

MEMORANDUM

No 12/2013

When Can Environmental Profile and Emissions Reduction Be Optimized Independently of the Pollutant Level

The seal of the University of Oslo is a circular emblem. It features a central figure of a woman in classical attire, holding a lyre. The text 'UNIVERSITAS OSLOENSIS' is inscribed around the top inner edge of the circle, and 'MDCCCXXXIII' is at the bottom. The seal is rendered in a light gray tone.

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WHEN CAN ENVIRONMENTAL PROFILE AND EMISSIONS REDUCTIONS BE OPTIMIZED INDEPENDENTLY OF THE POLLUTANT LEVEL?*

Memo 12/2013–v1

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Abstract. Consider a model for optimal timing of emissions reduction, trading off the cost of the reduction against the time-additive aggregate of environmental damage, the disutility from the pollutant stock $M(t)$ the infrastructure contributes to. Intuitively, the optimal timing for an *infinitesimal* pollution source should reasonably not depend on its historical contribution to the stock, as this is negligible. Dropping the size assumption, we show how to reduce the minimization problem to one not depending on the history of M , under linear evolution and suitable linearity or additivity conditions on the damage functional. We employ a functional analysis framework which allows for delay equations, non-Markovian driving noise, a choice between discrete and continuous time, and a menu of integral concepts covering stochastic calculi less frequently used in resource and environmental economics. Examples are given under the common (Markovian Itô) stochastic analysis framework.

Keywords: Optimal control, optimal stopping, environmental policy, emissions reduction, linear model, Banach space, stochastic differential equations.

MSC (2010): 91B70, 91B76, 60H10.

JEL classification: Q52, C61.

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1 Introduction, and the simplest model

Consider an economic activity that leads to emissions of a stock pollutant, which again leads to a certain damage per time unit, depending on the stock. The activity can at any time be shut down (or possibly be retrofitted with clean technology) at a given cost k , which may incorporate direct costs, scrap values, and/or the loss of utility from the services the economic activity provides. The stock of pollutant will trend upwards as long as the activity persists, and then trend downwards towards zero contribution (or at least stay put) from the time of implementation. Consider then the problem of finding the (non-clairvoyant) timing which minimizes expected discounted total cost.

A formal model for such a problem was treated by Pindyck in [19] and [20], slightly more general than described above. Therein, the running cost was specified as the bilinear form $\Theta(t)M(t)$ where $M(t)$ models the pollution from the activity (as stock units) at time t , and $\Theta(t)$ models the impact cost factor, the damage per unit this (marginal) activity causes. The aggregate discounted economic and ecological cost, is then taken to be

$$\int_0^\infty e^{-rt}\Theta(t)M(t) dt + ke^{-r\tau} \quad (1)$$

where τ is the time of policy implementation, and the objective is to minimize the *expected value of* (1) over all stopping times τ ; in this model, the initial technology is exogenous, that is, we inherit the installation up and running. (As we shall work with formula (1)-type random variables directly, we will usually not write the expectation operator explicitly.) Pindyck models pollution stock by a linear deterministic differential equation

$$dM = (\beta\eta(t) - \delta M) dt, \quad M(0) = m \quad (2)$$

where $\eta(t)$ could be interpreted as emission rate and β is the fraction that finds its way to the environment. η will in the simple model be restricted to being a positive constant until implementation, and 0 thereafter (with continuous fit at the implementation time), and so β is then redundant, and can and will be assumed equal to 1. The initial state $M(0) = m$ is assumed nonnegative.

The impact cost factor $\Theta(t)$ is, for the time being, a diffusion process which does not depend on M . A key result of [19] is that when Θ is a geometric Brownian motion, the optimal rule for closedown is to wait until it exceeds a sufficiently high value θ^* , which *does not depend on M* or in other words, does not depend on how long the installation has been in operation. However, as later pointed out by [10] and [3], this property crucially depends of the form of the damage function, and does not carry over to the case where the running damage is replaced by ΘM^2 – but the linear dynamics kept. (Of course, nonlinearities in the damage function can be cancelled out by a corresponding nonlinearity in the dynamics.)

This leads to the question which is the main subject of this paper: if we generalize the model, then *under what conditions will the optimization depend on the state – or history –*

of M ? Using a Banach space framework, we shall see that the property of non-dependence upon history, will carry over when «everything but the evolution of Θ » is linear – but it shall also turn out that there are nonlinear cases which could be of interest. Our approach will not be restricted to the optimal stopping problem, but will cover continuous and discontinuous optimization over infrastructure. The question does not apply to flow pollutants; if it is only the flow that causes damage, then damage does obviously not depend on stock. Nevertheless, for actually solving the optimization problems, it may be useful to compare quickly decaying stocks with the limiting case of flows.

1.1 Removing M from the simple optimization problem

The optimization in the above simple model can be solved by dynamic programming. Leaving a formal setup for the next section 2, we shall now restrict ourselves to the result for the problem where $\eta(t)$ is a given constant up to intervention time and 0 thereafter – henceforth, a «one-shot problem»:

1.1.1 Proposition ([19]). *Consider M following the dynamics (2) with $\beta = 1$ and $\eta(t)$ restricted to the form $\eta(t) = \bar{\eta} \cdot 1_{t \in [0, \tau]}$, with M continuous at τ and C^1 elsewhere. Suppose furthermore that Θ obeys the (Itô) stochastic differential equation*

$$d\Theta(t) = \Theta(t) \cdot (\alpha dt + \sigma dZ(t)), \quad \Theta(0) = \theta > 0 \quad (3)$$

where Z is standard Brownian, and that $r - \alpha > 0$, $\delta > 0$ and $k = k(\bar{\eta}) > 0$. Then the problem of minimizing the expected value of (1) over all stopping times τ , is solved by stopping first time Θ exceeds

$$\theta^* = \frac{\gamma k(\bar{\eta})}{(\gamma - 1)\bar{\eta}}(r - \alpha)(r + \delta - \alpha) \quad \text{with} \quad \gamma = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left(\frac{\alpha}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \quad (4)$$

and the value function is

$$\frac{\theta m}{r + \delta - \alpha} + k(\bar{\eta}) \cdot \begin{cases} \left[\gamma \frac{\theta}{\theta^*} - \left(\frac{\theta}{\theta^*}\right)^\gamma \right] / (\gamma - 1) & \text{if } \theta < \theta^* \\ 1 & \text{if } \theta \geq \theta^*. \end{cases} \quad (5)$$

In particular, the optimal rule does not depend on m .

Notice that the m -dependent first term is the damage that would incur even without the installation ($\tau = k = 0$). The term $(\theta/\theta^*)k\gamma/(\gamma-1) = \theta\bar{\eta}/((r-\alpha)(r+\delta-\alpha))$ is the damage from running the installation forever from now, and the subtractive element represents the value of the option to stop. The latter two do not depend on m . Let us give an argument for this property without using dynamic programming nor the form (5) directly – although, we can later use dynamic programming on the one-dimensional problem we reduce the problem to. The solution for $M(t)$ is

$$M(t) = e^{-\delta t} m + e^{-\delta t} \int_0^t e^{\delta s} \eta(s) ds \quad (6)$$

so that when we restrict controls to $\eta(s) = \bar{\eta}1_{s \leq \tau}$ as above, with only $\tau \geq 0$ to choose, the damage becomes

$$m \int_0^\infty e^{-(r+\delta)t} \Theta(t) dt + \bar{\eta} \int_0^\infty e^{-(r+\delta)t} \Theta(t) \left(\int_0^{\min\{t, \tau\}} e^{\delta s} ds \right) dt + ke^{-r\tau} \quad (7)$$

Already before taking any expectation, we observe that the first term – the damage obtained by running forever – does not depend on τ , while the two others do not depend on m .

