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AN ECONOMETRIC MARKET MODEL OF CAPITAL AND INVESTMENT INSPIRED BY HAAVELMO

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ABSTRACT: In the paper steps are taken towards integration of two parts of Trygve Haavelmo's work: investment theory and econometrics of interrelated markets. Attempts are made to bring the duality in the representation of the capital service price and the capital quantity in relation to the investment price and quantity into the foreground, by confronting it with elements from simultaneous equation modeling of vector autoregressive systems with exogenous variables (VARX), using linear four-equation models. The role of the interest rate and the modeling of the expectation element in the capital service price and the capital's retirement pattern, and their joint effect on the model's investment quantity and price dynamics are discussed. Simulation experiments illustrate some of the theoretical points. An extension relaxing geometric decay is outlined.

KEYWORDS: Investment theory. Econometrics of investment. Stock-flow interaction. Dynamic stability. Capital retirement. Price expectation. Duality. Final form. ARMAX.

JEL CLASSIFICATION: C32, C62, E22, E27

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1 Introduction

The pioneering article by Trygve Haavelmo on statistical implications of an economic model, Haavelmo (1943), related to a single market. In Girshick and Haavelmo (1947, 1953) it was extended to an empirical five-equation model specifying *two interrelated markets*, its five equations representing demand for food by consumers, supply of foodstuffs by farmers, demand for farm food products by the commercial sector, supply of food to the retail market, and generation of consumers' income.

One of the chapters in Haavelmo's study of *investment theory*, Haavelmo (1960, Chapter 32), also dealt with two interrelated markets, from a theoretical viewpoint, in discussing stock-flow dynamics in a capital market. He there considered a model with four endogenous variables, two quantities and two prices. The two interrelated quantities are capital as a stock and investment as a flow, and the two interrelated prices are the price of the stock of the durable capital good and the cost of exploiting the service flow from this stock. Haavelmo's contributions to *econometric market models* and his contribution to the *theory of investment* by his insistence on the integration of investment-flow-capital-stock dynamics in explaining investment fluctuation have rarely been given attention in applied econometric work. Two quotes illustrate his approach:

> "...the rate of investment is determined by a conjunction of the cost of producing capital goods and the yield from its use as a factor of production.... it is, actually, not the users of capital who "demand" investment, it is the producers of capital goods who determine how much they want to produce at the current price of capital" [Haavelmo (1960, p. 196)], "The demand for *investment* cannot simply be derived from the demand for *capital*. Demand for a finite addition to the stock of capital can lead to any rate of investment, from almost zero to infinity, depending on the additional hypothesis we introduce regarding the speed of reaction of the capital-users"[Haavelmo (1960, p. 216)].

In this paper, by elaborating models and model sketches with linear functional forms, I take some steps towards integration of these two parts of Haavelmo's work. Relying, *inter alia*, on Jorgenson (1974) and Biørn (1989, Chapter 4), I attempt to bring the duality in the representation of the capital service price vis-à-vis the capital stock prices and the representation of the capital quantity in relation to the investment flow into focus. The interest rate and the capital's retirement pattern will 'interfere' in the dynamic process. The specific purpose is to elaborate one of Haavelmo's innovative elements in his investment theory for confrontation with elements from simultaneous equation modeling of vector autoregressive systems with exogenous variables (VARX systems), as exposed by, *inter alia*, Quenouille (1957), Zellner and Palm (1974), and extended by, *inter alia*, Hsiao (1997), for simplicity using linear functional forms.

The paper proceeds as follows. In Section 2, a general linear dynamic model is outlined as a background, and its three derived forms, to be exemplified in the following sections, are defined. Next, in Section 3, we present a four-equation prototype model with two interrelated quantities and two interrelated prices, supplemented by examples to illustrate dynamic properties. In Section 4 an econometric market model for investment and capital inspired by ideas from Haavelmo's investment theory is considered. Some simulations to illustrate theoretical points, including conditions for dynamic stability, are provided. In Section 5 consequences of relaxing the geometric decay specification of the capital retirement process, are discussed. Concluding remarks follow in Section 6.

2 Background: A linear dynamic model and its derived forms

Consider a dynamic model with an $(N\times 1)$-vector of endogenous variables, $\boldsymbol{y}_t$, a $(K\times 1)$-vector of exogenous variables, $\boldsymbol{x}_t$, and an $(N\times 1)$-vector of disturbances $\boldsymbol{u}_t$. Its *structural form (SF)* is

$$\boldsymbol{B}_0\,\boldsymbol{y}_t = \boldsymbol{B}_1\,\boldsymbol{y}_{t-1} + \boldsymbol{\Gamma}\,\boldsymbol{x}_t + \boldsymbol{u}_t, \qquad \boldsymbol{u}_t \sim \mathsf{IID}(\boldsymbol{0},\boldsymbol{\Sigma}), \tag{2.1}$$

where t denotes time, and $\boldsymbol{B}_0$, $\boldsymbol{B}_1$, $\boldsymbol{\Gamma}$ are $(N\times N)$, $(N\times N)$ and $(N\times K)$ coefficient matrices, respectively. The model's *reduced form (RF)* is:

$$\begin{aligned} \boldsymbol{y}_t &= \boldsymbol{\Pi}_1\boldsymbol{y}_{t-1} + \boldsymbol{\Pi}_0\boldsymbol{x}_t + \boldsymbol{\epsilon}_t, \quad \boldsymbol{\epsilon}_t = \boldsymbol{B}_0^{-1}\boldsymbol{u}_t \sim \mathsf{IID}(\boldsymbol{0},\boldsymbol{\Omega}),\\ \boldsymbol{\Pi}_1 &= \boldsymbol{B}_0^{-1}\boldsymbol{B}_1, \qquad \boldsymbol{\Pi}_0 = \boldsymbol{B}_0^{-1}\boldsymbol{\Gamma}, \qquad \boldsymbol{\Omega} = \boldsymbol{B}_0^{-1}\boldsymbol{\Sigma}(\boldsymbol{B}_0')^{-1}. \end{aligned} \tag{2.2}$$

Successive inserting yields

$$\boldsymbol{y}_t = \boldsymbol{\Pi}_1^{\theta}\boldsymbol{y}_{t-\theta} + \boldsymbol{\Pi}_0\boldsymbol{x}_t + \sum_{i=1}^{\theta-1}\boldsymbol{\Pi}_1^i\boldsymbol{\Pi}_0\boldsymbol{x}_{t-i} + \boldsymbol{\epsilon}_t + \sum_{i=1}^{\theta-1}\boldsymbol{\Pi}_1^i\boldsymbol{\epsilon}_{t-i}.$$

The model's *stability condition*,

$$\lim_{\theta\to\infty}\boldsymbol{\Pi}_1^{\theta} = \lim_{\theta\to\infty}\left[\boldsymbol{B}_0^{-1}\boldsymbol{B}_1\right]^{\theta} = \boldsymbol{0},$$

will be satisfied if and only if [see Lütkepohl (2005, Section 2.1)]

$$\left\{\begin{array}{c} \textit{The eigenvalues of } \boldsymbol{\Pi}_1,\ (\lambda_1,\dots,\lambda_N),\ \textit{solving } |\boldsymbol{\Pi}_1-\lambda\boldsymbol{I}_N|=0,\\ \textit{are all inside the unit circle} \iff \\ \textit{all } N \textit{ roots of } |\boldsymbol{B}_0-q\boldsymbol{B}_1|=0 \textit{ are outside the unit circle.} \end{array}\right\} \tag{2.3}$$

Since, if (2.3) holds, $\boldsymbol{I}_N+\sum_{i=1}^{\infty}(\boldsymbol{\Pi}_1\mathsf{L})^i = (\boldsymbol{I}_N-\boldsymbol{\Pi}_1\mathsf{L})^{-1}$, L denoting the lag operator, it follows that the model's *final form (FF)* can be written as:

$$\begin{aligned} \boldsymbol{y}_t &= \boldsymbol{\Pi}_0\boldsymbol{x}_t + \sum_{i=1}^{\infty}\boldsymbol{\Pi}_1^i\boldsymbol{\Pi}_0\boldsymbol{x}_{t-i} + \boldsymbol{\epsilon}_t + \sum_{i=1}^{\infty}\boldsymbol{\Pi}_1^i\boldsymbol{\epsilon}_{t-i}\\ &= (\boldsymbol{I}_N-\boldsymbol{\Pi}_1\mathsf{L})^{-1}\boldsymbol{\Pi}_0\boldsymbol{x}_t + (\boldsymbol{I}_N-\boldsymbol{\Pi}_1\mathsf{L})^{-1}\boldsymbol{\epsilon}_t. \end{aligned} \tag{2.4}$$

Then also $\sum_{i=0}^{\infty}\boldsymbol{\Pi}_1^i=(\boldsymbol{I}_N-\boldsymbol{\Pi}_1)^{-1}$, so that

$$\boldsymbol{\Pi} \equiv \sum_{i=0}^{\infty}\boldsymbol{\Pi}_1^i\boldsymbol{\Pi}_0=(\boldsymbol{I}_N-\boldsymbol{\Pi}_1)^{-1}\boldsymbol{\Pi}_0 = (\boldsymbol{B}_0-\boldsymbol{B}_1)^{-1}\boldsymbol{\Gamma}, \tag{2.5}$$

which is the RF coefficient matrix of the static counterpart to (2.1):

$$(\boldsymbol{B}_0-\boldsymbol{B}_1)\boldsymbol{y} = \boldsymbol{\Gamma}\boldsymbol{x} + \boldsymbol{u}. \tag{2.6}$$

Writing (2.1) and (2.2) as

$$\boldsymbol{B}(\mathsf{L})\boldsymbol{y}_t = \boldsymbol{\Gamma}\boldsymbol{x}_t + \boldsymbol{u}_t, \qquad \boldsymbol{B}(\mathsf{L}) = \boldsymbol{B}_0 - \boldsymbol{B}_1\mathsf{L}, \tag{2.7}$$

$$\boldsymbol{\Lambda}(\mathsf{L})\boldsymbol{y}_t = \boldsymbol{\Pi}_0\boldsymbol{x}_t + \boldsymbol{\epsilon}_t, \qquad \boldsymbol{\Lambda}(\mathsf{L}) = \boldsymbol{I}_N-\boldsymbol{\Pi}_1\mathsf{L}, \tag{2.8}$$

(2.5) can be rewritten as

(2.9) $\boldsymbol{\Pi} = \boldsymbol{\Lambda}(1)^{-1}\boldsymbol{\Pi}_0 = \boldsymbol{B}(1)^{-1}\boldsymbol{\Gamma}, \qquad \boldsymbol{\Lambda}(1) = \boldsymbol{I}_N - \boldsymbol{\Pi}_1,\ \boldsymbol{B}(1) = \boldsymbol{B}_0 - \boldsymbol{B}_1.$

Let $\boldsymbol{B}^*(\mathsf{L})$ be the adjoint of $\boldsymbol{B}(\mathsf{L})$, so that

(2.10) $$\boldsymbol{B}^{-1}(\mathsf{L}) = \frac{\boldsymbol{B}^*(\mathsf{L})}{|\boldsymbol{B}(\mathsf{L})|}.$$

