will as well. However, this change can not depend upon the realizations of the $\mathbf{Z}$ itself and its jumps. For pseudo-isotropic increments, there is furthermore (almost surely) no concern about jumps to zero in one of the S_i, which would otherwise require either the opportunity being reborn or it could disrupt separation properties (this author, [15]).

Working with discounted figures is a simplication without loss of generality. The increments $\boldsymbol{\Sigma}\,\mathrm{d}\mathbf{Z}$ now take the rôle of the $\mathbf{X}\mathrm{R}$ of the single period model. We remark that we could in principle have a more general drift process which is not absolutely continuous, but that can be accomplished by a change of time-scale. We shall therefore assume without loss of generality that $\mathbf{Z}$ can be decomposed into an absolutely continuous part and a purely discontinuous part.

We want to specify what strategies are admissible, but we can first form preferences on the (consumption, value) pairs. In line with the example at the beginning of the section, we shall use the following dominance criterion.

Assumption 16. Consider two strategies

$$(\gamma,\boldsymbol{\xi}) = \{(\gamma(t),\boldsymbol{\xi}(t))\}_{t\geq 0} \quad \text{with corresponding wealth process} \quad \mathrm{V}=\{\mathrm{V}(t)\}_{t\geq 0}$$
$$(\gamma^*,\boldsymbol{\xi}^*) = \{(\gamma^*(t),\boldsymbol{\xi}^*(t))\}_{t\geq 0} \quad \text{with corresponding wealth process} \quad \mathrm{V}^*=\{\mathrm{V}^*(t)\}_{t\geq 0}$$

If the process pairs (γ,V) and (γ^*,V^*) satisfy

$$(\gamma^*,\mathrm{V}^*) \sim (\gamma+\gamma_+,\mathrm{V}) \qquad \text{where } \gamma_+(0)\geq 0, \qquad \mathrm{d}\gamma_+\geq 0 \tag{30}$$

then any agent (weakly) prefers (γ^*,V^*). △

Notice that (30) does not only mean that cumulative consumption $\gamma^*(t)\geq\gamma(t)$ for all t, but also that – up to equivalence in probability law – the amount consumed on any time interval is no less. In other words, if the agent can have more to consume at each instant and the same wealth left, then that is preferred. We shall in this case say that the strategy $(\gamma^*,\boldsymbol{\xi}^*)$ dominates $(\gamma,\boldsymbol{\xi})$.

4.1 The discrete-time case

This subsection considers discrete-time markets of the following form:

Assumption 17. For the discrete-time model, fix a partition $0=t^{(0)}<t^{(1)}<\dots$, assume that the parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are constant on each interval, and restrict the portfolios to the left-continuous step processes

$$(\gamma(t), \boldsymbol{\xi}(t)) = (\gamma^{(j)}, \boldsymbol{\xi}^{(j)}) \quad \text{on} \quad (t^{(j)}, t^{(j+1)}] \tag{31}$$

(with $\boldsymbol{\xi}(0)$ arbitrary). △

Note that we do not assume Markovian processes nor controls – in particular, we are not bound by infinite divisibility, which is necessary for Lévy processes. Indeed, we can have $\mathbf{Z}$ constant except at the $t^{(j)}$.

We can now define risk-aversion under pseudo-isotropy, by the property that an agent will shun dispersion:

Definition 18. Assume that $\big(\mathbf{Z}(t^{(j+1)}) - \mathbf{Z}(t^{(j)})\big)\big|\mathcal{F}_{t^{(j)}}$ are pseudo-isotropic with standard $c^{(j)}$ such that for all j,

$$\boldsymbol{\xi}^{(j)\top}\boldsymbol{\Sigma}(t^{(j)})\,\big(\mathbf{Z}(t^{(j+1)}) - \mathbf{Z}(t^{(j)})\big)\big|\mathcal{F}_{t^{(j)}} \sim c^{(j)}(\boldsymbol{\Sigma}(t^{(j)})^\top\boldsymbol{\xi}^{(j)})\,\big(\mathrm{Z}_1(t^{(j+1)}) - \mathrm{Z}_1(t^{(j)})\big)\big|\mathcal{F}_{t^{(j)}}$$

