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Optimal Migration and Consumption Policies over an Individual's Random Lifetime

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Optimal migration and consumption policies over an individual's random lifetime

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Abstract

The aim of this study is to determine optimal migration policies for an individual of known age-dependent mortality who accumulates capital by compound interest and also by skill-dependent income. By a continuing choice of several environments, which vary in intrinsic preference and in the rates at which they improve and reward skills, the individual have the option of developing skills, current earnings and residential preference by moving between the environments. Specifying the expected future utility which recognizes the preference for residence as well as for consumption, the corresponding dynamic programming (DP) equations are derived. The optimality of return migration is particularly investigated by studying the movement between two environments. The analysis is then extended to allow for several environments. The DP-equations are also modified to examine the effect of monetary as well as non-monetary (psychic) costs. In combining the essential causes motivating migration in a tractable dynamic control model that can be used for analysing the impacts of the various factors, this paper is believed to be an contribution to the migration literature.

Keywords: Residential preferences, development of skills, reward of human capital, optimal migration, dynamic programming.

JEL Classification C61, J24, R23

1 Introduction

Human migration is an issue of increasing topicality and has now an extensive mainly empirical literature (see e.g. Borjas [1], Borjas and Bratsberg [2]). Reviewing the extensive data sets now available, the following patterns are clearly discernible. It is quite possible to have migration flows simultaneously in opposite direction; not all, but a large proportion of those who emigrate, will return to their home country after some years; some of the the migrants are multiple movers in that they do not settle in a single destination permanently, but are constantly moving on. It is very natural to conjecture that this regularity of migration data reflects a degree of optimization in migrants' choice of domicile. Although several one-period (static) optimization models purposing to explaining particular features of migration data have appeared (see e.g. Stark [14]), fully specified dynamic behavioral models studying migration are scarce. McCall and McCall [9], Pessino [12] study migration and job search by applying multi-armed bandit theory. Although their studies are interesting, in order that the migrant's problem of choice can be represented as a bandit problem, it is has to be simplified to such a degree that it hardly remains true to life. An extensive review of the present migration literature is given by Constant and Zimmermann [3].

In this paper we study migration, not as a social issue for the communities concerned, but from the point of an individual who wishes to better himself. Its dynamic character indicates that the migration process is best studied in a life-cycle model where the drive for better earnings, the desire for developing ones' human capital and the locational preferences are explicitly recognized and whose impact on the individual choice are determined by optimization. In addition to raising interesting problems in control theory a detailed dynamic analysis of the migration process is also helpful in interpreting conflicting empirical results. Statistical analysis of migration data often give empirical findings pointing in different directions.

Note that we use the terms 'region', 'environment', 'location', and also 'country' interchangeably to designate areas, regarded as both economically distinct and economically homogeneous, between which migration takes place.

The individual is labeled by his current wealth, skill level ('human capital') and age. Age enters the model in that the expected mortality and skill-retention rates are age-dependent and that the individual's choices concerning consumption and migration are shaped very much by the consciousness that life is finite. Reliable theoretical modeling of migration processes requires an explicit awareness of the essential motives initiating this process as well as the factors explaining the shifting between the environments. This is the concern of the present paper.

2 Formulation of the model

Although most studies on migration consider earnings to be the driving force behind the migration process, recent research recognizes the influence of other factors. Research on migration data by economists and historians recognized at an early date that as important as earnings is the acquisition of skills and competence (Dustman [4], Grönberg [5]). The movement between different environments often exposes migrants to new production technologies and practices and is a way of acquiring such skills, and so of self-investment. By striving for better earnings and betterments of competence, people are able to sustain a higher long run level of consumption. But environments also differ regarding living conditions for their inhabitants, e.g. climate, social institutions and regulations, the opportunities for leisure activities etc. which affect peoples' quality of life. Thus, it is almost self-evident that people also have locational preferences. Indeed, Myrdal [10] emphasized "all sorts of social ties" to explain the delayed black South-to-North migration after the Civil War despite a large North-South income gap.

Migration can be a demanding process requiring efforts which are more easily coped with by younger people. Thus, a notion of mortality and age is also important: that the migrant faces an unknown but finite horizon. The stage reached in one's life, roughly specified by age, is then also a significant variable. Hence, in addition to consumption an adequate model of migration should explicitly allow for the elements: locational preferences, the opportunities for migrants to develop skills and competence, the reward of the human capital acquired and finally that the migrant faces an unknown but diminishing horizon.

We specify a dynamic control model in which the individual optimises over the feasible consumption paths. The situation is then compounded in that the model is assumed to hold for each of several environments (regions) over which the individual may have varying locational preferences and which may also differ in their opportunities for the individuals to build up skills and the rates they reward these skills. A switching policy will usually affect all these variables. By the dynamic programming principle we want to derive an optimal migration policy. An optimal policy may prescribe switching from one region to another at any time, so it is natural to consider continuous time. Unfortunately, the Bellman equation for this problem turned out to be a non-linear partial differential equation for which there does not exist a general solution (Zauderer [18]).

In order to overcome this problem it proved convenient to start with a discrete-time formulation, in which the time variable t labels the age of the individual, and takes integer values. We assume that the age at death is a random variable governed by a known age-dependent mortality rate. Variables w_t, v_t and b_t denote the individual's wealth, human capital and the rate of return on human capital at age t. The rate of return is environmentally dependent, and so age-dependent if an environmental sequence has been specified. The individual's state-variables at age t is specified by t itself, by his wealth w_t and (later) also by his human capital v_t. If c_t denotes the level of consumption at age t the wealth evolves by the forward recursion

$$w_{t+1} = aw_t + b_t v_t - c_t \tag{2.1}$$

Here a is the factor by which compound interest multiplies wealth over unit time. The sequence $\{v_t\}$ of human capital is assumed to evolve by known deterministic rules. This facilitates the solution of the dynamic programming equation. Later we return to this point. In our model $b_t v_t$ represents the level of the individual's current earnings and is simply the return on his human capital. This decomposition allows both the actual return rate b_t and the rules governing the the evolution of of human capital v_t to be environmentally dependent. This is natural and makes the model more versatile. We shall suppose that the utility associated with consumption c_t is $\log c_t$, implying that c_t must be positive. We shall also suppose that there is a further component of utility denoted h_t which is 'enjoyment of life' at age t. First we suppose that this variable takes prescribed values, later it will be used to reflect the intrinsic preferences for living in the different environments and thus become a variable to be determined by optimisation (section 4). No further utility can be enjoyed once the individual dies.

