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Resource Depletion and Capital Accumulation under Catastrophic Risk: The Role of Stochastic Thresholds and Stock Pollution

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Resource Depletion and Capital Accumulation under Catastrophic Risk: The role of Stochastic Thresholds and Stock Pollution[1]

By

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September 7, 2012

Abstract:

An intertemporal optimal strategy for accumulation of reversible capital and management of an exhaustible resource is analyzed for a global economy when resource depletion generates discharges that add to a stock pollutant that affects the likelihood for hitting a tipping point or threshold of unknown location, causing a random"disembodied technical regress". We characterize the optimal strategy by imposing the notion "precautionary tax" on current extraction. Such a tax will internalize future expected damages or expected welfare loss should a threshold be hit. With reversible capital the presence of a stochastic threshold should speed up accumulation as long as no threshold is hit so as to build up a buffer or stock for future consumption should a threshold be hit.

Keywords: Catastrophic risk and stochastic thresholds, capital accumulation, precautionary taxation, stock pollution, resource extraction

JEL classification: C61, Q51, Q54

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1. Introduction

Environmental hazards or threats that have not yet materialized should be met by precautionary actions today so as to reduce the likelihood for catastrophic outcomes in the future. Whereas there is a close correspondence in time and space between some types of emissions and environmental damages, there are some emissions that enter into a stock of pollutant that might be harmful in the future. Emissions of GHGs like CO2 or methane into the atmosphere may increase the likelihood of a future environmental disaster. Because emissions of GHGs' do not necessarily add to current damages, and hence no traditional static externality to be accounted for, the long-run environmental hazards caused by, say, current consumption of fossil fuels, should be internalized by current decision makers. Under a system where emissions and damages interact (almost) simultaneously, taxes, quotas or permits, have been shown to improve on the efficiency of the resource allocation. However, when there is a time lag, usually of random length, between current emissions and damages of unknown magnitude, taxes cannot be directly related to current damages or costs, but must be related to expected costs in the future. In order to take account of these hazards, we introduce a term "precautionary taxation", which is nothing more than just a way of defining how to tax a random stock pollutant. (A similar approach has been taken by Tsur and Zemel (2008) who introduces the notion "Pigouvian hazard tax" on such preemptive taxation when future random damages are of main concern.[4])

A large number of environmental problems are caused by stock pollution; e.g., problems related to emissions of GHGs and anthropogenic climate change. There is a vast literature on such issues; see for instance Ayong Le Kama et al. (2010), Barrett (2011), Becker et al. (2010), Brito and Intriligator (1987), Clarke and Reed (1994), Cropper (1976), Gjerde et al. (1999), Hoel and Karp (2001, 2002), Keeler et al. (1971), Nævdal (2003, 2006), Pindyck (2007), Tahvonen and Withagen (1996), Torvanger (1997), Tsur and Zemel (1996, 1998, 2008), and, de Zeeuw and Zemel

[4] The concept "precautionary taxation" has been used in the public finance literature to illustrate tax instruments that improve on government budgets or increase government savings, when times are uncertain or when government debt is too high.

(2012). In addition, the issue of what rate of discount factor should be used for how to value these environmental costs or hazards that might be incurred far into the future, has been a hot issue in the literature; see Arrow (2009), Dasgupta (2008), Gollier (2002), Heal (2005), Nordhaus (2007), Nævdal and Vislie (2010), Stern (2007), Tsur and Zemel (2009) and Weitzman (2007, 2009), just to mention some of the participants in that debate.

Although costs or environmental damages do not appear until accumulated stocks have reached some specific levels or thresholds (tipping points), we are seldom in a position to know exactly the location of these levels or when these critical values are reached; an issue that is discussed thoroughly in some of the cited papers. An analogy is driving a car in darkness knowing that somewhere in front of you there is a hole, of unknown size, and with a location being unknown to the driver, as long as the car is still on the road. Being aware of this possibility should normally affect a rational driver's caution, say, by slowing down the speed. A similar danger or hazard might be in front of us as well. We believe that present economic activity generates emissions that will increase the stock of a number of pollutants. The amount of accumulated stocks may not be harmful today, but we have some qualified opinion or belief that if (and when) some stock reaches a critical level, a natural disaster might be triggered, and hence be harmful to subsequent generations. However, even though we have some opinion about the relationship between the size of a stock and the disaster, we don't know for sure what level of stock will trigger such an event, say through a sufficiently rise in temperature or in the sea-level. The knowledge that the well-being of future generations might be severely affected, or, in a worst case, their mere existence is threatened, should – from an ethical and normative point of view – affect current generation's behavior. The present paper is just about this issue within the context of optimal capital accumulation and resource extraction when future environmental costs are uncertain.

To study this problem we consider an economy at a very high level of aggregation where current consumption of a resource-intensive commodity is the only "input" in the generation of a stock of emissions. Along with a resource-intensive good, there is also a "composite" commodity that is produced by capital alone, which can be used for investment and consumption. Preferences are related only to consumption profiles, consisting of the two consumption goods. Accumulated stock of emissions could enter as an argument in the utility flow, but here we approach the issue of introducing a relationship between a stock pollutant and a catastrophe in a different way: We assume that the likelihood for a disaster (like a sharp rise in temperature, a flood or sea-level rise) is assumed to be a function of the accumulated stock of pollutant, generated from the consumption of the resource-intensive commodity. Hitting a critical value or a threshold, whose location is random, will trigger a disaster in the sense that the economy will be inflicted a cost through a real and persistent productivity shock; cf. Torvanger (op.cit.). This shock can be conceived of as a random technical disembodied regress, which lowers the production capacity in all sectors uniformly. Hence, because the sequence of consumption choices of the resource-intensive commodity will affect the likelihood for a future disaster, an infinitely-lived planner should take this hazard into account when balancing the preferences of current and future generations. In this simplified world, the only way of postponing a likely disaster, is through lowering current consumption of the resource-intensive commodity or deplete the exhaustible resource at a slower rate.[5] (This type of inaction might be seen an application of the precautionary principle, as discussed by Gollier et al. (2000), and by Weisbach (2011).) Because consumption of the resource-intensive good prior to a disaster does not cause any harm, only benefits, but will affect the likelihood of some future adverse event, there is no current or static externality in the traditional sense to take into account, but a stochastic future dynamic externality. Due to the assumption that current consumption of a specific good enhances future risk, current generation's consumption of that good should be taxed according to the expected future cost of a disaster or, in a worst case, a catastrophe. This tax on current consumption of the resource-intensive commodity is