An agent in the discrete-time market is called *risk-averse* if (s)he (weakly) chooses $(\gamma^*, \boldsymbol{\xi}^*)$ over $(\gamma, \boldsymbol{\xi})$ whenever the following holds for all times:

$$\gamma^* \sim \gamma, \qquad (\boldsymbol{\xi}^* - \boldsymbol{\xi})^\top\boldsymbol{\mu} \geq 0, \qquad \text{and} \qquad c^{(j)}(\boldsymbol{\Sigma}(t^{(j)})^\top\boldsymbol{\xi}^{*(j)}) \leq c^{(j)}(\boldsymbol{\Sigma}(t^{(j)})^\top\boldsymbol{\xi}^{(j)}) \tag{32}$$

△

From this we obtain the following, where the analogy to Theorem 11 might require an additional time-scaling linear transformation in order to get the *standard* unit spheres assumed therein:

Theorem 19. *Consider the market model (27) on discrete time $0 = t^{(0)} < t^{(1)} < \ldots$, and assume it to be free of arbitrage opportinities and redundant investment opportunities. Then on each time interval $(t^{(j)}, t^{(j+1)}]$, the market inherits those separation (/nondiversification) properties from Theorems 9 (for the case with risk-free opportunity) and 11 (for the case without) which apply to the distribution of $\mathbf{X} := \big(\mathbf{Z}(t^{(j+1)}) - \mathbf{Z}(t^{(j)})\big)\big|\mathcal{F}_{t^{(j)}}$.*

Proof. For each predictable step strategy $(\bar{\gamma}, \bar{\boldsymbol{\xi}})$, consider a time step j and put $\bar{c}^{(j)} = c^{(j)}(\boldsymbol{\Sigma}^\top\boldsymbol{\xi}^{(j)})$. Then generate a strategy $(\gamma^*, \boldsymbol{\xi}^*)$ piecewise such that

$$\boldsymbol{\xi}^* \text{ maximizes } \boldsymbol{\xi}^\top(t)\boldsymbol{\mu}(t) \quad \text{subject to} \quad c^{(j)}(\boldsymbol{\Sigma}^\top\boldsymbol{\xi}^{(j)}) = \bar{c}^{(j)} \quad \text{and (28)} \tag{33}$$

$$\text{and} \qquad \gamma^{*(j)} \sim \tilde{\gamma}^{(j)} + (\boldsymbol{\xi}^{*(j)} - \boldsymbol{\xi}^{(j)})^\top\boldsymbol{\mu}(t^{(j)}) \cdot (t^{(j+1)} - t^{(j)}) \tag{34}$$

– the latter «$\sim$» being an equality if we express the consumptions in feedback form; at step 0, consume $\gamma^{(0)}+$ the excess drift, at step 1 consume excess drift plus what you would have consumed under the $(\gamma, \boldsymbol{\xi})$ strategy plugging in your actual «starred» return at time $t^{(1)}$ into the feedback form, and inductively so. Thus, this strategy $(\gamma^*, \boldsymbol{\xi}^*)$ dominates $(\gamma, \boldsymbol{\xi})$ in the sense of assumption 16. To resolve the case for the risk-averse agents, generate the strategy $(\gamma^{**}, \boldsymbol{\xi}^{**})$ by keeping $\gamma^{**} = \gamma^*$, but such that

$$\boldsymbol{\xi}^{**} \text{ minimizes } c^{(j)}(\boldsymbol{\Sigma}^\top\boldsymbol{\xi}^{(j)}) \quad \text{subject to} \quad \boldsymbol{\xi}^\top(t)\boldsymbol{\mu}(t) = \boldsymbol{\xi}^{*\top}(t)\boldsymbol{\mu}(t) \quad \text{and (28)} \tag{35}$$

□

Observe that we have not (yet) assumed any time-homogeneity, so the funds may change from step to step. Indeed, in the case of $\mathbb{L}^\alpha$-norm symmetry without risk-free opportunity, the number of funds may change from step to step as well, as we have not assumed time-invariant α. We might very well require the full market in one period, and only two funds in the next if α shifts to 2. In the following discussion on continuous time, then we have the property that if α is continuously varying in $(1, 2]$, we may approximate the case *with* risk-free opportunity for a two-fund separation property, but not the case without risk-free opportunity. For simplicity, we are therefore going to assume α-symmetry with α constant.

4.2 Continuous time

As mentioned above, we shall make a few simplifying assumptions to the continuous-time case. First, we consider limits through equidistant partitions $t^{(j)} = j\delta$, for $\delta \searrow 0$. Second, we assume some time-homogeneity of the increments of $\mathbf{Z}$, in α-symmetry for common α (the $\boldsymbol{\Sigma}(t)$ taking care of linear transformations). For the continuous-time case, the Itô integral requires the semimartingale property of $\mathbf{Z}$ (in discrete time, all adapted processes are semimartingales, so this assumption was redundant in the above).