Death is a random event, and the only random element of the model. We shall assume that an individual alive at age t has a probability p_t of surviving to age $t + 1$. We shall find it convenient to use the notation

$$P_{ts} = p_t p_{t+1} p_{t+2} ... p_{s-1} \quad (s \geq t) \tag{2.2}$$

for the probability that an individual alive at age t survives at least to age s, with initial condition $P_{tt} = 1$.

Let $\pi = \pi_s$, $(s \geq t)$ denote an arbitrary future consumption and switching policy. Under this policy the expected future utility is given by

$$G^{\pi}(w,t) = \sum_{s=t}^{\infty} P_{ts}(log(c_s^{\pi_s}) + h_s^{\pi_s}) \tag{2.3}$$

Let $F(w,t)$ denote be the value function for this problem; i.e. the maximal expected future utility which an individual of age t and wealth w can enjoy over the remaining random lifetime. The value function is given by

$$F(w,t) = \sup_{\pi} G^{\pi}(w,t) \tag{2.4}$$

We shall derive the optimal strategy π by using the dynamic programming principle Krylov [8], Whittle [17]. This course quickly reveals certain regularity properties of the solution which then expose the solution of the continuous-time DP-equation.

3 The dynamic programming equation and its solution

We shall suppose initially that the sequences $\{b_t\}$, $\{h_t\}$ and $\{v_t\}$ are prescribed and known, so that we are in effect working in a single environment with age-dependent parameters. We shall later see the age-dependence of b and h as being brought about by environmental switching, which must be optimized, and we shall also see the sequence $\{v_t\}$ as being generated by dynamics which are also environmentally dependent. The variable v will then become a component of the state variable. The value function (2.4) satisfies the DP-equation

$$F(w,t) = \max_{c}[logc + h_t + p_t F(aw + b_t v_t - c, t+1)], \tag{3.1}$$

the maximizing value of c being the optimal value of c_t. By solving (3.1) we determine the optimal consumption sequence $\{c_t\}$ for prescribed age-dependencies of the parameters h, p, b and v. The probability terms P_{ts} appearing in the following theorem are defined in (2.2).

Theorem 3.1 *The unique proper solution of the dynamic programming equation (3.1) is given by*

$$F(w,t) = A_t log(w + B_t) + H_t + C_t \tag{3.2}$$

where

$$A_t = \sum_{s=t}^{\infty} P_{ts} \tag{3.3}$$

$$B_t = \sum_{s=t}^{\infty} a^{-(s-t+1)} b_s v_s \tag{3.4}$$

$$H_t = \sum_{s=t}^{\infty} P_{ts} h_s \tag{3.5}$$

$$C_t = -A_t log A_t + A_t log(a) + \sum_{s=t}^{\infty} P_{t(s+1)} A_{s+1} log(a p_s) \tag{3.6}$$

provided that these sums are convergent.

The optimal rate of consumption at state (w,t) *is*

$$c_t = A_t^{-1}[a w_t + b_t v_t + B_{t+1}] \tag{3.7}$$

Proof. One verifies that (3.2) satisfies equation (3.1) if the coefficients A_t, B_t, H_t and C_t satisfy the backward recursions

$$A_t = 1 + p_t A_{t+1} \tag{3.8}$$
$$B_t = a^{-1}(b_t v_t + B_{t+1}) \tag{3.9}$$
$$H_t = h_t + p_t H_{t+1} \tag{3.10}$$
$$C_t = A_t log(a A_t^{-1}) + (p_t A_{t+1}) log(p_t A_{t+1}) + p_t C_{t+1} \tag{3.11}$$

The expressions (3.3)-(3.6) are what would be obtained by solving these relations from a finite horizon value. Expression (3.7) is just the maximizing value of c in (3.1).

More explicitly, suppose that the survival rate p_T is zero for some T, so that lifetime can not exceed T, implying $F(w, T+1) = 0$ since no further utility can be enjoyed once the individual dies. That is, T is an effective horizon for the problem. Then the sums (3.3)-(3.6) terminate at a finite value, and the expression (3.2) is just what one would obtain by solving the backward equation (3.1) recursively with the starting condition $F(w, T+1) = 0$. This solution is unique and the decision rule (3.7) associated with it is optimal. If there is no such T, but the sums (3.3)-(3.6) converge, then one can arbitrarily set $p_T = 0$ and determine the associated solution, which will then converge to that asserted in the theorem as one lets the horizon T tend to infinity. Note that this procedure expresses a regularity condition implied in the model itself: that the contribution to the solution from events after age T should converge to zero with increasing T. $\square$

The array of equations (3.3)-(3.6) look rather intimidating but they have all natural interpretations. A_t is the expected residual lifetime of an individual being alive at time t, B_t is the present value of the return from future incomes $b_s v_s$, $(s \geq t)$. Further, H_t is the expected value of the future 'residential' utility and finally C_t is the expected utility of future consumption. Expression (3.7) for the optimal consumption rate has an immediate interpretation: as the ratio of the present value of all anticipated assets to the expected lifetime given that one is alive at time t.

Consider now the continuous-time version of the problem. Utilities incurred at each stage becomes utility rates and, if we consider the progression in infinitesimal small time steps Δ, then a and p_t are respectively replaced by $(1 + r\Delta)$ and $1 - m(t)\Delta$, where r is the interest rate and $m(t)$ the age-dependent death rate. We shall now write the time (or age) variable t as the argument of a function rather than as a subscript, so that, for example h_t will now be written $h(t)$. A subscript i will later be used to label environment. We shall also be more specific about the evolution of the skill variable $v(t)$ in that we shall introduce it explicitly among the state variables. Hence, in addition to the individuals age the state variables are now

$$\dot{w} = rw(t) + b(t)v(t) - c(t) \tag{3.12}$$

$$\dot{v} = d(t) \tag{3.13}$$

The dynamic programming equation (3.1) now becomes

$$\max_c [\log c + h(t) + (rw(t) + b(t)v(t) - c)F_w + d(t)F_v + F_t - m(t)F] = 0 \tag{3.14}$$