[5] We rule out any option for abatement. Also the stock itself is not subject to any natural decay.

explicitly derived; see also Tsur and Zemel (2008). In addition to curbing current extraction of fossil fuels, so as to reduce the likelihood for a future disaster, the future productivity shock will induce the planner to increase capital accumulation, as compared with no shock, as long as no threshold is hit. The rationale for such excessive capital accumulation is found in the assumption that capital is reversible[6]. When anticipating that a fraction of the production capacity can be lost, the planner will have a strong motive to create a buffer so as to secure future consumption opportunities for the period after a threshold has been hit. (This motive is obviously affected by the magnitude of the expected productivity shock.)

The paper is organized as follows: In section 2 we present the model. The optimal contingency consumption-investment-extraction plan is derived in section 3 and expressed as modified Ramsey and Hotelling-rules. These rules will appear as decision rules for capital accumulation and resource depletion as long as no threshold is hit, and will of course be affected by the expected future shadow values of the capital stock and the remaining resource. In section 4 we derive explicitly the tax as related to the dynamic nature of the shadow price of the stock of pollutant prior to a disaster. Section 5 concludes and points to some extensions.

2. The Model

The global planner's objective is to derive an intertemporal optimum for an economy with two production sectors and one resource-extracting sector. One of the sectors – the one that produces a resource-intensive commodity – uses a flow of a non-reproducible natural resource (oil or fossil fuel), v, to produce a consumption good (as denoted c_2). The other sector – the one we call the capital-intensive sector – uses only capital to produce a composite good, according to a production function, $Af(k)$. This output is allocated to consumption (c_1) and gross investment (J). The resource-intensive sector's output is assumed to be of the linear type, as given by $Ag(v) \equiv Av$,

[6] Capital is reversible in the sense that capital equipment can eventually be transformed into ordinary consumption goods.

with v as the input-flow of a non-reproducible natural resource (equal to the output-flow), whose remaining stock at t is given by $R(t)$. The parameter $A \in [0,1]$ is a random variable and is related to a post-event permanent downgrading of the productivity, common to both sectors. The "kernel" production function f is bounded, twice continuously differentiable, strictly increasing and strictly concave, satisfying the Inada-conditions, with $f(0) = 0$.

At some future point in time the economy might suffer from a natural disaster or a catastrophe that is triggered by hitting a threshold whose location is not perfectly known. The hitting outcome is caused by having reached a critical amount of accumulated waste generated from extracting the resource. Once the disaster occurs the variable A is randomly drawn from a known distribution with a realized value $a \in [0,1]$, while $A \equiv 1$ prior to the shock.

The following relations characterize the economy, where $\dot{k}(t) := \frac{dk(t)}{dt}$ and δ is a fixed depreciation rate per unit capital equipment:

(1) $$Af(k(t)) = c_1(t) + J(t) = c_1(t) + \dot{k}(t) + \delta k(t) \quad \forall t \in [0, \infty) \; and \; k(0) = k_0$$

(2) $$c_2(t) = Av(t) \quad and \quad \dot{R}(t) = -v(t) \; with \; R(t) = R_0 - \int_0^t v(s)ds, \; R_0 \; given$$

The production technology in (1) of the "composite" good shows that it can either be consumed or allocated to gross investment.

The population is for simplicity assumed to be of a fixed size throughout, with stable preferences at any point in time, given by an additive, separable utility function $U(c_1(t), c_2(t)) = u(c_1(t)) + w(c_2(t))$. Both functions, (u, w), are twice continuously differentiable, strictly increasing and strictly concave, with $\lim_{c_1 \to 0} u'(c_1) = \infty = \lim_{c_2 \to 0} w'(c_2)$.

Production of the resource-intensive good generates waste or a flow of emissions, say of CO_2 per unit of time that accumulates into a stock of pollutant that is the main cause of a future environmental hazard. If the stock of waste at t is given by $z(t)$, we have the growth function,

(3) $\dot{z}(t) := \frac{dz(t)}{dt} = D(v(t)),\ with\ z(0) = 0$

saying that the growth per unit of time at t in the stock of pollutant is related to the use of the non-renewable resource in producing the resource-intensive commodity at t. We assume that D is twice continuously differentiable, strictly increasing and convex, with $D(0) = 0$. Because the stock is subject neither to any natural decay nor any abatement, the stock is non-degradable, according to Dasgupta (1982a). (Again this might be regarded as a too restrictive assumption, but "no natural decay" might be justified by assuming the average lifetime of the pollutant exceeds the relevant time scale.)

In this stylized economy the global planner has an objective to maximize expected present discounted utility; $E\left[\int_0^\infty e^{-rt}\left[u(c_1(t)) + w(c_2(t))\right]dt\right]$, subject to the relevant constraints, where r is a non-negative pure rate of time preference.

Define a probability function $F(z(t)) := \Pr(Y \leq z(t))$ for the location of the stochastic threshold, given by the random variable Y, with F assumed to be invariant to calendar time, increasing and twice continuously differentiable. We have $F(0) = 0$ and also, as a consequence of the precautionary principle or pessimistic beliefs, that a threshold exists; hence we assume that $\lim_{z\to\infty} F(z) = 1$.