Premultiplying (2.7) by $\boldsymbol{B}^*(\mathsf{L}) = |\boldsymbol{B}(\mathsf{L})|\boldsymbol{B}^{-1}(\mathsf{L})$ yields

(2.11) $$|\boldsymbol{B}(\mathsf{L})|\boldsymbol{y}_t = \boldsymbol{B}^*(\mathsf{L})\,\boldsymbol{\Gamma}\,\boldsymbol{x}_t + \boldsymbol{B}^*(\mathsf{L})\boldsymbol{u}_t.$$

We call this transformation of (2.1) the model's *autoregressive form*, or its *ARMAX-form (AF)*. All of its equations have the same autoregressive part, with $|\boldsymbol{B}(\mathsf{L})|$ as the common lag-polynomial of the endogenous variables, and with disturbance vector, $\boldsymbol{B}^*(\mathsf{L})\boldsymbol{u}_t$, whose elements are MA(1)-transformations of the SF disturbances, and lag-distributions on the exogenous variables. This model form exists if $|\boldsymbol{B}(\mathsf{L})| \neq 0$, but (2.3) is not required. From (2.10) and (2.11), if (2.3) holds, we obtain

(2.12) $$\boldsymbol{y}_t = \frac{\boldsymbol{B}^*(\mathsf{L})}{|\boldsymbol{B}(\mathsf{L})|}\boldsymbol{\Gamma}\boldsymbol{x}_t + \frac{\boldsymbol{B}^*(\mathsf{L})}{|\boldsymbol{B}(\mathsf{L})|}\boldsymbol{u}_t = \boldsymbol{B}^{-1}(\mathsf{L})\boldsymbol{\Gamma}\boldsymbol{x}_t + \boldsymbol{B}^{-1}(\mathsf{L})\boldsymbol{u}_t.$$

This equation expresses the FF in terms of the SF coefficients, while in (2.4) it is expressed by the RF coefficients.

The transformations which carry one model form into another can be summarized as follows, condition (2.3) being required for (c), (d) and (e) only:

(a) *From SF to RF:* Premultiply SF by $\boldsymbol{B}_0^{-1}$.
(b) *From SF to AF:* Premultiply SF by $\boldsymbol{B}^*(\mathsf{L})$.
(c) *From RF to FF:* Premultiply RF by $\boldsymbol{\Lambda}^{-1}(\mathsf{L})$.
(d) *From AF to FF:* Divide AF by $|\boldsymbol{B}(\mathsf{L})|$.
(e) *From SF to FF:* Premultiply SF by $\boldsymbol{B}^{-1}(\mathsf{L})$.

3 Two four-equation market models

As a preamble to and background for the presentation of the Haavelmo-inspired market model we consider two four-equation models ($N = 4$) which exemplify the general setup in Section 2. It has two quantities, (k_t, j_t), and two prices, (c_t, p_t). The first equation is a demand function which determines the quantity k_t from the price c_t and a shift variable z_t, which can contain exogenous terms and a disturbance. The second equation is a supply function which determines the quantity j_t from the price p_t and a (scalar) shift variable x_t, which can contain exogenous terms and a disturbance. The third equation connects the two quantity variables and can be interpreted as a supply function for k_t. The fourth equation connects the price variables, expressing c_t as a first-order lag-distribution in p_t.

Simple version: The simplest of the two models, exemplifying (2.1), is

(3.1) $$k_t = \alpha c_t + z_t,$$
(3.2) $$j_t = \beta p_t + x_t,$$
(3.3) $$k_t = \lambda k_{t-1} + j_t,$$
(3.4) $$c_t = \theta_0 p_t + \theta_1 p_{t-1},$$

where $\alpha < 0$, $\beta > 0$, $0 \leq \lambda < 1$. Eliminating j_t and c_t gives a system expressing the supply and the demand for the quantity k_t as functions of p_t and the shift variables:

$$(3.5)\quad \begin{aligned} k_t &= \lambda k_{t-1} + \beta p_t + x_t, \\ k_t &= \alpha\theta_0 p_t + \alpha\theta_1 p_{t-1} + z_t, \end{aligned} \iff \begin{bmatrix} 1 & -\beta \\ 1 & -\alpha\theta_0 \end{bmatrix} \begin{bmatrix} k_t \\ p_t \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \alpha\theta_1 \end{bmatrix} \begin{bmatrix} k_{t-1} \\ p_{t-1} \end{bmatrix} + \begin{bmatrix} x_t \\ z_t \end{bmatrix},$$

with

$$\boldsymbol{B}(\mathsf{L}) = \begin{bmatrix} 1-\lambda\mathsf{L} & -\beta \\ 1 & -\alpha(\theta_0+\theta_1\mathsf{L}) \end{bmatrix}, \quad \boldsymbol{B}^*(\mathsf{L}) = \begin{bmatrix} -\alpha(\theta_0+\theta_1\mathsf{L}) & \beta \\ -1 & 1-\lambda\mathsf{L} \end{bmatrix},$$

$$\boldsymbol{\Gamma}(\mathsf{L}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad |\boldsymbol{B}(\mathsf{L})| = \beta - \alpha\theta_0 + \alpha(\lambda\theta_0 - \theta_1)\mathsf{L} + \lambda\alpha\theta_1\mathsf{L}^2.$$

The roots of the polynomial equation $|\boldsymbol{B}(q)| = 0$, which is of the second order if $\lambda \neq 0$ and $\theta_1 \neq 0$, are:

$$(3.6)\quad \begin{aligned} q &= \frac{-\alpha(\lambda\theta_0 - \theta_1) \pm \sqrt{\alpha^2(\lambda\theta_0+\theta_1)^2 - 4\alpha\beta\lambda\theta_1}}{2\lambda\alpha\theta_1} \\ &= \frac{-(\lambda\theta_0 - \theta_1) \pm \sqrt{(\lambda\theta_0+\theta_1)^2 - 4(\beta/\alpha)\lambda\theta_1}}{2\lambda\theta_1}. \end{aligned}$$

They can be inside or outside the unit circle. Since $\alpha < 0$, the radicand is positive for $\theta_1 > 0$. If $\theta_1 < 0$ and $|\lambda\theta_0 + \theta_1| < 2\sqrt{\lambda\theta_1\beta/\alpha}$, the roots are complex conjugate. If $\lambda\theta_1 = 0$, the lag polynomial is of the first order, with

$$\begin{aligned} \theta_1 = 0,\ \lambda > 0 &\implies q = \frac{\theta_0 - \beta/(-\alpha)}{\lambda\theta_0}, \\ \lambda = 0,\ \theta_1 \neq 0 &\implies q = \frac{\theta_0 - \beta/(-\alpha)}{(-\theta_1)}. \end{aligned}$$

Consider next the derived forms for this simple model.

Reduced form: $$\begin{bmatrix} k_t \\ p_t \end{bmatrix} = \frac{1}{\beta - \alpha\theta_0} \begin{bmatrix} -\alpha\theta_0 & \beta \\ -1 & 1 \end{bmatrix} \left\{ \begin{bmatrix} \lambda & 0 \\ 0 & \alpha\theta_1 \end{bmatrix} \begin{bmatrix} k_{t-1} \\ p_{t-1} \end{bmatrix} + \begin{bmatrix} x_t \\ z_t \end{bmatrix} \right\} \implies$$

$$\begin{aligned} k_t &= -\frac{\alpha\theta_0\lambda}{\beta-\alpha\theta_0}k_{t-1} + \frac{\beta\alpha\theta_1}{\beta-\alpha\theta_0}p_{t-1} - \frac{\alpha\theta_0}{\beta-\alpha\theta_0}x_t + \frac{\beta}{\beta-\alpha\theta_0}z_t, \\ p_t &= -\frac{\lambda}{\beta-\alpha\theta_0}k_{t-1} + \frac{\alpha\theta_1}{\beta-\alpha\theta_0}p_{t-1} - \frac{1}{\beta-\alpha\theta_0}x_t + \frac{1}{\beta-\alpha\theta_0}z_t, \\ j_t &= -\frac{\beta\lambda}{\beta-\alpha\theta_0}k_{t-1} + \frac{\beta\alpha\theta_1}{\beta-\alpha\theta_0}p_{t-1} + \frac{\alpha\theta_0}{\beta-\alpha\theta_0}x_t + \frac{\beta}{\beta-\alpha\theta_0}z_t, \\ c_t &= -\frac{\theta_0\lambda}{\beta-\alpha\theta_0}k_{t-1} + \frac{\beta\theta_1}{\beta-\alpha\theta_0}p_{t-1} - \frac{\theta_0}{\beta-\alpha\theta_0}x_t + \frac{\theta_0}{\beta-\alpha\theta_0}z_t. \end{aligned}$$

ARMAX-form: $$[\beta - \alpha\theta_0 + \alpha(\lambda\theta_0 - \theta_1)\mathsf{L} + \lambda\alpha\theta_1\mathsf{L}^2] \begin{bmatrix} k_t \\ p_t \end{bmatrix} = \begin{bmatrix} -\alpha(\theta_0+\theta_1\mathsf{L}) & \beta \\ -1 & 1-\lambda\mathsf{L} \end{bmatrix} \begin{bmatrix} x_t \\ z_t \end{bmatrix} \iff$$

$$(3.7)\qquad k_t = -\frac{\alpha(\lambda\theta_0-\theta_1)}{\beta-\alpha\theta_0}k_{t-1} - \frac{\lambda\alpha\theta_1}{\beta-\alpha\theta_0}k_{t-2} + \frac{1}{\beta-\alpha\theta_0}[\beta z_t - \alpha(\theta_0+\theta_1\mathsf{L})x_t],$$

$$(3.8)\qquad p_t = -\frac{\alpha(\lambda\theta_0-\theta_1)}{\beta-\alpha\theta_0}p_{t-1} - \frac{\lambda\alpha\theta_1}{\beta-\alpha\theta_0}p_{t-2} + \frac{1}{\beta-\alpha\theta_0}[(1-\lambda\mathsf{L})z_t - x_t].$$

Since (3.3)–(3.4) imply $j_t = (1-\lambda\mathsf{L})k_t$ and $c_t = (\theta_0+\theta_1\mathsf{L})p_t$, it follows that

$$(3.9)\qquad j_t = -\frac{\alpha(\lambda\theta_0-\theta_1)}{\beta-\alpha\theta_0}j_{t-1} - \frac{\lambda\alpha\theta_1}{\beta-\alpha\theta_0}j_{t-2} + \frac{1-\lambda\mathsf{L}}{\beta-\alpha\theta_0}[\beta z_t - \alpha(\theta_0+\theta_1\mathsf{L})x_t],$$

$$(3.10)\qquad c_t = -\frac{\alpha(\lambda\theta_0-\theta_1)}{\beta-\alpha\theta_0}c_{t-1} - \frac{\lambda\alpha\theta_1}{\beta-\alpha\theta_0}c_{t-2} + \frac{\theta_0+\theta_1\mathsf{L}}{\beta-\alpha\theta_0}[(1-\lambda\mathsf{L})z_t - x_t].$$

Final form (if both roots of $|\boldsymbol{B}(q)|$ are outside the unit circle):

$$k_t = \frac{1}{|\boldsymbol{B}(\mathsf{L})|}[-\alpha(\theta_0+\theta_1\mathsf{L})x_t + \beta z_t],$$

$$p_t = \frac{1}{|\boldsymbol{B}(\mathsf{L})|}[-x_t + (1-\lambda\mathsf{L})z_t],$$

$$j_t = \frac{1-\lambda\mathsf{L}}{|\boldsymbol{B}(\mathsf{L})|}[-\alpha(\theta_0+\theta_1\mathsf{L})x_t + \beta z_t],$$

$$c_t = \frac{\theta_0+\theta_1\mathsf{L}}{|\boldsymbol{B}(\mathsf{L})|}[-x_t + (1-\lambda\mathsf{L})z_t].$$

Equations (3.7) and (3.9) are neither supply, nor demand functions, but *confluent* relations, containing elements from both, and therefore having a lower degree of *autonomy* than either.