Here F_w, F_v and F_t denote the partial differentials of $F(w, v, t)$ with respect to the variables indicated. The expected effects of the mortality rate is represented in (3.14) by the term $-m(t)F$. We allow $d(t)$ to be age-dependent and so later environmentally dependent. From (3.14) we find that the optimal value of consumption is now

$$c(t) = \frac{1}{F_w} \tag{3.15}$$

Using (3.15), (3.14) reduces to

$$-log(F_w) + (rw + b(t)v)F_w - 1 + d(t)F_v - m(t)F = 0 \tag{3.16}$$

The term $log(F_w)$ makes this non-linear and equation (3.16) will resist general methods of solving partial differential equations (Zauderer [18]). However, we know the solution of this by analogy from (3.2). The solution becomes

$$F(w, v, t) = A(t)log[w(t) + B(t)] + H(t) + C(t) \tag{3.17}$$

where

$$A(t) = \int_{s=t}^{\infty} P(t, s)ds \tag{3.18}$$

$$B(t) = \int_{s=t}^{\infty} e^{-r(s-t)}b(s)v(s)ds \tag{3.19}$$

$$H(t) = \int_{s=t}^{\infty} P(t, s)h(s)ds \tag{3.20}$$

$$C(t) = -A(t)logA(t) + \int_{s=t}^{\infty} P(t, s)A(s)[r - m(s)]ds \tag{3.21}$$

and

$$P(t, s) = e^{-\int_t^s m(v)dv} \tag{3.22}$$

Expression (3.17) is a valid solution of (3.16) provided that the integrals (3.18)-(3.21)are convergent. From (3.15) and (3.17) we deduce directly that the optimal consumption rate is given by

$$c(t) = \frac{w(t) + B(t)}{A(t)} \tag{3.23}$$

The co-state variable

$$\lambda(t) = F_w = \frac{A(t)}{w(t) + B(t)} \tag{3.24}$$

turns out to be an important variable. Using the equations (3.12), (3.18) and (3.19) one can show that $\lambda(t)$ obeys thee differential equation

$$\dot{\lambda} = (m(t) - r)\lambda(t) \tag{3.25}$$

whose solution is given by

$$\lambda(t) = \lambda(0)e^{\int_0^t (m(s)-r)ds} \tag{3.26}$$

for some constant $\lambda(0)$. (3.26) implies that

$$\lambda(s) = \frac{e^{-r(s-t)}}{P(t,s)}\lambda(t) \qquad (s \geq t). \tag{3.27}$$

Equation (3.25) in fact holds generally as a consequence of the Pontryagin formalism. This indicates that the continuous-time case can be treated by the maximum principle. The explicit solution (3.17)-(3.22) of (3.16) makes such recourse unnecessary.

Since $c(t) = \lambda(t)^{-1}$ under an optimal consumption policy, relation (3.27) can be written

$$c(t) = \frac{e^{-r(s-t)}}{P(t,s)}c(s) \qquad (s \geq t). \tag{3.28}$$

Since relation(3.28) is independent of the time-dependent quantities b, d, h, w and v it holds, remarkably enough, whatever the switching policy. That is, the switching policy affects consumption $c(t)$ only by the time-independent multiplicative factor $c(0)$. Equation (3.23) shows that the equation for $c(0)$ is given by

$$c(0) = \frac{w(0) + B(0)}{A(0)} \tag{3.29}$$

where $w(0)$ is the given initial capital, $A(0)$ is the expected value of residual lifetime at time 0. However, we note from (3.19) that the present value of anticipated future earnings $B(0)$ depends crucially on the future migration policy. Thus, although the relation (3.28) is independent of the time-dependent quantities, the quantity $B(0)$ relies heavily on these variables. That is, the optimal migration policy adjusts the overall level of the consumption-path but leaves it otherwise time-invariant. This is a feature well known to economists as 'consumption smoothing'. Thus, relation (3.28) makes sense.

It also states that $c(s)$ is inflated by relative to $c(t)$ by a factor $exp[r(s-t)]$, reflecting the growth of capital by compound interest, but that $c(t)$ is inflated relative to $c(s)$ by a factor $1/P(s,t)$. An increase in the death rate seems to have the effect of lowering consumption which is counter-intuitive. The prospect of approaching death would surely encourage immediate consumption. However, the point is that an increase in future death rate affects the whole course of the process, since $A(0)$ will be decreased and thereby $c(0)$ increased. If the mortality rate $m(t)$ is constant and is equal to the interest rate r, the consumption process (3.28) reduces to $(c(s) = c(t)$ $(s \geq t)$ reflecting the 'martingale' feature of optimal consumption.

4 Environmental dependence

Suppose now that the individual can move between n environments E_1, E_2,...,E_n. We shall make the environmental dependence explicit by assuming that $b(t)$, $h(t)$ and $d(t)$ adopt constant values b_i, h_i and d_i when the individual is in E_i. The valuation of h_i and d_i are personal to the individual showing the attachment and the individual's ability to develop his/her human capital while b_i reflects the level of reward of skills in E_i.

Thus for environment E_i the state equations become

$$\dot{w}(t) = rw(t) + b_i v(t) - c(t) \tag{4.1}$$

$$\dot{v}(t) = d_i - f(t)v(t) \tag{4.2}$$

Equation (4.2) summarizes the fairly realistic assumptions that human capital improves at a constant rate d_i which is assumed to be environmental dependent, but also declines at an age-dependent rate $f(t)v$ which is independent of the environment one is staying. Later when the i-dependence of policy is in the forefront we often set $f(t) = 0$ to simplify expressions.

Above we have solved the optimal consumption process in the case of a single environment, now we wish to optimise also with respect to the migration pattern. Starting the state variables $w(t)$ and $v(t)$ at w and v respectively, the DP equation (3.14) modifies to

$$\max_i \max_c [logc + h_i + (rw + b_i v - c)F_w + (d_i - f(t)v)F_v + F_t - m(t)F] = 0 \tag{4.3}$$

We now have a double optimization, with respect to the consumption path $c(t)$ and then with respect to choice of environment E_i. The first yields the rule (3.23) as before, while the second gives the rule for optimising the migration pattern.

We have the theorem

Theorem 4.1 *The dynamic programming equation for the optimization of environment is equivalent to choosing the environment with the largest value of the criterion $D_i(t)$ given by*

$$D_i(t) = h_i + b_i v\lambda(t) + d_i\mu(t) \tag{4.4}$$

where $\lambda(t) = F_w$ and $\mu(t) = F_v$.