Because consumption of the resource-intensive good is positive at any point in time as long as capital equipment is intact, z will be strictly increasing over time, due to non-degradibility. Hence we have

(3) $\Pr(Y \leq z(t)) = \Pr(z^{-1}(Y) \leq t) := \Pr(T \leq t) = \Omega(t)$

where T is the random point in time of hitting the threshold. The probability density for the event T ("the point in time of hitting the threshold") to occur in a small interval of length dt, is then $F'(z(t))\cdot \dot{z}(t)dt = F'(z(t))\cdot D(v(t))dt := \Omega'(t)dt$. If the threshold is hit, a new (permanent) regime will commence with a realized value of the productivity parameter, characterized by the distribution function $G(a;z(t)) = \Pr(A \le a \big| Y \le z(t))$, and a corresponding positive density $g(a;z(t)) := \dfrac{\partial G(a;z(t))}{\partial a}$ for all $a \in \left[0,1\right]$. To simplify even further we assume, perhaps a bit unrealistic, that G and g will be independent of accumulated waste; hence in the subsequent discussion we ignore the argument z in these functions. In that case only the expected value of A, conditional on hitting the threshold, will matter.

3. The contingency plan

We look at the problem from the perspective of a global planner with the objective to maximize expected welfare, when taking account of all relevant constraints. The real big issue is what kind of precautionary actions can or should be taken so as to prevent the occurrence or postpone the catastrophe as far out into the future as possible. (It seems hard to justify the opposite action.) With only one possible catastrophe the time line has two disjoint intervals; the last one, extending over $\left[\tau,\infty\right)$, is characterized by having all uncertainty resolved when the threshold should be hit at some arbitrary point in time, τ, and with a new and permanent value, a, of the stochastic shift parameter A being realized. Once the threshold is hit, the stock of waste has no longer any economic cost – it does no longer play any role in the cost-benefit calculations because no more damage can be made and all risk will from now on be eliminated. (This is a very special assumption, but could be justified by having more thresholds bunched closely together on the time line; cf. Barrett (op.cit.).) The first issue is therefore to characterize optimality in this continuation period, called the continuation regime. Thereafter we derive the full strategy.

3-i The continuation regime

The value function for the continuation regime, called the "continuation payoff", with an initial state as given by $\left\{k_\tau, R_\tau; a, \tau\right\}$, starting at τ with a realized productivity parameter as given by a, with capital stock and remaining resource stock as given by $k_\tau = k(\tau)$ and $R_\tau = R(\tau)$, respectively, is found as the solution to the following standard dynamic optimization problem: For any pair $\{\tau, a\}$ being realized, with a corresponding pair of state variables $\{k, R\}$, the solution will prescribe what to do in an optimal manner from τ and onwards, as summarized by the value function:

$$V(\tau, a, k_\tau, R_\tau) = Max_{(c_1, v)} \int_\tau^\infty e^{-r(t-\tau)} \left[u(c_1(t)) + w(av(t)) \right] dt$$

$$s.t. \begin{cases} k(\tau) = k_\tau, R(\tau) = R_\tau \\ \dot{k}(t) = af(k(t)) - c_1(t) - \delta k(t), \lim_{t\to\infty} k(t) \geq 0 \\ \dot{R}(t) = -v(t), \lim_{t\to\infty} R(t) \geq 0 \end{cases}$$

The planner has consumption of the capital-intensive good, c_1, and extraction rate, v, as control variables, with $\{k, R\}$ as state variables. Because the problem is a standard control problem it is not necessary to go into details. With p as the current shadow value of capital and q the current shadow value of the resource stock, we have that the continuation regime – represented by a sample path marked by a hat above the variable, is characterized by a Ramsey-rule and a Hotelling-rule, as given by:

$$(4-i) \quad af'(\hat{k}(t)) - \delta = r + \hat{\omega}(\hat{c}_1(t)) \cdot \frac{\dot{\hat{c}}_1(t)}{\hat{c}_1(t)}$$

$$(4-ii) \quad aw'(\hat{c}_2(t)) = q(\tau; a) e^{r(t-\tau)}$$

(Here $\hat{\omega}(\hat{c}_1)$ is the absolute value of the elasticity of $u'(\hat{c}_1)$. We have used that optimality requires $u'(\hat{c}_1(t)) = p(t; a)$ and that the shadow price has to obey: $r - \frac{\dot{p}(t;a)}{p(t)} = af'(\hat{k}(t)) - \delta$, with $\dot{p}(t;a) = \frac{\partial p(t;a)}{\partial t}$.) We recognize the first condition in *(4-i)* as the standard Ramsey-rule for determining the social rate of discount, when a proper choice of numéraire has been made, whereas the second one is the Hotelling-rule.

For any realized pair (a, τ), the shadow prices have the following well-known interpretations:

$$(5) \qquad \frac{\partial V(\tau, a, k_\tau, R_\tau)}{\partial k_\tau} := p(\tau; a), \qquad \frac{\partial V(\tau, a, k_\tau, R_\tau)}{\partial R_\tau} := q(\tau; a)$$

For any given initial state (k_τ, R_τ) at the date when a threshold is hit, we claim that $p(\tau; a)$ will be higher and $q(\tau; a)$ smaller, the smaller is the realized value of a. In the extreme "doomsday"-scenario, with $a = 0$, the capital stock, which from now on never is supplemented, will be turned into a non-renewable resource, used, net of depreciation, as input in the production the capital-intensive commodity. Also, in this extreme state the existing stock of the resource becomes worthless, with $q(\tau; 0) = 0$, because the production opportunity set of the resource-intensive sector is now entirely destroyed, as the marginal productivity of the resource used as input in the production of the resource-intensive commodity becomes zero.

At the ex ante-stage, A is stochastic. Hence, before we enter the continuation regime when the true value of A is realized, only the expected future benefit will be relevant for evaluating what to do as long as no disaster has yet occurred. Therefore we have to define the expected maximal future benefit or expected value function, as seen from some point in time τ. Define then:

$$(6) \qquad W(\tau, k_\tau, R_\tau) = \int_0^1 V(\tau, a, k_\tau, R_\tau) g(a) da$$

and let $\frac{\partial W(\tau, k_\tau, R_\tau)}{\partial k_\tau} = \int_0^1 \frac{\partial V(\tau, a, k_\tau, R_\tau)}{\partial k_\tau} g(a) da := \overline{p}(\tau, k_\tau, R_\tau)$, for short written as $\overline{p}(\tau)$ or $\overline{p}$, be the expected marginal shadow value of capital at the beginning of the continuation period, with the shadow value being declining in the capital stock available at that date. Also, $\frac{\partial W(\tau, k_\tau, R_\tau)}{\partial R_\tau} = \overline{q}(\tau, k_\tau, R_\tau)$, written for short as $\overline{q}(\tau)$ or $\overline{q}$, as the expected marginal shadow value of the resource at the beginning of the continuation period. These shadow prices, which are time- and state-dependent, will play an important role for the determination of the full strategy to which we now turn.

