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Ragnar Frisch's Axiomatic Approach in Econometrics

by

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Abstract

Ragnar Frisch's concept of econometrics was broader in scope than the more restricted connotation it has today as a sub-discipline of economics, it may be more properly rendered as a reconstruction of economics along principles inspired and drawn from natural sciences. In this reconstruction an axiomatic approach played a key role. In his 1926 essay, *Sur un problème d'économie pure,* Frisch set out what may have been the first axiomatic approach towards modelling consumer behaviour.

Frisch's axiomatic approach was driven by two ambitions. The general aim was to argue in favour of using axiomatics as a basis for theorizing in economics, not least for the modelling of individual behaviour. A more specific ambition was to provide a basis for a quantitative notion of marginal utility.

In 1933 Frisch presented an extended set of axioms for quantifying utility in a lecture series in Paris. The lecture series was supposed to be published, but Frisch failed to submit a final version of the manuscript and the lectures were never published. The set of axioms has been retrieved from Frisch's archival remains and is presented here for the first time.

Frisch returned yet another time to the same topic when he attended a research conference at Cowles Commission in 1937. He presented then a more general *choice-field theory*, the core of which was a system of axioms for individual behaviour, published only as a (long) abstract.

After this latter attempt Frisch's interest in explicitly axiomatization seem to have waned, although his promotion of axiomatics in economics is related to other methodological innovations that he promoted, such as the use of structural equations for modelling micro- and macroeconomic systems, and the use of structured interviews as a source of information when passive observations fail to provide a basis for estimating relationships. As a national accounting pioneer Frisch also argued for an axiomatic approach towards national accounting.

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JSL classification: B23, B31, B41, D11

1. Introduction

Ragnar Frisch is well known for his contribution towards establishing *econometrics*, a term he coined in 1926. Frisch's concept of econometrics was broader in scope than the more restricted connotation it has today as a sub-discipline of economics, it may be more properly rendered as a reconstruction of economics as a science along principles inspired and drawn from natural sciences. In this reconstruction an axiomatic approach played a key role. In Frisch's very first essay in economics, *Sur un problème d'économie pure* (Frisch, 1926a, English edition 1971), indeed the same paper in which he coined "econometrics," he demonstrated some of his key tenets. Frisch noted that despite the efforts of Edgeworth, Pareto and Fisher "an objective definition of utility has…not…been obtained", as "the axioms on which one must base oneself to establish the definition of utility as a quantity have not been displayed (Frisch, 1971:387). He then set out axioms "to establish an objective definition of utility in the economic sense" (Frisch, 1971:388).

Frisch's definition of econometrics was motivated by a desire to "turn economics … into a science" (Frisch, 1971:386). His concept of a science was contemporary physics. Scattered remarks he made various places indicated the view that for economics to become a science it should adapt and adhere to the principles and procedures of modern physics, which in Frisch's formative years had made spectacular advances, not least Einstein's unique achievement of the theory of relativity.[1] The proposed new term of "econometrics" should serve to delineate what was scientific in economics from what was not.

Econometrics aimed at subjecting "abstract laws … to experimental and numerical verification" (Frisch, 1971:386), as is, indeed, the current denotation of "econometrics". But that was only half of Frisch intended connotation. He was as much concerned with how theory should be formulated in economics to fulfil positivistic scientific requirements. That meant concepts to be given empirical meaning through operational definitions, and economic relationships to be formulated in ways we today describe as "modelling". When Frisch drafted the constitution for the Econometric Society in 1930 he called this part of his program for "theoretical quantification."

We can analyse the 1926 essay as being driven by two ambitions: the first one was to show how econometrics, this new branch of economics, should articulate mathematical analysis and statistical analysis. Secondly, a clearly stated aim was to achieve a measurable concept of the marginal utility of money, implying a cardinal interpretation. Also his terminology, e.g. "quantification" of utility seemed to suggest a close affinity to natural science. For a more comprehensive discussion of why Frisch turned to axiomatics see Dupont-Kieffer (2003).

In the 1926 essay Frisch expressed the hope to "realise the dream of Jevons" of measuring marginal utility (Frisch, 1971:386), as if that was part of the motivation of his axiomatic approach to individual choice behaviour.[2] Frisch was much influenced and inspired by Irving Fisher (1925), siding with Fisher on the issue of avoiding a psychological approach in the pursuit of a concept of utility appropriate for economic reasoning, not least for inter-personal utility comparisons, in which they both had an interest. The role of the econometrician was to

[1] A source of inspiration for Frisch in this regard may have been a lecture series given by Albert Einstein at the University of Oslo in 1920, shortly after observations of the solar eclipse in 1919 had confirmed in a spectacular way the predictions of the general theory of relativity. It was Einstein's first lecture series on the general theory of relativity outside Germany (Johansen, 2005).

[2] Cf. Jevons (1871): "[T]he price of a commodity is the only test we have of the [marginal] utility of the commodity to the purchaser, and if we could tell exactly how much people reduce their consumption of each important article when the price rises, we could determine, at least approximately, the variation in the final degree of utility – the all-important element in Economics." (Quoted from Chipman (1998:59).

measure the consequences of the individual choices driven by utility for the consumption of specific goods within the constraints present. So the task is to define precisely the "choice field" of a consumer, from which marginal utility can be determined as a "choice coefficient." For that purpose, Frisch equated rigorous definition with axiomatization of the individual behaviour. He favoured a cardinal approach, as having an appeal to "everyday experience".

A second step of the analysis in Frisch (1971) was to derive a functional representation of marginal utility of money. The essay thus also comprised an empirical part with estimation of marginal utility as a function of income. It is notable, not so much for its estimation methods as for the principle of letting theory guide the empirical analysis, notwithstanding the fact that Frisch had blundered somewhat in his analysis, cf. Chipman (1998). Frisch's specific simplifying assumptions for estimating the marginal utility of money later came under critical scrutiny by some of his fellow econometricians, i.a. Burk (1936) who showed that they were more restrictive than Frisch apparently had realized.

Frisch continued his utility studies in the monograph *New Methods of Measuring Marginal Utility* (Frisch, 1932), in which he did not reiterate the axiomatics, but worked directly from a functional representation of marginal utility. At the time Frisch's approach attracted considerable attention among fellow econometricians. After the reconstitution of demand theory in the 1930s following the rediscovery of Eugene Slutsky's 1915 article (Slutsky, 1952) by Allen, Hicks, and others cardinalism fell in disrepute. Logical positivism also exerted an influence, it allowed only empirically testable claims, and cardinality asserted to be non-testable (Köbberling, 2006). The demand revolution in the mid-1930s thus sidelined cardinality as an unwarranted assumption.

Frisch continued, however, his work on an axiomatic approach. While he visited Yale University in 1930, a visit arranged by Irving Fisher partly to get an opportunity to work closely with Frisch, he gave a lecture series in the autumn term 1930 called *A Dynamic Approach to Economic Theory* (Frisch, 1930), covering briefly most of his methodological innovations, in short his *econometrics*, including his axiomatic ideas.[3] Another opportunity for presenting his axiomatic ideas arose when Frisch in 1933 was invited by the Henri Poincaré Institute in Paris to give a series of eight lectures on *Problems and Methods of Econometrics* (Frisch, 1933a). He made his axiomatic approach the key theme in the opening lecture of the series. In the Paris lecture he introduced an extended axiom system and addressed the integrability issue and the question of a measure of total utility. The lectures never got published as they were originally meant to be.[4] Frisch also made an attempt at generalizing the "choice field" approach a few years as later at a presentation at the Third Cowles Commission Research Conference in 1937.

The main emphasis in this paper is to present the hitherto unpublished axiomatic approach in the Poincaré lectures, set out in the context of the earlier 1926 axioms and the later 1937 paper. The paper aims primarily at conveying Frisch's axiomatic approach much as he himself presented it, as relatively few have had opportunity to access the original sources. The presentation of the axiomatics of the Poincaré lectures is thus quite close to Frisch's own version, supported by quotes and a paraphrasing of Frisch's own argument. Already touched upon is the influence from Irving Fisher and from physics, and the role of axiomatics within Frisch's overall scientific conception of economics. An effort to trace the impact Frisch's axiomatics upon others and their reaction to his work has not been possible to include in this

[3] The Yale lecture notes were distributed in a small number of carbon copies. They have been retrieved from te Frisch archive and are being prepared for publication.

[4] The lecture series has been reconstructed from Frisch's handwritten notes retrieved from his archival remains and is being prepared for publication.

context. After a brief biographical background (section 2), the paper presents and discusses in succession his axiomatic approach of 1926 (section 3), excerpts from his remarks on utility in the 1930 lectures at Yale University (section 4), his extended axiom system presented in Paris in 1933 (section 5), and, more briefly, his further attempt at generalizing the approach in 1937 (section 6). Frisch's axiomatic approach was echoed in other of his methodological innovations, including his emphasis on interviews as virtual experiments. Another field proposed by Frisch for axiomatization was national accounting. These links are briefly elaborated in the concluding section 7.

2. Brief biographical background

Frisch was an only child and destined to take over the Frisch family's jeweller shop in Oslo. He excelled in school and while he was in training to become a silversmith he also followed, at his mother's suggestion, the two-year program in economics at the University of Oslo. Frisch's ambition to measure the marginal utility of income was curiously foreshadowed in on of his exam papers (on taxation issues) from 1919, containing the following bold passage: *Man must not be afraid of what seems impossible to do. History has shown that human beings possess a wonderful gift of being able to obey the saying of Aristotle : "measure the nonmeasurable!"* (Andvig & Thonstad, 1998:6). After completing his probation work as a silversmith his father made him partner in the business, which made Frisch relatively well off and allowed him to be a silent partner while pursuing his scientific interests. He was abroad for almost three years, mostly in Paris, where he without being enrolled in any study delved deeply into mathematics and statistics and mingled with French mathematicians, statisticians and economists.

At the end of his stay in Paris Frisch may have regarded himself more as a statistician than as an economist. He published some papers in mathematical statistics in the early 1920s and took part in an international mathematical congress in 1925. After his return to Oslo he submitted a doctoral dissertation in mathematical statistics (Frisch, 1926c). The same year he also in addition to his axiomatic approach (Frisch, 1926a) published an essay about the quantitative formulation of the theoretical economic laws (Frisch, 1926b). This concomitance illustrates perfectly Frisch's concern about finding a new – *econometric* – way of joining mathematics, statistics with economics in the study of economic phenomena.