Proof. It follows from (4.3) that the DP equation for optimizing environment is given by

$$\max_i[-log(F_w) + rwF_w - f(t)vF_v + F_t - m(t)F - 1 + (h_i + b_i vF_w + d_i F_v)] = 0 \tag{4.5}$$

Maximizing (4.5) with respect to environment i is equivalent to maximizing the expression $D_i(t)$ $\square$

From the general solution (3.17) we see that there is a definite relation between the partial derivatives F_v and F_w. We find $F_v = k(t)F_w$ where

$$k(t) = \frac{\partial B(t)}{\partial v} = \int_{s=t}^{\infty} \frac{R(s)}{R(t)} b(s)ds \tag{4.6}$$

and

$$R(t) = \exp[-rt - \int_0^t f(\tau)d\tau] \tag{4.7}$$

Thus, $k(t)$ is the marginal change of the present value of future earnings $B(t)$ generated by a marginal change of human capital at time t. It is natural to interpret $k(t)$ as a future marginal value of human capital (skills).

Since $\mu(t) = \lambda k(t)$ the criterion (4.4) reduces to

$$D_i(t) = h_i + [b_i v_i(t) + d_i k(t)]\lambda(t) \tag{4.8}$$

The expression $[b_i v + d_i k(t)]$ is the sum of current wages $(b_i v)$ plus the marginal effect of increases in future reward $d_i k(t)$ offered in environment E_i. Thus the choice criterion of environment E_i depends on its residential utility (h_i) but also on an economic quantity combining current wages with the present value of increases in future reward.

Theorem 4.2 *Let us assume that the depreciation rate of human capital $f(t)$ is non-decreasing with age and that the reward rate of skill $b(t)$ is bounded from above, $b(t) \le \bar{b}$. Then $k(t)$ given by (4.6) has an upper bound*

$$k(t) \le \frac{\bar{b}}{(r + f(t))} \tag{4.9}$$

Proof. Since $b(t) \le \bar{b}$ the integral (4.6) satisfies

$$k(t) \le \bar{b} \int_t^\infty e^{-r(s-t) - \int_t^s f(\tau)d\tau} ds \tag{4.10}$$

Since $f(\tau)$ is non-decreasing and substituting $u = s - t$ we readily deduce the inequality

$$\int_t^s f(\tau)d\tau = \int_t^{u+t} f(\tau)d\tau \ge uf(t) \tag{4.11}$$

Putting these details together we attain

$$k(t) \le \bar{b} \int_0^\infty e^{-ru - \int_t^{t+u} f(\tau)d\tau} du \le \bar{b} \int_0^\infty e^{-(r+f(t))u} du = \frac{\bar{b}}{(r + f(t))} \tag{4.12}$$

which proves inequality (4.9) $\square$

The following result explains the rationality of return migration.

Theorem 4.3 *Let us assume that the depreciation rate $f(t)$ is non-decreasing and that the mortality rate $m(t)$ is less than the interest rate r ($m(t) < r$), then the decision criterion $D_i(t)$*

$$D_i(t) = h_i + [b_i v(t) + d_i k(t)]\lambda(t) \tag{4.13}$$

will converge exponentially fast to h_i. That is, for large t the individual will essentially choose that environment which maximizes h_i.

Proof. Since $k(t)$ has a finite upper bound, the terms in the square bracket of (4.13) are finite for all t. Furthermore, since $m(t) - r \le -\delta$ for some positive constant δ, it follows from (3.26) that

$$\lambda(t) = \lambda(0)e^{\int_0^t (m(s)-r)ds} \le \lambda(0)e^{-\delta t} \tag{4.14}$$

for some constant $\lambda(0)$. This means that $\lambda(t)$ tends exponential fast to zero. Since an environment's earning potential $[b_i v + d_i k(t)]$ increases linearly while $\lambda(t)$ decreases exponentially to zero, the criterion $D_i(t)$ will eventually be dominated by the attachment rate h_i. $\square$

The theorems (4.1)-(4.3) provide useful information regarding individuals' migration behaviour. First, they show that as the migrant gets older the decision criterion is dominated by the locational utility rate (h_i). This implies that the migrant will eventually settle in the environment with largest rate h_i. If we label the most-favoured environment by $i = 1$; it is likely, although not necessarily, that this is the home environment (country) from which the migrant started. Theorem (4.3) thus substantiates the frequently observed behaviour: that migrants often return to their home countries.

In deriving theorem (4.3) we assumed that the mortality rate is less than the interest rate. Since $m(t)$ will be increasing in adult life, it is possible that this assumption is violated at the very end. However, by that stage there will be no time for further adventures.

5 Optimization between two environments

A considerable part of contemporary migration takes place between countries which are relatively rich and at a similar level of development. Almost all statistics on these migrations show the same pattern: that the previous permanent emigration is replaced by temporary emigration. Of those who emigrate a decreasing proportion will stay abroad permanently, and an increasing proportion will return to their home country. We call this behaviour return or circular migration.

Generally, individuals can move between n environments but when we consider return migration it is sufficient to consider the case $n = 2$. We shall now apply the theory developed above to determine the optimal switching between the environments E_1 and E_2. In order to be specific we shall regard E_1 as the home country for the individual and E_2 as the host country - 'home' and 'away'. If we assume that the home country is the most-favoured environment, the 'residential' utility rates satisfy $h_1 > h_2$. For economy we shall use the notation $\Delta h = h_1 - h_2$, $\Delta b = b_1 - b_2$ and $\Delta d = d_1 - d_2$. Since we are mainly interested in the effects of the parameters h_i, b_i and d_i for the two environments we put the depreciation rate of human capital $f(t)$ equal to zero. Then we shall use the criterion (4.8) to determine the optimal policy π for different signs of Δh, Δb and Δd

Theorem 5.1 *Suppose that the home country is preferred ($\Delta h > 0$) and that the optimal policy prescribes a period in the host country (E_2) followed by an infinite sojourn in the home country (E_1). That is, there is a critical age s, dependent upon capital and skill levels attained, such that one is in E_2 for $t < s$ and retires to E_1 for $t \geq s$. Then: (i) The optimal value of s is determined the equation*

$$\Delta h + \left[\Delta b v(s) + \frac{b_1 \Delta d}{r}\right] \lambda(s) = 0, \tag{5.1}$$

(ii) $D_1(t) < D_2(t)$ before the switch and $D_1(t) > D_2(t)$ after the switch and that there is only one switch.

