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IDENTIFYING SUSPICIOUS EFFICIENT UNITS IN DEA MODELS [*]

by

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Abstract: Applications of the DEA models show that inadequate results may arise in some cases, two of these inadequacies being: a) too many efficient units may appear in some DEA models; b) a DEA model may show an inefficient unit from the point of view of experts as an efficient one. The purpose of this paper is to identify suspicious units that may unduly become efficient. The concept of a terminal unit is introduced for such units. It is shown by establishing theorems how units can be identified as terminal units and how different definitions of suspicious units are related. An approach for improving the adequacy of DEA models based on terminal units is suggested, and an example shown based on a real-life data set for Russian banks.

Keywords: Data Envelopment Analysis (DEA); Terminal units; Efficiency; Weight restrictions; Domination cones

JEL classifications: C44, C61, C67, D24

[*] This paper is an extension of the analysis in Krivonozhko et al. (2011).

1. Introduction

After a decade of applications of Data Envelopment Analysis (DEA), originating in Farrell (1957) and Farrell and Fieldhouse (1962) and generalised and put into the linear programming format we use today by Charnes et al. (1978) (hereafter called the CCR model), it was recognised that results both concerning efficiency scores and shape of the frontier production function, on which Farrell efficiency measures are based, were not always adequate when confronted with expert knowledge of the units to which DEA was applied. Types of inadequacies discussed in the literature have been that too many efficient units may appear in some DEA models, a DEA model may show an inefficient unit from the point of view of experts as an efficient one, too many zeros appear as solutions for the multipliers (weights), and units are not properly enveloped.

The first attempts in the literature (Thompson et al., 1986; Dyson and Thanassoulis, 1988) to restrict the estimation of the frontier function and consequently the efficiency scores, took two different types of failings as their point of departure. The problem of Thompson et al. (1986) was that the number of units under investigation was so small (only six) that all but one of the units was rated efficient using conventional DEA. In order to increase the discrimination restrictions on the so-called weights (appearing in the dual solution if the primal model is the envelopment model formulated in the space of inputs and outputs) were enforced. This approach was followed up in Thompson et al. (1990), and Charnes et al. (1989, 1990), the latter papers introducing the cone-ratio approach of basing the shape of the frontier on a few efficient units selected by experts by restricting the weights (multipliers) to be within cones in the dual space.

Dyson and Thanassoulis (1988) were taking a different tack. They were preoccupied with the consequence of zero weights (or weights of value ε where ε is a non-Archimedean number) leading to "some DMUs being assessed only on a small subset of their inputs and outputs, while their remaining inputs and outputs are all but ignored (p. 563)." Restriction on weights should be based on expert opinion, but the purpose was to eliminate zero weights, and not to reduce the efficiency of units being 100% efficient within the conventional DEA model.

The development of this literature is reviewed in Allen et al. (1997) and Pedraja-Chaporro et al. (1997) (a critical assessment of the literature is offered in Førsund, 2012). An interesting new line of introducing restrictions directly in the input – output space and not in the dual space started with Bessent et al. (1988) and Lang et al. (1995) of extending the faces. This was followed up in Thanassoulis and Allen (1998) by explicitly reducing the number of zero weight for inefficient observations by introducing new unobserved units based on expert opinions using units called anchor units as point of departure. A formal attempt to define anchor units and to introduce ways of finding them was done in Allen and Thanassoulis (2004) in the case of constant returns to scale and a single input. This definition was generalised in Bougnol and Dulá (2009) to multiple inputs and outputs and variable returns to scale. An elaborate algorithm for finding anchor points was introduced. The empirical applications gave the somewhat surprising result that almost all extreme efficient units are in fact anchor points. The situation of some zeros for weights seems to be the normal situation for DEA applications. However, their algorithms may produce units that are just usual efficient units (vertices) in DEA models.

Thanassoulis et al. (2012) elaborated further the super-efficiency approach for finding anchor units in a general model exhibiting variable returns to scale. However, their approach does not reveal all efficient units that may be the point of departure for improving envelopment in the Banker et al. (1984) model of variable returns to scale.

Edvardsen et al. (2008) suggested an empirical witty method for discovering "suspicious" units; they call them "exterior units". However, their method cannot discover all suspicious units.

An elegant and subtle approach was proposed in the DEA area to deal with the problems of inadequacies of the DEA models. This approach is based on incorporating domination cones (Yu, 1974) in DEA models. A number of outstanding papers were devoted to substantiation, development and applications of domination cones to DEA models (Brockett et al., 1997; Charnes et al., 1989; Charnes et al., 1990; Thompson et al., 1997; Wei et al., 2008; Yu et al., 1996). Cones are usually determined in the dual space of multipliers.

It is rather difficult, however, for a manager (the decision-maker) to determine cones in the multipliers space that is dual to the space of inputs and outputs where a production possibility set is constructed (Cooper et al., 2000). For this very reason only two particular DEA models with cones are widely used in practice at present: the assurance region model and the cone-ratio model (Cooper et al., 2000).

The purpose of this paper is to identify units that may unduly become efficient by making use of a new concept: a terminal unit. The plan of the paper is to go into the background in Section 2, using key elements from the cone-ratio approach developed in Charnes et al. (1990), based on the Banker et al. (1984) model of variable returns to scale, and establish necessary definitions. The main results are presented in Section 3, including the definition of a terminal unit and illustrating its difference from the term anchor unit and using domination cones to establish that terminal production units exist if some production units become inefficient if cones are inserted in the model. Some numerical experiments on data for Russian banks are carried out in Section 4, showing how to find a terminal unit and how to use experts to indicate an artificial efficient unit using a visual interactive graphical technique. Section 5 concludes and offer ideas for further research.

2. Background

It was shown in the DEA scientific literature (see, Krivonozhko et al., 2009) that the model in Banker et al. (1984) exhibiting variable returns to scale (hereafter termed the BCC model) can approximate any DEA model from a large family of DEA models. For this reason, we consider the BCC model as a basic model in our exposition.

Consider a set of n observations of actual production units (X_j, Y_j), $j = 1,\dots,n$, where the vector of outputs $Y_j = (y_{1j},\dots,y_{rj}) \geq 0$, $j = 1,\dots,n$, is produced from the vector of inputs $X_j = (x_{1j},\dots,x_{mj}) \geq 0$. The production possibility set T is the set $\{(X,Y) \mid$ *the outputs* $Y \geq 0$ *can be produced from the inputs* $X \geq 0\}$. The primal input-oriented BCC model can be written in the form

$$\min \theta$$

subject to

$$\sum_{j=1}^{n} X_j \lambda_j + S^- = \theta X_o,$$
$$\sum_{j=1}^{n} Y_j \lambda_j - S^+ = Y_o,$$
$$\sum_{j=1}^{n} \lambda_j = 1, \tag{1a}$$
$$\lambda_j \geq 0, \quad j = 1,\dots,n,$$
$$s_k^- \geq 0, \quad k = 1,\dots,m,$$
$$s_i^+ \geq 0, \quad i = 1,\dots,r,$$

where $X_j = (x_{1j},\ldots,x_{mj})$ and $Y_j = (y_{1j},\ldots,y_{rj})$ represent the observed inputs and outputs of production units $j = 1,\ldots,n$, $S^- = (s_1^-,\ldots,s_m^-)$ and $S^+ = (s_1^+,\ldots,s_r^+)$ are vectors of slack variables. In this primal model the efficiency score θ of production unit (X_o, Y_o) is found; (X_o, Y_o) is any unit from the set of production units (X_j, Y_j), $j = 1,\ldots,n$.

Notice that we do not use an infinitesimal constant ε (a non-Archimedean quantity) explicitly in the DEA models, since we suppose that each model is solved in two stages in order to separate efficient and weakly efficient units.

The dual multiplier form of the BCC model (1a) is expressed as

$$\max\,(u^T Y_0 - u_0)$$

subject to

$$u^T Y_j - v^T X_j - u_0 \le 0, \quad j = 1,\ldots,n \tag{1b}$$
$$v^T X_0 = 1,$$
$$v_k \ge 0, \quad k = 1,\ldots,m, \quad u_i \ge 0, \quad i = 1,\ldots,r$$

where (v, u, u_0) is a vector of dual variables, $v \in E^m, u \in E^r$, u_0 is an unconstrained scalar variable associated with the convexity constraint.