Assumption 20. $\mathbf{Z}$ is a semimartingale whose increments $\big(\mathbf{Z}(T) - \mathbf{Z}(t)\big)\big|\mathcal{F}_t$ are pseudo-isotropic and such that for some $\alpha \in (0, 2]$ and some $h : (0, \infty) \to (0, \infty)$ we have for any $T > t \geq 0$, the equality in distribution

$$\boldsymbol{\zeta}^\top \big(\mathbf{Z}(T) - \mathbf{Z}(t)\big)\big|\mathcal{F}_t \ \sim \ h(T - t)||\boldsymbol{\zeta}||_\alpha (Z_1(1) - Z_1(0)). \tag{36}$$

Furthermore, $\boldsymbol{\Sigma}(t)$ is invertible for each t, with $t \mapsto \boldsymbol{\Sigma}(t)^{-1}\boldsymbol{\mu}(t)$ left-continuous and locally bounded, and – if the assumption of no risk-free investment opportunity is made – also $t \mapsto \boldsymbol{\eta}(t) := \boldsymbol{\Sigma}(t)^{-1}\mathbf{1}$ left-continuous and locally bounded.

Finally, consumption paths γ are restricted to be of finite variation on compacts. △

Again, no Markov property is assumed. The latter restriction likely requires a comment: if $\mathbf{Z}$ is a has a Brownian (continuous!) component, then an agent could affect the instantaneous volatility of the process by letting γ fluctuate in a correlating way. This would of course lead to the objectionable property that *cumulative* consumption fluctuates – by intuition, it should be non-decreasing. However, we may allow negative «consumption» rates from e.g. independent income from other sources. For jumps, it is not the same issue, as jumps in $\mathbf{Z}$ take place «between t^- and t» while jumps in the control are only effective the next instant.

By assumption 20, we can just like in the single-period model work as if $\boldsymbol{\Sigma}$ were the identity, at the cost of transforming the no risk-free opportunity condition in terms of $\boldsymbol{\eta}$ like in (8). We shall in Theorem 22 below do that, and writing the transformed portfolios as in Assumption 10 with $\boldsymbol{\zeta}(t) = \boldsymbol{\Sigma}(t)^\top \boldsymbol{\xi}(t)$.

Given $\mathbf{Z}$, we have now introduced the integral over a dense set of the class of integrands bounded in probability. The continuous linear extension, applying convergence *uniformly* *on compacts in probability* (ucp) of sequences of integrands, yields the Itô integral defined as the limit taken in probability. This general approach to the integral can be found at least as early as Meyer [27] and Bichteler [3], or for a now-classic reference see Protter's book [34, section II.4]; however, a reader who wants a brief account on the concept, might very well browse Williams' book review in the AMS Bulletin, [44]. We shall give conditions where we can discretize, apply Theorem 19, and then apply the limit. Our conditions are not going to be sharp; for convenience, we assume continuity to pick straigthtforward representative values for discretization, although generalizations to measurability are available.

First, we shall introduce an ad hoc concept of risk-aversion for α-symmetric markets (27) under Assumption 20.

Definition 21. An agent in a continuous-time market under Assumption 20, is called *risk-averse* if (s)he (weakly) chooses $(\gamma^*, \boldsymbol{\xi}^*)$ over $(\gamma, \boldsymbol{\xi})$ whenever the following holds for all $t \geq 0$:

$$\gamma^* \sim \gamma, \qquad (\boldsymbol{\xi}^* - \boldsymbol{\xi})^\top \boldsymbol{\mu} \geq 0, \qquad \text{and} \qquad ||\boldsymbol{\Sigma}^\top \boldsymbol{\xi}^*||_\alpha \leq ||\boldsymbol{\Sigma}^\top \boldsymbol{\xi}||_\alpha \tag{37}$$

△

On the other hand, the full domain of definition of the Itô integral is rarely used in finance, as it allows for doubling strategies in finite time (cf. Dudley's theorem, [9]). This is not a problem to this model – we are seeking strategies such that the centered parts are equivalent in probability law but one pays better than the other, so any restriction which forbids $\boldsymbol{\xi}$ (resp. $\boldsymbol{\zeta}$) whenever it forbids our constructed dominating $\boldsymbol{\xi}^*$, will be OK. This is the reason for the somewhat vague «admissibility» (which might impose further restrictions than (28)!) in the following:

Theorem 22. *Consider the continuous-time market (27), and denote $\boldsymbol{\zeta}(t) = \boldsymbol{\Sigma}(t)^\top \boldsymbol{\xi}(t)$. Suppose that there applies an admissibility constraint on the portfolios restricting to left-continuity with right limits, and such that any additional constraints are such that if one $\boldsymbol{\zeta}$ is admissible, then $\boldsymbol{\zeta}^*$ is admissible provided that $\boldsymbol{\zeta}^\top \mathrm{d}\mathbf{Z}$ and $\boldsymbol{\zeta}^{*\top} \mathrm{d}\mathbf{Z}$ coincide in probability law.*

Then in the continuous-time market, any strategy will be (weakly) dominated by one whose portfolio solves

$$\max_{\boldsymbol{\zeta}} \ \boldsymbol{\zeta}(t)^\top (\boldsymbol{\Sigma}(t)^{-1} \boldsymbol{\mu}(t)) \quad \textit{subject to} \quad ||\boldsymbol{\zeta}(t)||_\alpha = \bar{c} \quad \textit{and } (28) \tag{38}$$

and the continuous-time optimization problem inherits any applicable separation / non-diversification properties from Theorem 9 (for the case with risk-free opportunity) and Theorem 11 (for the case without) with $\mathbf{X} := \big(\mathbf{Z}(1) - \mathbf{Z}(0)\big) | \mathcal{F}_0$.

Proof. Partition the timeline into $t^{(j)} = j 2^{-\varpi}$, and consider the discretization with processes represented by the right limit at the left endpoint of the interval – e.g. in terms of (31), $(\gamma^{(j)}, \boldsymbol{\xi}^{(j)}) = \lim_{t \searrow t^{(j)}}(\gamma(t), \boldsymbol{\xi}(t))$. Then at each interval in each partition, the discrete-time Theorem 19 applies, and we can construct a dominating strategy from the funds.

Suppose first $\alpha = 1$, where the unit ball is then strictly convex and the unique dominating strategy varies continuous with the parameters. Then the following argument holds: The discretized strategies converge ucp to the continuous-time strategies, by this continuity and local boundedness of the parameters preserving boundedness in probability; hence the funds (being merely linear subspaces) do as well. Finally, the (consumption,wealth) pairs, being Itô integrals, converge in probability, hence in finite-dimensional distributions. By Assumption 16, the limiting $(\gamma^*(t), \boldsymbol{\xi}^*(t))$, with $\mathrm{d}\gamma^* \sim \mathrm{d}\gamma + (\boldsymbol{\xi}^* - \boldsymbol{\xi})^\top \boldsymbol{\mu}\,\mathrm{d}t \geq \mathrm{d}\gamma$, will dominate, and we have assumed it to be admissible.

For $\alpha \leq 1$, the above could fail at certain times, when the agent, depending on $\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$, swiches from one position to another or reaches indifference. Should we not have convergence, observe that if there are two distinct combinations of opportunities occurring in respective infinite subsequences as $t + 2^{-\varpi} \searrow t$, then dispersion and drift will converge to the same for both subsequences, and each subsequence will in the limit obtain a solution to problem (38).

□

5 Concluding remarks

The standard α-symmetric random variables behave quite a bit like the Fama [11] case of i.i.d. α-stables; indeed, by inspecting Fama's equations (14)–(17), we see that they do not depend upon the specific form of the $\mathbb{L}^\alpha$-norm symmetric characteristic function. As a matter of fact, it is the index of symmetry that is decisive – a special case of this follows, at least in the integrable case, from Owen and Rabinovitch [33] and the observation that scaling the uniform on the unit sphere by a univariate symmetric $\bar{\alpha}$-stable, yields ellipticity, i.e. $\mathbb{L}^2$-norm symmetry, and hence they admit the classical 2-fund separation property even without risk-free opportunity. As we have seen, this holds even in the nonintegrable case $\bar{\alpha} \leq 1$, where the i.i.d. case would lead to non-diversification and discontinuity wrt. the excess drift vector of the optimal adaptation – it is the index of symmetry that is the crucial, not the index of stability. The separation result holds of course for non-stable $\mathbb{L}^\alpha$-norm-symmetric distributions as well, and for any other pseudo-isotropics should the Misiewicz conjecture fail. Furthermore, we have shown CAPM for the case $\alpha > 1$ with an explicit form for the beta.