Proof. (i) Optimization of environment requires that for a given value of the state variables (w, v, t) we shall choose that environment maximizing the criterion $D_i(t)$ given by (4.8). From theorem (4.3) we know that $D_i(t)$ converges to h_i as t increases. Since $h_1 > h_2$ the individual will eventually settle down in the home country. Since the optimal policy requires a period in the host country there is a critical age s, dependent upon capital and skill level attained, such that one is E_2 for $t < s$ and retires to E_1 for $t \geq s$. Let $B^1(t)$, $k^1(t)$ and $\lambda^1(t)$ denote the values of $B(t)$, $k(t)$ and $\lambda(t)$ when $t \geq s$ and by $B^{(2)}(t)$, $k^{(2)}(t)$ and $\lambda^{(2)}(t)$ when $t < s$. The choice criteria for E_1 and E_2 are respectively

$$D_1(t) = h_1 + [b_1 v(t) + d_1 k^{(1)}(t)]\lambda^{(1)}(t) \tag{5.2}$$

$$D_2(t) = h_2 + [b_2 v(t) + d_2 k^{(2)}(t)]\lambda^{(2)}(t) \tag{5.3}$$

so that $D_2(t) > D_1(t)$ for $t < s$ and $D_1(t) \geq D_2(t)$ for $t \geq s$. On the boundary $t = s$ the values of the choice criteria are equal. In order to find an explicit form of the equation $D_1(s) = D_2(s)$ we calculate directly the quantities $B^{(i)}(t)$, $k^{(i)}(t)$ and $\lambda^{(i)}(t)$ ($i = 1, 2$).

For $t > s$ we calculate $B^{(1)}(t)$ by using (3.19) with E_1 parameters

$$B^{(1)}(t) = \frac{b_1 v}{r} + \frac{b_1 d_1}{r^2} \tag{5.4}$$

$$k^{(1)}(t) = \frac{\partial B^{(1)}(v)}{\partial v} = \frac{b_1}{r} \tag{5.5}$$

$$\lambda^{(1)}(t) = \frac{A(t)}{w + B^{(1)}(t)} \tag{5.6}$$

where w and v denote wealth and human capital at time $t \geq s$.

For $t < s$ we calculate $B^{(2)}(t)$ by using (3.19) with E_2 parameters on the interval $(t < s)$ and E_1 parameters on the interval $[s, \infty)$

$$B^{(2)}(t) = \frac{v[b_2 + e\Delta b]}{r} + \frac{d_2 e(s-t)\Delta b}{r} + \frac{[b_2 d_2 + (b_1 d_1 - b_2 d_2)e]}{r^2} \tag{5.7}$$

$$k^{(2)}(t) = \frac{\partial B^{(2)}(t)}{\partial v} = \frac{b_2 + \Delta b}{r} \tag{5.8}$$

$$\lambda^{(2)}(t) = \frac{A(t)}{w + B^{(2)}(t)} \tag{5.9}$$

where

$$e = exp[-r(s-t)] \tag{5.10}$$

here w and v denote wealth and human capital at time $t < s$.

From the equations (5.4)-(5.9) we observe that on the boundary $t = s$ we have

$$B^{(1)}(s) = B^{(2)}(s) = \frac{b_1 v}{r} + \frac{b_1 d_1}{r^2} \tag{5.11}$$

$$k^{(1)}(s) = k^{(2)}(s) = \frac{b_1}{r} \tag{5.12}$$

$$\lambda^{(1)}(s) = \lambda^{(2)}(s) = \frac{A(s)}{w + B^{(1)}(s)} \tag{5.13}$$

where w and v denote the values of wealth and human capital on the boundary. Inserting these values into the criteria on the boundary $D_1(s) = D_2(s)$ we deduce (5.1)

(ii) Define the function

$$L(s) = D_1(s) - D_2(s) = \Delta h + \left[\Delta b v(s) + \frac{b_1 \Delta d}{r}\right] \lambda^{(1)}(s) \tag{5.14}$$

Since Δh and $\lambda^{(1)}(s)$ are positive, $\left[\Delta b v(s) + \frac{b_1 \Delta d}{r}\right]$ must be negative at the switch-point $L(s) = 0$. Since $\dot{v}(s)$ is positive and $\dot{\lambda}^{(1)}(s)$ is negative according to (3.25) when $m(t) < r$, the derivative $\dot{L}(s)$ is positive. This means that $L(s)$ is always increasing which implies that $L(s)$ is negative before the switch and positive after the switch. This proves theorem (5.1) (ii). $\square$

It follows from this theorem that if it is optimal to spend a period in the host country one should go immediately. This means that the value of s solving (5.1) shows the duration in the host country. Equation (5.1) can be used to show the partial effects of the differential rates Δb, Δd and Δh on the duration of stay s. We find

$$\frac{\partial s}{\partial(\Delta b)} = \frac{-v(s)\lambda(s)}{(\Delta b v(s) + b_1 \Delta d/r)\dot{\lambda}(s)} \tag{5.15}$$

$$\frac{\partial s}{\partial(\Delta d)} = \frac{-b/r\lambda(s)}{(\Delta b v(s) + b_1 \Delta d/r)\dot{\lambda}(s)} \tag{5.16}$$

$$\frac{\partial s}{\partial(\Delta h)} = \frac{-1}{(\Delta b v(s) + b_1 \Delta d/r)\dot{\lambda}(s)} \tag{5.17}$$

Since $(\Delta b v(s) + b_1 \Delta d/r)$ is negative at the switch-point and $\dot{\lambda}(s)$ is always negative, the denominator of the above equations are all positive. This means that a small increase in the differential rates favouring the home country (E_1) tend to shorten the length of time spent in the host country. Remembering the definitions of the rates $\Delta b = b_1 - b_2$, $\Delta d = d_1 - d_2$ and $\Delta h = h_1 - h_2$ the results are what we would expect.

Equation (5.1) is also very useful for characterizing optimal migration behaviour under various signs of Δh, Δb and Δd.