3-ii The Full Strategy

As seen from the beginning of the planning period, an optimal strategy is found as the solution to the following constrained optimization problem:

$$Max_{(c_1,v)} \int_0^\infty \Omega'(\tau)\left\{\int_0^\tau e^{-rt}[u(c_1(t))+w(v(t))]dt + e^{-r\tau}W(\tau,k_\tau,R_\tau)\right\}d\tau$$

s.t.

$\dot{k}(t) = f(k(t)) - c_1(t) - \delta k(t) \quad \forall t \in [0,\tau],\ k(0) = k_0,\ f'(k_0) > r+\delta$

$\dot{R}(t) = -v(t);\ \forall t \in [0,\tau],\ R(0) = R_0$

$\dot{z}(t) = D(v(t)),\ \forall t \in [0,\tau],\ z(0) = 0$

$\Omega'(\tau)d\tau = F'(z(\tau)) \cdot D(v(\tau))d\tau$, where $z(s) = \int_0^s D(v(t))dt$,

No conditions on $z(\infty)$; with k_τ, R_τ both given, while $k(\infty) \geq 0, R(\infty) \geq 0$. We also define the hazard rate (or the conditional density for a disaster to occur in a short interval $[t, t+dt]$, conditional on the non-occurrence prior to t) as $\Lambda(t)dt := \frac{\Omega'(t)dt}{1-\Omega(t)}$, as well as the hazard rate in the domain of z, as given by $\lambda(z)dz := \frac{F'(z)dz}{1-F(z)}$.

Before solving this problem, it might be useful to consider what kind of trade-offs the planner is facing. Given the probability beliefs about when a threshold is hit – related to accumulated emissions – and the anticipated belief about the productivity shock, the planner should recognize the following: If the beliefs are very pessimistic, in the sense that the expected value of A is small – say close to zero – the planner should first of all induce a high capital accumulation so as to create a buffer for future consumption opportunities in the continuation regime. Also, resource depletion should be slowed down so as to postpone hitting the threshold. However, there might be a counteracting force, leading to higher extraction, caused by the low anticipated value of the remaining resource stock, should the threshold be hit. We show that the "fear" of hitting the threshold is so strong that this latter force on resource depletion is dominated.

Let us consider the optimal strategy in some more detail: Integrating the objective function by parts, while using $\Omega(t) = F(z(t))$, we can write our objective function as:

$$\int_0^\infty \Omega'(\tau)\left\{\int_0^\tau e^{-rt}[u(c_1(t)) + w(v(t))]dt + e^{-r\tau}W(\tau, k_\tau, R_\tau)\right\}d\tau$$
$$= \int_0^\infty e^{-rt}\left[(1 - F(z(t)))\cdot[u(c_1(t)) + w(v(t))] + F'(z(t))\cdot D(v(t))\cdot W(t, k_t, R_t)\right]dt$$

This transformed objective function is to be maximized subject to the constraints above, with (c_1, v) as non-negative control variables, and (k, R, z) as state variables in the pre-disaster regime, with z is a public bad that affects the likelihood for the occurrence and the associated magnitude of the productivity shock. The integrand can be regarded as the planner's expected utility during a small interval $\left[t, t + dt\right]$, where the first part is the expected utility of consumption at t, as long as no disaster has occurred, whereas the second term is the expected utility if the threshold should be crossed during $\left[t, t + dt\right]$, with a payoff given by the expected continuation payoff. (We have marginal damage being equal to zero both before and after the adverse event, with the constant post-event damage being included in the $W-$function.)

The consumption flows are the immediate contribution to current welfare. However, the higher are current consumption flows, the smaller are the future stocks of both types of capital with, as well, accumulated stock of waste being higher. For any given future value of the parameter a, the more constrained will future consumption opportunities be as more is being consumed early. Also, because current waste accumulation will increase, the likely occurrence of a disaster is accelerated. However there is a way out, depending on what expectations the planner should have about the seriousness of hitting a threshold: postpone or delay consumption of the resource-intensive commodity. In that case a likely disaster is pushed farther into the future.

To accomplish such a goal the global planner might impose what we have coined a *precautionary consumption tax* imposed on the resource-intensive commodity or a tax on future environmental threats or hazards, rather than *actual* damages; cf. Tsur and Zemel (2008). The environmental threat or hazard can be associated with a random dynamic externality as current consumption of the resource-intensive commodity affects future welfare, not current welfare or damage, through the likelihood of future bad or adverse states of the world to happen. We then have an ordinary intertemporal trade-off between present and future consumption, but this trade-off is supplemented (and complicated) by having to take into account how current consumption of one commodity is affecting the likelihood for an adverse regime-switching in the future, and also how to create a buffer against future loss in production capacity through capital accumulation.