For the case without risk-free opportunity, we note again that it is the geometry of the standard which is crucial. We have established a $k + 1$-fund separation result for $1 + 1/k$-norm-symmetric variables when k is odd, generalizing the elliptical case $k = 1$. Although infinitely, they form a discrete set in a continuum. The result thus appears much more robust from a central limit theorem considerations, than from a generalized CLT point of view; if tails are heavy and $\alpha < 2$ should be considered, then we have no result for $\alpha \in (4/3, 2)$, and $\alpha \leq 4/3$ implies infinite 4/3-order moment. Then on the other hand, the exact tail index is not necessarily the scope of application for a financial model. Indeed, with the emergence of quantile measures (the infamous *value-at-risk*), the risk of a portfolio is often measured in a way that totally disregards the order of integrability, and may in certain cases penalize diversification – a property which under our framework is only compatible with the even heavier-tailed non-integrabile distributions.

References

[1] R. Agnew, *Counter-examples to an assertion concerning the normal distribution and a new stochastic price fluctuation model.*, Rev. Econ. Stud, 38 (1971), pp. 381–383.

[2] L. Belkacem, J. L. Véhel, and C. Walter, *CAPM, risk and portfolio selection in "α-stable markets"*, Fractals, 08 (2000), pp. 99–115.

[3] K. Bichteler, *Stochastic integration and L^p-theory of semimartingales*, Ann. Probab., 9 (1981), pp. 49–89.

[4] K. Borch, *A note on uncertainty and indifference curves*, Rev. Econ. Stud, 36 (1969), pp. 1–4.

[5] S. Cambanis, R. Keener, and G. Simons, *On α-symmetric multivariate distributions*, J. Multivariate Anal., 13 (1983), pp. 213–233.

[6] D. Cass and J. E. Stiglitz, *The structure of investor preferences and asset returns, and separability in portfolio allocation: a contribution to the pure theory of mutual funds*, J. Econom. Theory, 2 (1970), pp. 122–160.

[7] G. Chamberlain, *A characterization of the distributions that imply mean–variance utility functions*, J. Econom. Theory, 29 (1983), pp. 185–201.

[8] B. de Finetti, *Il problema dei "pieni".*, Giorn. Ist. Ital. Attuari, 11 (1940), pp. 1–88.

[9] R. M. Dudley, *Wiener functionals as Itô integrals*, Ann. Probability, 5 (1977), pp. 140–141.

[10] M. L. Eaton, *On the projections of isotropic distributions*, Ann. Statist., 9 (1981), pp. 391–400.

[11] E. Fama, *Portfolio analysis in a stable Paretian market*, Management Sci., 11 (1965), pp. 404–419.

[12] M. S. Feldstein, *Mean-variance analysis in the theory of liquidity preference and portfolio selection*, Rev. Econ. Stud, 36 (1969), pp. 5–12.

[13] N. C. Framstad, *Portfolio separation without stochastic calculus (almost)*, University of Oslo: Preprint Pure Mathematics, (2001).

[14] ——, *Coherent portfolio separation—inherent systemic risk?*, Int. J. Theor. Appl. Finance, 7 (2004), pp. 909–917.

[15] ——, *On portfolio separation in the Merton problem with bankruptcy or default*, in Proceedings of the International Conference on Stochastic Analysis and Applications, Dordrecht, 2004, Kluwer Acad. Publ., pp. 249–265.

[16] E. D. Giorgi, T. Hens, and J. Mayer, *A note on reward-risk portfolio selection and two-fund separation*, Finance Research Letters, 8 (2011), pp. 52 – 58.

[17] T. Gneiting, *Criteria of Pólya type for radial positive definite functions*, Proc. Amer. Math. Soc., 129 (2001), pp. 2309–2318 (electronic).

[18] P. Hall, *A Comedy of Errors: The Canonical Form for a Stable Characteristic Function*, Bull. Lond. Math. Soc., 13 (1981), pp. 23–27.

[19] B. H. Jasiulis and J. K. Misiewicz, *On the connections between weakly stable and pseudo-isotropic distributions*, Statist. Probab. Lett., 78 (2008), pp. 2751–2755.

[20] A. Khanna and M. Kulldorff, *A generalization of the mutual fund theorem*, Finance Stoch., 3 (1999), pp. 167–185.