Theorem 5.2 *(i) Suppose an individual is characterized by $\Delta h > 0$, $\Delta b \geq 0$ and $\Delta d > 0$. That is, the home country is intrinsically preferred, values skills at least as highly, and develops these skills more rapidly. Then it is optimal to stay permanently in the home country. (ii) Suppose an individual is characterized by $\Delta h > 0$, $\Delta b \geq 0$ but $\Delta d < 0$. That is, the home country* *is preferred values skill at least as highly but develops these skill less rapidly. In this case the optimal migration behaviour depends upon the initial value of human capital $v(0)$. If $v(0)$ is sufficiently small the optimal policy may require a period in the host country while if $v(0)$ is sufficiently large it is optimal to stay permanently in the home country.*

Proof. (i) The assertion is obvious and follows directly from (5.1). Under the stated conditions the left-hand side of this equation is positive for all values of s implying $D_1(t) > D_2(t)$ which means that it is optimal to stay permanently in the home country. (*ii*) Since Δd is negative, the square bracket in (5.1) is also negative for small values of $v(0)$. This implies that for small values of human capital $D_1(t) < D_2(t)$ indicating that it is optimal to stay in the host country for a while. Similarly, if the initial value of human capital is sufficiently large e.g.

$$v(0) \geq \left[\frac{-b_1 \Delta d}{r \Delta b}\right] = \bar{v} \tag{5.18}$$

the left-hand side of (5.1) is always positive implying that $D_1(t) > D_2(t)$. That is, for this individual it is again optimal to stay permanently in the home country. $\square$

We shall now relate the theory just presented to some well known issues in the migration literature. Our first example concerns Scandinavia. Interpreting Scandinavia as environment E_1 and the rest of the world as E_2, it is reasonable to assume that Scandinavians whose education has not been continued past the compulsory period will be characterized by $\Delta h > 0$, $\Delta b \geq 0$ and $\Delta d \geq 0$. This reflects the Scandinavian traits of a benevolent welfare state, high minimum wages and a generous unemployment benefits, plus the difficulty that people of limited eduction would experience in gaining skills on the international labour market. In this case the signs of case (*i*) of theorem 5.1 are fulfilled indicating that for this category of persons emigration is non-optimal. Statistics on migration show that the propensity to emigrate in this group is almost negligible (Pedersen et. al. [11]).

As a second example we consider the inflow of people from low-income countries (collectively denoted E_1) to high-income Western countries (collectively denoted E_2). With the slackening of restrictions on international mobility there is growing concern that this inflow will challenge the foundation of the modern welfare state in the high-income countries, not to mention the character of the state itself (Boeri, Hanson and McCormick [15]). For this category of potential emigrants it seems obvious that $h_1 < h_2$, $b_1 < b_2$ and $d_1 < d_2$. Since now $\Delta h < 0$, $\Delta b < 0$ and $\Delta d < 0$ the left-hand side of (5.1) is always negative indicating that $D_2(t) > D_1(t)$ for all t. This implies that it is optimal for people in low-income countries to move to high-income countries and stay there permanently. In particular, there is no incentive for these people to return to their home countries. Hence, the high-income countries can do scarcely anything to control this inflow except try to restrict it. Which is, in fact, what happens.

Our final application consider an other important issue facing well-developed welfare states with immigration (Jensen and Pedersen [7]). When the welfare benefits are based on residence in the country and not on previous labour market participation, the interaction between the labour market and the welfare system may show that the economic incentives to work may be small or even negative. In this way a benevolent state may act as a 'Welfare Magnet' in particular on low-qualified immigrants. If $\beta(t)$ represents the welfare benefits, our model can easily be adapted to the immigrant's decision problem. The wealth equation (3.12) generalizes to

$$\dot{w} = rw(t) + b(t)v(t) + \beta(t) - c(t) \tag{5.19}$$

The solution (3.17) of the DP-equation still holds, but now with

$$B(t) = \int_{s=t}^{\infty} e^{-r(s-t)}(b(s)v(s) + \beta(s))ds \tag{5.20}$$

Since this simple situation involves comparing two states: work E_1, and not work E_2, we can write the choice criterion (4.8) as

$$D_1(t) = h_1 + [b(t)v(t) + d(t)k(t)]\lambda(t) \tag{5.21}$$

$$D_2(t) = h_2 + \beta(t)\lambda(t) \tag{5.22}$$

where $b(t)$ and $d(t)$ represent the reward and growth of human capital and $k(t)$ is given by (4.6). The optimal decision is determined by comparing $D_1(t)$ and $D_2(t)$ and the immigrant will choose the category whose criterion is the greater, i.e. the immigrant will work only if $D_1(t) - D_2(t) > 0$, or

$$h_1 - h_2 + (b(t)v(t) + d(t)k(t) - \beta(t))\lambda(t) > 0 \tag{5.23}$$

If we interpret h_1 and h_2 as the amount of leisure in the two states and assume $h_1 < h_2$ which is natural, the immigrant will work only if the income accruing from work more than compensates for the loss in the amount of leisure.

The difference $(b(t)v(t) + d(t)k(t) - \beta(t))$ shows the sum of current wages plus a marginal effect of increases in future rewards acquired by working over receiving social benefits and is naturally interpreted as the 'Welfare Magnet'. The strength of this 'magnet' depends on the positive size of this difference. If it is small or even negative the migrant has hardly any economic incentives to work. The reward rate $b(t)$ will often show strong cyclical variations, increasing during cyclical upswings but decreasing in downturns, $\beta(t)$ on the other hand is politically decided and will vary much more slowly. The 'Welfare Magnet' will thus to a great extent mimic the cyclical variation of $b(t)$ which means that these immigrants will tend to go in and out of the labour market.

It is fair to admit that the factors at work in these examples are fairly obvious, as then also are the explanations and conclusions. However, it is reassuring to see these conclusions as immediate implications of our formal analysis. We will argue that our analysis can also be used to improve insight and understanding of empirical results of migration studies in general.