We can actually compute (7) without dynamic programming, looking up the distributional properties of the geometric Brownian motion and its stopping times in e.g. Borodin and Salminen [4]. However, we could just as well reformulate in terms of a minimization problem which does not depend on m , and then guess and verify by means of the Bellman equation. Let us actually write out this, reorganizing (7) as

$$\left(m - \frac{\bar{\eta}}{\delta}\right) \int_0^\infty e^{-(r+\delta)t} \Theta(t) dt + \int_0^\infty e^{-rt} \underbrace{\Theta(t) e^{\delta \min\{0, \tau-t\}} \cdot \bar{\eta}/\delta}_{=: F(t)} dt + ke^{-r\tau} \quad (8)$$

We then have that F satisfies

$$dF(t) = F(t) \cdot \left[(\alpha - \delta 1_{t \in [0, \tau]}) dt + \sigma dZ(t) \right], \quad F(0) = \theta \bar{\eta} / \delta \quad (9)$$

For the minimization of the expectation of the latter two terms, the Bellman equation takes the form

$$-rV + \alpha qV' + \frac{1}{2} \sigma^2 q^2 V'' + q = \begin{cases} \delta qV' & \text{before intervention} \\ 0 & \text{after intervention} \end{cases} \quad (10)$$

The rest is routine: solve, find the strategy as a trigger q^* by a C^1 fit, and if one wishes a fully rigorous proof, do the limit transition to infinite horizon.

1.1.2 Remark. It is easy to see that (at least under the appropriate integrability conditions), the optimization will not depend of m even under the following generalizations:

- (a) The argument on M uses, essentially, only an integrating factor approach, i.e. linearity of the differential equation.
- (b) Under linearity, it does not matter whether M models the total stock of the pollutant, or if it models the project's contribution: suppose the latter, and denote everyone else's contributions by L also driven by a linear differential equation with the same decay rate, then we can either formulate in terms of the total $(L + M)$ (also following a linear differential equation with the same decay, though likely with different emission levels) or split up. In either case, the optimization will separate into a term not depending

on the decision and one not depending on L nor M (which enter only through the sum). In the problem of Proposition 1.1.1, it means that the optimization amounts to minimizing $k(\eta) \cdot [\gamma \min\{1, \theta/\hat{\theta}\} - \min\{1, \theta/\hat{\theta}\}^\gamma]/(\gamma - 1)$ – which is solved by choosing $\hat{\theta} = \theta^*$ according to (4).

- (c) We need not restrict Θ to be geometric Brownian. It can be any exogeneously given stochastic process which does not depend on M , and the optimal rule will still only depend on the history and future law of Θ (that is, its state if it is autonomous Markovian).
- (d) k can be allowed to depend on Θ as long as it does not depend on M .
- (e) The discount rate need not be constant nor deterministic, as long as it does not depend on M nor our control. In the above case, $e^{-rt}\Theta$ is geometric Brownian too, and we could therefore merge the discount factor into the Θ process – but then the intervention cost would have to be represented as a process too. That is however no issue:
- (f) The optimization problem need not be restricted to optimal stopping – we can replace the cost $ke^{-r\tau}$ by a cost *process* associated to $\eta(t)$ and Θ (possibly history-dependent), as long as this does not depend on M . △

Let us give an example of the latter.

1.1.3 Example. A work in progress by Jon Strand and this author [11] extends the Pindyck model by endogenizing the initial infrastructure, then subject to a utility and an investment cost. In particular, the initial investment will, as shown above, not depend on m . The paper also discusses extensions like endogenizing *timing* of the initial investment, and availability of emissions-reducing technology to be retro-fitted to the installation. As long as these quantities do not depend on m nor the subsequent development of M , then neither will those decisions. Also extending to a model with gradual build-up of infrastructure and subsequent reduction of emissions, will have decision rules not depending of M . △

The next section will formalize the property discussed, as well as more general models for which it carries over.

2 Linear evolutionary equations in a Banach space framework

We shall consider a more general setup than section 1. We replace the initial state m by a function H exogeneously given, unaffected by the control. This will cover e.g. delay equations for which the evolution depends on the past, in which case H is given initially as the history (hence the letter), the path $t \mapsto \{M(t)\}_{t \leq 0}$. Analogously, we replace the initial state θ for Θ , by an arbitrarily dimensional parameter denoted G – for example, this could be the history $t \mapsto \{\Theta(t)\}_{t \leq 0}$.

First some terminology:

2.0.4 Definition. We shall use the term «*does not depend on*» to mean invariance under partial shift, e.g. functional independence (contrasted to stochastic independence). Terms like «*might depend on*», «*will usually depend on*» and «*does depend on*» should be self-explanatory. \triangle

The property is now given slightly informally:

2.0.5 Definition. Consider a minimization problem indexed by $(G, H) \in \mathbb{G} \times \mathbb{H}$. We shall say that the problem *does not depend on* H if, for each $G \in \mathbb{G}$, the ordering of the controls according to performance does not depend on $H \in \mathbb{H}$. \triangle

We shall in practice look for decompositions of the form

$$\begin{aligned} & \text{[something which does not depend on the control]} \\ & + \text{[nonnegative functional of } H\text{]} \cdot \text{[functional which does not depend on } M\text{]} \end{aligned} \quad (11)$$

(where again, «does not depend on M » means functional independence, even when certain values for H may deduced from M without knowing the control). The first line will be analogous to the « $\theta m / (r + \delta - \alpha)$ » damage which incurs even without the pollutant source in question. Usually, the «nonnegative functional of H » will be a constant, however an example where it is not, will be given in (17)–(19). It should be noted that Definition 2.0.5 is weaker than the property that the *optimal strategy* not depending on H ; consider the example from the previous section, if one replaces $ke^{-r\tau}$ by some function nondecreasing in τ , then the optimal choice will be $\tau = 0$ for all $m \geq 0$, even when the M under the integral is replaced by any positive nondecreasing function of M .

2.1 Sufficient conditions for the optimization problem not to depend on history

The following simple application of the Banach fixed-point theorem essentially sums up why the linear cases behave as they do.

2.1.1 Lemma. Consider a Banach space \mathbb{M} , with a bounded linear operator $\Xi : \mathbb{M} \rightarrow \mathbb{M}$ such that some power is a contraction, and a linear functional $\Phi : \mathbb{M} \rightarrow \mathbb{R}$. Then for $X \in \mathbb{M}$, the unique solution M of the functional equation

$$M = X + \Xi M \quad (12)$$

$$\text{is } M = \Psi X, \quad \text{where} \quad (13)$$

$$\Psi := \mathbf{I} + \sum_{j=1}^{\infty} \Xi^j = (\mathbf{I} - \Xi)^{-1} \quad (14)$$

is a well-defined bounded linear operator from \mathbb{M} onto \mathbb{M} . Furthermore, $\Phi M = (\Phi \Psi)X$, a linear functional of X .

Assume a linear structure $X = A + H$, where A is a (fully or partially) controllable component (in dynamic systems, interpretable as controlling the future evolution) and H does not depend on the control chosen (interpreted as a given, uncontrollable «history»), and the following consequence is immediate:

2.1.2 Proposition. *Given a set $\mathbb{A} \subseteq \mathbb{M}$, a (possibly nonlinear) functional $\Gamma : \mathbb{A} \rightarrow \mathbb{R}$, a linear operator $\Xi : \mathbb{M} \rightarrow \mathbb{M}$ with some power being a contraction, and some fixed $H \in \mathbb{M}$. Then the problem*

$$\inf_{A \in \mathbb{A}} \{\Phi M + \Gamma(A)\} \quad \text{subject to} \quad (12) \text{ and } X = A + H \quad (15)$$

can be rewritten as

$$(\Phi\Psi)H + \inf_{A \in \mathbb{A}} \{(\Phi\Psi)A + \Gamma(A)\} \quad (16)$$

where the latter optimization problem does not depend on H .