The BCC primal output-oriented model can be written in the following form

$$\max \eta$$

subject to

$$\sum_{j=1}^{n} X_j \lambda_j + S^- = X_0,$$
$$\sum_{j=1}^{n} Y_j \lambda_j - S^+ = \eta Y_0,$$
$$\sum_{j=1}^{n} \lambda_j = 1, \tag{1c}$$
$$\lambda_j \ge 0, \quad j = 1,\ldots,n,$$
$$s_k^- \ge 0, \quad k = 1,\ldots,m,$$
$$s_i^+ \ge 0, \quad i = 1,\ldots,r.$$

The dual multiplier form of the BCC output-oriented model (1c) is written in the form

$$\min\,(v^T X_0 + u_0)$$

subject to

$$u^T Y_j - v^T X_j - u_0 \leq 0\,,\quad j = 1,\dots,n \tag{1d}$$

$$u^T Y_0 = 1,$$

$$v_k \geq 0\,,\; k = 1,\dots,m\,,\; u_i \geq 0\,,\; i = 1,\dots,r\,,$$

where (v,u,u_0) is a vector of dual variables, $v \in E^m$, $u \in E^r$, u_0 is a scalar variable associated with the convex constraint (the same symbols for dual variables are used as for models (1b)).

Definition 1. (Cooper et al. 2000). Unit $(X_o, Y_o) \in T$ is called efficient with respect to the input-oriented BCC model if and only if any optimal solution of (1a) satisfies: a) $\theta^* = 1$, b) all slacks s_k^-, s_i^+, $k = 1,\dots,m$, $i = 1,\dots,r$ are zero.

If the first condition (a) in Definition 1 is satisfied, then unit (X_o, Y_o) is called input weakly efficient with respect to the BCC input-oriented model. We denote the set of these weakly efficient points by $WEff_I T$. In the DEA literature (Banker and Thrall, 1992; Seiford and Thrall, 1990) this set is also called the input boundary.

Definition 2. (Cooper et al. 2000). Unit $(X_o, Y_o) \in T$ is called efficient with respect to the output-oriented BCC model if and only if any optimal solution of (1c) satisfies: a) $\eta^* = 1$, b) all slacks s_k^-, s_i^+, $k = 1,\dots,m$, $i = 1,\dots,r$ are zero.

If the first condition in Definition 2 is satisfied, then unit (X_o, Y_o) is called output weakly efficient with respect to the BCC model. We denote the set of these weakly efficient points by $WEff_O T$. In the DEA literature (Banker and Thrall 1992; Seiford and Thrall 1990), this set is also called the output boundary.

Definition 3. Activity $(X', Y') \in T$ is weakly Pareto efficient if and only if there is no $(X, Y) \in T$ such that $X < X'$ and $Y > Y'$. We denote the set of weakly Pareto efficient activities by $WEff_P T$.

We denote the set of efficient points of T with respect to the BCC model (1) by $Eff\ T$. Krivonozhko et al. (2005) have proved that the following relations hold:

$$Eff\ T \subseteq WEff_I T \cap WEff_O T\,,\quad WEff_I T \cup WEff_O T \subseteq WEff_P T = Bound\ T\,,$$

where the boundary of T is designated as $Bound\ T$.

The production possibility set T_B for the BCC model can be written in the form (Banker et al., 1984)

$$T_B = \left\{ (X,Y) \,\middle|\, \sum_{j=1}^{n} X_j \lambda_j \le X,\ \sum_{j=1}^{n} Y_j \lambda_j \ge Y,\ \sum_{j=1}^{n} \lambda_j = 1,\ \lambda_j \ge 0,\ j = 1,\ldots,n \right\}. \quad (2)$$

In this paper we will mainly consider production possibility sets of this type.

3. Main results

The use of cones will play a crucial role in our search for terminal units. The main idea of incorporating domination cones in DEA models is to reduce the domain of multipliers. For this purpose, additional constraints on multipliers are incorporated in the DEA models.

In the assurance region method, constraints on the multipliers are added to the CCR model in the following manner; see Charnes et al. (1990),

$$\begin{aligned} l_{1i} \le \frac{v_i}{v_1} \le k_{1i}, \quad (i = 2,\ldots,m), \\ L_{1s} \le \frac{u_s}{u_1} \le K_{1s}, \quad (s = 2,\ldots,r), \end{aligned} \quad (3)$$

where l_{1i}, k_{1i}, L_{1s}, K_{1s} are given low and upper bounds on the ratios of multipliers.

Assertion 1. *There exist polyhedral cones in multidimensional space E^{m+r} that cannot be described by relations (3).*

Thus, formulas (3) describe only some subset of possible polyhedral cones in multidimensional space E^{m+r}. The next model enables one to use more general form of domination cones in the DEA models.

The dual multiplier form of the cone-ratio model is expressed as (Charnes et al., 1989; 1990; Yu et al., 1996)

$$\max\ (uY_o - u_o)$$

subject to

$$\begin{gathered} v^T X_o = 1, \\ -v^T X_j + u^T Y_j - u_o \le 0, \quad j = 1,\ldots,n, \\ u \in U, \quad v \in V, \end{gathered} \quad (4a)$$

where variables $v \in E^m$, $u \in E^r$, $u_o \in E^1$ and $U \subseteq E_+^r$, $V \subseteq E_+^m$ are given polyhedral cones.

The primal problem of (4a) is written as

$$\min \theta$$

subject to

$$\begin{aligned} &\sum_{j=1}^{n} X_j \lambda_j - \theta X_o \in V^*, \\ &Y_o - \sum_{j=1}^{n} Y_j \lambda_j \in U^*, \\ &\sum_{j=1}^{n} \lambda_j = 1, \\ &\lambda_j \geq 0, \quad j = 1, \ldots, n, \end{aligned} \tag{4b}$$

where V^* and U^* are negative polar cones of sets V and U, respectively.

In practice (Charnes et al., 1990; Cooper et al., 2000), polyhedral cones U and V are constructed as follows: a) some excellent units are chosen from the point of view of experts; b) averages of the optimal multipliers u_i^*, v_i^* are computed for every excellent unit $i \in Ex$, where Ex denotes the set of excellent units (a subset of *Eff T*). Vectors u_i^*, v_i^*, $i \in Ex$ form polyhedral cones U and V.

Cones U and V reduce the feasible domain of multipliers, while the feasible domains of inputs, see Fig. 1, and outputs, see Fig. 2, are expanding.

Now, we make an attempt to reveal the causes of inadequacies in DEA models.

Assumption. *Let the cone-ratio method allows one to reduce the number of "suspicious" production units, i.e. the units that are efficient, but should be inefficient from the point of view of experts.*

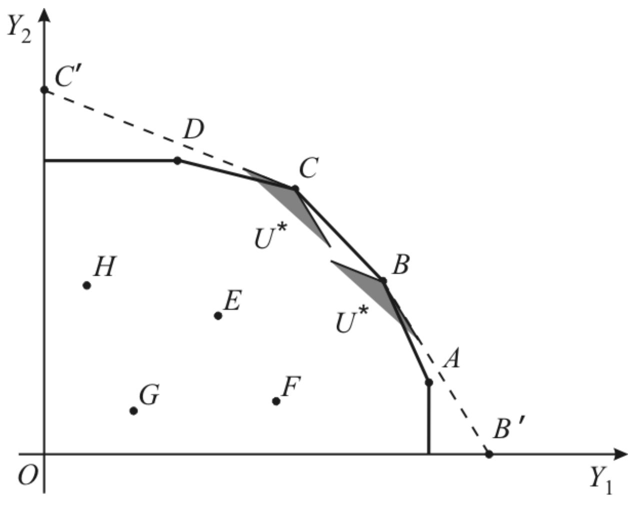

Fig. 1. Transformation of the frontier in the output subspace in the cone-ratio method

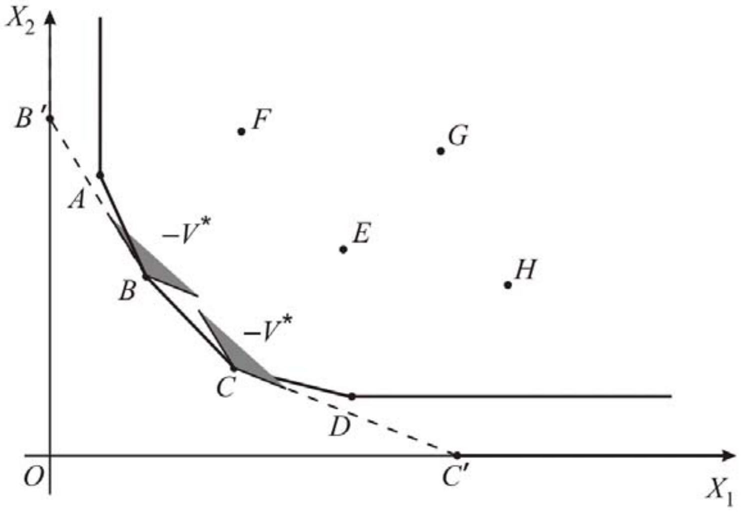

Fig. 2. Transformation of the frontier in the input subspace in the cone-ratio method

Production possibility set T_B (2) is a convex polyhedral set. According to the classical theorems of Goldman (1956) and Motzkin (1936) any convex polyhedral set can be represented as a vector sum of convex combination of vertices and the non-negative linear combination of vectors (rays).