Statistical analyses often show that movers experience a negative income gain. Stark [14] summarizes the empirical findings of the popular expected-income hypothesis of the rural-to urban migration by concluding that "it does not fare well in terms of either the sign of the coefficients or their statistical significance". Tunali's interesting study (Tunali [16]) of the rural-to-urban migration in Turkey adds to these results. Taking earnings as the decision variable and applying self-selection modeling Tunali deduced a variable 'return to migration' comparing the earnings for an urban migrant with the earnings he/she would have earned in a rural district. A thorough analysis showed that a great majority of the migrants realized a negative return. Tunali is not happy with these results and offers two explanations:"...migrants engage in a lottery of sorts. If they are lucky, they do extremely well. However, a very a large majority of them may have to settle for a loss. Another possible interpretation is that the individuals are making mistakes and are moving when they should not". However, having carefully worked out testable hypotheses and then finds that three-fourth of the individuals in the sample do not comply with the main hypothesis, it would be more natural to discuss the adequacy of the econometric model used than simply concluding that sampled people do not behave rationally.

To be specific we denote rural districts by E_1 and urban areas by E_2.

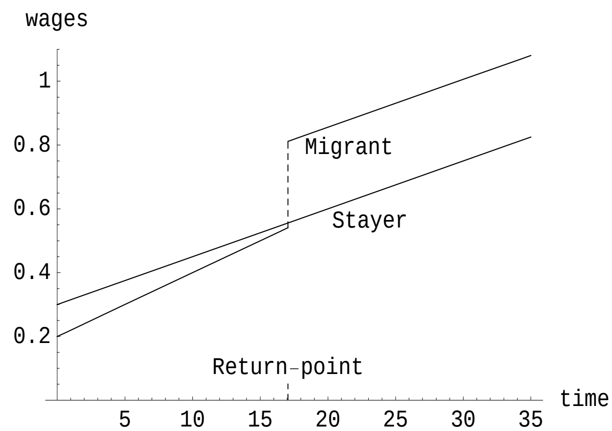

Figure 5.1: Comparing wages for a return-migrant and a stayer

First, if one has an intrinsic preference for living in an urban area enjoying the diversity of facilities and social life there, i.e. $h_2 > h_1$, it can be optimal to move from E_1 to E_2 even though the reward in E_1 is higher ($b_1 > b_2$). Since $h_2 > h_1$ theorem (4.3) shows that eventually urban areas will be the optimal environment for these people. Indeed, there is nothing peculiar about such behaviour, in his old study Sjaastad [13] writes "some people, for example, may be indifferent between earnings at one level in Minnesota and a lower level in California owing to a preference for the latter's climate".

Second, if rural districts are the most-favoured environment ($h_1 > h_2$) and one is better paid there ($b_1 > b_2$), theorem 5.2 (ii) indicates that it can still be optimal to spend a certain period in an urban area since one develops ones skill more rapidly there ($d_2 > d_1$). Since $h_1 > h_2$ we know that eventually it is optimal to live in a rural district. Thus, assuming $\Delta h > 0$, $\Delta b > 0$ but $\Delta d = (d_1 - d_2) < 0$, we know from theorem (5.2) that

$$D_1(s) - D_2(s) = \Delta h + \left[\Delta b v(s) + \frac{b\Delta d}{r}\right] \lambda^{(1)}(s) \tag{5.24}$$

can be negative for suitable small values of $v(s)$ implying that the migrant is in E_2 for $s < s^*$(the switch-value).

It is instructive to illustrate this situation by a numerical example. We use the specification: $\Delta h > 0.04$, $b_1 = 0.15$, $b_2 = 0.1$, $d_1 = 0.1$, $d_2 = 0.2$, $m = 0.03$, $r = 0.05$, the initial values of wealth and human capital are respectively $w(0) = 5$ and $v(0) = 2$. We assume that the mortality rate $m(t)$ takes the constant value m, implying that $A(t) = 1/m$. The value of $\lambda(s)$ at the switch-point s^* is given by (5.13). Here $A(s) = 1/m$, the wealth $w(s)$ is determined by solving the differential equation (4.1) with the E_2 parameters (b_2, d_2) and $w(0)$, $v(s)$ is obtained by solving $\dot{v} = d_2$ implying $v(s) = d_2 s + v(0)$. Since the expression in the square bracket of (5.24) is increasing and we know that $\lambda(s)$ is decreasing, we realize that (5.24) has only one solution. The value of s solving (5.24) is found by the 'Mathematica' program, which with the above parameter values gives the return time $s^* = 17.0542$.

Figure (5.1) compares the wages $(b_i v)$ for a migrant for whom it is optimal to spend a certain period in urban areas to take advantage of the wider range of opportunities there for developing ones skills, with a stayer who will not leave the rural districts. Compared to a stayer the wages of a rural-to-urban mover start at a lower level but increase faster since the human capital develops more rapidly in urban areas. Just before the wage lines intersect it is optimal for the mover to return to rural districts to take advantage of the higher reward rates there. It is evident from this figure that by using cross-section data, as is often the case, the sample will often include individuals reporting wages that are smaller than they would have been if they had remained in the rural districts. But to say that these people "..are moving when they should not" misses the point that a strong motive for people to move is to develop their skills more rapidly. Return migration is best analysed statistically by using long panel data.

6 Switching costs

Simple passage from one environment to another may well incur immediate money as well as non-money costs, representing the expenses and inconveniences of the move. These are switching costs that usually all prospective migrants will have to encounter in one form or another. Now we wish to carry over the preceding analysis to allow for switching costs. Since the analysis necessary to handle the two types of immediate costs are different, it is best to treat them separately.

Since people are often genuinely reluctant to leave familiar surroundings, family and friends, moving between environments will usually involve a penalty in utility. In the old study Sjaastad [13] discussed the effects of 'psychic' costs of migration. If a penalty u_{12} in utility is incurred in the passage from E_1 to E_2 at a future time point τ, then a term $u_{12}P(t,\tau)$ must be subtracted from $H(t)$ in the general solution (3.17). The effect of this as far as optimization is concerned, is to modify h_2 to $h_2 - m(\tau)u_{12}$. Similarly, if u_{21} is the penalty in passing from E_2 to E_1, h_1 has to be modified to $h_1 - m(\tau)u_{21}$.

Let assume for simplicity that skills are rewarded equally in the two environments. Then the decision criterion(4.8) reduces to

$$D_i = h_i + d_i k(t)\lambda(t) \qquad (i = 1, 2) \tag{6.1}$$

Using (4.6) - (4.7) the future marginal value of skills $k(t)$ become

$$k(t) = b\int_t^\infty e^{-r(s-t)-\int_t^s f(\tau))d\tau} ds = b\int_0^\infty e^{-rx-\int_t^{x+t} f(\tau))d\tau} dx \tag{6.2}$$

where we have used the substitution $x = s - t$.