Notice again that it does not matter whether M models the project's emissions or the total emissions, as long as the evolution is modelled by the action of a linear operator. Under linearity, it does not matter whether this «evolution» is actually in (univariate) time:

2.1.3 Remark. The motivating framework of section 1 concerned aggregate damage over time. However, there is nothing in Proposition 2.1.2 that precludes time–space aggregates; the vector M could be of arbitrary dimension, including space indexing dimensions, and the canonical model for the dissemination of a pollutant in physical space – the heat/diffusion equation – is of course linear. △

It is however crucial that the Ξ operator is not controlled:

2.1.4 Remark. Attempting to «fix» linearity by augmenting with more terms, will violate the crucial exogeneity of the Ξ operator, which we need to keep the first term of (16) outside the optimization. Let for example the model be $\dot{M} = \bar{\eta}1_{[0,\tau]} - \delta_2 M^2 - \delta_1 M$, and so that M is trapped in the unit interval. Then we attempt to introduce an infinite-dimensional linear model with coordinates $M_i(t) =$ the i th power of $M(t)$; it easily follows by induction that each \dot{M}_i can be written as a polynomial in M , hence as a finite linear combination of the coordinates. However, then $\dot{M}_2 = 2M\dot{M} = 2M_1\bar{\eta}1_{[0,\tau]} - \delta_2 M_3 - \delta_1 M_2$, and the first term makes the new infinite-dimensional Ξ dependent on control – and that ruins the non-dependence argument even if said dependence occurs only in coordinates which do not enter the running damage! △

Let us work out how to fit the model of Proposition 1.1.1 into the applicability of Proposition 2.1.2. The key is the contraction property established in the usual Picard–Lindelöf iteration to hold locally, and just as in that argument, we can apply the following piecewise:

- Our control A is now the *cumulative* emissions, the function $t \mapsto \int_0^t \eta(s) ds$. For the problem of Proposition 1.1.1, \mathbb{A} is the set of functions of the form $\bar{\eta} \min\{t, \tau\}$ for some stopping time $\tau \geq 0$.
- The H function is the constant m .
- Ξ takes as input the function $t \mapsto M(t)$ and returns the function $t \mapsto -\delta \int_0^t M(s) ds$.
- The aggregate damage is Φ , whose dependence on the initial condition $G = \theta \in [0, \infty)$ is notationally suppressed.

The «piecewise» version of this is carried out on the partition $0 = t_0 < t_1 < \dots$, where t_i defined as first time contraction failed in the previous step, by shrinking \mathbb{M} to the subspace¹ of functions with the known past $t \mapsto M(\min\{t, t_i\})$.

2.1.5 Remark. As mentioned in section 1, a strategy for flow pollutants will not depend on state. In the case where the flow incurs a cost – ecological damage or Pigouvian tax – then this is covered by the Γ functional, as it takes as input the entire emissions path and hence can depend on the time-derivative. Nevertheless, it could be of interest to consider a flow as a limit of a fast-decaying stock. One can then let Ξ represent a fast decay, and renormalize X . Letting $X = H + \frac{1}{\epsilon} \tilde{A}$ and $\Xi = \frac{1}{\epsilon} \tilde{\Xi}$, M is solved by

$$M = -(\tilde{\Xi} - \epsilon I)^{-1}(\epsilon H + \tilde{A})$$

in terms of the *resolvent* $(\tilde{\Xi} - \epsilon I)^{-1}$ of $\tilde{\Xi}$. If 0 is in the closure of the resolvent set, we can then let $\epsilon \rightarrow 0$ through an appropriate sequence. Using again the problem of Proposition 1.1.1 as example, we can take $\delta = 1/\epsilon$ and $\tilde{\eta} = \bar{\eta}/\epsilon$; then $\tilde{A} = -\tilde{\eta} \tilde{\Xi} 1_{[0, \tau]} = \tilde{\eta} \cdot ((\epsilon I - \tilde{\Xi}) 1_{[0, \tau]} - \epsilon 1_{[0, \tau]})$, so that

$$\begin{aligned} M &= \epsilon(\epsilon I - \tilde{\Xi})^{-1}(H + 1_{[0, \tau]}) + (\epsilon I - \tilde{\Xi})^{-1}(\epsilon I - \tilde{\Xi})\tilde{\eta} 1_{[0, \tau]} \\ &\rightarrow \tilde{\eta} 1_{[0, \tau]} \quad \text{as } \epsilon \searrow 0 \text{ i.e. as } \delta \rightarrow +\infty \end{aligned}$$

Thus in the limit, M is precisely the flow expressed as a limit of normalized stocks. △

3 Some nonlinear cases

Linearity turns out not to be a necessary condition for the property of Definition 2.0.5; we have merely used that H splits out additively upon application of the $(\Phi\Psi)$ functional. Furthermore, there are examples where not even this additivity holds, and where still the optimization does not depend on history. The question is rather, whether these are to be considered merely degeneracies. Of course, that is a matter of definition and opinion –

¹behind the scenes, the null element of this space is the history; indeed, we could have performed that translation by m in the first place

for example, one would likely consider it a degeneracy if one ad hoc, for the purpose of creating an example, restricts the set of controls in just in order to satisfy the requirement of Definition 2.0.5. Also, there are Φ functionals which do depend on H explicitly; for example one can construct a cancelling of H by $\Phi = \tilde{\Phi}(\mathbf{I} - \Xi)$, i.e. $\Phi X = \tilde{\Phi}M$, and $M \mapsto \tilde{\Phi}M$ need not depend on H . For example, in the language of Proposition 1.1.1, a functional that takes as input the path of $e^{\delta t}M(t) - M(0)$ will yield an expression which does not depend on $M(0) = m$. The integral criteria of sections 1 and 4 would not be prone to these kinds of constructed degeneracy, though.

The following will consider some cases which are nonlinear, but where the optimization still does not depend on initial stock. The first quadratic case is arguably the more «degenerate»:

3.1 A quadratic case

There turns out to be quadratic cases where the Γ cancels out the part which does *not* depend on H , leaving one which does, but in a way that might leave the *optimization* not depending. Suppose that the objective to be minimized is no longer linear, but involving a quadratic: $\Phi M + \langle M, M \rangle + \Gamma(A)$, for some suitable bilinear form $\langle \cdot, \cdot \rangle$. Suppose now the particular form where $\Gamma = \Gamma_0 - \Phi\Psi A - \langle A, A \rangle$, where Γ_0 does not depend on A . We still assume the linear $M = \Psi(A + H)$. Then the minimization problem becomes

$$\Phi\Psi H + \langle \Psi H, \Psi H \rangle + 2 \inf_{A \in \mathbb{A}} \langle \Psi A, \Psi H \rangle \quad (17)$$

and although the minimization usually depends upon H , there could be counterexamples if the dependence on H separates out as a single multiplicative factor. Let us again take the example from Proposition 1.1.1 as starting point. We modify the objective function (before applying the expectation) by replacing the linear integrand by the quadratic $e^{-rt} \Theta(t)(M(t))^2$ – as long as we assume $m \geq 0$, this is increasing in M – and then replace $ke^{-r\tau}$ by the functional

$$\Gamma = \bar{\eta}^2 \int_0^\infty e^{-(r+2\delta)t} \Theta(t) \left[\left(\int_0^t e^{\delta s} ds \right)^2 - \left(\int_0^{\min\{t,\tau\}} e^{\delta s} ds \right)^2 \right] dt \quad (18)$$

assuming r big enough to keep everything finite. It has some properties in common with the problem of Proposition 1.1.1; it is decreasing in τ , but positive whenever $\tau < \infty$ and $\theta > 0$. We can simplify the minimand to

$$\begin{aligned} & \int_0^\infty e^{-rt} \Theta(t)(M(t))^2 dt + \Gamma \\ &= \bar{\eta}^2 \int_0^\infty e^{-(r+2\delta)t} \Theta(t) \left(\int_0^t e^{\delta s} ds \right)^2 dt + 2m\bar{\eta} \int_0^\infty e^{-(r+2\delta)t} \Theta(t) \left(\int_0^{\min\{t,\tau\}} e^{\delta s} ds \right) dt \end{aligned} \quad (19)$$

and the minimization does not depend upon *nonnegative* m . Neither does the minimum, trivially obtained by $\tau = 0$. Again, we need to emphasize that establishing the «does not depend» property of Definition 2.0.5 is more modest than the claim that the corresponding property of the optimal strategy: $\mathbf{\Gamma}$ and $\mathbf{\Phi}$ could have a common minimand without Definition 2.0.5 applying. And, there could be other cases where there is a «corner solution» (i.e. for one-shot timing problems: such that the optimal τ is a.s. 0 or a.s. $+\infty$).