Rather than committing to one consumption path, as would be optimal with no intertemporal risk, it is well known that the optimal solution in the present context is given by a strategy that balances the current marginal benefit at any instant of time, conditional on not yet having entered the continuation phase, while taking into account new information about an adverse event that might occur in the very near future. This adverse state of the world can be avoided by consuming less or slow down the extraction of the exhaustible resource. When and if the adverse event will occur, the results from the preceding paragraph prescribes in detail what to do from the start of the continuation regime; i.e., once a value of τ and a have been realized.[7]

To derive an optimal strategy we introduce (P,Q,m) as a triple of current value adjoint variables associated with the dynamics of the state variables (k,R,z), with an associated current value Hamiltonian as given by:

[7] Similar problems have been analyzed in the R&D-literature – see e.g. Kamien & Schwartz (1977, 1978), Dasgupta (1982b), and Dasgupta & Stiglitz (1981).

$$H = (1 - F(z)) \cdot [u(c_1) + w(v)] + F'(z) \cdot D(v) \cdot W(t, k_t, R_t) + P \cdot (f(k) - c_1 - \delta k)$$
$$-Q \cdot v - m \cdot D(v)$$

Here P is a "current shadow value" of capital, Q is the current shadow value of the remaining resource, whereas m is the shadow *cost* associated with accumulated waste or the cost of the public bad. An optimal solution, marked by a star, given that no disaster has yet occurred, with strictly positive consumption flows at any point in time, has to obey the following conditions that follow directly from Pontryagin's maximum principle:

$$(7-i) \qquad (1 - F(z(t)) \cdot u'(c_1^*(t)) - P(t) = 0$$

$$(7-ii) \qquad r - \frac{\dot{P}(t)}{P(t)} = f'(k^*(t)) - \delta + F'(z^*(t)) \cdot D(v^*(t)) \cdot \frac{\overline{p}(t, k^*(t), R^*(t))}{P(t)}$$

$$(7-iii) \qquad (1 - F(z^*(t)) \cdot w'(v^*(t)) + F'(z^*(t)) \cdot D'(v^*(t)) \cdot W - Q(t) - m(t) \cdot D'(v^*(t)) = 0$$

$$(7-iv) \qquad r - \frac{\dot{Q}(t)}{Q(t)} = F'(z^*(t)) \cdot D(v^*(t)) \cdot \frac{\overline{q}(t, k^*(t), R^*(t))}{Q(t)}$$

$$(7-v) \qquad r - \frac{\dot{m}(t)}{m(t)} = -\frac{F'(z^*(t))}{m(t)} \cdot \left[u(c_1^*(t)) + w(c_2^*(t))\right] + \frac{F''(z^*(t)) \cdot D(v^*(t)) \cdot W}{m(t)}$$

(Here we have used the previously defined shadow values of real capital and resource stock, respectively, at the beginning of the continuation regime.) We now will use these conditions to make more precise the true nature of an optimal strategy.

3-iii Discussion and Results

The conditions above can be interpreted in different ways: One way is to look at, say, $e^{-rt}P(t)$ in (7-i) as the contingent price paid at $t = 0$ for delivery of a unit of c_1 at t, contingent on no threshold being crossed by t. An alternative interpretation is to consider what to pay for each commodity at some point in time t as long as no threshold has been hit or crossed prior to that date. Define therefore

$\pi(t) := \dfrac{P(t)}{1 - F(z(t))}$ as the "spot" price per unit of c_1 at t, conditional on no threshold

being crossed prior to t, balancing the immediate utility gain from a unit increase in consumption of the capital-intensive commodity and the expected cost due to lower capital accumulation at t, conditional on not having entered the continuation regime by t, and $\mu(t) := \frac{Q(t)}{1-F(z(t))}$ as the conditional shadow value of the remaining resource at t, given that no threshold has been hit prior to that point in time. Making use of these spot prices, an immediate consequence of introducing a randomly located threshold into the capital-accumulation-resource-extraction story is: On differentiating (*7-i*) with respect to time and combining this with (*7-ii*), we first get a modified Ramsey-rule:

$$(8) \quad f'(k^*(t)) - \delta + \lambda(z^*(t)) \cdot D(v^*(t)) \cdot \frac{\overline{p}(t)}{\pi(t)} = r + \hat{\omega}(c_1^*(t)) \frac{\dot{c}_1^*(t)}{c_1^*(t)} + \lambda(z^*(t)) \cdot D(v^*(t))$$

The left hand side of (8) shows the expected rate of return from a marginal investment made at t as long as no threshold has been hit so far, whereas the right hand side shows the required rate of return from a marginal increase in saving at t, as long as the economy has not yet entered the continuation regime. Introducing a randomly located threshold will modify the rate of return from investment with an additional term capturing the expected rate of return from higher real investment at t should the economy enter the continuation regime in $\left[t, t+dt\right]$. The conditional density of hitting the threshold in this interval, is $\lambda(z)\dot{z}(t)dt$, with an expected rate of return as given by $\frac{\overline{p}(t)}{\pi(t)}$. Also the required rate of return from saving is risk-adjusted, as given by the last term on the RHS of (8). The latter upwards adjustment can be seen as if we impose a higher rate of time-preference, which, cet.par., will lead to less saving because tomorrow it might be too late to reap the return from current saving. (This latter type of modifying the decision rule for capital accumulation is the one found in the literature on saving under uncertain horizon, with an expected value function, like W, being equal to zero; as discussed among others, by Levhari and Mirman (1977).)

Within the present context the expected marginal value of capital from the beginning of the continuation regime will not fall below the one prior to that regime; i.e., we have $\overline{p}(\tau) \geq \pi(\tau^-)$, if the threshold should be hit or crossed at τ. Only if the planner holds very optimistic beliefs about the shock, saying that A stays equal to one (so that crossing the threshold is not even identified and expected not to cause any harm), then we have $\overline{p} = \pi$. In that case the consumption path prior to "a regime shift" is exactly the one that would have been derived for standard optimal growth models. One line of reasoning to rationalize that the expected shadow price will in general shift upwards in the continuation regime is to consider the case with A being expected to be $1-\varepsilon$, with ε "small". In that case $\overline{p}(\tau)$ will increase, as compared to the case with $A = 1$, due to concavity of the value function, whereas the opportunity cost of consumption "prior to τ" will be adjusted downwards. Hence we have:

Proposition 1: If the planner holds very optimistic beliefs about the productivity shock for the continuation regime, with $EA \equiv 1$, then $\overline{p} = \pi$, and saving and consumption of the capital-intensive commodity will satisfy the standard Ramsey-rule, as given by; $f'(k(t)) - \delta = r + \hat{\omega}(c_1(t))\frac{\dot{c}_1(t)}{c_1(t)}$. On the other hand, if a shock is anticipated, with $EA < 1$, then $\overline{p} > \pi$. In that case initial consumption is lower than under optimistic beliefs, with a higher growth rate in c_1. The greater is the anticipated shock, the more is initial consumption reduced. Higher initial capital accumulation will therefore serve as an insurance or buffer against a future random shock or income loss. The more pessimistic beliefs held by the planner, the higher is capital accumulation in the initial phase prior to a disaster, leaving a larger stock of capital to serve as a reserve for future consumption once the threshold is hit.