Before going further, let us recall some notions from convex analysis. Faces are formed by an intersection of the supporting hyperplane and the polyhedral set. In the DEA models, the dimension of face may vary from 0 up to $(m+r-1)$, the maximal dimension. Faces of maximal dimension are called facets. Faces of 0-dimension are known as vertices, 1-dimension as edges.

Definition 4. *We call an efficient (vertex) unit terminal unit if an infinite edge is going out from this unit.*

We denote the set of terminal units with respect to the production possibility set (2) by T_{term}.

Definition 5. *We call a face* $\Gamma \subset WEff_I T \cup WEff_O T$ *of set* T_B *a terminal face if this face contains an infinite edge.*

Then the following assertion can be proved if the Assumption above is valid.

Theorem 1. *If some efficient production units in model (1) become inefficient in model (4) as a result of inserting cones in the BCC model (1), then it is necessary that there exist terminal production units among such inefficient units.*

Proof. See the proof of Theorem 8 in Appendix A for a more general case.

Observe that the results of Theorem 1 was proved for the cone-ratio model at first, since this model was thoroughly elaborated from theoretical and practical points of view in the scientific literature on the DEA models, see Charnes et al. (1990), Brockett et al. (1997) and Cooper et al. (2000). However, the cone-ratio model cannot cover all possible cases where inadequate results may appear in the DEA models. Therefore theorem 6 is proven in this paper for the generalized DEA model with domination cones.

Thus, Theorem 1 shows that terminal points are the first "suspicious" units which may cause inadequate results in the DEA models.

The following optimization models enable us to find terminal units or units belonging to terminal faces of the production possibility set. Let *EF* designate the set of observed efficient units (vertices) of the BCC model (1). For this purpose two types of models are solved for every efficient unit (vertex) $q \in EF$.

Problem $P_k(q)$ ($k = 1, \ldots, m$)

$$\begin{gathered}\max J_{1k} = \eta \\ \sum_{j=1}^{n} X_j \lambda_j \le X_q + \tau\, d_k, \\ \sum_{j=1}^{n} Y_j \lambda_j \ge \eta\, Y_q, \\ \sum_{j=1}^{n} \lambda_j = 1, \\ \lambda_j \ge 0,\ \tau \ge 0,\end{gathered} \tag{5}$$

where $d_k = (0, \ldots, 1, \ldots, 0) \in E^m$, the unity is in k-th position.

Variable τ provides that ray $(X_q + \tau d_k, Y_q)$ going out from unit (X_q, Y_q) belongs to the feasible set of the model (5).

Theorem 2. *If in problem (5) the optimal value* $J_{1k}^* = 1$*, then unit* (X_q, Y_q) *is a terminal one or belongs to a terminal face* $\Gamma \subset$ *WEff*$_O T$.

Proof. Under any $\tau \ge 0$ ray $(X_q + \tau d_k, Y_q)$ is a feasible subset for production possibility set T_B (2) due to monotonicity of T_B. As it follows from model (5), if $J_{1k}^* = 1$, then this ray belongs to the set *WEff*$_O T_B$. Hence this ray is an unbounded edge or belongs to an unbounded face of set T_B.

This completes the proof.

The following models determine infinite edges emanating along direction g_i, where $g_i = (0, \ldots, 1, \ldots, 0) \in E^r$ (the unity is in i-th position).

Problem $R_i(q)$ $(i = 1,\dots, r)$

$$\begin{aligned}
&\min J_{2i} = \theta \\
&\sum_{j=1}^{n} X_j \lambda_j \le \theta X_q, \\
&\sum_{j=1}^{n} Y_j \lambda_j \ge Y_q - \tau g_i, \\
&\sum_{j=1}^{n} \lambda_j = 1, \\
&\lambda_j \ge 0, \tau \ge 0.
\end{aligned} \tag{6}$$

Theorem 3. *If in problem (6) the optimal value* $J_{2i}^* = 1$*, then unit* (X_q, Y_q) *is a terminal one or belongs to a terminal face* $\Gamma \subset WEff_I T$.

The proof of Theorem 3 is very similar to the proof of the previous theorem and is therefore skipped.

Thus, models (5) and (6) enable us to reveal terminal units or efficient units belonging to unbounded faces and also directions of infinite edges going out from efficient units.

The following problems enable one to discover only terminal units.

Problem $\overline{P}_k(q)$ $(k = 1,\dots, m)$

$$\begin{aligned}
&\min \overline{J}_{1k} = \lambda_q \\
&\sum_{j=1}^{n} X_j \lambda_j \le X_q + \tau d_k, \\
&\sum_{j=1}^{n} Y_j \lambda_j \ge Y_q, \\
&\sum_{j=1}^{n} \lambda_j = 1, \\
&\lambda_j \ge 0,\ j = 1,\dots, n;\ \tau \ge 0,
\end{aligned} \tag{7}$$

Theorem 4. Unit (X_q, Y_q) is a terminal one if the optimal value of problem (7) $\overline{J}_{1i}^* = \lambda_q^* = 1$.

Proof. Observe that solution $\lambda_j = 0$, $j = 1,\dots,n$, $j \ne q$, $\lambda_q = 1$, $\tau = 0$ is a feasible solution of problem (7). So, if $\lambda_q^* = 1$, this implies that any interior point belonging to the ray $(X_q + \tau d_k, Y_q)$ cannot be represented as a convex combination of some other points of production possibility set T_B. Hence this ray is an infinite edge and point (X_q, Y_q) is a terminal unit.

However, if $\lambda_q^* < 1$, this implies that some interior points of the ray $(X_q + \tau d_k, Y_q)$ can be represented as a convex combination of some other points of set T_B.

This completes the proof.

The following problems allow one to find only terminal units and infinite edges emanating along directions g_i, $i = 1,\ldots,r$.

Problem $\overline{R}_i(q)$ ($i = 1,\ldots,r$)

$$\begin{aligned} &\min \overline{J}_{2i} = \lambda_q \\ &\sum_{j=1}^{n} X_j \lambda_j \le X_q, \\ &\sum_{j=1}^{n} Y_j \lambda_j \ge Y_q - \tau g_i, \\ &\sum_{j=1}^{n} \lambda_j = 1, \\ &\lambda_j \ge 0,\ j = 1,\ldots,n; \tau \ge 0, \end{aligned} \tag{8}$$

Theorem 5. *Unit (X_q, Y_q) is a terminal one if the optimal value of problem (8)* $\overline{J}_{2i}^* = \lambda_q^* = 1$.

The proof of Theorem 5 is very similar to the proof of Theorem 4 and is therefore skipped.

Corollary 1. Unit (X_q, Y_q) is a terminal one if and only if it turns out to be terminal at least in one of the problems $\overline{P}_k(q)$, $k = 1,\ldots,m$ in (7) and $\overline{R}_i(q)$, $i = 1,\ldots,r$ in (8).

Bougnol and Dulá (2009) determined an anchor point as an efficient vertex belonging to an unbounded face of set T_B. They proposed algorithms for discovering anchor points. However, their algorithms may produce units that are just usual efficient units (vertices) in the DEA models. Moreover, such units are not suitable as points of departure for considering improving the frontier. Indeed, consider the following illustrative example. In Figure 3, a two-inputs/one-output BCC model is depicted. Units A, B, C, D, E are the observed efficient production units that determine set T_B.