It seems tempting to guess that nonlinear $\mathbf{\Phi}$, will lead to an optimization problem depending on H except degenerate cases – for a suitable opinion on «degenerate». The next subsection will consider a class where the linearity condition is weakened.

3.2 Additivity over history or over control: worst-case scenario optimization and a connection to risk measures

Suppose that we are not optimizing a problem like (16), but, rather than with one fixed linear functional ($\mathbf{\Phi\Psi}$), we are given a family of functionals with the criterion being to optimize over the «worst-case». In the following, a range of functionals $\mathbf{\Lambda}$ will replace the single $\mathbf{\Phi\Psi}$, with you playing against a worst-case opponent $\mathbf{\Lambda} \in \mathbb{L}$. Modify the setup of Proposition 2.1.2 such that the objective is

$$\inf_{A \in \mathbb{A}} \sup_{\mathbf{\Lambda} \in \mathbb{L}} \{ \mathbf{\Lambda}(H + A) + \mathbf{\Gamma}_{\mathbf{\Lambda}}(A) \} \quad (20)$$

This criterion is, usually, nonlinear, but the optimization – the one wrt. A – need still not depend on H . One example, stated in the last part of the next Proposition, would be if a minimax theorem applies; reversing order and splitting off the $\mathbf{\Lambda}H$ term, we get – for each $\mathbf{\Lambda}$ – an optimization problem which does not depend on H . In that case, though, we need some structure on the $\mathbf{\Gamma}$ functional.

Of course we do not need $\mathbf{\Lambda}$ to be linear either, as long as $\mathbf{\Lambda}(H + A) = \mathbf{\Lambda}H + \mathbf{\Lambda}A$; this is reflected in the next Proposition, which is formulated somewhat ad hoc; the purpose of this subsection is to point out that there are such cases which – arguably – could be considered reasonable modelling criteria, not to give the full extent of those.

3.2.1 Proposition. *Consider the problem*

$$\inf_{A \in \mathbb{A}} \sup_{\mathbf{\Lambda} \in \mathbb{L}} \{ \mathbf{\Lambda}H + \mathbf{\Lambda}A + \mathbf{\Gamma}_{\mathbf{\Lambda}}(A) \} \quad (21)$$

where \mathbb{L} is a given family of functionals $\mathbf{\Lambda} : \mathbb{M} \rightarrow \mathbb{R}$ and A is our control. The functional $\mathbf{\Gamma}$ can depend on $\mathbf{\Lambda}$, although in part (i) below we will assume it does not:

- (i) If $\mathbf{\Lambda} \mapsto \mathbf{\Lambda}A$ is constant on \mathbb{L} for each A , and furthermore that $\mathbf{\Gamma}_{\mathbf{\Lambda}}$ is constant wrt. $\mathbf{\Lambda}$ for each A , the optimization problem (21) reduces to

$$\inf_{A \in \mathbb{A}} \{ \mathbf{\Lambda}_0 A + \mathbf{\Gamma}_{\mathbf{\Lambda}_0}(A) \} + \sup_{\mathbf{\Lambda} \in \mathbb{L}} \mathbf{\Lambda}H \quad (22)$$

for an arbitrarily chosen $\mathbf{\Lambda}_0 \in \mathbb{L}$.

(ii) If instead $\Lambda \mapsto \Lambda H$ is constant on \mathbb{L} for each H , then (21) reduces to

$$\Lambda_0 H + \inf_{A \in \mathbb{A}} \sup_{\Lambda \in \mathbb{L}} \{ \Lambda A + \Gamma_\Lambda(A) \} \quad (23)$$

again, for an arbitrarily chosen $\Lambda_0 \in \mathbb{L}$.

(iii) Alternatively, suppose that \mathbb{L} is convex and \mathbb{A} is convex and compact, and that $\Lambda \mapsto \Gamma_\Lambda(A)$ is upper semicontinuous and quasiconcave and $A \mapsto \Gamma_\Lambda(A)$ is lower semicontinuous and quasiconvex. Then (21) can be written

$$\sup_{\Lambda \in \mathbb{L}} \{ \Lambda H + \min_{A \in \mathbb{A}} \{ \Lambda A + \Gamma_\Lambda(A) \} \} \quad (24)$$

provided the inner minimum is attained.

Neither of these three optimization problems wrt. A , depend on H .

Proof. The first two parts are self-evident. The assumptions for part (iii), are those Sion's generalization of the celebrated von Neumann minimax theorem (cf. e.g. [14]). \square

It should be remarked that the convexity of \mathbb{A} could be a significant restriction; for the one-shot case, the set of functions of the form $A(t) = \bar{\eta} \min\{t, \tau\}$ is not convex, and one will have to extend the problem for a hope to apply part (iii).

The respective conditions of invariance wrt. Λ may at first glance seem artificial, but there are obvious examples where they are reasonable extensions of the model. For an obvious example: if current stock m (from everyone else's contributions) is the only thing uncertain and disagreed upon, then we are obviously in case (i). More generally, the invariance wrt. Λ , in particular the one in (ii), will show up if the optimization criterion is a certain special case of a *risk measure* as introduced in financial mathematics by Artzner and co-authors [1], [2] (the «coherent» case). Following the more general (convex) case of Föllmer and Schied [8], [9], we make the following definition, where the U is a «good» – such that $-U$ represents loss, cost or damage.

3.2.2 Definition. A *convex risk measure* ρ is a functional on a family of uncertain outcomes U , satisfying the below properties, where a is real-valued and constant:

- (i) Translation invariance over constants: $\rho(U + a) = \rho(U) - a$, all real a .
- (ii) Monotonicity: If $U_1 \geq U_0$ then $\rho(U_1) \leq \rho(U_0)$.
- (iii) Convexity: $\rho(aU_0 + (1 - a)U_1) \leq a\rho(U_0) + (1 - a)\rho(U_1)$, all $a \in [0, 1]$.

Furthermore, a convex risk measure is called *coherent* if (iii) is strengthened into (iv)–(v):

- (iv) Positive homogeneity: $\rho(aU) = a\rho(U)$ for all $a \geq 0$.
- (v) Subadditivity: $\rho(U_0 + U_1) \leq \rho(U_0) + \rho(U_1)$. \triangle

The «uncertain outcomes» are uncertain in the Knightian sense: there is no ex ante given probability space, and the probabilistic representation to follow below – representing Knightian uncertainty through Knightian risk – is a theorem and not an assumption. However, the definition assumes constants to be known, hence null sets, which also explains the rôle of the below dominating reference measure \mathbf{P}_0 – which nobody needs to actually believe in, apart from the sigma-ring of null sets it generates.

The theory of risk measures establishes a connection between the quantification ρ , and sets of *acceptable risks*, not unlike preferences represented by utility functions. For convex risk measures, the acceptance sets are convex (analogous to convex preferences), by [8, Proposition 2]. Under suitable regularity conditions on the family of null sets, there is a representation theorem, [8, subsection 2.2], where any convex risk measure ρ can be represented in terms of varying probability measures as follows, where $\mathbf{E}_{\mathbf{P}}$ denotes expectation wrt. the \mathbf{P} measure:

$$\rho(U) = \sup_{\mathbf{P} \in \mathbb{P}} \{ -\zeta(\mathbf{P}) + \mathbf{E}_{\mathbf{P}}[-U] \} \quad (25)$$

for some family \mathbb{P} of probability measures which are absolutely continuous wrt. some given common \mathbf{P}_0 , and some extended real functional $\zeta : \mathbb{P} \rightarrow (-\infty, +\infty]$ with $\zeta(\mathbf{P}) \geq -\rho(0)$, all $\mathbf{P} \in \mathbb{P}$. Further properties of \mathbb{P} and ζ are well established: under mild regularity conditions, ζ can be chosen to be convex and lower semicontinuous; \mathbb{P} can be chosen as the full family of \mathbf{P}_0 -absolutely continuous measures; and if ρ is also coherent, we can instead restrict \mathbb{P} in a way that admits $\zeta \equiv 0$ (this by putting $\zeta = +\infty$ outside the restriction).