[21] A. Koldobsky, *Fourier analysis in convex geometry*, vol. 116 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 2005.

[22] ——, *A note on positive definite norm dependent functions*, in High dimensional probability V: The Luminy Volume, C. Houdré, V. Koltchinskii, D. M. Mason, and M. Peligrad, eds., Institute of Mathematical Statistics Lecture Notes—Collections, volume 5, Institute of Mathematical Statistics, Beachwood, OH, 2009, pp. 30–36.

[23] ——, *Positive definite functions and stable random vectors*, Israel J. Math., 185 (2011), pp. 277–292.

[24] A. D. Lisitskiĭ, *The Eaton problem and multiplicative properties of multidimensional distributions*, Teor. Veroyatnost. i Primenen., 42 (1997), pp. 696–714.

[25] H. Markowitz, *Portfolio selection*, J. Finance, 7 (1952), pp. 77–91.

[26] ——, *de Finetti scoops Markowitz*, Journal of Investment Management, 4 (2006), pp. 3–18.

[27] P.-A. Meyer, *Caractérisation des semimartingales, d'après Dellacherie*, in Séminaire de Probabilités, XIII (Univ. Strasbourg, Strasbourg, 1977/78), vol. 721 of Lecture Notes in Math., Springer, Berlin, 1979, pp. 620–623.

[28] J. K. Misiewicz, *Substable and pseudo-isotropic processes – connections with the geometry of subspaces of L_α-spaces*, Dissertationes Math. (Rozprawy Mat.), 358 (1996), p. 91.

[29] J. K. Misiewicz and C. Ryll-Nardzewski, *Norm dependent positive definite functions and measures on vector spaces*, in Probability theory on vector spaces, IV (Łańcut, 1987), vol. 1391 of Lecture Notes in Math., Springer, Berlin, 1989, pp. 284–292.

[30] J. K. Misiewicz and C. L. Scheffer, *Pseudo isotropic measures*, Nieuw Arch. Wisk. (4), 8 (1990), pp. 111–152.

[31] S. Ortobelli, I. Huber, S. T. Rachev, and E. S. Schwartz, *Portfolio choice theory with non-Gaussian distributed returns*, in Handbook of Heavy Tailed Distributions in Finance, S. T. Rachev, ed., North-Holland, Amsterdam, February 2003, pp. 547 – 594.

[32] L. P. Østerdal, *The mass transfer approach to multivariate discrete first order stochastic dominance: Direct proof and implications*, J. Math. Econom., 46 (2010), pp. 1222 – 1228.

[33] J. Owen and R. Rabinovitch, *On the class of elliptical distributions and their applications to the theory of portfolio choice*, J. Finance, 38 (1983), pp. 745–52.

[34] P. Protter, *Stochastic integration and differential equations*, Springer-Verlag, Berlin, 1990.

[35] S. A. Ross, *Mutual fund separation in financial theory—the separating distributions*, J. Econom. Theory, 17 (1978), pp. 254–286.

[36] A. D. Roy, *Safety first and the holding of assets*, Econometrica, 20 (1952), pp. 431–449.

[37] G. Samorodnitsky and M. S. Taqqu, *Stable non-Gaussian random processes*, Chapman & Hall, New York, 1994. Stochastic models with infinite variance.

[38] P. A. Samuelson, *Efficient portfolio selection for Pareto-Lévy investments*, Journal of

Financial and Quantitative Analysis, 2 (1967), pp. 107–122.

[39] P. A. Samuelson, *General proof that diversification pays*, Journal of Financial and Quantitative Analysis, 2 (1967), pp. 1–13.

[40] W. Schachermayer, M. Sîrbu, and E. Taflin, *In which financial markets do mutual fund theorems hold true?*, Finance Stoch., 13 (2009), pp. 49–77.

[41] I. J. Schoenberg, *Metric spaces and completely monotone functions*, Ann. of Math. (2), 39 (1938), pp. 811–841.

[42] ——, *Metric spaces and positive definite functions*, Trans. Amer. Math. Soc., 44 (1938), pp. 522–536.

[43] J. Tobin, *Liquidity preference as behavior toward risk*, Rev. Econ. Stud., 25 (1958), pp. 65–86.

[44] R. J. Williams, *Book Review: Stochastic integration and differential equations—a new approach*, Bull. Amer. Math. Soc. (N.S.), 25 (1991), pp. 170–180.