Four *boundary cases* are worth a closer examination.

[a]. $\lambda=1, \theta_0=1, \theta_1=0 \Longrightarrow j_t=\Delta k_t, c_t=p_t$. This implies

$$k_t = \frac{\beta z_t - \alpha x_t}{\beta-\alpha(1-\mathsf{L})},\quad j_t = \frac{(1-\mathsf{L})[\beta z_t - \alpha x_t]}{\beta-\alpha(1-\mathsf{L})},\quad c_t = p_t = \frac{(1-\mathsf{L})z_t - x_t}{\beta-\alpha(1-\mathsf{L})}.$$

The equation for j_t has unit roots for z_t and x_t, $c_t = p_t$ has a unit root for z_t. The root of the common AR polynomial is $\beta/(-\alpha)+1>1$.

[b]. $\lambda=0, \theta_0=1, \theta_1=-1 \Longrightarrow j_t = k_t, c_t = \Delta p_t$. This implies

$$j_t = k_t = \frac{\beta z_t - \alpha(1-\mathsf{L})x_t}{\beta-\alpha(1-\mathsf{L})},\quad p_t = \frac{z_t - x_t}{\beta-\alpha(1-\mathsf{L})},\quad c_t = \frac{(1-\mathsf{L})[z_t - x_t]}{\beta-\alpha(1-\mathsf{L})}.$$

The equation for c_t has unit roots for both z_t and x_t, $j_t = k_t$ has a unit root for x_t. The root of the common AR polynomial is the same as in example [a].

[c]. $\lambda=0, \theta_0=-1, \theta_1=1 \Longrightarrow j_t = k_t, c_t = -\Delta p_t$. This implies

$$j_t = k_t = \frac{\beta z_t + \alpha(1-\mathsf{L})x_t}{\beta+\alpha(1-\mathsf{L})},\quad p_t = \frac{z_t - x_t}{\beta+\alpha(1-\mathsf{L})},\quad c_t = -\frac{(1-\mathsf{L})[z_t - x_t]}{\beta+\alpha(1-\mathsf{L})}.$$

The equation for c_t has unit roots for both z_t and x_t, $j_t = k_t$ has a unit root for x_t only. The root of the common AR polynomial is $1-\beta/(-\alpha)$, which is negative for $\beta > |\alpha|$, but may be both inside and outside the $(-1,+1)$ interval. Even this simple example has the potential for giving oscillating and non-oscillating and stable and non-stable solutions.

[d]. $\lambda=1, \theta_0=-1, \theta_1=1 \Longrightarrow j_t = \Delta k_t,\ c_t = -\Delta p_t$. Then we get

$$
\begin{aligned}
[\beta+\alpha(1-\mathsf{L})^2]k_t &= \beta z_t + \alpha(1-\mathsf{L})x_t,\\
[\beta+\alpha(1-\mathsf{L})^2]p_t &= (1-\mathsf{L})z_t - x_t,\\
[\beta+\alpha(1-\mathsf{L})^2]j_t &= \beta(1-\mathsf{L})z_t + \alpha(1-\mathsf{L})^2 x_t,\\
[\beta+\alpha(1-\mathsf{L})^2]c_t &= -(1-\mathsf{L})^2 z_t + (1-\mathsf{L})x_t.
\end{aligned}
$$

The equation for j_t has a single unit root for z_t and a double unit root for x_t. The converse holds for the equation for c_t. The roots of the AR polynomial, obtained from (3.6), are $1 \pm \sqrt{\beta/(-\alpha)}$, which are real and add to 2. Hence, the model has the potential for giving oscillating and non-oscillating as well as stable and non-stable solutions.

Generalized version: A generalized model follows from (3.1)–(3.4) by extending $\alpha, \beta, \lambda\mathsf{L}, \theta_0+\theta_1\mathsf{L}$ to *lag polynomials* $\alpha(\mathsf{L}), \beta(\mathsf{L}), \lambda(\mathsf{L}), \theta(\mathsf{L})$ and attaching the polynomial $\eta(\mathsf{L})$ to j_t in (3.3):

$$k_t = \alpha(\mathsf{L})c_t + z_t, \tag{3.11}$$
$$j_t = \beta(\mathsf{L})p_t + x_t, \tag{3.12}$$
$$k_t = \lambda(\mathsf{L})k_t + \eta(\mathsf{L})j_t, \tag{3.13}$$
$$c_t = \theta(\mathsf{L})p_t. \tag{3.14}$$

Equations (3.11) and (3.12) can be interpreted as generalized cobweb demand and supply equations, respectively, (3.13) connects the two quantities by a rational lag-distribution the two quantities, and (3.14) connects the two prices by a finite lag-distribution. Eliminating c_t and j_t leads to the following generalization of (3.5):

$$\begin{bmatrix} 1-\lambda(\mathsf{L}) & -\beta(\mathsf{L})\eta(\mathsf{L}) \\ 1 & -\alpha(\mathsf{L})\theta(\mathsf{L}) \end{bmatrix} \begin{bmatrix} k_t \\ p_t \end{bmatrix} = \begin{bmatrix} \eta(\mathsf{L}) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ z_t \end{bmatrix}, \tag{3.15}$$

with

$$\boldsymbol{B}(\mathsf{L}) = \begin{bmatrix} 1-\lambda(\mathsf{L}) & -\beta(\mathsf{L})\eta(\mathsf{L}) \\ 1 & -\alpha(\mathsf{L})\theta(\mathsf{L}) \end{bmatrix}, \qquad \boldsymbol{\Gamma}(\mathsf{L}) = \begin{bmatrix} \eta(\mathsf{L}) & 0 \\ 0 & 1 \end{bmatrix},$$
$$\boldsymbol{B}^*(\mathsf{L}) = \begin{bmatrix} -\alpha(\mathsf{L})\theta(\mathsf{L}) & \beta(\mathsf{L})\eta(\mathsf{L}) \\ -1 & 1-\lambda(\mathsf{L}) \end{bmatrix}, \qquad |\boldsymbol{B}(\mathsf{L})| = \beta(\mathsf{L})\eta(\mathsf{L}) - \alpha(\mathsf{L})\theta(\mathsf{L})[1-\lambda(\mathsf{L})],$$

and the following generalizations of the ARMAX-form (3.7)–(3.10):

$$|\boldsymbol{B}(\mathsf{L})|k_t = -\alpha(\mathsf{L})\theta(\mathsf{L})\eta(\mathsf{L})x_t + \beta(\mathsf{L})\eta(\mathsf{L})z_t, \tag{3.16}$$
$$|\boldsymbol{B}(\mathsf{L})|p_t = -\eta(\mathsf{L})x_t + [1-\lambda(\mathsf{L})]z_t, \tag{3.17}$$
$$|\boldsymbol{B}(\mathsf{L})|j_t = -\alpha(\mathsf{L})\theta(\mathsf{L})[1-\lambda(\mathsf{L})]x_t + \beta(\mathsf{L})[1-\lambda(\mathsf{L})]z_t, \tag{3.18}$$
$$|\boldsymbol{B}(\mathsf{L})|c_t = -\theta(\mathsf{L})\eta(\mathsf{L})x_t + \theta(\mathsf{L})[1-\lambda(\mathsf{L})]z_t. \tag{3.19}$$

If the disturbance parts of x_t and z_t are white noise, the disturbances of this model form will be MA processes of order determined by the lag-polynomial orders.

The common lag-polynomial in the AR-part of the equations for quantities, (3.16) and (3.18), and for prices, (3.17) and (3.19), has elements which originate from the demand and the supply response mechanisms, the equation connecting the quantities, and the lag-distribution between the prices. This generalized model exemplifies the simultaneous equation time-series analysis (SEMTSA) approach; see Zellner and Palm (1974), Zellner (1979), Palm (1986), and Hsiao (1997).

We can, letting $\bar{\alpha} = \alpha(1)$, $\bar{\beta} = \beta(1)$, $\bar{\lambda} = \lambda(1)$, $\bar{\eta} = \eta(1)$, $\bar{\theta} = \theta(1)$, write the corresponding static RF, satisfying $c_t = \bar{\theta} p_t$, $j_t = [(1-\bar{\lambda})/\bar{\eta}]k_t$ and corresponding to (2.6), as

$$\begin{aligned}
k_t &= \frac{\bar{\eta}(\bar{\beta} z_t - \bar{\alpha}\bar{\theta} x_t)}{\bar{\beta}\bar{\eta} - \bar{\alpha}\bar{\theta}(1-\bar{\lambda})} = \bar{\alpha} c_t + z_t, \\
p_t &= \frac{(1-\bar{\lambda}) z_t - \bar{\eta} x_t}{\bar{\beta}\bar{\eta} - \bar{\alpha}\bar{\theta}(1-\bar{\lambda})}, \\
j_t &= \frac{(1-\bar{\lambda})(\bar{\beta} z_t - \bar{\alpha}\bar{\theta} x_t)}{\bar{\beta}\bar{\eta} - \bar{\alpha}\bar{\theta}(1-\bar{\lambda})} = \bar{\beta} p_t + x_t, \\
c_t &= \frac{\bar{\theta}[(1-\bar{\lambda}) z_t - \bar{\eta} x_t]}{\bar{\beta}\bar{\eta} - \bar{\alpha}\bar{\theta}(1-\bar{\lambda})}.
\end{aligned}$$

Below five models exemplifying (3.11)–(3.14) are given.