Compare now to the various cases of Proposition 3.2.1: For part (ii), note the similarity in form of the braced expressions of (25) and (23) – though the properties of $\mathbf{P} \mapsto -\zeta$ need not in general apply to $\mathbf{\Lambda} \mapsto \mathbf{\Gamma}$; should the latter however be concave and upper semicontinuous, we are in the scope of part (iii). For part (i), note in the coherent case ($\zeta = 0$) the similarity to the rightmost term of (22). We have the following observation which connects a special case of coherent risk measures to the nondependence of history in the problem of Proposition 3.2.1:

3.2.3 Remark. Suppose that each $\mathbf{\Lambda}$ is a negative expectation $-\mathbf{E}_{\mathbf{P}}$ and that $\mathbf{\Gamma}$ does not depend on $\mathbf{\Lambda}$, i.e. not on \mathbf{P} . With the $\sup_{\mathbf{\Lambda}} \{ \mathbf{\Lambda}H + \mathbf{\Lambda}A \}$ of (21) being a coherent risk measure, make the additional assumption that part (i) of the definition admits strengthening from *translation invariance over constants* to *additivity over the damage $\mathbf{\Lambda}H$ from the history*. Then (20) does admit the form (21), and

$$\inf_{A \in \mathbb{A}} \sup_{\mathbf{\Lambda} \in \mathbb{L}} \{ \mathbf{\Lambda}(H + A) + \mathbf{\Gamma}_{\mathbf{\Lambda}}(A) \} = \inf_{A \in \mathbb{A}} \{ \mathbf{\Gamma}(A) + \rho(H) + \rho(A) \} \quad (26)$$

where the latter optimization does not depend on H . △

Thus if the criterion is to minimize a functional where the ecological damage can be written as a coherent risk measure, then we need a bit more – but, it could be argued, not drastically more – structure to obtain the non-dependence.

There is a canonical interpretation to the representation (25). A coherent risk measure corresponds to evaluating risk by picking the worst-case among a fixed set of probability measures (interpretable as «worst of the reference scenarios», or «most pessimistic committee member’s expectation»); a convex risk measure also introduces a penalty ζ , interpretable as a trade-off to account for the scenarios not being equally credible. Applying this thinking to Proposition 3.2.1, we can give the cases the following respective interpretations:

- (i) Proposition 3.2.1 part (i) fits a case where the expected damage from our actions – for a given strategy – is the same in all scenarios; everyone agrees on the expectation of this quantity. Fitting this to the case of Proposition 1.1.1, it would mean that there should not be any disagreement over the probability distribution of Θ , as that would affect the damage from our project; however, with m being the stock from everyone else’s activity, that could be unknown. Polling a panel of experts to each give their expected value, does not affect the optimization, only the optimal value. Notice that this interpretation is completely at odds with the «small agent» model, where m is *this* project’s history and Θ models the marginal cost, a result of everyone else’s actions.
- (ii) Proposition 3.2.1 part (ii) fits a case where the expected damage from the history is the same in all scenarios, but the expected damage or cost from the infrastructure under consideration, is not. It does not seem natural to consider this option in the problem of Proposition 1.1.1, but example 3.2.4 will elaborate on a case where such a criterion could be employed for the optimization of the initial infrastructure like in [11].

Before the specific example, let us make some more general considerations: Specialize first to the expectation form, where for each \mathbf{P} – the class being such that all members are absolutely continuous wrt. some common probability measure \mathbf{P}_0 – we have

$$\begin{aligned} \Phi M &= \mathbf{E}_{\mathbf{P}} \left[\int_0^{\infty} e^{-rt} \Theta(t) M(t) dt \right] \\ &= \mathbf{E}_{\mathbf{P}} \int_0^{\infty} e^{-rt} \Theta(t) \Psi A(t) dt + \mathbf{E}_{\mathbf{P}} \int_0^{\infty} e^{-rt} \Theta(t) \Psi H(t) dt \end{aligned} \quad (27)$$

and assume that all $\mathbf{P} \in \mathbb{P}$ agree on the rightmost term, i.e. the expected future damage from the history. Consider the following criterion:

$$\inf_{A \in \mathbb{A}} \sup_{\mathbf{P} \in \mathbb{P}} \left\{ \mathbf{E}_{\mathbf{P}} \int_0^{\infty} e^{-rt} \Theta(t) M(t) dt + \Gamma_{\mathbf{P}}(A) \right\} \quad (28)$$

Even without linearity, the invariance over history yields sufficient structure to write

$$\mathbf{E}_{\mathbf{P}_0} \int_0^\infty e^{-rt} \Theta(t) \Psi H(t) dt + \inf_{A \in \mathcal{A}} \sup_{\mathbf{P} \in \mathbb{P}} \left\{ \mathbf{E}_{\mathbf{P}_0} \int_0^\infty e^{-rt} \Theta(t) \Psi A(t) dt \frac{d\mathbf{P}}{d\mathbf{P}_0} + \Gamma_{\mathbf{P}}(A) \right\} \quad (29)$$

Moving the Radon–Nikodým derivative under the integral sign and applying t -conditional expectation (where the integrand is assumed adapted to the filtration), we can put $Q(t) = \mathbf{E}_t[d\mathbf{P}/d\mathbf{P}_0]$. If the Γ functional admits of a similar expected integral form, we can make the similar operation, and formulate as a sequential optimal control problem formulated under a single measure, namely \mathbf{P}_0 (despite that nobody needs actually believe in it). The next example will be a bit simpler, however.

3.2.4 Example. In the Framstad and Strand [11] extension to the model of Proposition 1.1.1, there are two choice variables. One is the initial infrastructure, represented by its emission level $\bar{\eta}$, chosen subject to an immediate cost of setting up / benefit from the infrastructure $k_0(\bar{\eta})$ – we assume no uncertainty about the benefit from the infrastructure. The other is the time τ to cease emissions. The model is otherwise quite similar to the one of Proposition 1.1.1, where Θ is geometric Brownian – let us assume that everyone agrees it will be that kind of process. Now the expected damage from current stock, from (7), is $\Theta_0 m / (r + \delta - \alpha)$. Let us imagine that all those five parameters are known and agreed upon, but that the volatility σ^2 is unknown and disagreed upon; notice that this uncertainty would unwind immediately as Θ starts, as volatility can be calculated from any positive time-interval. Let us therefore assume that the next observation of Θ is at a future deterministic time, where a continuous observation of the time series will start, and that implementation of the cleaning technology will certainly not take place before that date. Shifting time, call that date zero, so that we now are at time $-T$.

Suppose now that starting from time 0, then τ will be optimized according to the scenario which as been revealed, but the initial choice of $\bar{\eta}$ should be made in order to minimize total damages and cost for the worst-case scenario to be revealed at time zero. As the optimization problem does not depend on the current stock, we drop out the $m\Theta_{-T}/(r + \delta - \alpha)$ term; we shall optimize

$$\inf_{\bar{\eta}} \left\{ k_0(\bar{\eta}) + \bar{\eta} e^{-rT} \sup_{\mathbf{P} \in \mathbb{P}} \mathbf{E}_{\mathbf{P}} \left[\int_0^\infty e^{-(r+\delta)t} \Theta(t) \left(\int_0^{\min\{t, \tau_{\mathbf{P}}\}} e^{\delta s} ds \right) dt + k e^{-r\tau_{\mathbf{P}}} \right] \right\} \quad (30)$$

where $\tau_{\mathbf{P}}$ is the optimized stopping time. \mathbf{P} can however be indexed by the possible values of σ^2 , and we pick the one that yields the highest value function (5). The aggregated damages and costs if running forever, do not depend on σ , and can be reduced by an *option* to stop – this reduction term increases in σ^2 , so the smallest possible choice yields the worst-case (5) – in particular, if all σ^2 could be arbitrarily small, the optimization wrt. $\bar{\eta}$ should assume the *deterministic* scenario. In [11], comparative statics wrt. volatility are considered for more general cost structures, assuming k_0 decreasing and convex. Lower volatility is shown to reduce both initial emission level $\bar{\eta}$ and the trigger value θ^* , as well as the expected total emissions, provided that the implementation cost $k = k(\bar{\eta})$ has elasticity between 1 and $\gamma/(\gamma - 1)$. △

4 Linear (stochastic) differential equations.

This section concerns cover the case where M is governed by a stochastic differential equation. The standard existence and uniqueness result is a Picard–Lindelöf argument, which of course applies under linearity, so it is a special case of Proposition 2.1.2, but if we want to *solve* the problem (15) from the representation (16), we would want to write out the latter. Notice though, that it is not necessarily desirable to work with (16) – especially as Proposition 2.1.2 provides us with the information that we can look for an optimal rule not depending on the state of M . The next subsection will cover the semimartingale Itô differential case, and then other integrals will be sketched in subsection 4.2.