What will happen should a threshold be crossed? Suppose the economy hits the threshold at τ. Conditional on what has been expected, the consumption at τ will jump downwards or upwards, depending on whether the realized value of a is above or below what was expected. Suppose the realized value of a is below EA. In that case, the shadow price will jump upwards, with $p(\tau;a) > \overline{p} \geq \pi(\tau^-)$, to meet the non-negativity constraint on the capital stock. (This upwards jump in the shadow price signals that, ex post, we have saved too little.) Given the realized value of a, we then have a downwards jump in consumption, $\hat{c}(\tau) < c^*(\tau^-)$, with the new consumption path increasing at a smaller rate. (In fact this will be true only if $\hat{\omega}(c_1)$ is constant.) From then on the shadow price of capital will behave according to $p(t;a) = p(\tau;a)e^{-\int_\tau^t [af'(\hat{k}(s))-r-\delta]ds}$, which is increasing over time if the capital stock, along with the value of a, is such that $af'(k^*(\tau^-)) < r + \delta$. Under these circumstances the consumption path is declining. On the other hand, if $af'(k^*(\tau^-)) > r + \delta$, the consumption path is increasing.

The excessive capital accumulation is one precautionary action taken by the planner in the face of a disaster caused by a random regime switch. The more pessimistic beliefs the planner holds as to the expected productivity shock, the more capital will be accumulated and less will be consumed before the threshold is hit. As mentioned above this way of building up a buffer for future consumption is one way current generations can accommodate to a future disaster, leaving the generations of the continuation regime with a higher stock of capital that (in the worst possible case) can be turned into consumption c_1.

Next let us turn to "the beast of our story"; the rate of extraction of the resource used as input in producing the resource-intensive commodity. Current extraction generates emissions or discharges that add to a stock of pollution which affects the likelihood for crossing a threshold. When taking account of the growth in accumulated emissions and the associated increase in the likelihood for a disaster, optimal extraction during a short interval $\left[t, t+dt\right]$, conditional on no threshold being hit until t, follows directly from (7-iii):

$$(9) \quad w'(v^*(t)) = \mu(t) + \frac{D'(v^*(t))[m(t) - F'(z^*(t))W(t,k^*(t),R^*(t))]}{1 - F(z^*(t))}$$

where, according to (7-iv) and the definition of "the conditional spot price of the resource", $\mu(t)$, we have:

(10) $$\frac{\dot{\mu}(t)}{\mu(t)} = r + \lambda(z^*(t))D(v^*(t))\left[\frac{\mu(t) - \overline{q}(t)}{\mu(t)}\right]$$

Consider (10) first. Here will $\overline{q}(t)$ indicate the expected shadow value of the remaining resource at the start of the continuation regime, as long as no threshod has been crossed prior to t. As long as no threshold is crossed, we conjecture that $\mu \geq \overline{q}$, with strict inequality if a productivity shock is anticipated. The reason is that when the threshold is crossed, the true marginal productivity of the resource as input in the resource-intensive sector will normally fall below the current productivity; hence $\overline{q} < \mu$. In that case the conditional expected shadow value of the resource will increase at a rate above the utility discount rate as compared to what will be the case in the continuation regime, and with a corresponding "low" initial value $\mu(0)$. We therefore have:

Proposition 2: On interpreting the shadow price $\mu(t)$ as the "producer's spot price" of the resource at t, conditional on not having entered the continuation regime by t, we have: If the planner holds very optimistic beliefs, anticipating an insignificant change in the productivity parameter, then $\mu = \overline{q}$, and the depletion profile obeys the standard Hotelling-rule. On the other hand, if the beliefs are pessimistic, anticipating a large or severe shock, then $\overline{q} < \mu$, and the spot price will increase at a rate above the utility discount rate as long as $\lambda(z)D(v)$ is positive, so as to delay extraction. The initial spot price $\mu(0)$ is lowered as compared to the scenario with optimistic beliefs so as to avoid the possibility of depletion too early.

In (9) we have a decision rule for the consumption of the resource-intensive commodity. Current marginal utility from consuming and hence extracting an amount vdt during $\left[t, t+dt\right]$, interpreted as the spot price for the resource at t, conditional on no threshold yet being hit (the LHS of (9)), should be equal to the expected social utility cost of extracting a marginal unit, conditional on not yet having crossed the threshold that is triggering a productivity shock; as given by the RHS of (9). This social marginal utility cost consists of two terms; the first one, $\mu(t)$, shows the social marginal resource cost ("producer's spot price") in utility terms, given that no threshold so far has been crossed. The second term is the net marginal environmental utility cost due to higher extraction, conditional on no threshold yet being hit, and given by $\frac{D'(v^*)[m - F'(z^*)W(t,k^*,R^*)]}{1-F(z^*)}$. The first part, $\frac{mD'(v^*)dt}{1-F(z^*)}$, shows the gross marginal environmental utility cost caused by a higher stock of pollution due to a unit increase in resource extraction during $\left[t, t+dt\right]$, as long as we have not entered the continuation regime by t. A higher rate of extraction during this interval will increase the rate of growth of accumulated emissions, and thereby increase or shift upwards the trajectory of accumulated emissions, and hence increase the likelihood for entering the continuation regime. If a regime-switching should occur in this interval, the outcome is the expected future welfare, given by the change in expected continuation payoff, as given by $\lambda(z^*(t))D'(v^*(t))W(t,k^*(t),R^*(t))dt$. Once the economy moves into the continuation regime, the continuation payoff is reaped and the shadow cost of accumulated emissions (m) drops (permanently) to zero. The expected marginal benefit that is reaped when entering the continuation regime will therefore modify the overall marginal utility cost of higher emissions. In a regulated market regime, this net marginal environmental cost should be reflected in a precautionary tax, which ideally should capture the true welfare cost from a higher stock of accumulated waste caused by a marginal increase in resource extraction at some point in time, as long as no disaster has yet occurred. A tax on current emissions should therefore reflect future expected net damage or utility loss caused by hitting the randomly located threshold. Lowering resource extraction in the pre-disaster regime is therefore a second kind of precautionary action taken by the planner in this model.