Frisch was concerned about profound methodological issues and dealt with them at times by throwing around catchphrases, as e.g. "… theoretical economics is about to enter the phase of development at which natural sciences, particularly theoretical physics long have been, the phase in which the theory gets its concepts from the observational technique" (Frisch, 1926b; Frisch's emphasis).

Among scholars Frisch studied in Paris was Irving Fisher, whose 1891 dissertation he studied in a French edition (Fisher, 1917). He shared with Fisher the idea that to achieve a more scientific economics it was necessary to adapt the principles and methods of the natural sciences. They were on similar tracks in trying to make the elusive concept of utility subject to empirical estimation. Although Frisch (1926a) indicated a cue from Jevons, a more direct impetus for Frisch's utility study came from Irving Fisher, who in his dissertation had outlined how the measurement of utility could be solved, adding: "To do this statistically is of course a more difficult, though by no means hopeless proceeding."[5]

[5] Fisher (1925:20). Frisch read Fisher in a French edition, in which he had underlined the corresponding sentence: "Quant à l'établissement de cette courbe en conformité de données statistiques, c'est une autre affaire et bien plus difficile, quoiqu'il n'y ait nullement lieu de désespérer de sa réussite." Fisher (1917 :24), cf. Bjerkholt (1995 :xxiv-xxv).

Fisher published his own attempted at estimating marginal utility in a festschrift for John Bates Clark (Fisher, 1927). While Fisher was proofreading this paper in 1926 he received a reprint of Frisch (1926a). Fisher recognized immediately that Frisch had hit upon a more promising method than his own and hastily wrote a note to be attached to his own paper when it was distributed as a reprint, to notify American readers of the achievement of the unknown Frisch from Norway. Thus the parallel efforts of Fisher and Frisch both originated in Fisher's 1891 dissertation.

In February 1927 Frisch went to the U.S.A. with a Rockefeller fellowship. He met with Irving Fisher for the first time. During the year he stayed in the USA he completed two treatises, both reflected his emphasis on statistics as the key to a more scientific economics. One was *The Analysis of Statistical Time Series* (Frisch, 1927), a treatise on methods of time series analysis, critical of the current methods in use for of determining trends and cycles in economic time series data, such as Fourier analysis and periodograms, see Morgan (1990:83-90). The other was *Correlation and Scatter*, an essay on the analysis of multidimensional economic data (Frisch, 1929a). The two treatises pointed forwards towards later work on explaining business cycle studies (Frisch, 1933b) and on confluence analysis (Frisch, 1934) for which he would become better known than for his utility studies.

During his visit Frisch got an opportunity to express his view on the need for axiomatic economics. At the joint meetings of the American Economic Association and other associations in December 1927 Frisch took part from the floor in a panel discussion on the "present status and future prospects of quantitative economics." The panel was chaired by F.C. Mills who invited Frisch to submit his statement for publication. Frisch submitted but in the end the editor of the American Economic Review decided to publish only the statements of the invited panelists.[6] Frisch's prepared statement is of interest as he took the opportunity to argue in favour of an axiomatic approach, a topic not touched upon by any of the panelists:

> "We speak of one statistical procedure as giving a better result than another. The idea underlying this distinction is evidently that a statistical procedure is considered as a sort of approximation by which we try to determine the numerical magnitude or intensity of a certain phenomenon or the character of a certain function. … I cannot get rid of the impression that very often we engage in this kind of approximation work without knowing exactly what we are trying to approximate. We engage seriously in target shooting without having any target to shoot at. The target has to be furnished by axiomatic economics. The axiomatic process of target making must necessarily be rather abstract, a fact which accounts, perhaps, for its lack of popularity in these days when it is considered quite a virtue to disregard abstract thinking in economics. It is abstract, but neither in the sense of a logic game nor in the sense of metaphysical verbiage, of which we have had some in economics, at times. Axiomatic economics will construct its quantitative notions in the same way as theoretical physics has constructed its quantitative notions." (Frisch/Mills, 21 February, 1928, National Library of Norway).

Frisch's statement also conveyed his critical attitude towards empirical studies not based upon a proper theoretical foundation.

The Rockefeller fellowship had been awarded for three years, of which Frisch had planned to spend the last two in Europe. Shortly after Frisch's return to Europe Frisch's father became seriously ill and later died. This put Frisch's entire career in jeopardy. He surrendered the fellowship to take care of the family business which was in dire straits. He confided to Irving Fisher in the spring of 1929 that he was considering giving up his scientific career to take care of the family business and his economic responsibilities. Fisher responded by arranging for an

[6] The panelists included W.C. Mitchell, E.B. Wilson and others, see Mills et al. (1928).

invitation from Yale University as Visiting Professor.[7] Frisch arrived at Yale University in February 1930, a visit that would last until June 1931, with decisive importance for the development of Frisch's scientific ideas and for the emergence of econometrics, including the foundation of the Econometric Society at the end of the autumn term 1930.

In the spring of 1931 Frisch was by a special act of the Norwegian Storting (Parliament) called to a new chair in economics and statistics at the University of Oslo to prevent him from accepting an offer of a chair at Yale University, amply equipped with research resources. After his return to Oslo Frisch negotiated with Rockefeller Foundation about support for a research institute at the University of Oslo. From Rockefeller Foundation's point of view Frisch had excellent credentials for the kind of research the foundation supported and a minimal level of support was granted. Thus from 1932 Frisch was director of his own, admittedly small, Institute of Economics at the University of Oslo with research assistants and computing equipment according to what he could afford, an econometric laboratory it might well be called.

3. The 1926 axioms for utility measurement

Ragnar Frisch set out a system of axioms related to the measurement of utility in 1926. As it apparently was the first attempt at introducing an axiomatic approach in the theory of consumer behaviour it deserves to be retrieved from the shadow of history.[8] The presentation of the axiom system was preceded by one for Frisch's theory of science maxims: "The real advances in a science of the outside world begin on the day that it is realized that vague common sense notions must be replaced by notions capable of objective definition" (Frisch, 1926a). The vague notion Frisch referred to here was that of cardinal utility, a central, but not well founded concept in economics.

The concept of utility was rooted in the utilitarian school of philosophy but entered economics after Jevons (1871) introduced reasoning based on marginal utility. Utility reasoning became popular, although some economists shied away from using the concept. Several must have shared Frisch's opinion that it was a concept without a proper foundation and used in ways that were at variance with scientific criteria. As Frisch expressed it some years later:

> "There have been numerous attempts at getting rid of this concept in economics, but the concept has remained despite these efforts. We have given it other names, that is true, but we have not been able to dispense with it. When it has been thrown out of the main door, it has come back, so to say, by slipping in through the back door. In my opinion, we can just as well accept this situation, reconcile ourselves with the concept of utility and make a true effort at reformulating it as far as possible as a quantitative concept." (Frisch, 1933a, Lecture 1).

Frisch's conception of utility adhered to those of Fisher and Pareto, both of whom he found had contributed substantially to quantify the concept of utility.[9] But Frisch argued that an

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[7] It is still hard to believe that Frisch after his achievements up to 1929 would consider giving up his scientific career. Was he perhaps just fishing for Fishers' generosity. Fisher indeed a very wealthy man at the time, he gave Yale the money for inviting Frisch. Fisher became, however, ruined by the 1929 crash. His commitment to Frisch (and Yale) was given just before the crash.

[8] The main effort in this regard was undertaken by John S. Chipman who had Frisch's obscurely published essay translated and also commented in Chipman et al. (1971) volume that immediately became a classic in demand theory.

[9] Frisch's admiration for Jevons, as well as for Pareto, for their contributions towards turning economics into a science and thus as predecessors of econometrics, is apparent from the fact that they were both among a small group of pioneers *Econometrica*, edited by Frisch, paid homage to in early issues, in *Econometrica* **3**, 225-237 and **6**, 1-21, respectively. The only other past pioneers who were honoured similarly were Cournot, Wicksell, von Thünen, Edgeworth and Walras, all among Frisch's heroes. Irving Fisher, in Frisch's view a pioneer on the same level, was at the time the President of Econometric Society.

objective definition of utility had not yet been achieved, as the axioms on which such a definition must be based had not been displayed. That is what he set out to do in his essay, following Fisher in considering the issue one of logic of choice, not of psychology.[10] Inter-personal comparisons were not ruled out at the outset but the axiomatic approach helped to clarify also that issue.

The axioms were simple. A central feature of Frisch's approach to axioms for *homo œconomicus* was the distinction between the initial position given as vector of economic goods and a displacement (finite or infinitesimal) of the initial situation, also given as a vector.

The preference comparisons in Frisch's axiom system were between combinations of a position with a displacement. The symbol $\succ$ was used for preferred to and $\simeq$ for equivalent.

His axioms were as given in the table 1 below. The vector notation is slightly adapted and uses

$\boldsymbol{P} = (P_1,P_2,\ldots,P_n)$, $\boldsymbol{Q}$, $\boldsymbol{R}$, … for positions,
$\boldsymbol{a} = (a_1,a_2,\ldots,a_n)$, $\boldsymbol{b}$, $\boldsymbol{c}$,… for finite displacements and
$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots \alpha_n)$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, … for infinitesimal displacements.

Table 1. The 1926 axioms for utility measurement

Axioms relating to a given position	**Axioms relating to different positions**
Axiom of choice $(\boldsymbol{P},\boldsymbol{a}) \succ (\boldsymbol{P},\boldsymbol{b})$ or $(\boldsymbol{P},\boldsymbol{b}) \succ (\boldsymbol{P},\boldsymbol{a})$ or $(\boldsymbol{P},\boldsymbol{a}) \simeq (\boldsymbol{P},\boldsymbol{b})$	**Axiom of choice** $(\boldsymbol{P},\boldsymbol{a}) \succ (\boldsymbol{Q},\boldsymbol{b})$ or $(\boldsymbol{Q},\boldsymbol{b}) \succ (\boldsymbol{P},\boldsymbol{a})$ or $(\boldsymbol{P},\boldsymbol{a}) \simeq (\boldsymbol{Q},\boldsymbol{b})$
Axiom of coordination $(\boldsymbol{P},\boldsymbol{a}) \succ (\boldsymbol{P},\boldsymbol{b})$ & $(\boldsymbol{P},\boldsymbol{b}) \succ (\boldsymbol{P},\boldsymbol{c})$, implies $(\boldsymbol{P},\boldsymbol{a}) \succ (\boldsymbol{P},\boldsymbol{c})$	**Axiom of coordination** $(\boldsymbol{P},\boldsymbol{a}) \succ (\boldsymbol{Q},\boldsymbol{b})$ & $(\boldsymbol{Q},\boldsymbol{b}) \succ (\boldsymbol{R},\boldsymbol{c})$, implies $(\boldsymbol{P},\boldsymbol{a}) \succ (\boldsymbol{R},\boldsymbol{c})$
Axiom of addition $(\boldsymbol{P},\boldsymbol{\alpha}) \succ (\boldsymbol{P},\boldsymbol{\beta})$ & $(\boldsymbol{P},\boldsymbol{\gamma}) \succ (\boldsymbol{P},\boldsymbol{\delta})$, implies $(\boldsymbol{P},\boldsymbol{\alpha}+\boldsymbol{\gamma}) \succ (\boldsymbol{P},\boldsymbol{\beta}+\boldsymbol{\delta})$	**Axiom of addition** $(\boldsymbol{P},\boldsymbol{\alpha}) \succ (\boldsymbol{Q},\boldsymbol{\beta})$ & $(\boldsymbol{P},\boldsymbol{\gamma}) \succ (\boldsymbol{Q},\boldsymbol{\delta})$, implies $(\boldsymbol{P},\boldsymbol{\alpha}+\boldsymbol{\gamma}) \succ (\boldsymbol{Q},\boldsymbol{\beta}+\boldsymbol{\delta})$

Adapted from Frisch (1971:388-389).