Model A: Baseline static model
If $\alpha(\mathsf{L}) = \alpha$, $\beta(\mathsf{L}) = \beta$, $\lambda(\mathsf{L}) = 0$, $\eta(\mathsf{L}) = \theta(\mathsf{L}) = 1$, all dynamics is eliminated, the two quantities and the two prices coincide, and all derived model forms degenerate to

$$\begin{aligned}
k_t = j_t &= -\frac{\alpha}{\beta-\alpha} x_t + \frac{\beta}{\beta-\alpha} z_t, \\
p_t = c_t &= -\frac{1}{\beta-\alpha} x_t + \frac{1}{\beta-\alpha} z_t.
\end{aligned}$$

Model B: Introducing delayed supply response – supply cobweb
Let now $\beta(\mathsf{L}) = \beta_1 \mathsf{L}$, but retain the other assumptions in Model A. The two quantities and the two prices again coincide. The only dynamic element is the lagged supply response. The solution oscillates. Stability requires $|\alpha| > \beta_1$, *i.e.*, demand should respond more strongly to the current price than supply responds to the lagged price. We get

REDUCED FORM	ARMAX-FORM	FINAL FORM
$k_t = \beta_1 p_{t-1} + x_t,$	$k_t = \frac{\beta_1}{\alpha} k_{t-1} + x_t - \frac{\beta_1}{\alpha} z_{t-1},$	$k_t = \frac{1}{\alpha - \beta_1 \mathsf{L}}[\alpha x_t - \beta_1 \mathsf{L} z_t],$
$p_t = \frac{\beta_1}{\alpha} p_{t-1} + \frac{1}{\alpha}[x_t - z_t],$	$p_t = \frac{\beta_1}{\alpha} p_{t-1} + \frac{1}{\alpha}[x_t - z_t],$	$p_t = \frac{1}{\alpha - \beta_1 \mathsf{L}}[x_t - z_t].$

The RF equation of k_t and the supply function coincide. The RF- and the AF-equations for p_t coincide.

Model C: Introducing delayed demand response – demand cobweb
Let $\alpha(\mathsf{L}) = \alpha_1 \mathsf{L}$, but retain the other assumptions of Model A. The two quantities and the two prices again coincide. The only dynamic element is the lagged demand response. The solution oscillates. Stability requires $\beta > |\alpha_1|$, *i.e.*, supply should respond more strongly to the current price than demand responds to the lagged price. We get

REDUCED FORM	ARMAX-FORM	FINAL FORM
$k_t = \alpha_1 p_{t-1} + z_t,$	$k_t = \frac{\alpha_1}{\beta} k_{t-1} + z_t - \frac{\alpha_1}{\beta} x_{t-1},$	$k_t = \frac{1}{\beta - \alpha_1 \mathsf{L}}[\beta z_t - \alpha_1 \mathsf{L} x_t],$
$p_t = \frac{\alpha_1}{\beta} p_{t-1} + \frac{1}{\beta}[z_t - x_t],$	$p_t = \frac{\alpha_1}{\beta} p_{t-1} + \frac{1}{\beta}[z_t - x_t],$	$p_t = \frac{1}{\beta - \alpha_1 \mathsf{L}}[z_t - x_t].$

The RF equation of k_t and the demand function coincide. Again, the RF-equation and the AF-equation for p_t coincide.

Model D: Introducing an autoregression connecting quantities
Let $\lambda(\mathsf{L}) = \lambda_1 \mathsf{L}$, but retain the other assumptions in Model A. Then

REDUCED FORM	ARMAX FORM	FINAL FORM
$k_t = \dfrac{-\alpha\lambda_1 k_{t-1} + \beta z_t - \alpha x_t}{\beta - \alpha}$,	$k_t = \dfrac{-\alpha\lambda_1 k_{t-1} + \beta z_t - \alpha x_t}{\beta - \alpha}$,	$k_t = \dfrac{\beta z_t - \alpha x_t}{\beta - \alpha(1 - \lambda_1 \mathsf{L})}$,
$p_t = \dfrac{-\lambda_1 k_{t-1} + z_t - x_t}{\beta - \alpha}$,	$p_t = \dfrac{-\alpha\lambda_1 p_{t-1} + z_t - \lambda_1 z_{t-1} - x_t}{\beta - \alpha}$,	$p_t = \dfrac{z_t - \lambda_1 z_{t-1} - x_t}{\beta - \alpha(1 - \lambda_1 \mathsf{L})}$.

The RF and AF equations for k_t coincide. Stability is ensured for $\lambda_1 \in (0,1)$, $\alpha < 0$, $\beta > 0$, since then $\beta - \alpha > (-\alpha)\lambda_1$.

Model E: Introducing a lag-distribution connecting prices
Let $\theta(\mathsf{L}) = \theta_0 + \theta_1 \mathsf{L}$, but retain the other assumptions in Model A. Then

REDUCED FORM	ARMAX FORM	FINAL FORM
$k_t = \dfrac{\beta\alpha\theta_1 p_{t-1} + \beta z_t - \alpha\theta_0 x_t}{\beta - \alpha\theta_0}$,	$k_t = \dfrac{\alpha\theta_1 k_{t-1} + \beta z_t - \alpha(\theta_0 + \theta_1 \mathsf{L}) x_t}{\beta - \alpha\theta_0}$,	$k_t = \dfrac{\beta z_t - \alpha(\theta_0 + \theta_1 \mathsf{L}) x_t}{\beta - \alpha(\theta_0 + \theta_1 \mathsf{L})}$,
$p_t = \dfrac{\alpha\theta_1 p_{t-1} + z_t - x_t}{\beta - \alpha\theta_0}$,	$p_t = \dfrac{\alpha\theta_1 p_{t-1} + z_t - x_t}{\beta - \alpha\theta_0}$,	$p_t = \dfrac{z_t - x_t}{\beta - \alpha(\theta_0 + \theta_1 \mathsf{L})}$.

The RF and AF equations for p_t coincide. Stability requires $\beta - \alpha\theta_0 > (-\alpha)\theta_1 \iff \theta_1 < \theta_0 + \beta/(-\alpha)$.

4 Simple capital-investment model à la Haavelmo

We now give model (3.11)–(3.14) a specific interpretation, as a simplified linearized version of the market model for investment of Haavelmo (1960, Chapter 32). We interpret k_t as the capital stock, j_t as the quantity of gross investment flow, p_t as the price of investment, and c_t as the capital service price, and assume $\alpha(\mathsf{L}) = \alpha$, $\beta(\mathsf{L}) = \beta$. Also, we let $\lambda(\mathsf{L}) = \lambda \mathsf{L} = (1-\delta)\mathsf{L}$, interpreting $\delta \in (0,1]$ as a retirement (depreciation) rate, let ρ be a market interest rate and $\widehat{p}_{t+1|t}$ a forecast for the investment price in period $t+1$ formed by the capital users in period t:

$$k_t = \alpha c_t + z_t, \tag{4.1}$$
$$j_t = \beta p_t + x_t, \tag{4.2}$$
$$k_t = (1-\delta) k_{t-1} + j_t, \tag{4.3}$$
$$c_t = p_t - \tfrac{1-\delta}{1+\rho} \widehat{p}_{t+1|t}. \tag{4.4}$$

Equation (4.4) connects the capital service price with the investment price, the retirement rate and the interest rate. Its rationale is that the per unit cost of capital service is obtained by deducting from p_t the present value of the remaining share $1-\delta$ of one unit next period, evaluated at the expected price $\widehat{p}_{t+1|t}$. How can (4.4) be related to (3.14)?

The price forecast may be connected to p_t in several ways and the specific choice here may crucially affect the model's dynamic properties. The model has no *explicit* cobweb mechanism, as model examples B and C in Section 3. It has two 'static' equations, one autoregressive backward-looking equation and one forward-looking equation.

Haavelmo (1960, Chapter 28) states explicitly the importance of the effect of changes in the capital price on the demand for capital, which he calls the *'speculative element in the holding of capital'*:

> "A producer must pay attention, not only to the price of capital in relation to its productivity and its interest cost, but also to the possibility of price *changes* over his planning period. There is, of course, a great difference in this respect between the case of perfect mobile capital in a perfect capital market and the case where the producer has to tie himself to immobile....capital. But it is not so that a producer can free himself from losses due to price falls even in a perfect capital market. This he could do only if he knew *the time* at which the price will change...."[Haavelmo (1960, p. 168)].

His treatise, being basically theoretical, is not concerned with econometric *modelling* of the price increase term in the capital service price, *i.e.*, how to connect this 'speculative element' with observable variables. Let $\widehat{p}_{t+1|t} = p_t + \widehat{\Delta p}_{t+1|t}$ and

$$\mu = \tfrac{1}{1+\rho}, \qquad \lambda = 1-\delta \implies \mu\lambda = \tfrac{1-\delta}{1+\rho}, \qquad 1-\mu\lambda = \tfrac{\rho+\delta}{1+\rho}.$$

Consider Table 1, giving some boundary values for (ρ, δ) and the implied (j_t, c_t).

TABLE 1. EQUATIONS (4.3) AND (4.4) FOR BOUNDARY VALUES

	$\mu\lambda$	j_t	c_t
Case 1: $\delta=1$	0	k_t	p_t
Case 2: $\rho\to\infty$	0	$\Delta k_t+\delta k_{t-1}$	p_t
Case 3: $\delta=\rho=0$	1	Δk_t	$-\widehat{\Delta p}_{t+1\|t}$
Case 4: $\delta=0,\ \rho>0$	$1/(1+\rho)$	Δk_t	$[\rho p_t-\widehat{\Delta p}_{t+1\|t}]/(1+\rho)$
Case 5: $\rho=0,\ 0<\delta<1$	$1-\delta$	$\Delta k_t+\delta k_{t-1}$	$\delta p_t-(1-\delta)\widehat{\Delta p}_{t+1\|t}$

Case 1, with immediate retirement, takes us back to the static model. Case 3, with zero capital retirement and zero interest cost, represents 'costless capital use', with a service price equal to minus the increase in the expected capital price. Case 5 is the only case in which j_t depends on both the level and the increase of k_t while c_t depends on both the level and the (expected) increase of p_t. Case 2 removes $\widehat{\Delta p}_{t+1|t}$ from c_t, while Case 3 removes k_{t-1} from j_t. In Case 4 the capital service price is positive (negative) if the real interest rate $\rho-\widehat{\Delta p}_{t+1|t}/p_t$ is positive (negative).