An optimal strategy, i.e., as long as the economy has not entered the continuation regime, will exhibit features that are related to the dynamics of the shadow prices.

First let us consider the shadow price of accumulated emissions as long as no threshold is hit.

From (7-v), with $\Delta(z) := (1 - F(z)) \cdot \left[u(c_1) + w(v)\right] + F'(z) \cdot D(v) \cdot W(t,k,R)$, we have $\dot{m}(t) - rm(t) = \Delta'(z^*(t))$, with $\Delta(z)$ as a measure of expected current welfare flow when the stock of pollutant has reached the level z. We assume that this measure is adversely affected by the stock of pollutant; i.e. $\Delta'(z) < 0$. Then we observe that the current (unconditional) shadow cost of accumulated stock, m, should increase at a rate below the utility discount rate, with a high initial value $m(0)$, which should indicate a high future expected cost of early waste accumulation.

Define the current utility intensity or welfare flow at t, as $\phi(t) := u(c_1^*(t)) + w(v^*(t))$, as long as no threshold has been hit. Then solving for the shadow cost, $m(t)$, while using a transversality condition, $\lim_{t\to\infty} e^{-rt} m(t) = 0$, and that $F'(\infty) = 0$, in (*7-v*) we get:

$$
\begin{aligned}
m(t) &= \int_t^{\infty} e^{-r(s-t)} \left[F'(z(s))\phi(s) - F''(z(s))D(v(s))W(s,k_s,R_s)\right] ds \\
&= F'(z(t)) \cdot W(t,k_t,R_t) + \int_t^{\infty} e^{-r(s-t)} F'(z(s)) \left[\phi(s) + \frac{\partial W}{\partial s} - rW(s,k_s,R_s)\right] ds
\end{aligned}
\tag{11}
$$

As seen from point in time t, outside the continuation regime, we can define the expected continuation flow of welfare at some future point in time s as $\gamma(s) := \int_0^1 [u(\hat{c}_1(s)) + w(a\hat{v}(s))] g(a) da$, cf. section 3-i.Then, by definition, we have $W(\tau,k_\tau,R_\tau) = \int_\tau^{\infty} e^{-r(t-\tau)} \gamma(t) dt$, and also $\dfrac{\partial W(\tau,k_\tau,R_\tau)}{\partial \tau} + rW(\tau,k_\tau,R_\tau) = -\gamma(\tau)$. Making use of this relationship along with (11) in (9), we can rewrite the central optimality condition as a decision rule for resource extraction at some point in time t as long as no threshold has been crossed by then, as:

$$
w'(v^*(t)) = \mu(t) + \frac{D'(v^*(t)) \cdot \int_t^{\infty} e^{-r(s-t)} F'(z^*(s)) \left[\phi(s) - \gamma(s)\right] ds}{1 - F(z^*(t))}
\tag{12}
$$

This condition offers a nice representation of the true marginal utility cost of resource extraction at t, conditional on no threshold being hit by that point in time, as given by the RHS. In addition to the expected shadow value of the remaining resource or the direct utility cost of resource extraction at t, $\mu(t)$, cf. (10), there is a term showing the change in present discounted value of expected welfare loss from a future regime-switching due to a marginal increase in extraction at t, given that no switch has occurred by t. A higher rate of extraction during a small period of length dt, given by vdt, will increase the stock of pollutant for the entire future, as the $z-$trajectory is shifted upwards according to $D'(v)dt$. The future expected cost of a higher stock of pollutant is captured by the present discounted value of expected welfare loss from crossing the threshold. The loss in welfare at some future point in time s is given by the difference in utility flows, $\phi(s)-\gamma(s)$. These future outcomes at t are weighted by the truncated or conditional probability density function, $\frac{F'(z(s))ds}{1-F(z(t))}$, conditional on not having crossed the threshold by t. (See also Reed and Heras (1992).)

In order to implement the optimal solution, the planner can impose a time-dependent precautionary tax on resource extraction. This tax will slow down the pace of extraction in the early phase of the planning period, prior to a disaster, and hence encourage saving. The stock of accumulated waste will then not increase too fast, and a likely catastrophe is postponed. A strategy for delaying the catastrophe is accomplished by changing the price structure through taxing the consumption of the resource-intensive commodity, which will lower the producers' price and increase the consumers' price, relative to an unregulated situation.

4. The Precautionary Tax

Let us take a brief look at the precautionary tax which in the present context will appear as a tax on resource extraction. Our optimal strategy can be supported by a contingent price of the capital-intensive commodity $\pi(t)$, a contingent producer price of the resource $\mu(t)$ as well as a time-dependent precautionary tax per unit of the resource $\sigma(t)$, imposed at some point in time t as long as no regime-switching has yet occurred, so as to confront buyers of the resource flow with a unit price at t, equal to $\mu(t)+\sigma(t)$, so as to get them to internalize the present discounted value of future expected welfare loss from a marginally higher consumption or extraction rate at t, as long as no threshold is crossed. The precautionary tax rate is then:

$$\sigma(t) := \frac{D'(v^*(t))\int_t^{\infty} e^{-r(s-t)}F'(z^*(s))\left[\phi(s)-\gamma(s)\right]ds}{1-F(z^*(t))} \tag{13}$$

Without such a tax, the non-renewable (fossil) resource will be extracted too fast, generating a too fast increase in accumulated stock of emissions and hence, accelerating the occurrence of a disaster through an upwards shift in the likelihood of hitting the threshold. The purpose of the tax is therefore to get agents to internalize the future expected welfare loss from a more intensive resource extraction at t, while also induce the non-renewable resource to be managed optimally over time.