The displacements could be of either sign. An individual would then traverse an *acquisition path*, assumed to be piecewise continuous. The individual's consumption would likewise be represented by a *consumption path*, not necessarily identical with the acquisition path.

Frisch's "axioms relating to a given position" make binary comparisons between commodity bundles considered as displacements from a given initial bundle. The axioms of choice and coordination (more commonly called connexity and transitivity) imply a ranking of all such displacements. The *addition axiom* in terms of infinitesimal displacements allow the

[10] The first section in Frisch (1926a), where the axioms were given, was titled "Utility as a quantity", i.e. exactly the same title as Chapter one of Fisher (1925). Hardly a coincidence!

approximation of indifference varieties in small neighbourhoods by their supporting hyperplanes. The vector normal to the supporting hyperplane at a given point Frisch called - following Fisher's terminology - the *maximum direction*. Let $\boldsymbol{x}$ now denote a position and let the vector normal to the supporting hyperplane through $\boldsymbol{x}$ be $\boldsymbol{u}$ (with $\boldsymbol{u}$ of arbitrary length). Then small displacements from **x**, say $\boldsymbol{\delta x}$, can be compared by the inner product $\boldsymbol{u\delta x}$:

$$(\boldsymbol{x},\boldsymbol{\delta^{(1)}x}) \begin{pmatrix} \succ \\ \sim \\ \prec \end{pmatrix} (\boldsymbol{x},\boldsymbol{\delta^{(2)}x}) \quad \text{according as} \quad \boldsymbol{u\delta^{(1)}x} \begin{pmatrix} > \\ = \\ < \end{pmatrix} \boldsymbol{u\delta^{(2)}x}$$

Thus the outcome of the axioms relating to a given position is an ordinal measure of marginal utility. That would suffice to elaborate the theory of the static equilibrium of exchange but not for a quantitative conception of utility. Thus for Frisch's purpose this was not enough, an "objective definition of the length of $\boldsymbol{u}$" was needed (Frisch, 1971:391).

The axioms "relating to different positions" similarly made binary comparisons of displacements from different initial bundles and asserted connexity and transitivity. The idea of making such comparisons can be traced back to Pareto, from which Frisch most likely got the inspiration for these axioms.[11] The addition axiom allowed comparisons to be made for comparing displacements from different initial bundles, leading to the determination of the normal for any point $\boldsymbol{x}$, i.e. $\boldsymbol{u=u(x)}$. Then "the values of the inner product $\boldsymbol{u\delta x}$ is distributed throughout the entire space in a manner analogous to individual preferences" (Frisch, 1971:393). Hence, if $\boldsymbol{\delta x}$ and $\boldsymbol{\delta y}$ were two arbitrary infinitesimal displacements around $\boldsymbol{x}$ and $\boldsymbol{y}$, respectively, then the choice would be

$$(\boldsymbol{x},\boldsymbol{\delta x}) \begin{pmatrix} \succ \\ \sim \\ \prec \end{pmatrix} (\boldsymbol{y},\boldsymbol{\delta y}) \quad \text{according as} \quad \boldsymbol{u(x)\delta x} \begin{pmatrix} > \\ = \\ < \end{pmatrix} \boldsymbol{u(y)\delta y}$$

By this result Frisch's target was achieved, the inner product $\boldsymbol{u(x)\delta x}$ was a measure of the utility of the displacement. The components $(\boldsymbol{u_1, u_2,\ldots, u_n})$ of $\boldsymbol{u(x)}$ were the marginal utilities of the goods 1,2,…,*n*. From the first set of axioms the ratio of two marginal utilities could only be defined with reference to one and the same position, while from the second set the ratio of two arbitrary marginal utilities could be determined. The vector field thus defined Frisch called the choice field of the individual considered.

But the length of the vector $\boldsymbol{u}(\boldsymbol{x})$ still had an arbitrary factor, not determined by axioms. This would be a common factor for the marginal utilities all over the choice field. The remaining factor could be identified with the individual under consideration, a factor Frisch found it impossible to define in an objective manner. In this sense the definition of marginal utility achieved was not universal: "This lack of universality in the objective definition of marginal utility is nevertheless not essential to the aim we pursue. What matters for us is to have defined the field of choice field of a specific individual" (Frisch, 1971:394).

Thus in principle the choice field could be determined for any individual by interview questioning according to the axioms. Frisch called it experimental determination, clearly considering questioning as a (virtual) experiment. He also tried to get away from the tainted "utility" term by redefining the marginal utility as a *choice coefficient,* perhaps to underline the operative the nature of the concept, "when equilibrium actually occurs in the market our individual… will find himself at a point in his field of choice where [the choice coefficients]

[11] Pareto's sketch of an argument can be found in Pareto (1909, Ch. IV:264-265), see Chipman et al. (1971:326).

are proportional to the prices" (Frisch, 1971:394-395). He considered also Fisher's term, *desirability*, or Pareto's term, *utility*, as alternatives to his own, and seemed to vacillate a little between them.

Frisch considered only very briefly at the end of his discussion of the axiom system the integrability issue, i.e. whether a definition of total utility could be sought by considering the integral of $\boldsymbol{u(x)}$ along the consumption path, noting that "such a passage from marginal utility to total utility is not as simple as it might seem at first glance … one must not only distinguish two cases according as $\boldsymbol{u(x)}$ is or is not the derivative of a potential, one must also make more delicate distinctions" (Frisch, 1971:395).[12]

Frisch's 1926 attempt at axiomatization is discussed in by John S. Chipman in Chipman et al. (1971, 326-327). Frisch's axiomatic logic was convincing but perhaps with shortcomings in the lack of mathematical regularity conditions added to the axioms. A mathematically proficient treatment of the measurability of utility starting from a set of postulates, was given a few years later by Franz Alt, see Alt (1971).[13]

The axiomatic contribution of Frisch (1926a) was not much referred to and hardly discussed at all before WWII. One reason may have been the limited distribution of the journal it appeared in, although Frisch surely tried to remedy this by distributing reprints. This is in contrast to Frisch (1932) which became well known and got many reviews.[14] Frisch's empirical approach is thoroughly discussed in Chipman (1998). In a recent survey of integrability in demand theory Hands (2006) reviews Frisch (1926a) and compares his framework with those of Pareto, Hicks and Allen, Georgescu-Roegen and others. Hands misinterprets Frisch, however, when he asserts that Frisch's framework was "a version of contingent, or position-dependent, preferences" (Hands, 2006:163) which is hardly the case.

The 1932 monograph did not comprise the axiom system, but its approach to measurement of marginal utility rested ultimately upon the measurability proved in Frisch (1926a). The attention Frisch attracted for his utility work in the interwar period thus relied not on his innovative approach to the "theoretical quantification" of utility formulation but on his equally inventive definition of empirical measurement procedures.

The econometric research program that Frisch had pursued since the early 1920s with regard to utility comprised three steps, stated in 1932 as follows:

> "(1) List the choice axioms which are necessary for considering utility as a quantity and define utility in a rigorous manner on the basis of such axioms;
> (2) Develop a method for measuring utility statistically;
> (3) Apply this method to given data"
> (Frisch, 1932:2-3).

The axioms defined at the first step of work would then provide the foundation for step two and ultimately be the basis for the results in step three. The 1932 monograph was mainly devoted to the second and last steps of the Frischian agenda in order to get empirical demand curves according to the availability of data on prices, incomes, purchased quantities. And it is this empirical research result that Schultz paid homage to in 1938:

[12] Frisch gave some attention to the "passage from marginal utility to total utility" in Frisch (1929b), but this article is in Norwegian and the relevant part not in the abbreviated translation that appeared in 1992.
[13] Alt's essay was inspired by ongoing, but largely non-axiomatic, discussion of the measurability issue, see Lange (1934a, 1934b), Phelps Brown (1934) and Bernardelli (1934).
[14] Of seminal contributions on the measurement of utility in the mid-1930s such as Lange (1934a), Samuelson (1937), and Alt (1936) only Lange referred to Frisch (1926a), but without mentioning the axioms.

> "Frisch's contributions have for their primary object the measurement of the degree of utility of money and not the derivation of statistical demand curves. But his fundamental procedure yields as a by-product the Cournot-Marshall demand curve. (…) Frisch must, therefore, be considered as the first economist after Pigou to suggest a method for deriving demand curves from family-budget data." (Schultz, 1938: 111).

Schultz then explained that the method developed by Frisch in order to derive the empirical demand curves from household budget data together with the work of Irving Fisher (1927), of René Roy (1932) and of Jacob Marschak (1931) marked the "revival and the development" of the statistical research in demand field (Schultz, 1938:64-65). Other contemporaries concerned with empirical demand curves were Moore, Ezekiel, Bean, Warren and Working. But the econometricians were still trying to reach a consensus on the definition of the theoretical object to be measured, cf. Wulwich (1995).