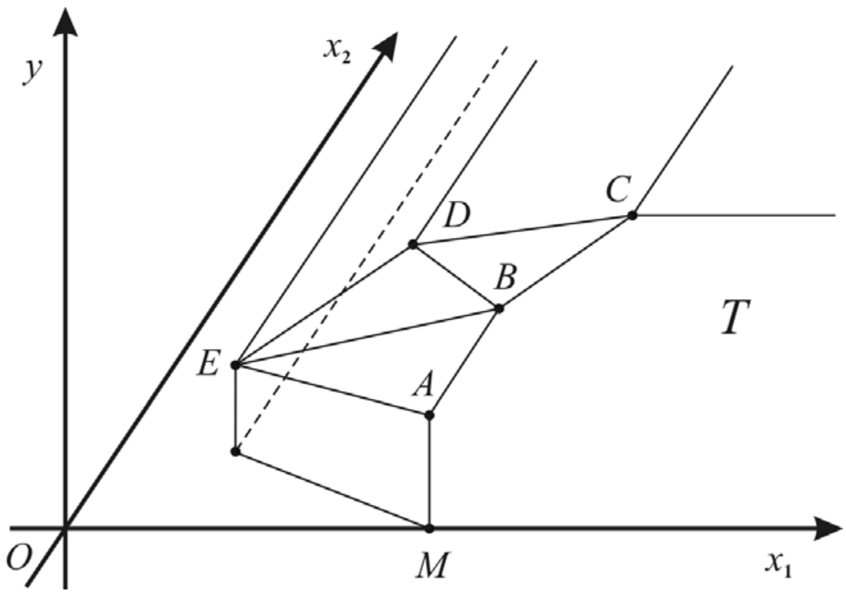

Fig. 3. Unit B is an anchor point, but not a terminal point

Units M, A, B, C, L form the face of set T_B. This face belongs to the orthant X_1OY. Hence efficient unit B belongs to the unbounded face. However, unit B is just a common vertex of the BCC model. Increasing an input or decreasing an output of unit B generates a new inefficient point that does not belong to set $(WEff_I T \cup WEff_O T)$. This unit cannot be used to reduce the DEA-inefficient part of the production possibility set, see Allen and Thanassolius (2004), hence this unit is not suitable for the frontier improvement, and contradicts the notion of an anchor point.

Let us denote the set of anchor units in the BCC model (1) with respect to the definition of Bougnol and Dulá (2009) by T^1_{anc}.

Theorem 6. *Unit (X_q, Y_q) belongs to the set T^1_{anc} in the BCC model (1) if and only if the optimal value $J^*_{1k} = 1$ and/or $J^*_{2i} = 1$ at least in one of the problems $P_k(q)$, $k = 1,\ldots,m$ in (5) and $R_i(q)$, $i = 1,\ldots,r$ in (6).*

Proof. The result follows from Theorems 2 and 3. Indeed, if $J^*_{1k} = 1$ and/or $J^*_{2i} = 1$ at least in one of the problems $P_k(q)$, $k = 1,\ldots,m$ (5) and $R_i(q)$, $i = 1,\ldots,r$ (6), then unit (X_q, Y_q) belongs to an edge or to a terminal face of the production possibility set T_B (2). Remember from the convex analysis that an edge of the polyhedral set represents also a face of 1-dimension. Hence unit (X_q, Y_q) is an anchor point with respect to the definition of Bougnol and Dulá (2009). This completes the proof.

Thus Theorem 6 gives us a constructive way to reveal whether unit (X_q, Y_q) belongs to the set T^1_{anc} or not. For this purpose one has to solve problems $P_k(q)$, $k = 1,\dots,m$ (5) and/or $R_i(q)$, $i = 1,\dots,r$ (6).

Now we can formulate the following result.

Corollary 2. *The set of terminal units T_{term} of the BCC model (1) belongs to the set of anchor points T^1_{anc} with respect to the definition of Bougnol and Dulá, i.e. $T_{term} \subseteq T^1_{anc}$.*

This result immediately follows from Theorems 2, 3 and 6.

Edvardsen et al. (2008) suggested an empirical method for discovering "suspicious" units, they call them "exterior units". Let T_{ext} denote the set of exterior units in the BCC model (1). However, their method cannot discover all suspicious units. Indeed, consider the following illustrative example, Figure 4, panel (a). The three dimensional BCC model is determined by units A, B, C, D. Again consider the BCC model with the same units, but now y is an input variable, x_1 and x_2 are output variables, see Figure 4, panel (b), points *E* and *F* are projections of points *B* and *C* onto planes x_1Oy and x_2Oy, respectively. Point D is not efficient in this case. Hence this point is not exterior. At the same time this point is a "suspicious" unit, since it may cause inadequacy in the DEA model.

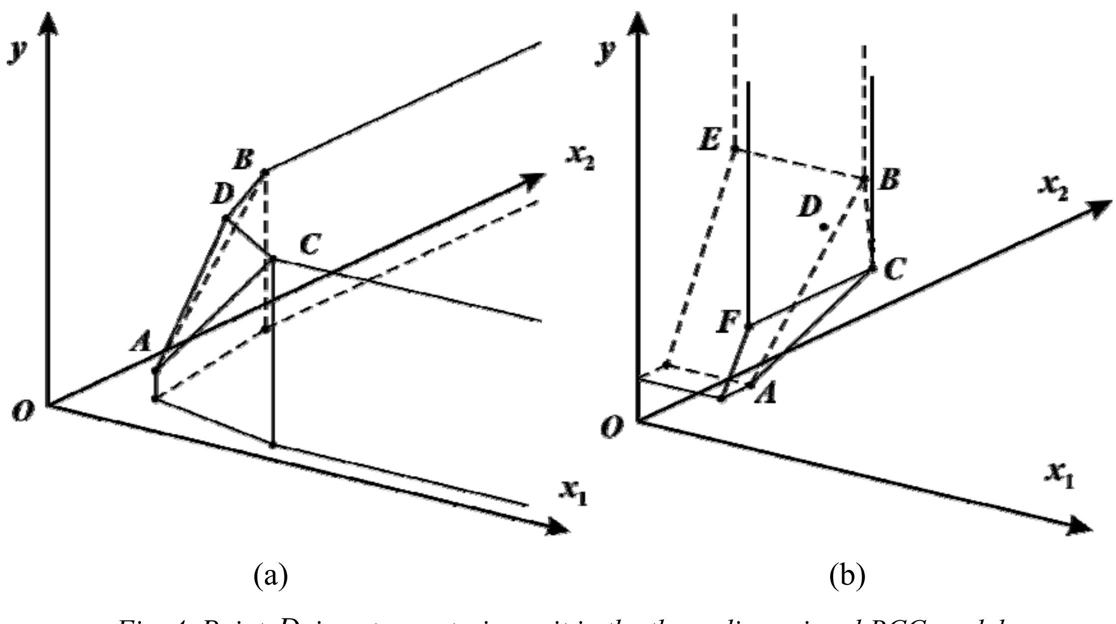

Fig. 4. Point D is not an exterior unit in the three-dimensional BCC model

Units, like point D, belonging to unbounded faces and not being terminal units are also "suspicious" points. These units may also cause inadequate results in the DEA models. However our computational experience shows that the number of such units in real-life data sets is very small in comparison with the number of terminal units.

The next theorem establishes that the set of terminal units includes the set of exterior units.

Corollary 3. The s*et of terminal units* T_{term} *of the BCC model contains the set of exterior units* T_{ext}, *i.e.* $T_{ext} \subseteq T_{term}$.

Proof. See Appendix A.

However some terminal units may not belong to the set of exterior units. Indeed, consider the following illustrative example. Figure 5 depicts a three-dimensional BCC model, points A-F (efficient units) determine the production possibility set T_B.