So far $\widehat{\Delta p}_{t+1|t}$ has been unspecified. We represent this term by a lag-distribution on the *realized* increase over the current and last M periods, with polynomial $\pi(\mathsf{L}) = \pi_0 + \pi_1\mathsf{L} + \cdots + \pi_M\mathsf{L}^M$, given by

$$\begin{aligned} \widehat{\Delta p}_{t+1|t} = \pi(\mathsf{L})\Delta p_t = \pi(\mathsf{L})(1-\mathsf{L})p_t &\implies \widehat{p}_{t+1|t} = [1+\pi(\mathsf{L})(1-\mathsf{L})]p_t \\ &\implies c_t = \{1-\mu\lambda[1+\pi(\mathsf{L})(1-\mathsf{L})]\}p_t. \end{aligned}$$

This implies that we in (3.14) let

$$(4.5)\quad \theta(\mathsf{L}) = 1-\mu\lambda[1+\pi(\mathsf{L})(1-\mathsf{L})] \iff \theta_m = \begin{cases} 1-\mu\lambda(1+\pi_0), & m=0, \\ -\mu\lambda(\pi_m-\pi_{m-1}), & m=1,\dots,M-1, \\ +\mu\lambda\pi_{M-1}, & m=M, \end{cases}$$

so that (3.15) becomes:

$$\begin{bmatrix} 1-\lambda\mathsf{L} & -\beta \\ 1 & -\alpha\{1-\mu\lambda[1+\pi(\mathsf{L})(1-\mathsf{L})]\} \end{bmatrix} \begin{bmatrix} k_t \\ p_t \end{bmatrix} = \begin{bmatrix} x_t \\ z_t \end{bmatrix}.$$

The model version obtained represents the following special case of (3.11)–(3.14):

$$\alpha(\mathsf{L})=\alpha,\ \beta(\mathsf{L})=\beta,\ \lambda(\mathsf{L})=\lambda\mathsf{L},\ \eta(\mathsf{L})=1,$$
$$\theta(\mathsf{L})=1-\mu\lambda[1+\pi(\mathsf{L})(1-\mathsf{L})],$$
$$|\boldsymbol{B}(\mathsf{L})| = \beta - \alpha(1-\lambda\mathsf{L})\theta(\mathsf{L}) = \beta - \alpha(1-\lambda\mathsf{L})\{1-\mu\lambda[1+\pi(\mathsf{L})(1-\mathsf{L})]\},$$

with ARMAX-form, exemplifying (3.16)–(3.19),

$$|\boldsymbol{B}(\mathsf{L})|k_t = -\alpha\theta(\mathsf{L})x_t + \beta z_t, \tag{4.6}$$
$$|\boldsymbol{B}(\mathsf{L})|p_t = -x_t + (1-\lambda\mathsf{L})z_t, \tag{4.7}$$
$$|\boldsymbol{B}(\mathsf{L})|j_t = -\alpha\theta(\mathsf{L})(1-\lambda\mathsf{L})x_t + \beta(1-\lambda\mathsf{L})z_t, \tag{4.8}$$
$$|\boldsymbol{B}(\mathsf{L})|c_t = -\theta(\mathsf{L})x_t + \theta(\mathsf{L})(1-\lambda\mathsf{L})z_t. \tag{4.9}$$

In the Haavelmo-type investment equation exemplified by (4.8), the interest and retirement rates, ρ and $\delta = 1-\lambda$, interact with the demand and supply slopes α and β, as well as with the form of the price expectation process $\pi(\mathsf{L})$, via $\theta(\mathsf{L})$. The model has *implicit* cobweb elements due to capital accumulation and the way the capital service price is connected to the capital stock (investment) price. It has a lower degree of autonomy than the capital demand and the investment supply equations; see Aldrich (1989) on Haavelmo and the autonomy concept.

Since $\theta(1) = 1-\mu\lambda = \frac{\rho+\delta}{1+\rho}$ and $|\boldsymbol{B}(1)| = \beta - \alpha(1-\mu\lambda)(1-\lambda)$, irrespective of $\pi(\mathsf{L})$, the *static reduced form counterpart* to (4.6)–(4.9) is

$$k_t = \frac{\beta z_t - \alpha(1-\mu\lambda)x_t}{\beta-\alpha(1-\mu\lambda)(1-\lambda)},$$
$$p_t = \frac{(1-\lambda)z_t - x_t}{\beta-\alpha(1-\mu\lambda)(1-\lambda)},$$
$$j_t = \frac{(1-\lambda)[\beta z_t - \alpha(1-\mu\lambda)x_t]}{\beta-\alpha(1-\mu\lambda)(1-\lambda)} = (1-\lambda)k_t,$$
$$c_t = \frac{(1-\mu\lambda)[(1-\lambda)z_t - x_t]}{\beta-\alpha(1-\mu\lambda)(1-\lambda)} = (1-\mu\lambda)p_t.$$

An interesting question is which parameter combinations can ensure *stability* of this dynamic four-equation system. Consider the cases $M=1$ and $M=2$:

$$M=1: \quad |\boldsymbol{B}(\mathsf{L})| = \beta - \alpha[\theta_0 + \theta_1\mathsf{L}](1-\lambda\mathsf{L})$$
$$= \beta-\alpha\theta_0 - \alpha(\theta_1-\lambda\theta_0)\mathsf{L} + \alpha\lambda\theta_1\mathsf{L}^2,$$
$$\text{where } \theta_0 = 1-\mu\lambda(1+\pi_0),\ \theta_1 = \mu\lambda\pi_0.$$
$$M=2: \quad |\boldsymbol{B}(\mathsf{L})| = \beta - \alpha[\theta_0 + \theta_1\mathsf{L} + \theta_2\mathsf{L}^2](1-\lambda\mathsf{L})$$
$$= \beta-\alpha\theta_0 - \alpha(\theta_1-\lambda\theta_0)\mathsf{L} - \alpha(\theta_2-\lambda\theta_1)\mathsf{L}^2 + \alpha\lambda\theta_2\mathsf{L}^3,$$
$$\text{where } \theta_0 = 1-\mu\lambda(1+\pi_0),\ \theta_1 = \mu\lambda(\pi_0-\pi_1),\ \theta_2 = \mu\lambda\pi_1.$$

Table 2 shows for selected cases, the *roots of the characteristic equation* of the associated difference equation [corresponding to $1/q$ in (2.3)], all of which are required to be *inside* the unit circle for stability to be ensured.

Table 2. Characteristic equation of Haavelmo-type model (4.6)–(4.9)

Roots depending on demand-slope/supply-slope α/β, horizon M, δ and ρ.

$\mu\lambda = \frac{1-\delta}{1+\rho}$ $i = \sqrt{-1}$. *Figures in boldface signalizes unstable roots*

$M = 1$: $\widehat{\Delta p}_{t+1|t} = p_t - p_{t-1}$, $\theta(\mathsf{L}) = 1 - 2\mu\lambda + \mu\lambda\mathsf{L}$

a. Short-lived and moderately long-lived capital

	α/β	δ	ρ	$\mu\lambda$	Roots of char.eqn.	Modulus
a1:	−0.5	0.20	0.05	0.76190	0.11613 ± 0.63199 i	0.64258
a2:	−0.5	0.80	0.05	0.19048	0.06000 ± 0.10462 i	0.12060
a3:	−0.8	0.20	0.05	0.76190	0.23607 ± 0.88522 i	0.91616
a4:	−0.8	0.80	0.05	0.19048	0.08408 ± 0.11538 i	0.14277

b. Very long-lived capital

	α/β	δ	ρ	$\mu\lambda$	Roots of char.eqn.	Modulus
b1:	−0.5	0.02	0.02	0.96078	0.02673 ± 0.93401 i	0.93439
b2:	−0.8	0.02	0.02	0.96078	0.08776 ± 1.69090 i	**1.69320**
b3:	−0.5	0.01	0.01	0.98020	0.01414 ± 0.96604 i	0.96614
b4:	−0.8	0.01	0.01	0.98020	0.05077 ± 1.82980 i	**1.83050**

c. Demand slope close to supply slope

	α/β	δ	ρ	$\mu\lambda$	Roots of char.eqn.	Modulus
c1:	−0.9	0.01	0.50	0.66000	0.21691 ± 0.88254 i	0.90881
c2:	−0.9	0.50	0.01	0.49505	0.22301 ± 0.41361 i	0.46990
c3:	−0.9	0.20	0.20	0.66667	0.25714 ± 0.78714 i	0.82808
c4:	−0.9	0.15	0.15	0.73913	0.26279 ± 0.96109 i	0.99637
c5:	−0.9	0.10	0.10	0.81818	0.25851 ± 1.21830 i	**1.24540**

$M = 2$: $\widehat{\Delta p}_{t+1|t} = \frac{1}{2}(p_t - p_{t-2})$, $\theta(\mathsf{L}) = 1 - \frac{3}{2}\mu\lambda + \frac{1}{2}\mu\lambda\mathsf{L}^2$

b. Very long-lived capital

	α/β	δ	ρ	$\mu\lambda$	Roots of char.eqn.	Modulus
b1:	−0.5	0.02	0.02	0.96078	−0.37043 ± 0.71719 i	0.80720
					0.46351	0.46351
b2:	−0.8	0.02	0.02	0.96078	−0.52421 ± 0.92623 i	**1.06430**
					0.51387	0.51387
b3:	−0.5	0.01	0.01	0.98020	−0.38452 ± 0.72898 i	0.82460
					0.46647	0.46647
b4:	−0.8	0.01	0.01	0.98020	−0.55695 ± 0.94552 i	**1.09740**
					0.51676	0.51676

c. Demand slope close to supply slope

	α/β	δ	ρ	$\mu\lambda$	Roots of char.eqn.	Modulus
c3:	−0.9	0.20	0.20	0.66667	−0.23248 ± 0.67981 i	0.71846
					0.46495	0.46495
c4:	−0.9	0.15	0.15	0.73913	−0.28807 ± 0.75135 i	0.80468
					0.48397	0.48397
c5:	−0.9	0.10	0.10	0.81818	−0.36650 ± 0.83438 i	0.91133
					0.50158	0.50158
c6:	−0.9	0.05	0.05	0.90476	−0.48386 ± 0.93106 i	**1.04930**
					0.51771	0.51771
c7:	−0.9	0.01	0.01	0.98020	−0.62804 ± 1.01760 i	**1.19580**
					0.52951	0.52951

Thirteen parameter combinations for $M=1$, with $\pi_0=1 \Longrightarrow \theta(\mathsf{L})=1-2\mu\lambda+\mu\lambda\mathsf{L}$, and nine combinations for $M=2$, with $\pi_0=\pi_1=\frac{1}{2} \Longrightarrow \theta(\mathsf{L})=1-\frac{3}{2}\mu\lambda+\frac{1}{2}\mu\lambda\mathsf{L}^2$, and the corresponding characteristic roots are given in Table 2. These examples show that whether or not the system is dynamically stable, depends strongly on the relative demand and supply slopes and on the retirement and depreciation rates. The steeper the demand slope relative to the supply slope, the more long-lived the capital, and the lower the interest rate, the stronger is the tendency for the system to be unstable. This is characterized by at least one root of the characteristic equation having modulus larger than one (equivalent to the roots of the autoregressive polynomial $|\boldsymbol{B}(\mathsf{L})|$ being inside the unit circle), confer condition (2.3) and the two b-panels. From the point of view of stability, long-lived capital (δ low) can be 'compensated by' a high interest rate, and a low interest rate can be 'compensated' by capital being short-lived (δ high). Compare, for $M=1$, alternatives c1 and c2 with b4: $\alpha/\beta=-0.8, \rho=\delta=0.01$ is an unstable constellation, while $\alpha/\beta=-0.9, \rho=0.50, \delta=0.01$ as well as $\alpha/\beta=-0.9, \rho=0.01, \delta=0.50$ ensure stability. The latter conclusion relies on the assumption that the demand and supply slopes are unchanged when the durability of the capital is changed.

The effect of a change in the capital users' price expectation horizon on the model's dynamic stability can be illustrated by comparing the b and c panels in Table 2 for $M=1$ with those for $M=2$. The latter case, smoothing the price increases, should be expected to stabilize the model. This is confirmed: all roots have, *cet.par.*, smaller moduli for $M=2$ than for $M=1$, and there is a stronger tendency for stability to be ensured. A notable example is c5, with the relative slopes equal to $\alpha/\beta=-0.9$ and both the interest and the retirement rates equal to 0.1. In the $M=1$ case $[\widehat{\Delta p}_{t+1|t}=\Delta p_t]$ the system is unstable (modulus=1.25), while in the $M=2$ case $[\widehat{\Delta p}_{t+1|t}=\frac{1}{2}(\Delta p_t+\Delta p_{t-1})]$ it is stable (moduli=0.911 and 0.502). If the common value of δ and ρ decreases to 0.15 and 0.10, the system becomes unstable also in the $M=2$ case.