4. Frisch on utility measurement at Yale 1930.

At Yale 1930 Frisch lectured on specific topics such as production theory and time series analysis, but in addition he gave a lecture series in the autumn of 1930 called *A Dynamic Approach to Economic Theory* (Frisch, 1930), which covered other topics including the ones we are discussing here. The lectures were supported by lecture notes, written out but never published. In these lectures Frisch gave a lucid and well formulated account of his ideas, which we can convey here as excerpts from the manuscript as retrieved from Frisch's archive.[15] Frisch offers this account and rationale for the overall idea of the axiomatic foundation of utility:

> "The axiomatic definition of utility as a quantity is based on a system of fictitious *interrogation experiments* performed on an individual. We invent, so to speak, a series of situations, and imagine that we ask the individual questions as to what he would do in these various situations. These questions we may call the *choice-questions*. From the description of the situations involved in the choice-questions and from the answers given we try then to formulate the rigorous utility definition. The choice-questions must, of course, be such that both the situations and the answers can be formulated in objective terms. Sometimes it may even be necessary to require that they can be formulated in quantitative terms. But, if we only have the *definition* of utility in mind, it is not necessary that the interrogation experiments shall be actually possible in a technical and practical sense. It does not matter if the cost of, or the practical difficulties involved in an actual statistical survey would be prohibitive. It is sufficient that the experiments are possible in principle." (Frisch, 1930, Part II).

A parallel could be drawn with physics:

> "The point may perhaps be made clear by a comparison with some of the axiomatic "experiments" of physics. Take, for instance, the light signals of relativity axiomatics. These are not actually "possible" in a technical sense. Strictly speaking, they are only a way of thinking. They are only a theoretical tool used in order to give a precise and concrete significance to our ideas. When we look upon them and talk about them as "experiments" it is just because this similarity with actual experiments furnishes the preciseness and clearness of thought that are necessary in the logical construction of the science. A similar role is played by the interrogation experiments of utility theory."

The axioms represent the behaviour of *typical* rather than *living* individuals:

> "There is also this difference between an axiomatic and an actual interrogation experiment, that the latter must always be made on some living individual (or on a concrete group of individuals that are guided by some sort of joint action), while the former can be conceived of

[15] The lecture notes are now being prepared for publication.

> as made on some average or *typical* individual. In this latter case we have to adopt some general assumptions or *choice-axioms* regarding the way in which our typical individual will react when subjected to our "experiments." Otherwise our "experiments" would be a complete chaos without any meaning. From a formal point of view we are at liberty to choose any set of choice-axioms that we may favour, but in reality we are under a very severe restriction, that namely of adopting a set of axioms that will lead to a really fruitful theory fitting the facts, that is, a theory which will be able to "explain" the results of actual observations or actual experiments." (Frisch, 1930, Part II).

After this general sketch of the idea of an axiomatic approach Frisch elucidated the relation between the hypothetical choice-question setting and market transactions:

> "When we ask a choice-axiom we must take care to specify if the assumption is that the transaction involved in the choice-question shall be the only one which the individual is allowed to make, or if the assumption is that the individual, after having made this transaction, shall be allowed to make *other* transactions which a possible third party may offer him. It is obvious that his answer to the choice-question may be different in the two cases. Suppose for instance that a passionate smoker is asked if he wants 50 good Havana cigars or a carload of California peaches as a present. If it is understood that he may accept the peaches and *afterwards sell them*, he will, of course, choose the peaches, because the sale of these will bring money enough to buy both Havana cigars and many other things. But if it is understood that he must actually keep and consume the thing he chooses, then he would probably choose the cigars." (Frisch, 1930, Part II).

This again led to the distinction between *direct* and *indirect* utility:

> "Since the answer to a choice-question may differ accordingly as we assume the transaction involved in the question to be the only transaction allowed or not, the notion of utility which is defined by such question will, of course, also be different in the two cases. The kind of utility which is defined by assuming, in each choice question, that the transaction involved in the question is the only transaction allowed, we shall call a *direct* utility. The other kind we shall call an *indirect* utility." (Frisch, 1930, Part II).

The primacy of direct utility (a reversal of the *revealed preference* approach?) would finally lead to money utility, but not without rebuking Pareto:

> "From an axiomatic point of view the fundamental difference between a direct and indirect utility lies in the fact that the choice-question by which the latter is established *has no sense unless the whole set of exchange possibilities in the market (prices, etc.) are given*, while these data in general are irrelevant to the choice-question establishing the direct utility. In other words, direct utility can be defined without reference to the market situation but indirect utility cannot. The notion of direct utility therefore offers itself as a more fundamental tool of analysis than indirect utility. This fundamental character of direct utility is emphasised by the fact that the notion of indirect utility can be derived from an analysis of the exchange mechanism based on the notion of direct utility. …The marginal utility of money is essentially an indirect utility when the good serving as money is only a medium of exchange without any intrinsic usefulness to the individual. The definition of money utility therefore demands special care. So far as the utility definition is concerned, money cannot be treated in exactly the same way as the direct goods. Therefore the procedure used by Pareto and his followers, namely, to take any commodity as a "numéraire" and by convention put its price equal to unit, breaks down when the money is not a direct good. Pareto himself does not seem to have been aware of this fact." (Frisch, 1930, Part II).

Let these quotes just serve as a glimpse into a source that is not yet available, but is most likely the place where Frisch best explained in general terms the ideas underlying his axiomatic approach. Why was it not published? It is unfortunately far from being the only case of key Frisch ideas remaining in the shadow. In fact he published poorly.

But let us note also Frisch's beautiful elaboration of the psychological vs. the behaviouristic approach to utility measurement and his settling of accounts with the Austrian school of Eugen von Böhm-Bawerk:

> "The chain of reasoning by which the founders of the Austrian School in economics reached the notion of marginal utility, was somewhat like this: They took as a starting point the empirical fact that when a man gets more of a certain good there is a certain feature in his state of mind that *changes*. In precise objective terms, it is difficult to explain in what this change really consists. … Therefore it is possible to give some kind of description of the process, simply by referring to this internal experience and, so to speak, leave it to the reader or listener himself to supply the concrete meaning of the words. If, on this basis, we want a verbal definition of the process, we can say that the *want* or the *craving* for the good in question diminishes as one gets more of it. The satisfaction that is obtained by the consumption of the *last* unit of the good or if one prefers: the utility created by this satisfaction, may be called the "marginal utility"." (Frisch, 1930, Part II).

Here Frisch drew the borderline of what was scientific:

> "This definition has none of the objective criteria that we are accustomed to look for in the definitions of the natural sciences. It has a sense and is understood by other people only because these people themselves have had an internal experience, a want satisfaction, that corresponds to the definition given. It is a definition of the same kind as one we would obtain, say, by defining "force" as "the strain exerted when heavy things are lifted." In particular it should be noticed that this primitive utility definition does not establish marginal utility as a quantitative notion." (Frisch, 1930, Part II).

But scientific economics, i.e. econometrics, requires quantification:

> "Nevertheless, in the earlier days of marginal utility theory the notion was often handled as if it should have been quantitatively defined. It is easy to understand why it was handled this way. If marginal utility shall be of any use at all in the theoretical structure of economics, it must be drawn into a quantitative reasoning. Those phenomena which marginal utility shall explain are, indeed, by their very nature quantitative. For instance: In some way or another one has to introduce the idea that when equilibrium is reached the prices are *proportional* to the marginal utility. Such an idea evidently implies that marginal utility is quantitatively defined. Something which shall be *proportional* to something else must necessarily be quantitatively defined." (Frisch, 1930, Part II).

But deceptive shortcuts are not allowed:

> "This procedure of drawing a non-quantitatively defined notion into a quantitative argument is, of course, entirely illegitimate. It really amounts to give the notion a meaning by appealing to a certain mental association process in the reader or the listener, and then when a certain number of steps in the reasoning are passed and the attention of the reader or listener is dulled, to use the notion in a different meaning, to attribute to it features which do not lie in the given definition." (Frisch, 1930, Part II).

Frisch's point was indeed one for which the marginal utility theory in its original form was criticised, a criticism which led to a series of attempts at rigorously defining utility in a quantitative way. These attempts proceeded on two different lines, *psychological* and the *behaviouristic*, with Edgeworth and Fisher as respective exponents:

> "Edgeworth believed that he could make the notion of marginal utility more precise by an appeal to experimental psychology …a natural avenue of approach if one would follow up as closely as possible to the original idea of the Austrian School. In this theory the individuals appear as essentially passive beings, rejoicing or suffering. Fisher's point of view if different. He considers the individuals as active and acting. … Fisher himself takes the *choice acts* of the individuals as the basis for a quantitative definition of utility. … Pareto … called it *the theory of choice* ("la théorie des choix"). And the choice point of view in the analysis of utility is

perhaps best known in connection with Pareto's name. However, the theory of choice was, in fact, first introduced and developed with great consistency by Fisher." (Frisch, 1930, Part II).

Summing up the Austrian approach:

> "The question may arise: Have not the Austrian themselves adopted the idea of taking the choice acts of the individuals as a basis for the definition of utility? It is true that in several places in the works of Austrian economists we find extensive studies of the choice acts of the individuals. This is particularly true of Böhm-Bawerk, the great dialectician of the School. Böhm-Bawerk gives a whole series of arguments showing that in particular there exists certain definite regularities in the way in which people make up their minds about their choice acts. But this in itself is no sufficient to make his theory a behaviouristic theory of choice. There is one essential thing which is characteristic for the behaviouristic theory of choice and which we do not find in Böhm-Bawerk's approach, namely to take the observable regularities in the choice behaviouristic theory as a matter of fact, as something given, without digging very deeply into the psychological *motivation* which led to these regularities. …[But w]hen Böhm-Bawerk studies the observable choice regularities it is only in order *to use them as a tool in his motivation theory*." (Frisch, 1930, Part II).

The behaviouristic approach does in distinction an allowable shortcut:

> "The behaviouristic approach avoids these difficulties by … a sort of short-cut through a field which does not yet seem to be accessible to rigorous quantitative methods. By this short-cut a point is reached where it is possible to get down to positive work both in the sense of an abstract theory and in the sense of empirical verifications. By this short-cut we also avoid the difficulty which arises out of the fact that human behaviour is not built on abstract rational motives as was assumed in the earlier days of the Hedonistic School." (Frisch, 1930, Part II).

An approach when even encompasses more than *rational man*:

> "As a matter of fact, modern psychology has conclusively proved that if we want to explain human behaviour we have to take account of a whole series of irrational elements. The existence of such irrational elements makes it impossible to arrive at a rigorous quantitative utility definition on the old hedonistic and "psychological" lines, but it does not prevent the possibility of arriving at a quantitative utility definition by a behaviouristic approach to the problem. The utility definition along these lines must be based on *an axiomatic foundation*. We shall later intimate how this can be done." (Frisch, 1930, Part II).

We leave Frisch anno 1930 here to look into another still unpublished source for the progress of Frisch's ideas.