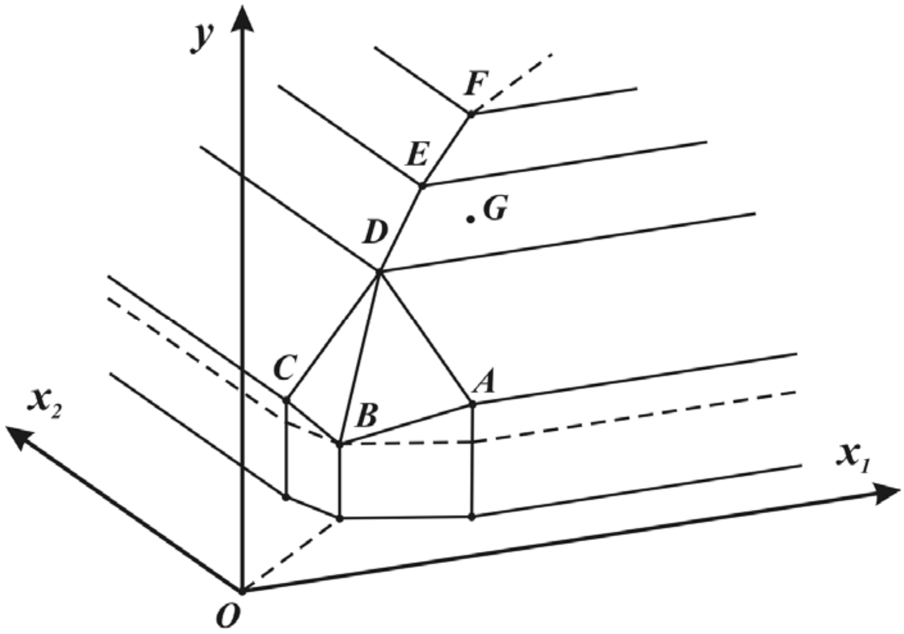

Fig. 5. Three-dimensional BCC model, units D and E are not exterior ones

Units D and E are terminal ones since these units belong to unbounded edges. However these units are not exterior ones since these units will be inefficient after "reversing the inputs and outputs".

The following theorem summarizes the results of Corollary 2 and 3.

Theorem 7. *For BCC model (1) the following relations hold* $T_{ext} \subseteq T_{term} \subseteq T^1_{anc}$.

Its proof is based on previously stated results.

Thanassoulis et al. (2012, p.178) proposed a new definition of anchor units for efficient units, "which with reference to the extreme-efficient DMUs [vertices] but excluding the evaluated DMU itself, can be rendered class F [weakly efficient units] by contracting radially their output levels, while keeping their input level constant OR by increasing inputs and keeping their output levels constant". However, their approach does not reveal all efficient units that may be used for improving envelopment in BCC models.

Consider again an illustrative example in Figure 5. Units D and E are not anchor units with respect to the definition of Thanassoulis et al. (2012). Figure 6 depicts an input isoquant for unit D. Unit G is inefficient. Its projection will be point G' on some slack face. However, their approach cannot improve this part of the boundary, unit D is not identified as an anchor unit, since unit D can be moved to the efficient part of the frontier by contracting radially its output level while keeping its input levels constant or by increasing inputs and keeping its output levels constant. So, their approach is incomplete.

The cone-ratio model (3) cannot help in every case where suspicious units appear in the DEA models. In Figure 5, point B is a terminal unit. However, it is impossible to transform the frontier with the help of cones U and V in such a way that terminal point B would be inefficient, see Fig. 6.

Only simultaneous transformation of the frontier in the space of inputs and outputs enables one to make suspicious unit B inefficient, see Fig. 7.

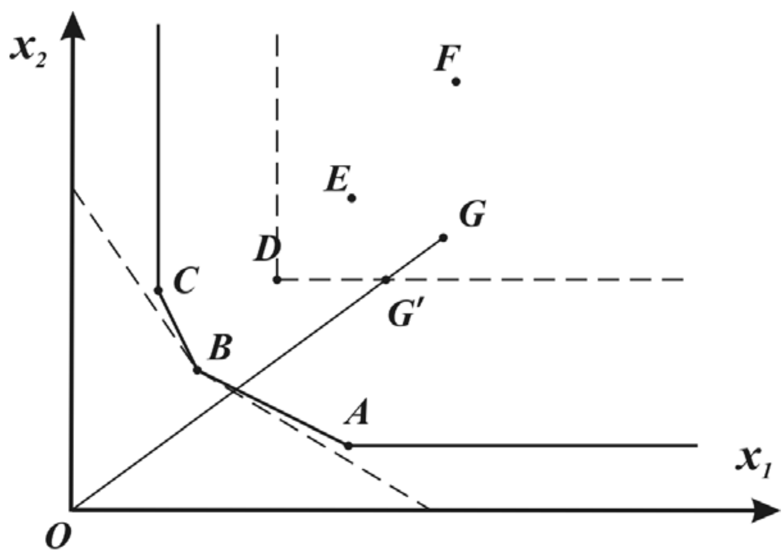

Fig. 6. Input isoquants for units B and D

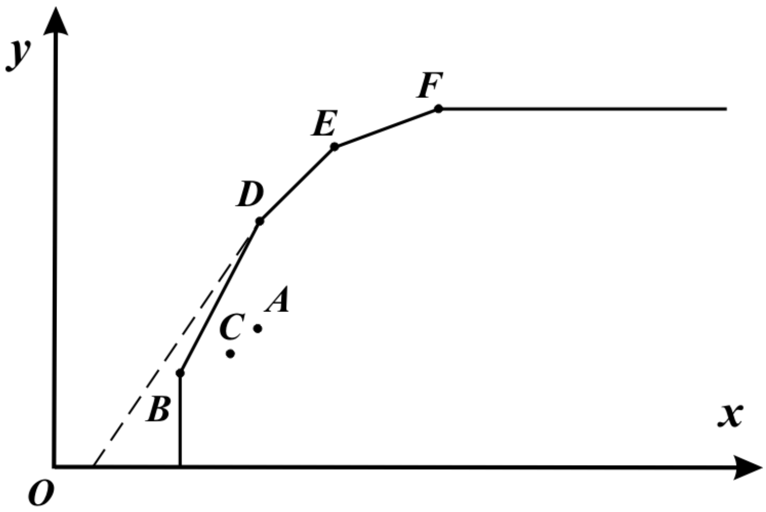

Fig. 7. Production function for unit B

Yu et al. (1996) proposed the following generalized DEA (GDEA) model that unifies and extends most the well-known DEA models based on using domination cones (see, e.g. Yu, 1974) in their constraint sets.

$$\max(u^T Y_0 - \delta_1 u_0)$$

subject to

$$\begin{aligned} & v^T \overline{X} - u^T \overline{Y} + u_0 \delta_1 e^T \in K, \\ & v^T X_0 = 1, \\ & \begin{pmatrix} v \\ u \end{pmatrix} \in W, \quad \delta_1 \delta_2 (-1)^{\delta_3} u_0 \ge 0, \\ & v \in E^m, u \in E^r, u_0 \in E^1. \end{aligned} \tag{9a}$$

The optimization dual problem to (9a) is written in the form (Yu et al, 1996):

$$\min \theta$$

subject to

$$\begin{aligned} & \begin{pmatrix} \overline{X}\lambda - \theta X_0 \\ -\overline{Y}\lambda + Y_0 \end{pmatrix} \in W^*, \\ & \delta_1 e^T \lambda + \delta_1 \delta_2 (-1)^{\delta_3} \lambda_{n+1} = \delta_1, \\ & \lambda \in -K^*, \quad \lambda_{n+1} \ge 0, \quad \theta \in E^1, \end{aligned} \tag{9b}$$

where $\overline{X} = (X_1, \ldots, X_n)$ is an $m \times n$ matrix, $X_j = (x_{1j}, \ldots, x_{mj}) \ge 0$ is the input vector for the j^{th} production unit $j = 1, \ldots, n$; $\overline{Y} = (Y_1, \ldots, Y_n)$ is an $r \times n$ matrix, $Y_j = (y_{1j}, \ldots, y_{rj}) \ge 0$ is the output vector for the j^{th} production unit $j = 1, \ldots, n$. Parameters $\delta_1, \delta_2, \delta_3$ are binary ones assuming only the values 0 and 1. Vector e is determined as $e = (1, \ldots, 1) \in E^n$. Sets $W \subseteq E^{m+r}$ and $K \subseteq E^n$ are the closed convex cones, where E^{m+r} and E^n are Euclidean spaces of the dimensions $(m+r)$ and n, respectively. W^* and K^* are the negative polar cones (Charnes et al, 1989; Yu et al, 1996) of sets W and K, respectively. It is usually assumed in the DEA models that the polyhedral cones $W \subseteq E_+^{m+r}$ and $K \subseteq E_+^n$ and $\text{int}\, W \neq \emptyset$ and $\text{int}\, K \neq \emptyset$, then we get $W^* \neq \emptyset$ and $K^* \neq \emptyset$.