The examples in Section 3 – notably the 'multiple unit roots' example [d], the 'two-price distributed lag' example E, and the expression for the two roots, (3.6), in the simple model – indicate that the system's tendency to be unstable is stronger the closer is the demand slope (in absolute value) to the supply slope. The two b panels in Table 2, representing very long-lived capital and low interest rate, illustrate this. In all cases, $\alpha/\beta=-0.5$ (b1 and b3) gives stability, even for retirement and interest rates as low as $\delta=\rho=0.01$ – although with modulus close to unity in the $M=1$ horizon case – while $\alpha/\beta=-0.8$ (b2 and b4) gives instability. In the $M=1$ case, the modulus of the two roots is as high as 1.83.

Results from four simple *simulation experiments*, with calibrated series for x_t and z_t for 120 periods (the first 10 retained for 'initializing' the process), performed by using modules in the PcGive modelling system (OxMetric 6.01), are illustrated graphically in Figures 1 through 4. The intention is to mimic effects of smoothly changing 'signals', exogenous variables, normally distributed 'noise', assuming

$$x_t = 10 + 0.2\,t + u_t, \qquad u_t \sim \mathsf{N}(0, 4^2),$$
$$z_t = 100 + 0.1\,t + 0.005\,t^2 + v_t, \qquad v_t \sim \mathsf{N}(0, 2^2).$$

Other potentially interesting alternatives might be profiles with jumps or transitory shocks to illustrate possible 'echo effects' in investment quantity and prices. The parameter values assumed for the slopes are in all cases $\alpha = -2.5, \beta = 5$. For (δ, ρ) two alternatives are considered: short-lived capital and somewhat low interest rate $(0.5, 0.05)$ (Figures 1 and 2 for $M = 1$ and $M = 2$, respectively) and medium long-lived capital and somewhat high interest rate $(0.2, 0.15)$ (Figures 3 and 4 for $M = 1$ and $M = 2$, respectively).

The four graphs in each figure primarily illustrate the sensitivity of the time profile of the endogenous variables (k_t, j_t, p_t, c_t) (c_t symbolized by `ucc`) to changes in the assumed price expectation horizon and the retirement and interest rates. The overall shape of simulated series for k_t and j_t is not very sensitive to changes in the parameter values, except that the former is, of course, closer to the latter when faster retirement is assumed. For the two price variables, we find striking differences across the four sets of graphs. As expected, c_t has a more jagged pattern when a one-year horizon for the price increase is assumed than when the normalized two-period difference for the price increase is assumed. In both $M = 1$ examples, c_t tend to take negative values. When a medium-lived capital is assumed ($\delta = 0.2$), the simulated c_t series fluctuates around zero.

5 Extension: Models allowing for non-geometric capital decay

Haavelmo (1960, Chapter 32) specifies the capital accumulation process and the capital service prices in his neo-classical market model by geometric decay. So far, we have followed this description, the popularity of which also in much later theoretical and empirical work probably reflects its simplicity – it is a one-parameter process, implying a constant hazard rate of capital deterioration, *i.e.*, an age-invariant retirement rate. Good reasons can be given to modify this assumption. Geometric decay eliminates very likely situations where capital survival follows concave functions, *i.e.*, increasing retirement with age. If we change the model in that direction, the capital service price and the retirement must be remodeled, with due regard to the duality between the two variables. How this influences the capital market model's dynamic properties is worth a closer examination. Below, we briefly sketch possible ways to proceed.

Simple cases where the capital is assumed to have a finite (maximal) service life N, are sudden retirement at age N, and linear retirement up to age N. To accommodate these, we can replace (4.3), which implies $k_t = \sum_{s=0}^{\infty}(1-\delta)j_{t-s}$, and (4.4) by, respectively:

Sudden retirement: $k_t = \sum_{s=0}^{N-1} j_{t-s}, \qquad c_t = p_t - \frac{1}{1+\rho}(1 - \frac{1}{N})\widehat{p}_{t+1|t},$

Linear retirement: $k_t = \sum_{s=0}^{N-1}(1-\frac{s}{N})j_{t-s}, \qquad c_t = p_t - \frac{1}{1+\rho}(1 - \frac{2}{N+1})\widehat{p}_{t+1|t}.$

Then, in the formula for the capital service price, we let δ under geometric decay be replaced by, respectively, $\frac{1}{N}$ and $\frac{2}{N+1}$, which equal the respective retirement rates in the hypothetic case with constant investment; see Biørn (2005, Section 5). If the capital lives one year only ($N=1$), we are back at the static model with $k=j_t$, $c_t=p_t$ (corresponding to $\delta=1$ under geometric decay).

More generally, we can let the *capital's declining efficiency* with age be indicated by a non-negative *sequence of survival rates* $\{b_s\}_{s=0}^{s=N-1}$; $b_0=1$, and replace (4.3)–(4.4) by

$$k_t = \textstyle\sum_{s=0}^{N-1} b_s j_{t-s}, \tag{5.1}$$

$$c_t = p_t - \frac{1}{1+\rho}\left(1-\frac{1}{\sum_{s=0}^{N-1} b_s}\right)\widehat{p}_{t+1|t} = p_t - \frac{1-\delta_N}{1+\rho}\widehat{p}_{t+1|t}, \tag{5.2}$$

where $\delta_N=1/\sum_{s=0}^{N-1} b_s$ is the retirement rate in a stationary situation with constant investment: $j_t=\bar{j}$, $k_t=\bar{k}=\bar{j}\sum_{s=0}^{N-1} b_s$. An example of a two-parametric survival function, which can exhibit both concavity and convexity, depending on whether the parameter σ is less than or larger than 1, is

$$b_s = (1-\tfrac{s}{N})^{\sigma}, \qquad N \geq 1, \sigma \geq 0.$$

Letting $b(\mathsf{L}) = b_1\mathsf{L} + b_2\mathsf{L}^2 + \cdots + b_{N-1}\mathsf{L}^{N-1}$, we can rewrite these relationships as

$$\begin{aligned} k_t &= [1+b(\mathsf{L})]j_t, \\ c_t &= p_t - \mu\tfrac{b(1)}{1+b(1)}\widehat{p}_{t+1|t} = p_t - \mu\lambda_N\widehat{p}_{t+1|t}, \end{aligned}$$

where $\mu=\frac{1}{1+\rho}$, $\delta_N=\frac{1}{1+b(1)}$ and $\lambda_N=1-\delta_N$. The form of the expectation process for the capital price and the form of the retirement process will then be entwined with the interest rate in the autoregressive polynomial. Let us take a closer look at this.

The specification (5.1)–(5.2) can be combined with the lag-distribution (4.5) for the price expectations in Model (3.11)–(3.14). We let $\alpha(\mathsf{L})=\alpha$, $\beta(\mathsf{L})=\beta$ and

$$\begin{aligned} \lambda(\mathsf{L}) &= 0, \quad \eta(\mathsf{L}) = 1 + b(\mathsf{L}), \\ \theta(\mathsf{L}) &= 1-\mu\lambda_N[1+\pi(\mathsf{L})(1-\mathsf{L})], \end{aligned}$$

and find that the equation which connects the investment quantity and the investment price, after inserting $c_t=\{1-\mu\lambda_N[1+\pi(\mathsf{L})(1-\mathsf{L})]\}p_t$ and $k_t=[1+b(\mathsf{L})]j_t$ in the stock demand function (4.2) becomes

$$[1+b(\mathsf{L})]j_t = \alpha\,\{1-\mu\lambda_N[1+\pi(\mathsf{L})(1-\mathsf{L})\}\,p_t + z_t. \tag{5.3}$$

Inserting the investment supply function (3.11), $j_t=\beta p_t + x_t$, and its inverse $p_t = (1/\beta)(j_t - x_t)$, we get, respectively,

$$\begin{aligned} [1+b(\mathsf{L})](\beta p_t + x_t) &= \alpha\,\{1-\mu\lambda_N[1+\pi(\mathsf{L})(1-\mathsf{L})\}\,p_t + z_t, \\ [1+b(\mathsf{L})]j_t &= \alpha\,\{1-\mu\lambda_N[1+\pi(\mathsf{L})(1-\mathsf{L})\}\,(1/\beta)(j_t - x_t) + z_t. \end{aligned}$$

which can be rewritten as *ARMAX-form equations for the investment quantity and the investment price*, respectively,

$$[\beta-\alpha\{1-\mu\lambda_N[1+\pi(\mathsf{L})(1-\mathsf{L})]\}+\beta b(\mathsf{L})]p_t = z_t - [1+b(\mathsf{L})]x_t, \tag{5.4}$$

$$[\beta-\alpha\{1-\mu\lambda_N[1+\pi(\mathsf{L})(1-\mathsf{L})]\}+\beta b(\mathsf{L})]j_t = \beta z_t - \alpha\{1-\mu\lambda_N[1+\pi(\mathsf{L})(1-\mathsf{L})]\}x_t. \tag{5.5}$$

Combining (5.4) with the capital service price equation (3.14), $c_t = \theta(\mathsf{L})p_t$, and (5.5) with $k_t = [1+b(\mathsf{L})]j_t$, we get

$$(5.6) \quad [\beta - \alpha\{1-\mu\lambda_N[1+\pi(\mathsf{L})(1-\mathsf{L})]\} + \beta b(\mathsf{L})]\, c_t = \theta(\mathsf{L})z_t - \theta(\mathsf{L})[1+b(\mathsf{L})]x_t,$$

$$(5.7) \quad [\beta - \alpha\{1-\mu\lambda_N[1+\pi(\mathsf{L})(1-\mathsf{L})]\} + \beta b(\mathsf{L})]\, k_t = \beta[1+b(\mathsf{L})]z_t - \alpha[1+b(\mathsf{L})]\{1-\mu\lambda_N[1+\pi(\mathsf{L})(1-\mathsf{L})]\}x_t.$$

Equations (5.5) and (5.7) can be interpreted as *Haavelmo-type investment and capital equations* under a generalized description of capital retirement. Again the form of the retirement and the price expectation processes and the interest rate interact in a rather complex way in the determination of the dynamic behaviour of the investment quantity and its price.