5. The extended axiom system of the Poincaré lectures 1933

Frisch was invited to give a series of eight lectures on *Problems and Methods of Econometrics* at the Poincaré Institute in Paris in the spring of 1933. He used the occasion to argue forcefully for of an axiomatic approach towards a measurable concept of utility. In the first of the eight lectures, *The philosophical foundations of econometrics, the axiomatic method, utility as quantity,* he presented a set of axioms which extended the 1926 axioms, and has never been published. The extension had to do with the integrability issue, or, more precisely, with the existence of a total utility indicator.

To Frisch econometrics meant quantification, and quantification had two aspects of equal importance. First, it meant to make theoretical concepts measurable. Then came the statistical aspect, which in the lingo of the 1920s was expressed as "to fill the boxes of abstract quantitative relationships" with real numbers based upon empirical data." For the quantification of theoretical relationships Frisch key recommendation was to approach this task through axioms, an entirely abstract approach consisting in establishing as far as possible logical and quantitative definitions and to construct from the definitions a quantitative theory

of economic relations. It was the combination (or "unification" as Frisch said) of these two aspects of abstract and concrete quantification that was the foundation for econometrics.

One of the objections against attempts of quantifying utility in precise mathematical terms that Frisch perceived and explicitly dealt with, was still that utility involved a psychological element which could not expressed in quantitative terms. Frisch had sided with Fisher on this issue and found the argument groundless and based on a confusion with regard to the nature of the conditions needed to elaborate an exact and quantitative theory for a given domain. The importance of psychological aspects for economic phenomena could not be denied and the econometrician had to take that into account. Indeed, modern scientific psychology could be of great value for the econometrician, but that did not at all imply that psychology constituted a barrier against the advancement of quantitative studies in economics.

In Frisch's view psychological phenomena of interest to the econometrician did not arise spontaneously, but were caused or intimately linked to objective external phenomena, which could be subjected to quantitative study and to a considerable degree measured, say, as an index or as a barometer, which then indirectly measured the psychological phenomena. The desire of human beings for various economic goods could for example be assessed by their actions, whether they were willing to work much to obtain them, willing to exchange them at certain rates, etc. Such actions were usually measurable objective facts.

In the lecture Frisch tried to clarify the relationship between psychological factors and objective facts also by the example of a banking crisis, created by the distrust of depositors withdrawing their deposits. Certainly there were psychological factors at play in a run on the banks but the whole phenomenon was driven by the fact that the public has observed over a period of time the development in objective and measurable circumstances characterizing the economic situation. It was in fact a regular phenomenon, which we could be predicted as occurring at a certain stage in the economic cycle, when the objective conditions were ripe. As in 1926, Frisch focused on the consequences of the choices and not on the determinants of the choices.

Hence Frisch concluded in line with what he had argued in the Yale lectures:

> "The essential condition which must be fulfilled for the econometrician to be able to formulate his quantitative laws, is not that the psychological element is present, but that it manifests itself with a certain regularity in the empirical phenomena he observes, whether psychological or not. Regularity – or lack of regularity – is not necessary linked to the absence or presence of a psychological factor." (Frisch, 1933a, Lecture 1.)

The underlying idea is that the utility for a given individual (or family) can be determined by a series of questions "which we suppose have been posed" to the given individual (or family). The questions are all about the consumer preferences and are called choice questions. The nature of the questions (and answers we can expect) is defined by a series of choice axioms. From the answers to the choice questions Frisch will show that we can deduce choice coefficients. From the choice coefficients it is then only a short step to define utility as function over vectors of consumer goods.

An essential element in the structuring of these choice questions was, as in 1926, the distinction between *choice situation* and *choice object*. These were the two key logical concepts. The choice object was something subject to a choice, while the choice situation on the contrary was given in the setting of the choice and could not be altered before deciding between the choice objects. By *choice situation* was meant a complete description of the individual's situation at the moment the transaction implied by the choice question is supposed to take place. The *choice object* was the description of the transaction itself. As an

example of choice objects Frisch offered more prosaic items than the flashy Havana cigars of the Yale lectures:

> "Let us take the example of an ordinary working family …in a situation where it has a well specified consumption budget and … someone offers it a gift:
>
> 1) either a pound of ham each month during the following year, or
> 2) 36 tickets to the cinema during that year.
>
> Here the specified budget is the choice situation and the ham and the cinema tickets are two optional choice objects." (Frisch, 1933a, Lecture 1).

Although the distinction between choice object and choice situation seemed fairly obvious, Frisch insisted on the need to point it out: "…this necessity exists. In the sophisticated and often complex analyses of the consequences we can draw from the axioms, we risk falling into graves of logical mistakes if we do not all the time remind ourselves about this distinction" (Frisch, 1933a, Lecture 1).

The *choice situation* can then be considered as a point in an abstract space with axes representing the various characteristics of choice situations.[16] This space was not necessarily a Euclidean space in *n* dimensions, but as a special case the choice situation could be represented by a point in such a space. The *choice objects* could be considered as points in another abstract space, the object space, normally assumed to be a Euclidean space. We shall assume in the following that both the choice situation and the choice object can be represented in the same *n*-dimensional space. Under this assumption the choice situation is referred to as *choice position*.

Then followed the system of axioms, which, as can easily be ascertained, was an extension of the 1926 system. Frisch also made some minor terminological changes. For the vector of marginal utilities was now used $\boldsymbol{\omega}$ instead of $\boldsymbol{u}$. The axioms relating to a given position vs. to different positions now had become *local* and *interlocal* axioms, respectively.

In setting out of the axioms the notation is as follows:

> Capital Latin letters - ***P, Q, R,*** … - denote choice situations.
>
> Small Latin letters - ***a, b, c, d,*** … - denote finite changes.
>
> Small Greek letters - $\boldsymbol{\alpha, \beta, \gamma, \delta,}$ … - denote infinitesimal changes.

Table 2. The 1933 axiom system

1. Local axioms

1.1 Determination

$$(P,a) \succ (P,b) \text{ or } (P,b) \succ (P,a) \text{ or } (P,a) \simeq (P,b)$$

1.2 Transitivity

$$\text{If } (P,a) \succ (P,b) \text{ and } (P,b) \succ (P,c), \text{ then } (P,a) \succ (P,c)$$

1.3 Additivity

[16] Frisch drew a distinction between the *choice situation* and the *decision situation*, the latter he described as "all the circumstances the individual finds himself in at the time we expose him to the choice questions. The *decision situation* might or might not be identical with the *choice situation*. If this was the case, the choice question would be said to be *immediate*. If not, the choice question would be said to be *mediate*, and it would be assumed that the individual possessed a capability of imagining more or less precisely which consequences for his well-being would follow from a change in his situation. Frisch's presentation was brief, the discussion of the axiom system was only a part of his first Poincaré lecture.

If $(P,\alpha) \succ (P,\beta)$ and $(P,\gamma) \succ (P,\delta)$, then $(P,\alpha+\gamma) \succ (P,\beta+\delta)$

2. Interlocal axioms

2.1 Determination

$(P,a) \succ (Q,b)$ or $(Q,b) \succ (P,a)$ or $(P,a) \simeq (Q,b)$

2.2 Transitivity

If $(P,a) \succ (Q,b)$ and $(Q,b) \succ (R,c)$, then $(P,a) \succ (R,c)$

2.3 Additivity

If $(P,\alpha) \succ (Q,\beta)$ and $(P,\gamma) \succ (Q,\delta)$, then $(P,\alpha+\gamma) \succ (Q,\beta+\delta)$

3. Affinity axioms

$$P + a = Q$$

3a. Contact axiom

$$\omega_i (\omega_{jk} - \omega_{kj}) + \omega_j (\omega_{ki} - \omega_{ik}) + \omega_k (\omega_{ij} - \omega_{ji})$$

3b. Potential axiom

$$\omega_{ij} = \omega_{ji}$$

4. Connectivity axiom

If $(P,a) \succ (Q,b)$ & $(P+a,\alpha) \succ (Q+b, \beta)$, implies $(P,a+\alpha) \succ (Q,b+\beta)$

5. Reversibility axiom

$(P,a-a) = (P,0)$

6. Roundabout indifference axiom

If c a closed curve, then $(P,c) = (P,0)$

7. Point determination axiom (or Position axiom)

If $(P \rightarrow R) \succ (P \rightarrow S)$, then $(Q \rightarrow R) \succ (Q \rightarrow S)$

8. Combination axiom

If $(P,a) \succ (Q,b)$ and $(R,c) \succ (S,d)$, then $(P,a) + (R,c) \succ (Q,b) + (S,d)$

As for the 1926 axioms Frisch showed that the local axioms allowed the definition in any choice position of the maximum direction determining the relative marginal utilities associated with any choice position. That was the *local point of view*. Then from the interlocal axioms also the magnitudes of the utilities, characteristic for the individual in question, could be determined apart from a constant, as components of a vector associated with each point in the abstract space of choice positions. That was the *interlocal point of view*.

Frisch then "let into the picture the idea of a certain affinity between the object space and the choice position space, i.e. considering the two spaces as being to some degree characterized by commensurable concepts." That was a new logical element not entering into the definition of the marginal utilities and meant to permit the definition of *total utility*. Frisch suggested further this would open up for "an even more complex concept of ophelimity from a dynamic point of view, i.e. ophelimity in its connection with the passage of time."

The 1933 Paris presentation did not, unlike the 1926 essay, comprise any application of the results in empirical measurement of utility, but dealt exclusively with the logical questions concerning the axioms and the consequences one may draw from them. The justification for the list of axioms, Frisch told the audience, should not "be judged purely by a priori considerations. It is not called for to argue that one could, by reference to actual life situations, find situations more or less bizarre where this or that axiom is not satisfied. It is always necessary to remind ourselves that in economics we are interested in the general phenomena and not in the isolated cases." The scientific attitude behind the axiomatic structure was to investigate which consequences we could deduce adopting this or that axiom and then see whether the consequences agreed with the observations: "It is by the subsequent agreement of the consequences of the axioms with reality that we can judge the plausibility of them."

As in the 1926 essay the choice object space was assumed to be Euclidian space of dimension equal to the number of goods, while the choice situation space on the other hand was an entirely abstract space. Thus choice positions and choice objects were at the outset inhomogeneous concepts which could not be compared. In most analyses on the concept of utility the idea that the objects and the choice positions were comparable was adopted or taken for granted from the very beginning of the analysis. It was simply assumed that the two concepts could be represented by points in the same quantitative space with identical axes. Frisch insisted on the distinction to avoid "the false impression that commensurability between object and choice situation is necessary for establishing the concepts of maximum direction and marginal ophelimity."