Theorem 8. *If some efficient units in model (1) become inefficient in model (9) as a result of inserting cones in the BCC model, then it is necessary that there exist terminal production units among such inefficient units.*

Proof. See Appendix A.

It is rather difficult for a manager (expert) to determine cones in the multipliers space that is dual to the space of inputs and outputs where a production possibility set is constructed.

For this very reason it is difficult to use the GDEA model in practice.

Krivonozhko et al. (2009) proposed a model that is more general than the GDEA model, on the one hand, as it covers situations that the GDEA model cannot describe. On the other hand, this model enables one to construct step-by-step any model from a large family of the DEA models by incorporating artificial units and rays in the space of inputs and outputs in the BCC model, which makes the process of model construction visible and more understandable.

The production possibility set of this model is written in the form

$$T_G = \left\{ (X,Y) \;\middle|\; X \geq \sum_{j=1}^{n} X_j \lambda_j + \sum_{i \in I} D_i \mu_i + \sum_{k \in J} A_k \rho_k, \right.$$
$$Y \leq \sum_{j=1}^{n} Y_j \lambda_j + \sum_{i \in I} G_i \mu_i + \sum_{k \in J} B_k \rho_k, \; \sum_{j=1}^{n} \lambda_j + \sum_{i \in I} \mu_i = 1, \tag{10}$$
$$\left. \lambda_j \geq 0,\; j = 1,\ldots,n,\; \mu_i \geq 0,\; i \in I,\; \rho_k \geq 0,\; k \in J \right\}.$$

where (D_i, G_i), $i \in I$, I is a set of artificial production units, (A_k, B_k), $k \in J$, J is a set of vectors (rays) added to the model.

Figure 8 shows the transformation of the frontier of the two-dimensional BCC model with the help of artificial units and rays. In the figure, cone Q is formed by artificial rays, point B' is an artificial unit.

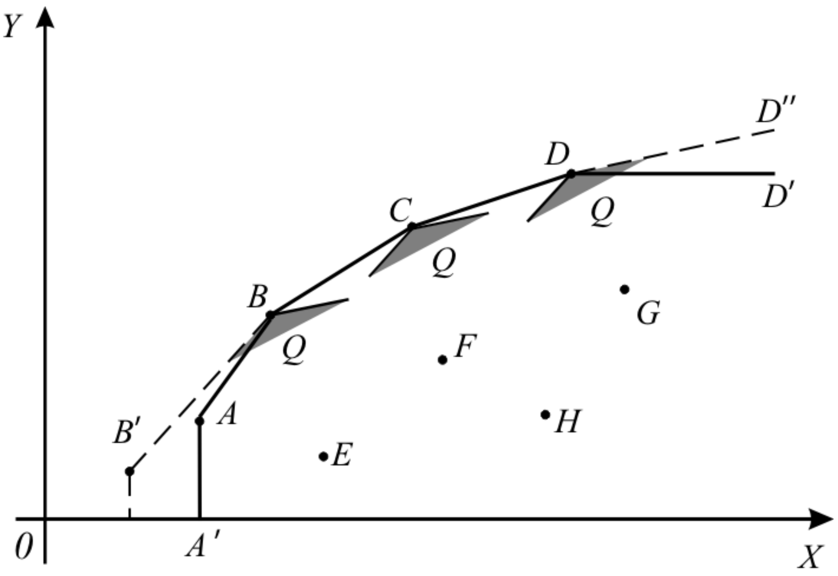

Fig. 8. Transformation of the frontier with the help of artificial units and rays

In addition to problem (7) and (8), we can also discover terminal (suspicious) production units with the help of constructions of two-dimensional and three-dimensional sections of the frontier.

Define three-dimensional affine subspace in space E^{m+r} as

$$\mathrm{Pl}(X_o, Y_o, d_1, d_2, d_3) = (X_o, Y_o) + \alpha d_1 + \beta d_2 + \gamma d_3, \tag{11}$$

where $(X_o, Y_o) \in T_B$, α , β and γ are any real numbers, directions $d_1, d_2, d_3 \in E^{m+r}$ are not parallel to each other.

Next, define intersections of the frontier with three-dimensional affine subspace

$$\begin{aligned} \mathrm{Sec}(X_o, Y_o, d_1, d_2, d_3) = \{(X, Y) \mid (X, Y) \in \mathrm{Pl}(X_o, Y_o, d_1, d_2, d_3) \cap \\ \cap\, WEff_P\, T(X, Y);\ d_1, d_2, d_3 \in E^{m+r}\}, \end{aligned} \tag{12}$$

where $WEff_P T$ is a set of weakly Pareto-efficient points. Krivonozhko et al. (2005) have proved that set $WEff_P T$ coincides with the boundary of T_B (2).

By choosing different directions d_1, d_2 and d_3 we can construct various two-dimensional and three-dimensional sections going through point (X_o, Y_o) and cutting the frontier. Parametric optimization algorithms for construction of sections of the type (11) are described in detail by Krivonozhko et al. (2004) and Volodin et al. (2004).

Moreover, thanks to our package FrontierVision, one can add to the DEA model any artificial units and rays on the computer screen interactively.

Assertion 2. *There always exists a section (11) that reveals any terminal unit and/or efficient units belonging to an unbounded face.*

However, the specific section may not reveal some terminal units. In the three-dimensional BCC model, see Figure 5, unit B is a terminal one. In Figure 6, unit B does not look like a terminal one. The section in Figure 7 reveals this unit as a terminal point.

Generally speaking, a two-dimensional section of the type (11) consists mainly of a number of segments and two rays. The first and the last vertices in the chain of segments are usually terminal units.

Next, a user (expert) can control the changes of efficiency scores of inefficient units as a result of inserting artificial rays and units with the help of our package FrontierVision. Indeed, the following assertion is valid.

Assertion 3. *There always exists a vicinity for any terminal unit such that any change of this unit's position within this vicinity will change only efficiency scores of some inefficient units within some small range.*

So, if some changes in efficiency scores are unacceptable according to expert's opinion, he/she can move artificial units closer to some small vicinity of a terminal unit just on the screen of the computer. Our computational experiments showed that it is sufficient to make two or three such iterations in order to adjust his/her model and obtain reliable computational results.

4. Computational results

In order to illustrate consequences of letting experts choose one unit from the terminal units and change data to more reasonable values we used a dataset for 920 Russia banks' financial accounts for January 1 of 2009. The following inputs and outputs for the BCC output-oriented model were used:

Inputs: working assets; time liabilities; demand liabilities.

Outputs: equity capital; liquid assets; fixed assets.

Max, min and mean statistics for banks are shown in Table 1.

Table 1. Data for banks Russia 2008

Variables	Mean	St. deviation	Min	Max
Outputs				
Liquid assets, ths roubles	4 279 490	30 304 201	73	717 402 532
Equity capital, ths roubles	2 205 806	23 572 632	423	632 286 730
Fixed assets, ths roubles	608 481	7 414 069	42	221 058 541
Inputs				
Demand liabilities, ths roubles	11 318 997	140 641 585	0	4 184 548 095
Time liabilities, ths roubles	18 289 244	162 725 433	1	4 213 176 749
Working assets, ths roubles	24 587 080	230 385 425	0	6 233 536 293

The data were financial accounts of Russian banks for the year 2008. Remember that this year was the first year of the world crisis. It was important at that time for financial experts to have reliable tools for forecasting the behavior of financial institutes and for warning about possible bankruptcies.

Figure 9 represents the dependence of the number of units on the range of efficiency scores according to the BCC output model (1c, 1d).

Notice that the number of efficient banks is very low; 42 units out of 920. The majority of banks have efficiency scores less than 50%. This situation is different from the situation reported in Charnes et al. (1990).