4.1 Itô stochastic differential equations

Let us fix the setup and notation for this subsection:

- We shall work on a (notationally suppressed) usual stochastic basis; namely, a probability space equipped with a right-continuous filtration complete at time zero.
- Vectors are column vectors, unless indicated by the transposition superscript \dagger . The symbol \cdot denotes the Euclidean inner product on \mathbb{R}^d , but will be used for products between scalars as well.
- For stochastic processes, denote by superscript c for continuous part, and for discontinuities: $\Delta^+Y = Y(t^+) - Y(t)$, $\Delta^-Y = Y(t) - Y(t^-)$. We will use Δ^\pm for formulas valid for both, and merely Δ when the interpretation is unambiguous. Furthermore, we use accents \dot{Y} , resp. \check{Y} for the left-continuous version, resp. right-continuous version of Y .
- The reader should be aware that as matrix products do not commute, notation like $dII(t)M(t)$ may be necessary even when M will be part of the integration.
- Differentials denote Itô type integration.

We now specify the objective function; we assume that the optimization problem is to minimize the expected value of the functional

$$\int_0^\infty \left(M(t^-)^\dagger dD(t) + dC(t) \right). \quad (31)$$

where we shall, ad hoc, assume integrability. Since we are in a multidimensional setting, the discount factor e^{-rt} has been incorporated into the processes D and C (mnemonics: *D*amage from the pollutant, *C*ost of control).

The modelling building blocks are the following entities

Measurability/continuity assumptions on the processes. We assume that all processes are adapted, and their sample paths possess both left and right limits. In addition, the following are standing assumptions:

- (a) The \mathbb{R}^d -valued process D , assumed right-continuous, is an exogeneously given (uncontrolled) process which aggregates the environmental damage of the pollutant stock. dD specializes Φ but generalizes $e^{-rt}\Theta(t) dt$.
- (b) M will be the pollutant stock, which we can affect through a predictable, hence assumed left-continuous, control denoted S ; we introduce this for the sake of interpretation, although we will not write down the explicit way it enters. $t \mapsto M$ need not be left nor right continuous. We shall introduce a driving process X for M , and the following measurability/continuity conditions will apply for X and for M :
 - On intervals where S is constant, M will be assumed right-continuous.
 - At discontinuity times T for S – henceforth *interventions*, M , does not affect $M(T)$, only $M(T^+)$. In other words, we have that $M(T)$ does not depend on $\Delta^+S = S(T^+) - S(T)$, which may in turn depend on the history up to and including T (the T -measurability of this difference is the assumed predictability), and which affects Δ^+M . Due to the assumed right-continuity of the filtration, $M(T^+)$ is T -measurable; at time T , we know our intervention Δ^+S , and there is no randomness drawing the right limit.
- (c) The process C – specializing the Γ functional but generalizing $ke^{-r\tau}$ – is the incurred cost of the control, allowed to depend on the history (subject to assumptions specified below), in particular, the entire history of S , but we shall below assume it does not depend on H . As C is only an integrator for the continuous discount factor, we do not need to worry about left-hand or right-hand jumps, as discontinuities contribute only through $C(T^+) - C(T^-)$. We can therefore work with any version.

The dynamics for M will in this subsection be assumed to obey the following form (in terms of transposes, to get the differential postmultiplied):

$$dM^\dagger(t) = M^\dagger(t^-) d\Xi^\dagger(t) + dX^\dagger(t), \quad M(0) = m \quad (32)$$

where H is fully represented through m , and the Ξ functional is specified as integration wrt. the given $\mathbb{R}^{d \times d}$ -valued right-continuous process Ξ . In order to fit to the setup, put $X(0) = m$ and $A = X - m$; then the \mathbb{R}^d -valued process $X - m$ can be influenced by the control (but shall not depend on M nor m).

We then have the following:

4.1.1 Proposition. *Suppose that for each given control S , the following holds: Ξ , D , X and C are given semimartingales, the two first right-continuous, and that the jumps satisfy, with probability 1,*

$$\Delta\Xi\Delta X \in \text{column space}(I + \Delta\Xi), \text{ all jumps.} \quad (33)$$

Suppose furthermore that M uniquely (up to version) solves the (Itô) stochastic differential equation (32). Then there exists some \mathbb{R}^d -valued semimartingale Π , given by

$$\Pi(0) = I, \quad d\Pi(t) = (d\xi(t)) \Pi(t^-) \quad (34)$$

²such that (31) equals, if at most one integral diverges,

$$m^\dagger \int_0^\infty \Pi(t)^\dagger dD + \int_0^\infty \left(\left[\Delta^+ X(0) + \int_{(0,t]} \Pi(s^-)^{-1} d(\dot{X}(s) - Y(s)) \right]^\dagger \Pi(t)^\dagger dD + dC(t) \right) \quad (35)$$

where Y is a right-continuous process such that, in terms of Itô differentials,

$$dY^c = d\xi^c dX^c, \quad (I + \Delta\xi)\Delta Y = \Delta\xi\Delta\dot{X} \quad (36)$$

In particular, if ξ and D do not depend on S , then the minimization over S does not depend on M .

Proof. Notice first that (33) ensures that (36) can be satisfied even when $\Delta\xi$ has an eigenvalue of -1 . Now M enters directly the objective only through the left-continuous version. We can therefore first do the differential calculus on the right-continuous version \dot{M} , which satisfies (32) except at intervention times. Notice that $\dot{M}(0) = m + \Delta^+ M(0) = m + \Delta^+ X(0)$, and $\Delta^+ X(0)$ depends solely on $\Delta^+ S(0)$. We claim that

$$\dot{M}(t) = \Pi(t)m + \Pi(t) \left[\Delta^+ X(0) + \int_{(0,t]} \Pi(s^-)^{-1} d(\dot{X}(s) - Y(s)) \right] \quad (37)$$

To see this, differentiate using the Itô formula. Suppressing time arguments,

$$d\dot{M} = (d\xi)\dot{M} + \Pi\Pi^{-1} d(\dot{X} - Y) + (d\xi)(d(\dot{X} - Y)) \quad (38)$$

where the latter term is the cross-variation expressed as Itô differentials. Now cancel terms using (36). \square

4.1.2 Remark. A few comments are appropriate.

- (a) The restriction (33) applies in the case where the jump amplitude could possibly have -1 as eigenvalue (in which case, at the eigenvector, ξ would cause a jump to null). It limits the possible jumps X could make at the same time. The condition will be satisfied if, at any jump time T , then if a coordinate of M jumps to a state not depending on $M(T^-)$, then this new post-jump state is zero. It thus covers cases where the pollutant should vanish at a jump (say, if we are modelling a case where some exogeneous agent could at some point T choose to clean up).