It is only when it was a need to push the analysis still longer that it was necessary to introduce the idea of a certain *affinity* between choice situations and choice objects. That became necessary for example to do a systematic study of the way the utility vector changed from one point to another in the choice position space, e.g. to compare the rate of the change of the vector with the rate of change of the position. This was above all necessary to push the analysis all the way to a problem implying the idea of total utility.

The idea of affinity between object and position expressed that the acceptance in a given choice position of a certain object changed in an unequivocal way the choice position, i.e. it changed in a well defined way the basis of which the individual evaluated any new small objects. Such a change could be conceived to be of different kinds, but most naturally it would change the choice position in a unique way and thus have overall consequences for the response the individual will give subsequently to local and interlocal questions. Thus from a choice position $\boldsymbol{P}$ accepting a certain object $\boldsymbol{a}$ is equivalent for individual to come to another position $\boldsymbol{Q}$, unequivocally defined by $\boldsymbol{P}$ and $\boldsymbol{a}$.

The affinity was a fundamentally new logical element in comparison with the ideas which served for the definitions of marginal utility. For those no affinity was needed, neither between the various choice positions, nor between choice positions and choice objects. The affinity axiom established such an affinity, and by that also the concept of a path in the choice position space, a path which might connect several choice positions, and the path itself a choice object.

The affinity axiom implied only that a position $\boldsymbol{P}$ and an object $\boldsymbol{a}$ determined another position $\boldsymbol{Q}$ uniquely, the inverse was not implied, i.e. two positions $\boldsymbol{P}$ and $\boldsymbol{Q}$ did not uniquely define an object. There could be several paths from $\boldsymbol{P}$ to $\boldsymbol{Q}$. The affinity axiom opened for studying way how the maximum direction as well as the length of the utility vector changed between positions.

The only changes in the choice position we should consider are those caused by the acceptance of an object, all other things are held constant. The choice position can then be

represented geometrically in the same quantitative space in *n* dimensions as the choice object. The axes measure the quantities of goods, each point in the space represents a choice position while any movement away from it represents a choice object. The movement can be either an infinitesimal change or given as a finite curve. To each point in this space a vector $\boldsymbol{\omega}$ – marginal utilities – is associated. The overall picture is analogous to the vector field in physics, it could be called a choice field.

In this vector field some special cases can be distinguished. First, there is the *local contact* case. That means the case when the maximum direction changes from one point to another in the field such that the normal planes, so to say, can be stuck together to make a family of surfaces traversing the choice situation space. The criterion for that case is that

$$\omega_i (\omega_{jk} - \omega_{kj}) + \omega_j (\omega_{ki} - \omega_{ik}) + \omega_k (\omega_{ij} + \omega_{ji}) = 0,$$

where again ω_i is the *i*-th component of the utility vector and $\omega_{ij} = \partial\omega_i/\partial x_j$. This is what is expressed by the contact axiom.

The existence of contact surfaces did not in the least imply the existence of indifference surfaces from the point of view of choosing between choice positions, and even less could we ascertain from the contact axiom that the contact surfaces represented levels of a function of total utility. In fact until this point we have not taken into the picture any choice axiom that could lead to the concept of indifference and even less to the concept of utility as an exact point function. The contact surfaces existing when the local axioms hold and also the affinity axiom and the contact axiom were thus surfaces with a purely marginal meaning. They characterized a certain way of variation of the maximum directions and have no significance beyond that.

By accepting the interlocal axioms we could consider an even more special case where the variation in the length of the vector fulfils the integrability criterion expressed by the potential axiom:

$$\omega_{ij} = \omega_{ji}$$

This criterion could be formulated, as one can see, without specifying the arbitrary universal component, but it could not be formulated without supposing that the variation in the length of the vector is defined in the field. This meant that the formulation of the potential axiom already presupposed the interlocal axioms.

When the potential axiom was satisfied, the vector integral taken along some path is, as we know, defined by the end points. There existed thus a *potential*, i.e. a function whose partial derivatives were exactly the components of the vector. In that case the contact axiom was a fortiori satisfied and the surfaces of the potential levels were identical with the contact surfaces existing according to the contact axiom. The contact surfaces now had a certain numerical meaning defined by the field vector and its length.

At this point Frisch emphasized the point that the concept of a potential function was not the same thing as an indicator of the total choice of the individual. The analysis had so far not introduced any axiom which permitted to establish the consequences of total choice. It was admittedly true that if the potential axiom was satisfied one could integrate the vector, but that did not imply an indicator for the total choice. Frisch warned about false prophets:

> "We are here at a crucial point where it seems to me that most authors who have dealt with the ophelimity problem have committed inexactness or even a fundamental error. They have assumed more or less implicitly that if the integrability criterion is satisfied in such a way that a potential exists, then that potential can be taken as total ophelimity." (Frisch, 1933a, Lecture 1).

But with integrability fulfilled the potential is necessarily a measure of total utility. To be able to interpret the vector integral as a fundamental choice index it would be necessary with a fundamentally new axiom, the *connectivity* axiom. It was only by accepting that axiom that we might interpret the vector integral as a total choice index.

What was then the content of the *connectivity* axiom? The affinity axiom said that the acceptance of an object defines a well-determined change in position. Frisch argued that it was necessary to make precise what choice characteristics were attached to the new situation. That was exactly the content of the connectivity axiom, which defined a sort of transitivity between position and object. If the individual preferred the object $\boldsymbol{a}$ in the position $\boldsymbol{P}$ to the object $\boldsymbol{b}$ in the position $\boldsymbol{Q}$, and if he further preferred the object $\boldsymbol{\alpha}$ in the position $\boldsymbol{P+a}$ to the object $\boldsymbol{\beta}$ in the position $\boldsymbol{Q+b}$, then he would also prefer the object $\boldsymbol{a+\alpha}$ in position $\boldsymbol{P}$ to $\boldsymbol{b+\beta}$ in position $\boldsymbol{Q}$. The object $\boldsymbol{a+\alpha}$ was here defined simply as the object obtained by joining the curves ($\boldsymbol{P}$,$\boldsymbol{a}$) and ($\boldsymbol{P+a}$, $\boldsymbol{\alpha}$) end to end.

The connectivity axiom presupposed by its formulation that the affinity axiom was satisfied. This could be indicated denoting the connectivity axiom instead for the *affine connectivity axiom*, but in the following it is called just the "connectivity" axiom. Adopting the connectivity axiom implied that the vector integral of utility along a given path could be taken as a "total choice coefficient" (= total utility). The connectivity axiom allowed us thus to define the total utility quantitatively. It should be noted that this definition was completely independent of the integrability condition. The only difference was that if the potential axiom holds, then total utility became a point function. If the potential axiom was not fulfilled, total utility became a functional, i.e. curve integral.

The decisive point about the existence of a total utility indicator was not that the marginal utility had a certain functional form as expressed by the potential axiom. The crux of the matter was rather to know, whether any conclusions about total aspects could be reached by starting from marginal considerations. This is just what was expressed by the connectivity axiom. On this point Frisch parted with Pareto who had attached the question of the existence of a total utility indicator to the integrability condition.

How then could total utility be defined, departing from the connectivity axiom? This possibility existed by virtue of the connectivity axiom alone, only as far as the paths along which the integration was taken were such that the increase in the ophelimities, i.e. $\boldsymbol{\omega\delta x}$ remained non-negative all along the path. But if the *reversibility axiom* is adopted the definition of total utility as a vector integral along a given path would apply also in the general case where the increase in the utility $\boldsymbol{\omega\delta x}$ did not remain non-negative.

As the reversibility axiom is nothing but a supplementary axiom to the connectivity axiom, on might say *reversible connectivity* and understand by that the affinity, connectivity, in the constrained sense, and reversibility axioms together.

The remaining axioms were of less importance and their logical content can easily be stated. The *indifference roundabout axiom* stated that the individual was indifferent with regard to transaction taking him back to the starting point. The *point determination axiom* stated that if position $\boldsymbol{R}$ was preferred to $\boldsymbol{S}$ when starting from $\boldsymbol{P}$, then $\boldsymbol{R}$ would be preferred to $\boldsymbol{S}$ also when starting from another position $\boldsymbol{Q}$. If this axiom was fulfilled then all the points in the field could be ranked in a unique order (allowing for multiplicities of indifference). Finally, the *combination axiom* expresses the possibility of enlarging the notion of an object. We consider now new objects, each of which composed by a simultaneous realization of a number of objects of the kind originally considered. It is thus an axiom, serving as basis for the definition of marginal and total utility, i.e. to help reach the point of view of internal growth and quenching.

Frisch then came to the natural question of whether the axioms were consistent and independent. Rather than systematic discussion of the issue he went straight to the core example of interest.

Connectivity, even reversible, did not imply the indifference roundabout axiom. For this one could simply imagine the case where the individual acted as if his marginal choice was determined by a vector field not derived from a potential and that his total choice was determined by the integration of that vector field along the transaction path. If connectivity was established then the indifference roundabout axiom became equivalent to the point determination axiom.

If we had connectivity, then it would also be the case that the point determination axiom and the integrability axiom were equivalent. But if connectivity was not satisfied, the integrability axiom and the point determination axiom expressed two different things. To see this more clearly one might consider the example of an individual in position $\boldsymbol{x}$ with choice coefficient for $\boldsymbol{\Delta x}$ given as:

$$coeff\ (\boldsymbol{x, \Delta x}) = \boldsymbol{\varphi(x)}\ [\boldsymbol{c_1\Delta x_1+c_2\Delta x_2+\ldots+c_n\Delta x_n}] = \boldsymbol{\varphi(x)\ \Sigma_i c_i\Delta x_i}$$

where $\boldsymbol{\varphi(x)}$ is a positively definite function and $\boldsymbol{c_1,c_2,\ldots,c_n}$ are constants. It is easy to see that we have here point determination, for, if V and W were arbitrary points we have

$$coef\ \boldsymbol{(X\rightarrow V) = \varphi(x)\ \Sigma_i c_i(V_i-X_i)}$$

and, likewise

$$coef\ \boldsymbol{(X\rightarrow W) = \varphi(x)\ \Sigma_i c_i(W_i-X_i)}$$

then

(1) $$coef\ \boldsymbol{(X\rightarrow V) - coef\ (X - W) = \varphi(x)\ \Sigma_i c_i(V_i-W_i)}$$

and, likewise

(2) $$coef\ \boldsymbol{(Y\rightarrow V) - coef\ (Y - W) = \varphi(y)\ \Sigma_i c_i(V_i-W_i)}$$

If (1) is positive, clearly (2) will also be positive, as $\boldsymbol{\varphi}$ is a positively definite function, and likewise for the negative case. The point determination axiom is thus satisfied.