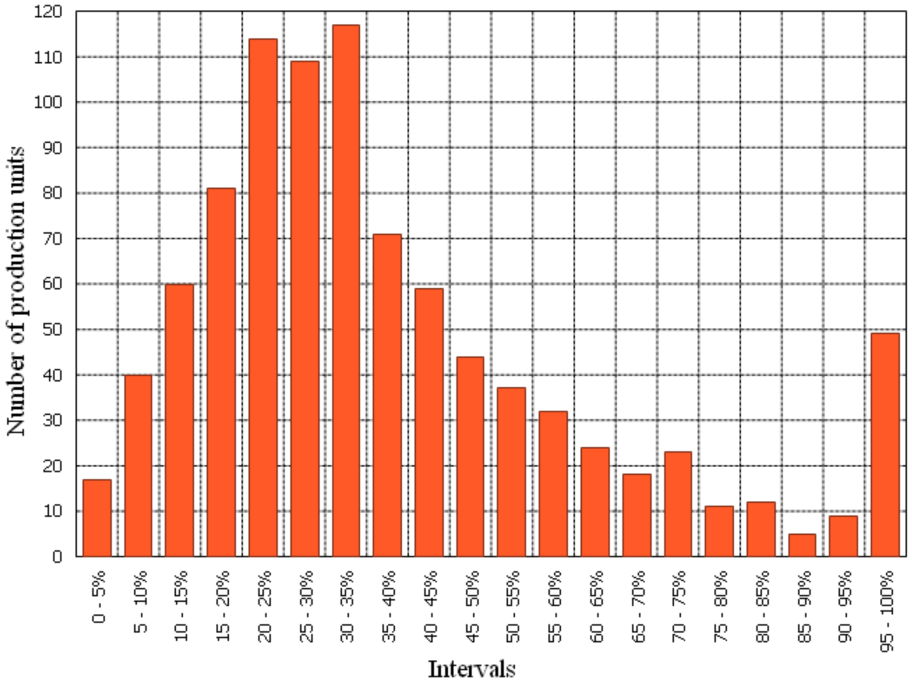

Fig. 9. Distribution of efficiency scores before the frontier transformation

However, financial experts expressed doubt about the result of full efficiency of some banks. For example, Figure 10 presents a cut of the frontier in a six-dimensional space by the two-dimensional plane for bank A; certainly we use legends instead of real names of banks. The directions of the plane are determined by two inputs: demand liabilities and working assets.

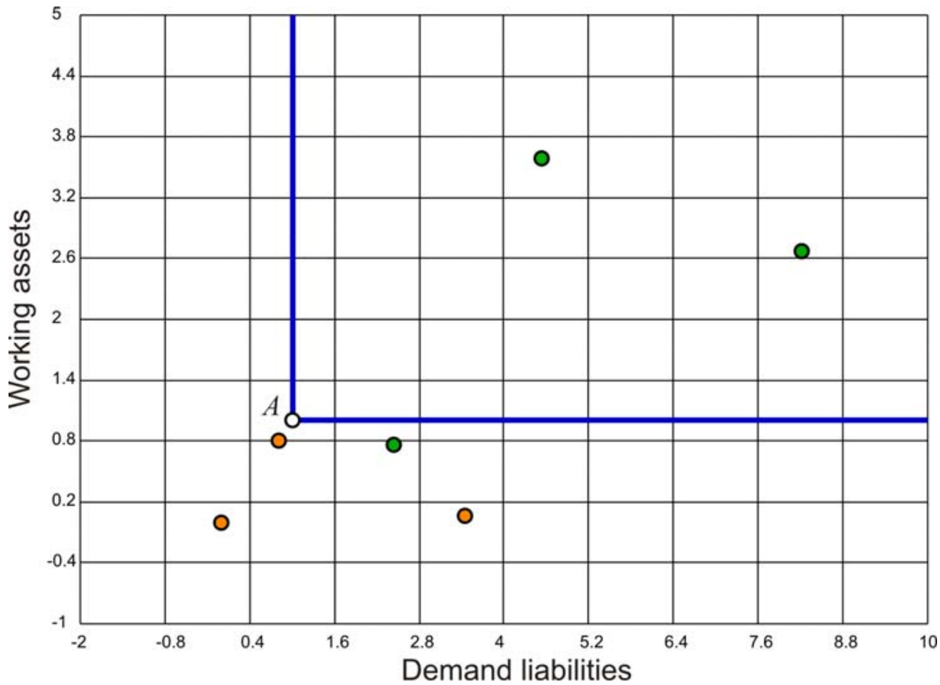

Fig. 10. Input isoquant for bank A

The scale is such that point $(1,1)$ in the figure corresponds to bank A. According to the BCC model bank A is 100% efficient. However experts did not agree with this evaluation, and they were right, since bank A was bankrupted in six months. In fact point A is a typical terminal unit, since unbounded edges go out from this unit. However, Figure 10 cannot help us to improve the frontier.

For this purpose we should use another section. Figure 11 shows a cut of the frontier in a six-dimensional space by the two-dimensional plane for bank A. The horizontal axis in the figure is determined by input vector X_o and the vertical axis corresponds to output vector Y_o of bank A, respectively. The scale is such that point $(1,1)$ corresponds to bank A. The solid line outlines the production function (the cut of the production possibility set) of the model. The balls in the figure denote projections of some other banks on the two-dimensional plane. Again, according to the model, bank A is efficient, which contradicts experts' opinion.

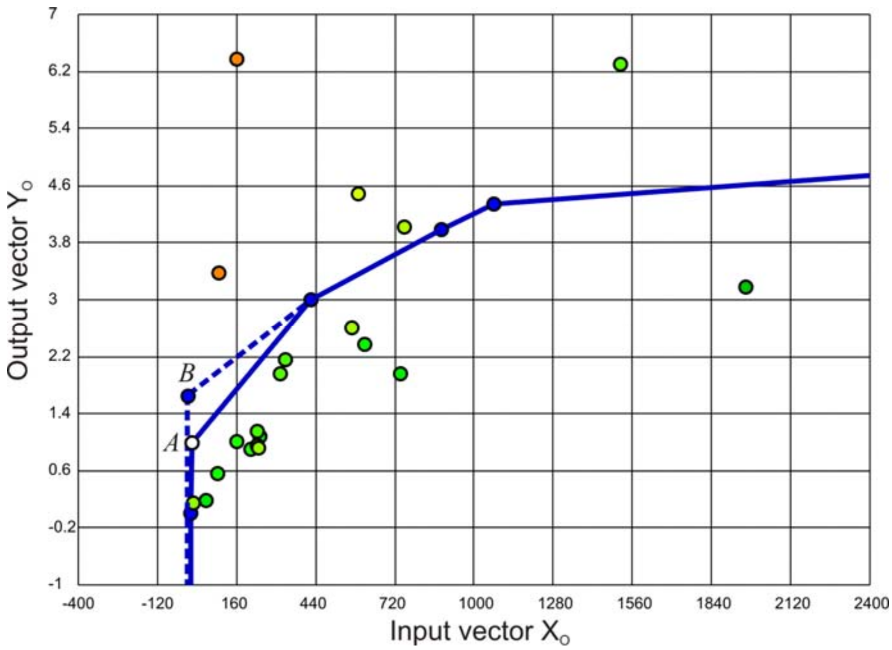

Fig. 11. Production function for bank A

However, Figure 11 can help us to improve the frontier. Experts were asked to insert an artificial efficient unit on the screen by phrasing the question how much outputs should be expected from an efficient unit using the observed inputs of unit A (implicitly assuming a proportional increase of the outputs). This artificial unit is denoted by B. In the figure, the dotted line together with the solid line after it shows the frontier of the modified model.

After the frontier transformation the efficiency score of bank A became 48.3%. Some other banks also changed their efficiency scores after inserting artificial unit in the model. Table 2 shows efficiency scores of some banks, which were bankrupted during six months, in the BCC model before and after frontier transformation.

After the second run of the model, the experts recognized the modelling results to be adequate and reliable.

Table 2. Changes of efficiency scores after the frontier transformation

Name	Efficiency score before frontier transformation, in %	Efficiency score after frontier transformation, in %	Date of bankruptcy
C	1.79	1.50	30.03.2009
D	46.34	41.87	16.04.2009
H	11.49	9.16	21.05.2009
A	100.00	48.36	25.06.2009
K	7.43	6.35	25.06.2009

We have presented an investigation for only one terminal unit, but demonstrated that the choice of terminal units as units that should be investigated using expert information worked out satisfactorily; reducing the efficient unit to an inefficient one and also reducing several other units' scores and improving the realism of the results. However, the working out of a more formal procedure for eliciting expert help in providing more realistic efficient units based on terminal units is still to be done.