²form with time-differentials postmultiplied: $(\Pi^\dagger d\xi^\dagger)^\dagger$

- (b) The model and result admits cases where we actually intervene in the pollution stock, but only through X – that is, in absolute numbers, not in percentages. The intervention does not depend on the level; if damages and costs are so that it pays off to remove 1 unit of the pollutant, then that decision does not depend on the stock level. This may be objectionable when M models cardinal level, as it could bring M outside the first orthant – and capping the cleansing operation to keep stocks nonnegative, would mean that the strategy takes the state of M into account. This objection does however not apply to a discrete emission which instantly *increases* one or more coordinates of M .
- (c) These discrete interventions in X may seem to be not captured in the above argument, but no information is lost. Even if the proof only uses the left- and right-continuous versions, then $\Delta^+ S(t)$ could in principle be based on the observation of $M(t)$ as well (as that is measurable), and not merely the left limit. Without welfare loss, it *will* actually not depend on $M(t)$ – that property is now proven, not merely assumed.
- (d) The dynamics of M can depend on D , but not the other way around; if D depends on our control then the m -dependent part may of course also do so. M and D may however be driven by common given processes – just augment D with these, and augment M with zero-valued coordinates to match the dimension for the dot product.
- (e) Even though D is assumed semimartingale, it is a generalization of the «no assumptions needed» Θ of Remark 1.1.2(c). Recall that $D(t)$ does not correspond $\Theta(t)$, but to $\int_0^t e^{-rs} \Theta(s) ds$.
- (f) Proposition 4.1.1 covers linear stochastic difference equations. To those who have only familiarized themselves with the stochastic integral with respect to Brownian motion and then maybe with respect to Lévy motions, Markov chains as Itô diffusion-type processes may look as a bit of an odd approach. However, the semimartingale concept does not require jump-times to be random, and processes could very well be constant between integer times – all such processes are in fact semimartingales, as long as they are adapted. △

The linearity of the evolutionary operator covers higher-order differential equations for M . Assuming ad hoc stability and generality, then the optimal strategy still does not depend on M if the model of Proposition 1.1.1 is modified to allow for an n th order linear differential equation

$$\sum_{i=0}^n \delta_i \cdot \left(\frac{d}{dt} \right)^i M(t) = \bar{\eta} \cdot 1_{t \in [0, \tau]} \quad \text{with} \quad M \in C^{n-1} \cap C^n([0, \infty) \setminus \{\tau\}) \quad (39)$$

In the following example, we shall cover the stable non-oscillating case with $n = 2$.

4.1.3 *Example.* Consider the model of Proposition 1.1.1, except that M obeys equation (39) with $\delta_2 = 1$ and $\delta_1 > 2\sqrt{\delta_0}$, $\delta_0 \geq 0$. With characteristic roots

$$\lambda_1 = -\frac{\delta_1}{2} - \sqrt{\frac{\delta_1^2}{4} - \delta_0} < \lambda_2 = -\frac{\delta_1}{2} + \sqrt{\frac{\delta_1^2}{4} - \delta_0} \quad (40)$$

and initial data $M(0) = m$, $\dot{M}(0) = \mu$, we get

$$M(t) = \frac{\bar{\eta}}{\delta_0} + \frac{1}{\lambda_2 - \lambda_1} \left(\left[\lambda_2(m - \frac{\bar{\eta}}{\delta_0}) - \mu \right] e^{\lambda_1 t} - \left[\lambda_1(m - \frac{\bar{\eta}}{\delta_0}) - \mu \right] e^{\lambda_2 t} - \frac{\bar{\eta}}{\delta_0} \left\{ \lambda_2(1 - e^{\lambda_1 \min\{0, t - \tau\}}) - \lambda_1(1 - e^{\lambda_2 \min\{0, t - \tau\}}) \right\} \right) \quad (41)$$

We see that the m and μ terms split out linearly in a way that does not depend on emissions. However, with $\bar{\eta}$ given, we can just as well split out the entire first line, which does not depend on τ . The first line yields the damage

$$\begin{aligned} & \frac{\theta \bar{\eta}}{\delta_0(r - \alpha)} + \frac{\theta}{\lambda_2 - \lambda_1} \left(\frac{\lambda_2(m - \frac{\bar{\eta}}{\delta_0}) - \mu}{r - \lambda_1 - \alpha} - \frac{\lambda_1(m - \frac{\bar{\eta}}{\delta_0}) - \mu}{r - \lambda_2 - \alpha} \right) \\ &= \left(\frac{\bar{\eta}}{(r - \alpha)} + (r - \lambda_1 - \lambda_2 - \alpha)m + \mu \right) \cdot \frac{\theta}{(r - \lambda_2 - \alpha)(r - \lambda_1 - \alpha)} \end{aligned} \quad (42)$$

while the contribution from the rest, including intervention cost, is

$$ke^{-r\tau} - \frac{\bar{\eta}}{\delta_0(\lambda_2 - \lambda_1)} \int_0^\infty e^{-rt} \Theta(t) \left\{ \lambda_2(1 - e^{\lambda_1 \min\{0, t - \tau\}}) - \lambda_1(1 - e^{\lambda_2 \min\{0, t - \tau\}}) \right\} dt \quad (43)$$

which has expectation

$$\mathbf{E} \left[ke^{-r\tau} - \frac{\bar{\eta}}{\delta_0} e^{-r\tau} \Theta(\tau) \int_0^\infty e^{-(r-\alpha)t} \left\{ 1 + \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_2 - \lambda_1} \right\} dt \right] \quad (44)$$

where we have used the strong Markov property to perform the time-change, and the multiplicative form of the geometric Brownian motion. Again, by the strong Markov property and the continuity of the gBm, it suffices to consider stopping times of the form $\hat{\tau} =$ first hitting time for $[\hat{\theta}, \infty)$, and then one can optimize over $\hat{\theta}$:

$$\begin{aligned} & \left(k - \hat{\theta} \cdot \frac{\bar{\eta}}{\delta_0} \cdot \left\{ \frac{1}{r - \alpha} - \frac{r - \lambda_1 - \lambda_2 - \alpha}{(r - \lambda_2 - \alpha)(r - \lambda_1 - \alpha)} \right\} \right) \cdot \mathbf{E}[e^{-r\hat{\tau}}] \\ &= \left(k - \frac{\bar{\eta} \hat{\theta}}{(r - \alpha)(r - \lambda_1 - \alpha)(r - \lambda_2 - \alpha)} \right) \cdot \min\{1, (\theta/\hat{\theta})^\gamma\} \end{aligned} \quad (45)$$

with γ given by (4). The minimizer is

$$\theta^* = \frac{\gamma}{\gamma - 1} \cdot \frac{k(\bar{\eta})}{\bar{\eta}} \cdot (r - \alpha)(r - \lambda_1 - \alpha)(r - \lambda_2 - \alpha) \quad (46)$$

– compare to (4) again – so that (45) becomes $-\min\{1, (\theta/\theta^*)^\gamma\} \cdot k/(\gamma - 1)$. To get the value function, and on a form comparable with (5), we add the (42) contribution:

$$\frac{(r - \lambda_1 - \lambda_2 - \alpha)m + \mu}{(r - \lambda_2 - \alpha)(r - \lambda_1 - \alpha)}\theta + k(\bar{\eta}) \cdot \begin{cases} \left[\gamma \frac{\theta}{\theta^*} - \left(\frac{\theta}{\theta^*}\right)^\gamma\right]/(\gamma - 1) & \text{if } \theta < \theta^* \\ 1 & \text{if } \theta \geq \theta^*. \end{cases} \quad (47)$$

The first term is the damage which incurs with or without the project in question, and we see that the contribution from the optimized project has precisely the same form, except with a modified formula for the optimal trigger level θ^* . Observe that λ_1 and λ_2 are both negative, so the condition $r > \alpha$ ensures that everything converges. \triangle

4.2 Some considerations beyond the Itô integral

The previous subsection employed the standard stochastic calculus setup: the Itô integral wrt. semimartingales. The approach does however apply to other integral concepts as well.

Let us first point out that *fractional integrals* – whether they are of Erdély–Kober type (unifying and generalizing both the Weyl and Riemann–Liouville types, see [18]) or Hadamard type – are linear and can be covered by the form (12). Fractional differential equations have been proposed to model anomalous diffusion («diffusion» here meaning the physical phenomenon, e.g. particle flows in hydrology), see e.g. [5]. Another use is to allow for non-semimartingales, e.g. the well-known fractional Brownian motion, as a driving noise in differential equations. The brief exposition on fractional calculus in what follows, is intended to facilitate the latter – the Stratonovich type and Hitsuda–Skorohod / Wick–Itô type integrals are valid also for the semimartingale framework as a special case.