So far, we have not addressed the problem of explicitly modelling a *forward-looking mechanism in the formation of price expectations*. One, somewhat simplistic, way of doing this could have been to replace $\pi(\mathsf{L})$ by $\pi(\mathsf{L}^{-1}) = \pi_0 + \pi_1\mathsf{L}^{-1} + \cdots + \pi_M\mathsf{L}^{-M}$, $\widehat{\Delta p}_{t+1|t} = \pi(\mathsf{L})\Delta p_t$ by $\widehat{\Delta p}_{t+1|t} = \pi(\mathsf{L}^{-1})\Delta p_t$ and (4.5) by

$$\theta(\mathsf{L}^{-1}) = 1 - \mu\lambda[1+\pi(\mathsf{L}^{-1})(1-\mathsf{L})].$$

Another way is to omit $\pi(\mathsf{L})$ and instead invoke a '(non-)arbitrage condition' frequently postulated as an equilibrium condition in the capital market literature, saying that the price of a capital asset should equal the present value of its future service prices weighted by the relevant efficiency at each age; confer Hotelling (1925), Hicks (1973, Chapter II), and Jorgenson (1989, section 1.2); see also Takayama (1985, p. 694), and Diewert (2005, Section 12.2). A simple implementation could be to replace (5.2) and $c_t = \{1-\mu\lambda_N[1+\pi(\mathsf{L})(1-\mathsf{L})]\}p_t$ by:

$$(5.8) \qquad p_t = c_t + \mu b_1 c_{t+1} + \mu^2 b_2 c_{t+2} + \cdots + \mu^{N-1} b_{N-1} c_{t+N-1} = [1+b(\mu\mathsf{L}^{-1})]c_t,$$

where still $\mu = 1/(1+\rho)$. Combining (5.8) with (4.1) and (4.2) gives, instead of equation (5.3), which connects j_t and p_t, the following system in j_t and c_t:

$$[1+b(\mathsf{L})]j_t = \alpha c_t + z_t,$$
$$j_t = \beta[1+b(\mu\mathsf{L}^{-1})]c_t + x_t,$$

to obtain

$$(5.9) \qquad \{\beta[1+b(\mathsf{L})][1+b(\mu\mathsf{L}^{-1})] - \alpha\}c_t = z_t - [1+b(\mathsf{L})]x_t,$$

$$(5.10) \qquad \{\beta[1+b(\mathsf{L})][1+b(\mu\mathsf{L}^{-1})] - \alpha\}j_t = \beta[1+b(\mu\mathsf{L}^{-1})]z_t - \alpha x_t.$$

Combining (5.9) with $p_t = [1+b(\mu\mathsf{L}^{-1})]c_t$ and (5.10) with $k_t = [1+b(\mathsf{L})]j_t$, we get

$$(5.11) \quad \{\beta[1+b(\mathsf{L})][1+b(\mu\mathsf{L}^{-1})] - \alpha\}p_t = [1+b(\mu\mathsf{L}^{-1})]z_t - [1+b(\mathsf{L})][1+b(\mu\mathsf{L}^{-1})]x_t,$$

$$(5.12) \quad \{\beta[1+b(\mathsf{L})][1+b(\mu\mathsf{L}^{-1})] - \alpha\}k_t = \beta[1+b(\mathsf{L})][1+b(\mu\mathsf{L}^{-1})]z_t - \alpha[1+b(\mathsf{L})]x_t.$$

The common polynomial of (5.9)–(5.12), $\beta[1+b(\mathsf{L})][1+b(\mu\mathsf{L}^{-1})] - \alpha$, accounts for both lags and leads up to order N. A closer examination of this polynomial and the polynomials attached to the exogenous shift and error variables, which once again captures the interaction of the interest rate and the retirement pattern, is required to describe the *Haavelmo-type investment and capital equations*, (5.10) and (5.12), in this forward-looking model.

6 Concluding remarks

The main conclusions can be summarized as follows:

First, the 'Haavelmo-type investment equation' – as elaborated through a number of somewhat simplistic linear four-equation models inspired from his *A Study in the Theory of Investment*, published more than 50 years ago, and condensed in the dynamic market models' final form equations – is a *system property of the market model*. Its investment equation is a confluent relation with a potentially low degree of autonomy, representing both demand and supply responses as well the form of the capital accumulation process and the expectation process for the investment price.

Second, the value of the (assumed constant) interest and the retirement rates are parts of the coefficients of the model's final form equations. The model has implicit cobweb elements due to the capital's survival process and the relation between the investment price and the capital service price, even if the demand and supply functions are seemingly static.

Third, the model's exogenous variables, in conjunction with its disturbance terms contribute to the fluctuations in capital accumulation and prices through the moving average elements in the final form equations. Both the capital deterioration process and the form of the price expectations have impact on the form and length of the moving average polynomials in the final form equations.

Fourth, the price expectation term in the capital service price and how it is modeled, are essential for the system's dynamic behaviour. This notably applies to the length of the capital users' horizon when forming expectations. The model framework can be modified to explicitly admit forward-looking behaviour.

Fifth, switching from 'regimes' with short-lived to long-lived capital or from high-interest 'regimes' to low-interest 'regimes', for given supply and demand slopes, tends to destabilize the system. This confirms that in a Haavelmo-like neo-classical capital market, the values of the interest and the retirement rates are important determinants for the system's dynamics.

Sixth, the Haavelmo type of model can be extended to allow for more flexible survival pattern of capital than geometric decay. The duality in the representation of the capital service price vis-à-vis the capital stock prices and the representation of the capital quantity in relation to the investment then comes into focus. Changes in the survival pattern clearly have the potential to change the systems dynamics.

Several *extensions* of the framework presented here could be envisaged. Some examples indicate the *potential* of the Haavelmo capital market model: (i) Including explicit cobweb elements in the demand and supply functions to 'refine' the dynamics of the system. (ii) Changing the status of the interest rate from being a constant 'parameter' to becoming an exogenous time function. (iii) Revising the specification of the expectation process for the capital price as suggested the 'rational expectations' literature. (iv) Extending the setup from linear to non-linear forms for the demand and supply functions. (v) Relaxing the model's implicit assumption of a closed capital goods market, in allowing for capital that is mobile between markets, 'forced' scrapping of capital, etc. (vi) Splitting the functions x_t and z_t into interpretable economic variables. (vii) Representing the noise terms of x_t and z_t by VAR or VARMA processes. (viii) Including multiple markets for capital good with different durability, as well as demand equations for consumption goods and other non-capital goods. (ix) Performing Monte-Carlo simulations on calibrated model versions to improve our learning about its dynamic properties. (x) Confronting the model with genuine empirical data by using appropriate econometric time-series methodology.

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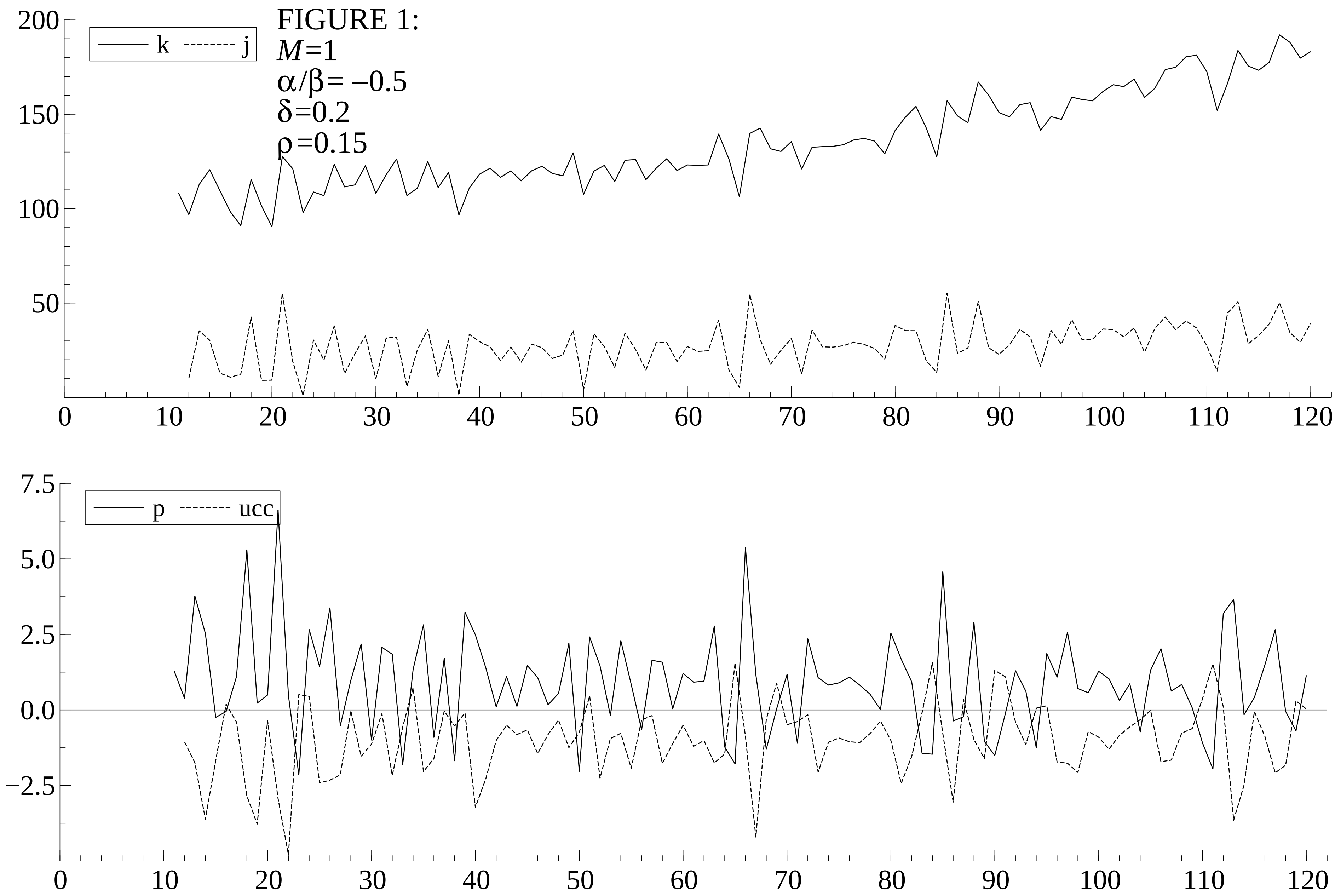

FIGURE 1:
M=1
α/β= –0.5
δ=0.2
ρ=0.15
k
j
p
ucc
200
150
100
50
7.5
5.0
2.5
0.0
−2.5
0
10
20
30
40
50
60
70
80
90
100
110
120