The marginal tax rate at t, $\sigma(t)$, captures the welfare-relevant expected cost from a marginal increase in extraction at t. We observe that the tax rate will reflect the impact of a marginal increase in extraction at t on the truncated probability distribution for a disaster to take place at some future point in time, as well as the expected loss in welfare from entering a new regime. We therefore have the rather obvious result:

Proposition 3: The precautionary tax rate, which should be time-dependent and being imposed at any point in time t as long as no threshold has been crossed, will be higher the more likely is a disaster to happen during $\left[t, t+dt\right]$ due to an increase in the stock pollutant at t. The tax rate will also be higher the greater is the expected future loss from crossing a threshold.

We can perhaps be more specific as to how the precautionary tax is in fact related to the planner's anticipations? If the planner holds very optimistic beliefs, then the expected welfare loss $\phi-\gamma$ is small and the tax rate should be low, as noted above. Resource extraction will then obey the standard Hotelling-rule leading to a fast depletion, a fast waste accumulation and if passing the threshold, which now becomes more likely, leads to a more severe shock than anticipated, capital accumulation has, from an ex post point of view, been too small. On the other hand, with very pessimistic beliefs, then the tax rate is high, with low resource extraction in the initial phase, low waste accumulation and high capital accumulation. If it turns out that the shock taking place at τ, which is now a less likely outcome, should happen with a more favorable shock than anticipated, then $q(\tau;a) < \mu(\tau^-)$, which induces an upwards jump in resource extraction, with $\hat{v}(\tau) > v^*(\tau^-)$. Also, from an ex post point of view, there has been excessive capital accumulation, and when the favorable shock (relative to what was anticipated) occurs, the shadow cost of consumption in the continuation regime jumps downwards, inducing a jump upwards in consumption of the capital-intensive commodity and with less saving in the continuation regime. One might therefore conclude that as a precautionary principle it seems favorable to adapt a pessimistic perspective.

We should perhaps be interested in the dynamic features of such a tax rate. But it is not easy to make any strong inferences as to how the tax rate should behave over time as long as no threshold has been crossed. (Once a threshold is crossed, the tax will drop to zero.) Should it be growing or declining over time? For instance, we might want to compare our precautionary tax rate with emission taxes derived from models where the stock of accumulated emissions itself enters current welfare as argument with a negative impact. In such a context a tax on current emissions is imposed to curtail future costs or damage from a too high stock of pollutant. With a finite horizon and no risk (and also under some other plausible assumptions) such a tax should be declining over time, with a high initial level so as to get low emissions and a correspondingly low stock of pollutants, in an early phase of the planning period. On the other hand, with infinite horizon, the emission tax should be constant. In the present model, with a randomly located threshold whose probability distribution is affected by the stock of pollutant with no direct impact on current welfare, the precautionary tax rate has to evolve over time, in a less transparent way. If we consider the price, properly measured, paid by the final users of the resource, we have noted that this price is made up of the sum of $\mu(t)$ and $\sigma(t)$. We have also seen that if a disaster is expected to be dramatic or expected to occur with a high hazard rate, then the shadow price $\mu(t)$ is adjusted downwards with a steeper slope; cf. (10), while the tax rate is adjusted upwards, with, perhaps, a high initial level, so as to punish early contributions to the stock of pollutant. The strength of these early punishments will depend on the magnitude of the expected welfare loss should a disaster occur.

However, even though we have identified the welfare-relevant informational requirements for designing such an optimal precautionary tax rate, it might be hard to get this kind of information.

5. Some conclusions and remarks about the model

The main contribution of the present paper has been the derivation of an optimal strategy for capital accumulation and resource extraction for a global economy that is facing a future random disaster – like a persistent productivity shock – caused by a stock pollutant which affects the future probability distribution for hitting a threshold. The problem is put into the context of optimal saving, with a standard Ramsey-model supplemented by resource extraction and the accumulation of emissions affecting the likelihood of switching regime. The various aspects interact in a complex and not very transparent way, but we have been able to identify what we have coined a "precautionary tax on current resource extraction", not to correct for a current externality, but to capture future expected costs of a stochastic dynamic externality. This tax rate is higher the stronger the truncated probability distribution for a disaster is affected by a marginal increase in the rate of extraction and the higher is the welfare loss should the economy move into the continuation regime. The optimal strategy prior to the continuation regime will show higher capital accumulation, so as to build up a buffer against future income loss in the continuation regime, and less resource extraction so as to reduce the contribution to the stock of pollutant. A disaster is then, hopefully, postponed. Due to the random dynamic externality caused by current consumption of a resource-intensive commodity, current generations have to pay the expected costs that are inflicted on future generations, as measured by the precautionary tax on resource extraction.

One important aspect of the preceding discussion has been the assumption that capital was reversible. What would be the outcome if capital is irreversible in the sense that gross investments cannot be negative, so that capital cannot be turned into consumption goods?[8] First, in the irreversibility case we expect there is no incentive for building a stock of capital that can be transformed to consumption in the continuation regime, should the shock be severe. Then we conjecture that capital accumulation will be slower in a phase prior to a disaster, as compared to the reversibility case. On the other hand, with pessimistic beliefs about the future shock, more capital should be required for the continuation regime so as to support future production of the capital-intensive commodity. Hence, with capital being irreversible there seem to be countervailing forces on capital accumulation. But one has to take into account when considering capital accumulation in the contingency phase, that if hitting a threshold with a shock more severe than anticipated – say with a close to zero – then one might find oneself with too much capital at the beginning of the continuation regime, as the benefits from previous savings then cannot be reaped.

Another critical assumption of the preceding model is the existence of only one threshold. If all these thresholds can be bunched together in time, as pointed out by Barrett (op.cit.), then our approach seems relevant. On the other hand if there is a sequence of possible thresholds, located at various points along the time line, and also depending on each other, then our modeling framework is of course too simple. We hope to come back to both the irreversibility case and a sequence of thresholds in forthcoming papers.

[8] See Arrow and Kurz (1970).

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