An example: Fractional Brownian motion The *fractional Brownian motion* (fBm) $X^{(h)}$ of Hurst parameter $h \in (0, 1)$ is a Gaussian process with zero mean, and covariance structure (for the univariate case) $\mathbf{E}[(X^{(h)}(T) - X^{(h)}(t))^2] = |T - t|^{2h}$. We shall work with the continuous-path version (ensured by the Kolmogorov continuity theorem). The term was coined by the seminal paper [17], defining it as the $(h - 1/2)$ -order Weyl fractional integral (/derivative) of ordinary Brownian motion X , as, for $h \neq 1/2$,

$$(\text{constant}) \times \int_{-\infty}^t \left[(t - s)^{h-1/2} - (\max\{0, -s\})^{h-1/2} \right] dX(s) \quad (48)$$

but fBm also admits finite-memory Erdély–Kober representations (e.g. [6]). fBm has negatively correlated increments for $h < 1/2$. For $h > 1/2$ it has positively correlated increments, and the *long memory* property that the covariance of the increments $X^{(h)}(1) - X^{(h)}(0)$ and $X^{(h)}(T) - X^{(h)}(1)$ diverges to $+\infty$ with T . The long memory has been a rationale to consider it as a model for various phenomena, including finance; however, not being semimartingale, it leads to arbitrages (i.e. riskless free lunches) in frictionless markets with continuous trading. Among the vast literature on the topic, see e.g.

Rogers' article [21], where he not only establishes an arbitrage strategy, but also how to fit the same long memory property into an arbitrage-free semimartingale model. Fractional Brownian motion has also been used in the modelling of pollution, e.g. [12].

The non-semimartingale property means that fractional Brownian motion as an integrator, behaves somewhat different from ordinary Brownian motion. We mention a few cases suited to allow these kinds of processes as integrators.

The integrals of Young and Stratonovich and beyond. For stochastic analysis wrt. Brownian motion, there is the well-known Stratonovich integral, formalized by choosing the midpoint time for the integrands, taking limits of sums $Y(\frac{1}{2}(t_{i+1} + t_i))(X(t_{i+1}) - X(t_i))$. The Stratonovich integral admits an ordinary chain rule, without second-order terms. It turns out that if the driving process is continuous (discontinuities may be handled jump-by-jump) and with paths of zero quadratic variation, the Itô and Stratonovich integrals coincide, and equal the Young integral, which is, in some sense, the only continuous pathwise integral under this regularity. The Stratonovich integral thus extends the Young integral while keeping its (ordinary) chain rule. If we assume continuous sample paths and ordinary chain rule in Proposition 4.1.1, we put $Y = 0$ and delete ΔX and cross terms. Even though the optimization problem could be cumbersome, we know that the state and history of M need not be taken into account, reducing the dimensionality of the optimization problem.

Further generalizations can be given through the theory of *rough paths*, see e.g. [16], and the linear differential equations that arise with those integrals, admit existence/uniqueness by Picard-type iteration.

The Wick–Itô and Hitsuda–Skorohod type integrals. Originating from white noise theory, these integrals are defined on distributions spaces, wherein Brownian motion is actually differentiable. The Wick–Itô formulation does in fact use the time-derivative of Brownian motion, allowing a Riemann-sum based integral (technically defined in the Bochner or even Pettis sense), written as

$$\int_0^T Y(t) \diamond \dot{X}(t) dt \quad (49)$$

where the \diamond denotes the (associative, commutative) so-called *Wick product*. This is not a pathwise (« ω -wise») integral, as the Wick product is not a product between the *realizations* of random variables, but a product of probability *distributions* (somehow in the sense that the convolution product is); it has the property that the expectation of a Wick product, is the product of expectations. Furthermore, the Wick–Itô integral admits a Wick product-version of the ordinary chain rule, on a certain closure of the set of Wick polynomials $\sum c_i U^{\diamond i}$ (for example, the Wick exponential $\exp^\diamond(X(t)) = 1 + \sum_{i \in \mathbb{N}} X(t)^{\diamond i} / i!$ will have the time-derivative $\dot{X}(t) \diamond \exp^\diamond(X(t))$). References for white noise theory with applications to

Wick–Itô differential equations include the book [13], and for fBm, [7].

Let us simply perform the formal algebraic manipulation using the Wick-type integral, to see the consequences for the model – assuming for simplicity continuous sample paths. First, we need a Wick-integrating factor: $\Pi = \exp^\diamond(\Xi)$. Then M becomes

$$M(t) = \Pi(t) \diamond m + \Pi(t) \diamond \int_0^t \Pi(s)^{\diamond(-1)} \diamond dX(s) \quad (50)$$

$$= \exp^\diamond(\Xi(t)) \diamond m + \exp^\diamond(\Xi(t)) \diamond \int_0^t \exp^\diamond(-\Xi(s)) \diamond \dot{X}(s) dt \quad (51)$$

Now the first term goes outside the minimization. Let us assume that we are again in a one-shot model where $\dot{X}(t)$ can be written $\bar{\eta}1_{[0,\tau]}$. If the damage functional is on Wick form – the differential being $\diamond dD = \diamond \Theta dt$ – then we are in a sense lucky, as we then have a pure Wick formulation, and one might apply expectations first and optimization afterwards. However, mixing the ω -wise product and the Wick product, will usually lead to intractabilities, and converting back and forth is certainly not trivial. For example, if we have pathwise differential dD , we would want to calculate the probability distribution $\exp^\diamond(\Xi(t) - \Xi(s)) \diamond 1_{[0,\tau]}$ and evaluate at ω ; the author is not aware of any tractable way to do this for general stopping times τ . And without doing this evaluation, we only have a distribution, not a response to path; without evaluation, the Wick product does not state how to respond to observation. Thus, the *modelling choice* at each «product» occurring in the model – Wick-type or ω -wise type – has nontrivial consequences to model behaviour. The linearity is still key to the property of Proposition 2.1.2 though, and the knowledge that the optimization can be carried out without regard to H could potentially help making the problem tractable.

The Wick–Itô integral is often employed in *anticipative* stochastic calculus – for example, in (51), the \diamond in front of the m allows it to be random, and it could even depend on future states of M , as long as we do not have the opposite (functional) dependence. Indeed, there are other integral concepts designed for anticipative stochastic calculus, see for example [15] for a fairly recent one.

Stochastic partial differential equations As pointed out in Remark 2.1.3, the theory of Proposition 2.1.2 covers (linear) time–space evolution modeled by the heat equation. The dissemination of pollutant in space could also be subject to randomness. Such models could be hard to accommodate under ordinary stochastic calculus, as one could easily encounter models where one would want multi-parameter Brownian motion and its second-order derivative. However, there is a well-developed theory based on white noise analysis, using the Wick–Itô approach – potentially leading to the same difficulties as for optimal stopping, in converting the model to one for response to actual observations. Again, see the book [13].

5 Closing remarks

For linear models for the decay of pollution, as considered in this paper, the optimization can be carried out without regard to the stock. This is a property often assumed as a valid approximation for idealized infinitesimal agents, but under linearity it holds exact regardless of size. The result admits general damage functionals as long as they do not depend on M explicitly and so that the functionals themselves are not controllable, and are linear or possess the appropriate additivity property.

In Itô stochastic differential equation models, the impact cost factor can even covariate with M in terms of Itô differentials, and the dynamics for M could depend on dD or Θ . Covariating Itô differentials seems natural from a small agent point of view, when the impact could in reality depend on the aggregate stock; then upwards fluctuations (or jumps) in the aggregate stock could cause Θ to increase, while it would still be a reasonable approximation to disregard a small agent's contribution to this effect. But the reverse causality should definitely be allowed in a multi-agent model, where there would be a feedback from the Θ level to the agents' behaviour, and the « θ^* » triggers will vary over the agents' cost structures represented by k or more generally the $\mathbf{\Gamma}$ functional.

For future research, the non-dependence result could make it easier to guess a solution form for problems with linear models, and then fit and verify by the tool of choice, e.g. dynamic programming. Furthermore, the reduction of dimensionality might be helpful for numerical solutions. The risk measure-alike criteria, on the other hand is a different issue, with possible room for modelling of heterogeneous beliefs; a question for the applicability of the representation herein, is when the additivity property is a realistic modeling assumption. And, finally, is the kind of transformations employed in this paper useful for more tractable optimization in models which include stochastic (physical) diffusion of pollutant?

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