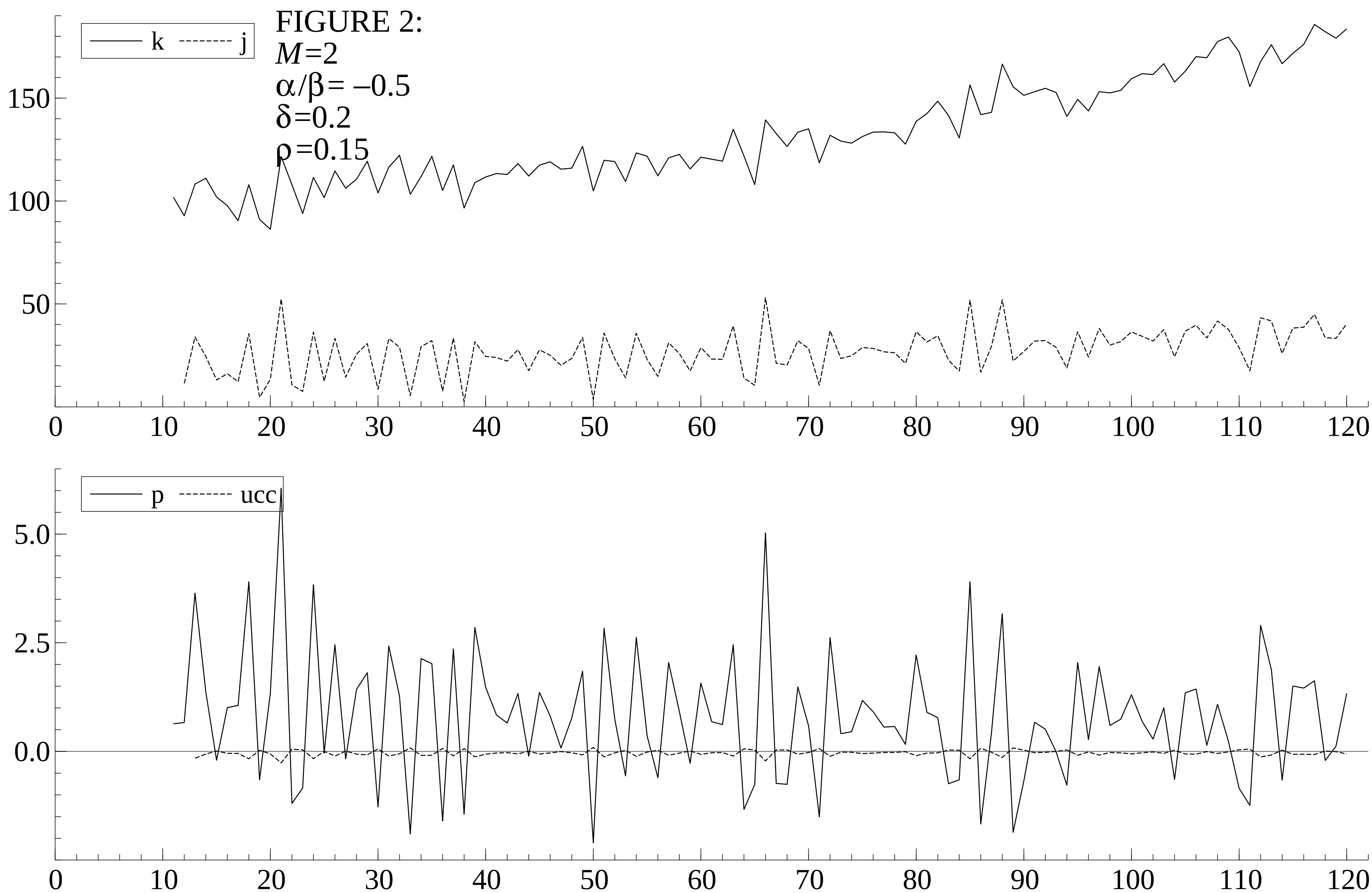
FIGURE 2:
M=2
α/β= –0.5
δ=0.2
ρ=0.15
k
j
150
100
50
0
10
20
30
40
50
60
70
80
90
100
110
120
p
ucc
5.0
2.5
0.0
0
10
20
30
40
50
60
70
80
90
100
110
120

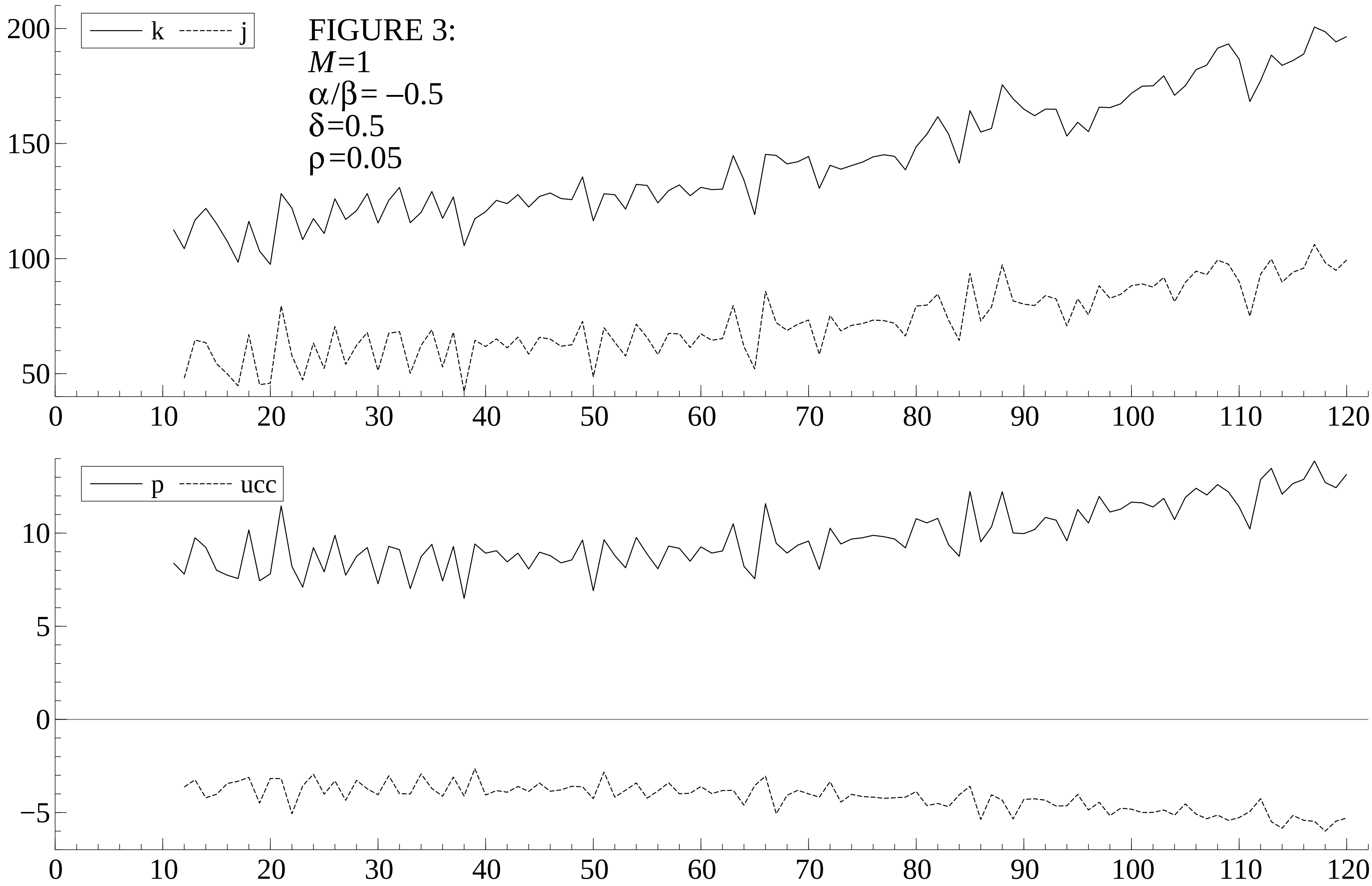
FIGURE 3:
M=1
α/β= –0.5
δ=0.5
ρ=0.05
k
j
p
ucc
200
150
100
50
10
5
0
−5
0 10 20 30 40 50 60 70 80 90 100 110 120

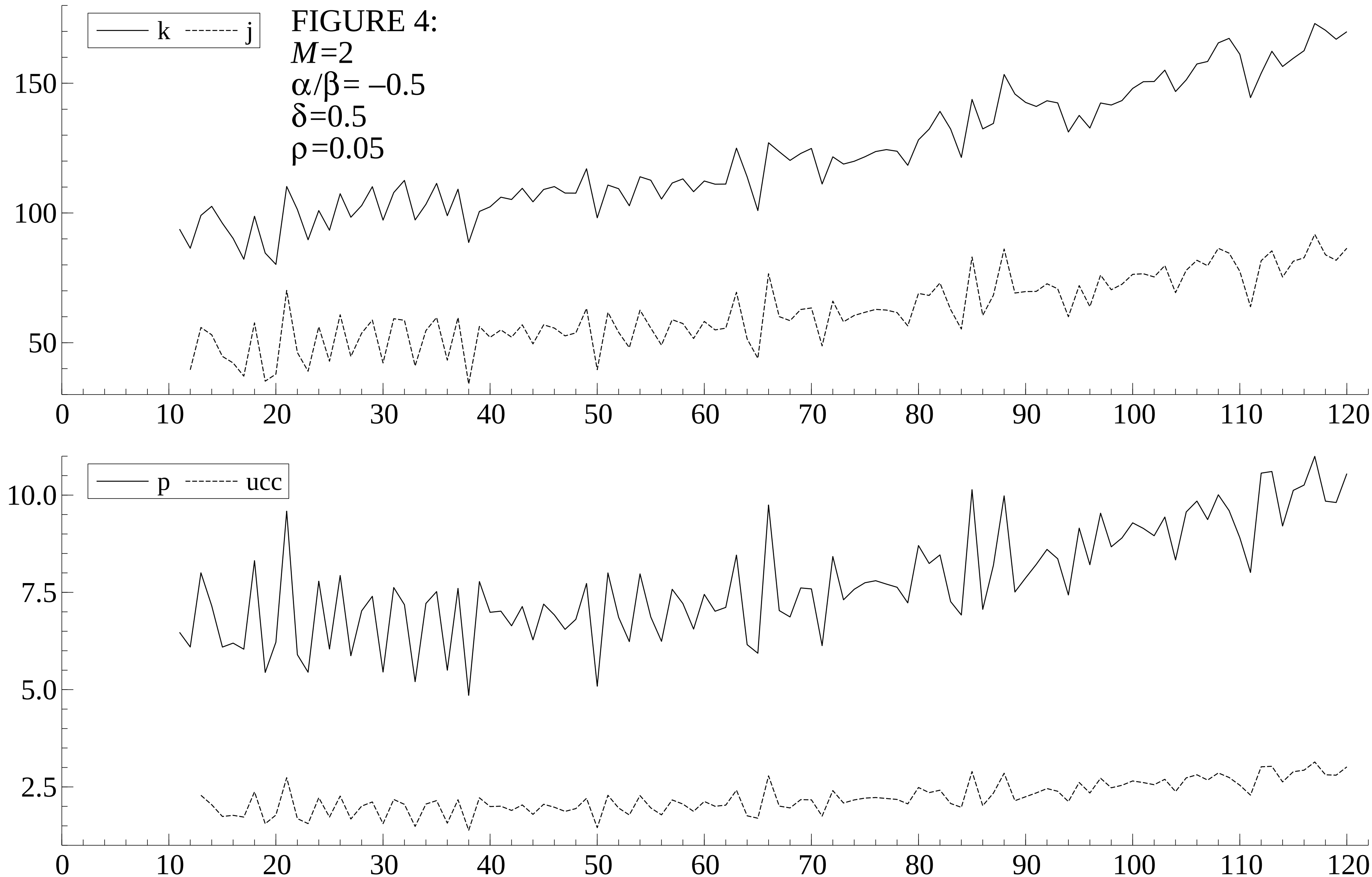

FIGURE 4:
M=2
α/β= –0.5
δ=0.5
ρ=0.05
k
j
150
100
50
0
10
20
30
40
50
60
70
80
90
100
110
120
p
ucc
10.0
7.5
5.0
2.5