5. Conclusions

In this paper, we proposed tools for discovering units which may cause inadequate results in the DEA models. It was shown that terminal units constitute "suspicious" points in the first place. If the graph of intersection of the frontier with a two-dimensional plane is constructed, then the first and the last vertices of the graph are usually terminal units. However, it is not necessarily the case that terminal units may cause inadequate results in the DEA models, such units may be quite normal efficient points. Only experts in the specific area can evaluate the adequacy of efficiency scores of terminal units.

Terminal units arise because a non-countable (continuous) production possibility set T is determined on the basis of a finite number of production units; some of these units turn out to be terminal ones. A gap between derivatives may take place at these points. For example, the left-hand side scale elasticity takes infinite value, and the right-hand side scale elasticity takes zero value at some terminal points, see Førsund et al. (2007).

Let us remember that Farrell (1957) introduced artificial units at infinity in order to smooth his model, see also Førsund et al. (2009).

We also propose how to deal with inadequacies in the DEA models with the help of incorporating artificial units and rays interactively on the screen of the computer by experts into some BCC model. This makes the DEA models more adequate and adjustable.

Only one case of eliciting information from experts suggesting artificial units that should be efficient based on terminal units has been shown. However, several other experiments were carried out. Carrying out systematic experiments on new real-life datasets will be attempted in a further research, together with developing more formal procedures for eliciting information from experts on the activities under investigation constructing artificial efficient units based on terminal units. The use of interactive graphical representation in a space of sufficiently reduced dimensions to be readily understandable and recognizable by experts seems a promising approach.

Acknowledgments

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Appendix A

Proof of Corollary 3. Take any point $Z_q = (X_q, Y_q) \in T_{ext}$. Assume that Z_q is not a terminal unit. Consider the following ray

$$R(Z_q) = \left\{ Z \middle| Z = Z_q + \tau\left(\sum_{k=1}^{m} \alpha_k \tilde{d}_k - \sum_{i=1}^{r} \beta_i \tilde{g}_i \right), \tau > 0 \right\}, \quad \text{(A.1)}$$

where $\alpha_k > 0$, $k = 1,\ldots,m$, and $\beta_i > 0$, $i = 1,\ldots,r$ are constants, $\tilde{d}_k = (d_k, 0) \in E^{m+r}$ ($k = 1,\ldots,m$), $\tilde{g}_i = (0, g_i) \in E^{m+r}$ ($i = 1,\ldots,r$).

Since Z_q is not a terminal unit, some part of ray (A.1) goes through the interior of the set

$$\overline{T}_{ext} = \left\{ Z \middle| Z = \sum_{s \in T_{ext}} \mu_s Z_s,\ \sum_{s \in T_{ext}} \mu_s = 1,\ \mu_s > 0,\ s \in T_{ext} \right\}. \quad \text{(A.2)}$$

Consider some point Z^* that belongs to set (A.1) and to set (A.2).

After “reversing the inputs and outputs” according to the procedure of Edvardsen et al. (2008) point Z^* will dominate point Z_q. Hence point Z_q cannot belong to the set of exterior points, contradicting the assumption.

This completes the proof.

Proof of Theorem 8. Consider efficient unit (X_1, Y_1) in model (1) under evaluation, that remains efficient in model (9) after inserting domination cones, and efficient unit (X_2, Y_2) in model (1) that becomes inefficient in model (9) after inserting domination cones. Assume, at first, that $K = E_+^n$, this implies that cone K coincides with the first non-negative orthant.

If unit (X_2, Y_2) is a terminal point, then the theorem is proved. Assume that unit (X_2, Y_2) is not a terminal unit.

Let (v_1, u_1) be optimal dual solution in model (9) and (v_2, u_2) be optimal dual solution in model (1) for units (X_1, Y_1) and (X_2, Y_2), respectively. It is known that dual optimal solution (v_1, u_1) is an orthogonal vector to some face of the frontier at point (X_1, Y_1) (see Cooper et al., page 120, theorem 5.1). Notice that we do not write here the third part u_0 of the dual vector, because we need only an orthogonal vector for our purpose.

Dual optimal solution (v_1, u_1) for problem (9) is also optimal solution for problem (1) since the inclusion of cones in model (1) may only decrease the feasible set of dual variables (Yu et al., 1996).

Denote vector of dual variables by $w = (v, u) \in E^{m+r}$. Next, let $\|w\|_2$ be the quadratic norm of vector w.

Consider the following linear programming problem with a parameter α in the objective function (see Dantzig 1997, 2003)

$$\begin{aligned} \max & \left[\bar{u}_1 + \alpha(\bar{u}_2 - \bar{u}_1)\right]^T Y - \left[\bar{v}_1 + \alpha(\bar{v}_2 - \bar{v}_1)\right]^T X \\ & X \geq \sum_{j=1}^{n} X_j \lambda_j, \\ & Y \leq \sum_{j=1}^{n} Y_j \lambda_j, \\ & \sum_{j=1}^{n} \lambda_j = 1,\ \lambda_j \geq 0, \end{aligned} \tag{A.3}$$

here

$$\bar{u}_1 = u_1 / \|w_1\|_2,\quad \bar{u}_2 = u_2 / \|w_2\|_2,\quad \bar{v}_1 = v_1 / \|w_1\|_2,\quad \bar{v}_2 = v_2 / \|w_2\|_2, \tag{A.4}$$

where $w_1 = (v_1, u_1)$ and $w_2 = (v_2, u_2)$.

According to the theory of linear programming with a parameter in the objective function when α is increasing, the optimal solution of problem (A.3) moves along the frontier from one face to another face.

It follows from (A.4), that some components of $\overline{w}_1 = (\overline{v}_1, \overline{u}_1)$ will be greater than the corresponding components of $\overline{w}_2 = (\overline{v}_2, \overline{u}_2)$ and vice versa. Some components of vector $\overline{w}_3 = \left(\overline{v}_1 + \alpha^*(\overline{v}_2 - \overline{v}_1), \overline{u}_1 + \alpha^*(\overline{u}_2 - \overline{u}_1)\right)$ will be equal to zero under some $\alpha^* > 1$.

Since vector $\overline{w}_3$ is orthogonal to some face $\Gamma \subset T_B$, this face contains at least one unbounded edge and one terminal point.

Points of face Γ are not efficient in model (9) with domination cones. Let us dwell on this in detail.

According to the assumption, unit (X_1, Y_1) is efficient in model (1) and in model (9). Optimal dual variables $w_1 = (v_1, u_1)$ are associated with efficient unit (X_1, Y_1) in both model (1) and model (9). Hence vectors w_1, $\overline{w}_1$ belongs to some cone W_1 that is included in problem (1) in order to get problem (9a). Unit (X_2, Y_2) is efficient in problem (1) and inefficient in problem (9). Hence vector $w_2 = (v_2, u_2)$ associated with unit (X_2, Y_2) does not belong to cone W_1.

Since vectors w_2, $\overline{w}_2$ and cone W_1 are convex sets, we can construct a hyper-plane

$$(\beta, w) = b, \tag{A.5}$$

where $\beta \in E^{m+r}$, $w \in E^{m+r}$, b is a scalar, that separates these two vectors and cone W_1, or, in other words,

$$(\beta, w) < b, \text{ under } w \in W_1, \tag{A.6}$$
$$(\beta, w_2) > b.$$

It follows from (A.6), that

$$(\beta, w_3) = \beta^T\left[w_1 - \alpha^* w_1 + \alpha^* w_2\right] = -(\alpha^* - 1)\beta^T w_1 + \alpha^* \beta^T w_2 > -(\alpha^* - 1)b + \alpha^* b = b$$

under $\alpha^* > 1$.

Hence vectors w_3 and $\overline{w}_3 = w_3 / \|w_3\|_2$ do not belong to cone W_1.

Thus, points of face Γ associated with dual vector w_3 are inefficient in model (9), and among units of face Γ there exist terminal points.

Now, let $K \subseteq E^n_+$. The inclusion of cone K in model (9) may only expand the production possibility set T_B, therefore the number of efficient units in model (1) that become inefficient in model (9) may only increased.

This completes the proof.

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