If on the other hand the local and interlocal axioms were satisfied for this individual with vector components are

$$\boldsymbol{\omega_i = \lambda\ \varphi(x_1,x_2,\ldots,x_n)\ c_i} \qquad (\boldsymbol{\lambda} \text{ an arbitrary constant})$$

with partial derivatives

$$\boldsymbol{\omega_{ij} = \lambda\ \varphi_j(x_1,x_2,\ldots,x_n)\ c_i} \qquad (\text{with } \boldsymbol{\varphi_j(x_1,x_2,\ldots,x_n) = \partial\varphi/\delta x_j})$$

With $\boldsymbol{\varphi}$ an arbitrary point function, obviously we do not have in general $\boldsymbol{\omega_{ij} = \omega_{ji}}$, i.e. we can have point determination without the integrability condition being satisfied. But as a special case we might also have integrability, e.g. if $\boldsymbol{\varphi}$ was given by

$$\varphi(x_1,\ldots,x_n) = c_1x_1 + \ldots + c_nx_n$$

then $\varphi_i = c_i$ and the integrability condition could be satisfied.

Let us then see whether the connectivity axiom is satisfied. The connectivity axiom can be rendered as

$$\boldsymbol{(x,\Delta x) \succ (y, \Delta y)\ \ and\ \ (x+\Delta x, \Delta' x) \succ (y+\Delta y, \Delta' y)}$$

implies $$\boldsymbol{(x, \Delta x+\Delta' x) \succ (y, \Delta y+\Delta' y)}$$

i.e.

(1) $\varphi(x)\,\Sigma_i c_i\,\Delta x_i > \varphi(y)\,\Sigma_i c_i\,\Delta y_i$

and

(2) $\varphi(x+\Delta x)\,\Sigma_i c_i\,\Delta' x_i > \varphi(y+\Delta)\,\Sigma_i c_i\,\Delta' y_i$

ought to imply

(3) $\varphi(x)\,\Sigma_i c_i\,(\Delta x_i + \Delta' x_i) > \varphi(y)\,\Sigma_i c_i\,(\Delta y_i + \Delta' y_i)$

Generally it is obviously possible to choose $\varphi(x)$ such that this inequality is not satisfied. We can thus have point determination without having connectivity and integrability. Thus we can have point determination with integrability, but without connectivity.

Frisch ended his presentation here, apparently as if he had reached the conclusion in an issue that had been much discussed:

> "This result is perhaps surprising, but has an entirely natural explanation if we keep in mind the fundamental role played by the connectivity axiom. That axiom suffices to show you the kind of analyses through which we can ensure the compatibility and independence of the various axioms." (Frisch, 1933a, Lecture 1)

Thus this was Frisch's effort at clarifying the issues of integrability and the existence of a total utility indicator. Unfortunately, the further discussion of this issue by other scholars chose alternative routes and thus a complete complete clarification via axioms may still be missing.

6. The General Choice-Field Theory of 1937

Frisch attended the Cowles Commission Third Annual Research Conference on Economics and Statistics in 1937. The conference took place as the preceding conferences in Colorado Springs, Colorado where the Cowles Commission had its offices.[17] The conference was more like a summer camp with only a couple of lectures each day and with outings to Pikes Peak, Mesa Verde and Cheyenne Mountain. It lasted from 28 June until 23 July.

Frisch gave a series of three lectures under the common title of General Choice-Field Theory. The theory of choice was introduced as very broad in scope "applying for instance to the behaviour of the entrepreneurs in a capitalistic society as well to the chief of production in a planned economy and to the housewife who decides how much to use on food, clothing, shelter, etc." The aim of the paper was to outline general principles for all such applications and the title of "choice-field theory" (obviously adapted from field theory in physics) was used because "frequently one is led to consider notions like vectors, potentials and the like." Thus it was a somewhat broader attempt at arguing for an axiomatic approach towards individual economic behaviour, the content of his lectures may be viewed as a generalization of his earlier system of axioms.[18]

The Cowles Commission conferences were not organized along common themes, hence it was most likely a coincidence that there were a several related contributions, particularly by Edward V. Huntington (Harvard University) on The Method of Postulates, Karl Menger (University of Notre Dame) on An Exact Theory on Social Relations and Groups, Harold T.

[17] The Cowles Commission was affiliated with the Econometric Society, in which Frisch was central figure as Council member of the Society and editor of its journal Econometrica. He was both a research consultant and member of the Advisory Council of the Commission.

[18] The discussion is based on the abstract of the lectures over six (small) pages in Cowles Commission (1937). A paper was apparently never completed, but Frisch left a file of notes in his archive related to the lecture series. The file has not been studied.

Davis (Indiana University) on The Fonctionnelle Nature of Utility, and Jakob Marschak (University of Oxford) on Utilities and Probabilities in Human Choice, which may have provided a basis for leisurely exchange on the topic of axiomatic approaches and utility measurement.[19]

Frisch again reiterated that only by axiomatics – or "a strict postulational method" – could a logically satisfactory foundation for a general choice-field theory be constructed. Again Frisch changed the conceptual definitions and the terminology slightly. An actor to be considered in a choice-field theory, say an individual (a family, a group of consumers) is someone to whom questions are put regarding his or her preferences. The decision point of the actor is the entity of "all the conditions that prevail at the moment when the questions are asked." The reception points are the "situations that are assumed to prevail when the goods or services involved in the questions are to be received." The comes the choice objects, which are the "complexes of goods involved in the question." A choice alternative is an association between a choice object and a reception point.

It is already indicated that Frisch's approach here is directed towards a structured approach to questioning. This is indeed an activity that he had experimented with from the mid-1920s and which he should come to pursue very seriously in the postwar period, see below.

The "logical base" Frisch outlined to his Colorado audience comprised three classes of elements:

K_1 was the class of all reception points (i.e. corresponding to the choice situations or positions in 1933)

K_2 was the class of all choice objects, which could be either finite or infinitesimal.

K_3 was the class of all choice alternatives, that is a combination of a choice object and a reception point.

A binary operator on an element of K_1 and an element of K_2 would then generate an element in K_3. The choice objects could be added with commutativity and associativity to generate other elements in K_2. The formulation somewhat more of set-theoretic logic than the earlier attempts.

To the class of choice alternatives, K_3, applied the dyadic relations of preference and equivalence. The set of *choice postulates* comprised in addition to *determinateness*, *transitivity* and *additivity*, other postulates denoted as *continuity*, *connectivity*, *reversibility* and *contractivity*.

The point of the presentation was then to prove, as in 1933, that "certain vectors and integrals exist and have the meaning of choice indicators" (Cowles Commission, 1937, 69).

We leave the 1937 effort at establishing a general choice-field theory here.[20]

7. Link from the axiomatization effort into other fields

There are various links for the axiomatization effort to other areas of Frisch's research. We have already touched upon some. The analogy to axioms for approaching the behaviour of economic man in how to approach the functioning of an economic (sub-)system is the structural model, typically in the form of a system of equations. Before Frisch (1933b) this

[19] Only abstracts were published in the conference report, see Cowles Commission (1937).

[20] Frisch also gave an additional lecture at the conference titled *Price-of-Living Comparisons Between Different Countries*, which he presented as an application of the theoretical apparatus of the *Choice-Field Theory* lectures and for which Frisch claimed that it offered a solution to the problem of how a price-of-living index could be constructed between markets that were "structurally different in the sense that they did not have the same kinds of commodities and services." The abstract/summary of this paper was only one sentence in the Report.

approach was not a very common one. Frisch noted himself with some surprise that among all those who tried to figure out the true answer to what causes economic fluctuations, no one approached the issue by means of a determinate system of equations. This was Frisch's structural equations approach, taken over by Haavelmo and mostly discussed in econometric contexts, but together with axiomatics it was his contribution to "theoretical quantification".

Frisch introduced the idea of modelling in a persuasive way, set out in simple language in the 1930 Yale lectures we have mentioned above. We cannot summarize his view here any better than again quoting directly a passage from the lectures:

> "The observational world itself, taken as a whole in its infinite complexity and with its infinite mass of detail, is impossible to grasp. Taken in its entirety, in its immediate form of sense impressions, it resembles, so to speak, a jelly-like mass on which the mind cannot get a grip. In order to create points where the mind can get a grip, we make an intellectual trick: In our mind we create a little model world of our own, a model world which is not too complicated to be overlooked, and which is equipped with points where the mind can get a grip … And then we analyse this little model world instead of the real world. This mental trick is the thing which constitutes the rational method, that is, theory. …When we create the model world it is up to ourselves to decide which features and characteristics the model world shall have and what kind of relations shall exist between the various phenomena and groups of phenomena in the model world. This we can do because we are sovereigns in the model world, so long as we do not break the rules of formal logic.
>
> This does not mean, of course, that our decisions regarding the constitution of the model world are ruled completely by free fantasy or caprice. The model world shall serve a purpose. It shall help to adopt a way of thinking that will ultimately be useful in our fight for control over nature and social institutions. It shall picture those undefinable things in the real world which we might call "essentials," meaning by that, of course, essentials with regard to our own ends.
>
> What kind of criterion have we then, by which to judge if our model world conforms to this idea? We have no such criterion, – none that can be formulated as a definite logical rule. We have nothing except a mysterious, inborn "sense of smell" which as a rule will guide as so that we finally get on the right track. This is precisely the reason why the scientist is to be considered a logical sovereign in his model world. He is just like a wise, absolute monarch. He uses his prerogatives with tact and care. He knows that this is the only way of ultimately obtaining his ends. He listens to the suggestions of facts but takes care to consider them non-obligatory.
>
> The laws of the model world will often consist only in typification, and idealisation of some observed empirical law. … But often the investigator will equip his model world with something more than this. By a heroical guess, he will add something which is entirely outside the body of observation at his disposal. It is exactly in this kind of heroical guesses, transgressing the observational facts, that the great constructive minds distinguished themselves from the average scientific worker.
>
> This is something which the constructive imaginative mind adds to the observations can either be a new kind of objects, not resembling anything which is known from actual observations, or it can be a new relation between phenomena which are by themselves well known from actual experiences but which have never been observationally related, because nobody has thought of it, or because the phenomena are of such a kind that they cannot be observed together directly with the given technique of observation." (Frisch, 1930, Part I).

The structural equation modelling approach was quickly embraced in the profession, unlike the axiomatization idea, but it can easily be seen as having come from the same source.

Another link is national accounting. Frisch argued on various occasions in the early post-WWII period for an axiomatic approach in national accounting. Frisch had been in charge of a pioneering effort to develop national accounts for Norway from 1937. Frisch concentrated on

methodological aspects and left the data work to his assistants who worked until the war interrupted the project. The appropriate references here is Frisch (1949,1950), but these are preliminary works which promoted the idea of an axiomatic approach rather than offering one. But this point can be summarized by noting that national accounting became increasingly axiomatized, although that term may not have been used, when the international cooperation under the auspices of OEEC and UN got under way. Frisch deserves credit for having argued in this direction relatively early.

Still another link is Frisch's interest in interview technique as a basis for information that is hard or impossible to get by passive observation. One interpretation Frisch offered of the axiomatic approach was that it represented hypothetical questioning and could be made basis for actual questioning. This was a line Frisch pursued in the post-war period for estimating preference functions of politicians and policy makers, see Frisch (1970). Frisch had a poor understanding of the functioning of the political system. He pursued ideas that the execution of economic policy should be put in the hands of experts with access to the most sophisticated models of the economy, to the carefully estimated preferences of policy makers, and to the required mathematical programming algorithms. But within the misconceived picture of the political realities Frisch displayed brilliant insights, among which was his interview technique for revealing the preferences of politicians and planners, see Bjerkholt and Strøm (2001), Frisch (1961).

Frisch in fact argued more generally from early on that interviews ought to be pursued for compiling economic data passive observations constrained to fulfil simultaneous relationships are unable to provide (the counterfactual) information about the underlying autonomous relationships, as he did in Frisch (1938).

Then there is the utility measurement which was part and parcel of the axiomatization approach in 1926 and for which the axioms were meant to provide a logical foundation. Frisch's applied empirical work in "utility measurement" which gained much more attention and critical review than his axiomatization. The connection between the axiomatization and the empirical application was hardly as close as asserted and Frisch's ad hoc assumptions received sharp criticism, as in Burk (1936). For the empirical analysis Frisch assumed that the price of the common good (the bundle of goods serving as a reference for measuring income) could be approximated by the statistical index measuring the general level of prices. But Allen showed in 1933 that this approximation amounted in fact to a restrictive assumption on the composition of the common good. The criticism led to Frisch's exploration of price index theory, but applied utility measurement never became a very flourishing field, although the Frisch's *Complete Scheme* for estimating demand systems (Frisch, 1959) was a late, but still a direct outcome of the axiomatic approach and the utility measurement effort. It became Frisch's most cited work.[21]

[21] Frisch (1959) had an appendix on integrability conditions, which was appended to the translation of Frisch (1926a) in Frisch (1971).

References

Alt, F. (1971[1936]), On the Measurability of Utility. In Chipman et al. (1971), 424-436. [Über die Messbarkeit des Nutzens, *Zeitschrift für Nationalökonomie* **7**, 161-169.]

Andvig, J.Chr and T. Thonstad (1998), Ragnar Frisch at the University of Oslo. In: Strøm (1998), 3-25.

Bernardelli, H. (1934), Notes on the determinateness of the utility function, II, *The Review of Economic Studies* **2**, 69-75.

Bjerkholt, O. ed. (1995), *Foundations of Modern Econom*ctrics. *The Selected Essays of Ragnar Frisch*, Two vols., Aldershot, UK: Edward Elgar.

Bjerkholt, O. and S. Strøm (2001), Decision Models and Preferences: The Pioneering Contributions of Ragnar Frisch. In A. S. Tangian and J. Gruber (eds.): *Constructing and Applying Objective Functions*, Berlin: Springer, 2001, 17-36.

Burk [Bergson], A. (1936), Real income, expenditure proportionality and Frisch's "New Methods of Measuring Marginal Utility", *Review of Economic Studies* **4**, 33-52.

Chipman, J.S, L. Hurwicz, M.K. Richter and H.F. Sonnenschein (eds.), 1971: *Preferences, Utility and Demand*. *A Minnesota Symposium*. New York: Harcourt Brace Jovanovich.

Chipman, J.S. (1998), The Contributions of Ragnar Frisch to Economics and Econometrics. In: Strøm (1998), 58-108.

Cowles Commission 1937: *Report of the Third Annual Research Conference in Economics and Statistic, Cowles Commission*, Colorado Springs, Colorado.

Dupont-Kieffer, A. (2003), *Ragnar Frisch et l'économétrie: l'invention de modèles et d'instruments à des fins normatives*, Thèse pour le doctorat en science économique, Université Paris 1 Pantheon-Sorbonne, Paris.

Fisher, I. (1917), *Recherches Mathématiques sur la Théorie de la Valeur et des Prix*, Paris : Giard & Briére.

Fisher, I. (1925[1892]), *The Mathematical Investigations in the Theory of Value and Prices*, New Haven; Yale University Press.

Fisher, I. (1927), A statistical method for measuring marginal utility and testing the justice of a progressive income tax. In: J.H. Hollander, ed.: *Economic Essays , Contributed in Honor of John Bates Clark*, New York: Macmillan, 157-193.

Frisch R. (1938), Statistical versus theoretical relations in economic macrodynamics, Mimeo. [Reprinted in Bjerkholt (1995, Vol. 1, 272-287)].

Frisch, R. (1926a), Sur un problème d'économie pure, *Norsk Matematisk Forenings Skrifter*, Series 1, No. 16, 1-40.

Frisch, R. (1926b), Kvantitativ formulering av den teoretiske økonomikks lover, *Statsøkonomisk Tidsskrift*, 40: 299-334.

Frisch, R. (1926c), *Sur les semi-invariants et moments employés dans l'étude des distributions statistiques*, Skrifter utgitt av Det Norske Videnskaps-Akademi i Oslo, II Historisk-Filosofisk Klasse, No. 3.

Frisch, R. (1927), *The Analysis of Statistical Time Series*, New York, Mimeographed.

Frisch, R. (1929a), Correlation and scatter in statistical variables, *Nordic Statistical Journal*, **1**, 36-102.

Frisch, R. (1929b), Statikk og dynamikk i den økonomiske teori, *Nationaløkonomisk Tidsskrift* **67**, 321-379. [Sections 1-3 published as: Statics and dynamics in economic theory, *Structural Change and Economic Dynamics*, 3 (1992), 391-401.]

Frisch, R. (1930), *A Dynamic Approach to Economic Theory. Lectures at ale University beginning September, 1930*, mimeographed. [Unpublished].

Frisch, R. (1932), *New methods of measuring marginal utility*, Tübingen: Verlag von J. C. B. Mohr (Paul Siebeck).

Frisch, R. (1933a), *Problèmes et méthodes de l'économétrie*, Eight lectures given at Institut Henri Poincaré, University of Paris, March-April 1933, Frisch Archive, Department of Economics, University of Oslo. [To be published by Routledge, 2008.]
Frisch, R. (1933b), Propagation problems and impulse problems in dynamic economics. In: *Economic Essays in honour of Gustav Cassel*, London: Allen & Unwin: 171-205.
Frisch, R. (1934), *Statistical confluence analysis by means of complete regression systems*, Publikasjon nr 5, Institute of Economics, Oslo.
Frisch, R. (1937), General Choice-Field Theory. In Cowles Commission (1937), 64-69.
Frisch, R. (1938), Statistical versus theoretical relations in economic macrodynamics, Mimeo. [Reprinted in Bjerkholt (1995, Vol. 1, 272-287)].
Frisch, R. (1949), *Axiomatic remarks on some national income concepts*, Memorandum from the Institute of Economics 16 October 1949, University of Oslo.
Frisch, R. (1950), *Aksiomatisk oppbygging av inntektsbegrepet og dets bestanddeler i en ikke-isolert sektor*, Memorandum from the Institute of Economics 6 February 1950, University of Oslo.
Frisch, R. (1959), A complete scheme for computing all direct and cross demand elasticities in a model with many sectors, *Econometrica* **27**, 177-196.
Frisch, R. (1961), Numerical determination of a quadratic preference function for use in macroeconomic programming, *Giornale degli Economisti e Annali di Economia* **20**, 3-43.
Frisch, R. (1970), *Co-operation between politicians and econometricians on the formalization of political preferences*, The Federation of Swedish Industries, Stockholm.
Frisch, R. (1971), On a Problem in Pure Economics. In: Chipman et al. (1971), 386-423. [Originally published as Frisch (1926a)].
Hands, D. Wade (2006), Integrability, rationalizability, and path-dependency in the history of demand theory. In: Mirowski, P and D. Wade Hands: *Agreement in Demand: Consumer Theory in the Twentieth Century*, Durham: Duke University Press, 153-185.
Jevons, S. (1871), *The Theory of Political Economy*, London: Macmillan.
Johansen, N.V. (2005), *Einstein i Norge*, Oslo: Cappelen.
Köbberling, V. (2006), Strength of preference and cardinal utility, *Economic Theory* **27**, 375-391.
Lange, O. (1934a), The determinateness of the utility function, *The Review of Economic Studies* **1**, 218-225.
Lange, O. (1934b), Notes on the determinateness of the utility function, III, *The Review of Economic Studies* **2**, 75-77.
Marschak, J., (1931), *Elastizität des Nachfrage*, Tübingen: Verlag von J. C. B. Mohr (Paul Siebeck).
Mills, F.C., J.H. Hollander, J. Viner, E.B. Wilson, W.C. Mitchell, F.W. Taussig, T.S. Adams, J.D. Black and J.C. Cobb (1928), The present status and future prospects of quantitative economics, *American Economic Review* **18**, 28-45.
Morgan, M. (1990), *The History of Econometric Ideas*. Cambridge, UK: Cambridge University Press.
Pareto, V. (1909), *Manuel d'Economie politique*, Paris, V. Giard & E. Briére.
Phelps Brown, E.H. (1934), Notes on the determinateness of the utility function, I, *The Review of Economic Studies* **2**, 66-69.
Roy René (1942), De l'utilité. Contribution à la théorie des choix, Paris: Hermann & Cie.
Samuelson, P.A. (1937), A note on the measurement of utility, *The Review of Economic Studies* **4**, 155-161.

Slutsky, E.E. (1952[1915]), On the Theory of the Budget of the Consumer. In: Stigler, G.J. and K.E. Boulding, eds.: *Readings in Price Theory*, Homewood, Ill.: Irwin. [Sulla teori del bilancio del consumatore, *Giornal degli Economisti* **51**, 1-26.]

Strøm, S., ed. (1998), *Econometrics and Economic Theory in the 20th Century. The Ragnar Frisch Centennial Symposium*, Cambridge University Press.

Wulwick, N. (1995), A reconstruction of Henry L. Moore's demand studies. In: Rima, I.H., ed.: *Measurement, Quantification, and Economic Analysis: Numeracy in Economics*, New York: Routledge, Chapter 8.