From the integral on the right-hand side of (6.2) we find

$$\dot{k}(t) = -b\int_0^\infty (f(x+t) - f(t))e^{-rx-\int_t^{x+t} f(\tau)d\tau} dx \tag{6.3}$$

Since the aging rate of human capital $f(\tau)$ is non-decreasing the 'coefficient' $f(x+t) - f(t)$ is always non-negative. This implies that the derivative $\dot{k}(t)$ is almost everywhere negative which means that $k(t)$ is decreasing.

In the absence of switching costs the switching locus for choice between the environments is

$$\Delta h = \Delta d k(t)\lambda(t) \tag{6.4}$$

where $\Delta h = h_1 - h_2 > 0$ and $\Delta d = d_2 - d_1 > 0$. According to (6.1) the environment to be chosen is E_1 or E_2 according as the left or the right hand member of (6.4) is greater.

Let us now bring in switching penalties and assume that the individual resides in E_1. Applying the criterion (6.1) the prospective migrant will stay in E_1 for all t satisfying

$$(h_1 + d_1 k(t)\lambda(t) \geq h_2 - u_{12}m(t) + d_2 k(t)\lambda(t) \tag{6.5}$$

i.e. for all t satisfying the inequality

$$\Delta h + u_{12}m(t) \geq \Delta d k(t)\lambda(t) \tag{6.6}$$

Comparing (6.4) and (6.6) we observe that the effect of introducing switching penalty in utility is to increase the left hand side of (6.4) while the right hand side is unaffected. Thus the penalty u_{12} discourages a $E_1 \to E_2$ transition.

Conversely, if the individual stays in E_2, it is optimal to stay there for all t satisfying the the inequality

$$\Delta h - u_{21}m(t) \leq \Delta dk(t)\lambda(t) \tag{6.7}$$

That is, if one is considering the transition $E_1 \to E_2$ the Δh in (6.4) should be modified to $\Delta h + u_{12}m(t)$, if one is considering the transition $E_2 \to E_1$ Δh should be modified to $\Delta h - u_{21}m(t)$.

If we assume that $m(t)$ increases linearly then the switch points will be determined as in Figure (6.1). If at time t one is in E_1 then one switches to E_2 only if $t \leq t_{12}$. Note that this interval may be empty if $\lambda(0)$ is small enough. If at time t one is in E_2 then one switches to E_1 only if $t \geq t_{21}$. The effect of switching penalty is then to rule out the $E_1 \to E_2$ switch if $t > t_{12}$. Figure (6.1) illustrates (6.7) when $u_{21} = 0$ i.e. there is no penalty incurred by moving back to ones home country. The penalty u_{21} can obviously be both positive and negative. It is positive if the move back entails inconveniences of some sorts, it is negative if one feels strongly for returning to ones home country. From (6.7) it is evident that compared to the case $u_{21} = 0$ showed in the figure, the effect of $u_{21} > 0$ is to delay the $E_2 \to E_1$ switch and to precipitate the $E_2 \to E_1$ move if $u_{21} < 0$. Penalty in utility due to switching can thus be incorporated by modifying the preference rate (h_i).

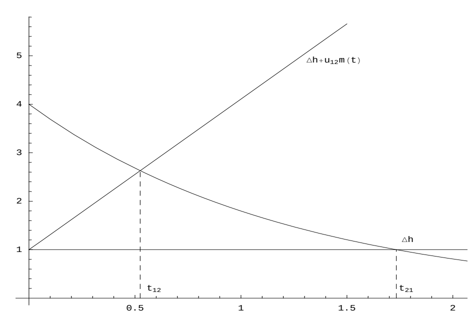

Figure 6.1: Switching penalties in utilities

It is possible that a switch from E_i to E_j implies not merely the penalty u_{ij} in utility but also a monetary cost σ_{ij}, so that the capital w changes discontinuously to $w - \sigma_{ij}$ in the transition. This discontinuity in w implies that that the analysis must be in terms of the value function F itself, rather than in terms of λ, which is essentially a differential of F. When monetary switching costs are involved, a move implies that the migrant have the capital to cover the necessary expenses. If one does not have this capital one can not move and have to settle for a less advantageous migration pattern. However, the situation would be eased if w were allowed to become negative, which means that the migrant is permitted to borrow.

For simplicity the assumptions above will be retained, so that environment E_1, the home country is intrinsically preferred $h_1 > h_2$, but one develops the skills more rapidly in the host country E_2, $d_2 > d_1$, and finally the two environments value skills equally i.e. $b_1 = b_2 = b$.

We shall use $F_i(x) = F_i(w, v, t)$ to denote the value function when the migrant begins and remains in environment E_i. We have then, in the notation of (3.17)-(3.21)

$$F_i = A(t)[log(w + B_i) + h_i] + C(t) \tag{6.8}$$

where

$$B_i = bvr^{-1} + bd_i r^{-2} \tag{6.9}$$

and the environment with the larger value function $F_i(w, v, t)$ is the optimal one.

Theorem 6.1 *Suppose that the individual stays in the host country E_2. If passage from E_2 to infinite stay in E_1 is optimal, then this will happen at the greatest value of t for which*

$$F_2(w, v, t) \leq F_1(w - \sigma_{21}, v, t) \tag{6.10}$$

In virtue of (6.8) this can be written

$$log\left(\frac{z + bd_2 r^{-2}}{z - \sigma_{21} + bd_1 r^{-2}}\right) \leq h_1 - h_2 \tag{6.11}$$

where $z = w + bvr^{-1}$

Proof. Inequalities (6.10)-(6.11) follow from (6.8) and the principle of dynamic programming. □

For this passage to be possible we must certainly have $z > \sigma_{21} - bd_1 r^{-2}$.

Passage into E_1 will then be permanent if the fraction on the left-hand side of (6.11) is decreasing with t. We find that

$$\dot{z} = (r - A(t)^{-1})(z + bd_1 r^{-2}) \tag{6.12}$$

implying that the fraction will be decreasing if $A(t)r > 1$. This will certainly be so if $m(t)$ is less than r and decreasing.

The transition rule (6.11) does not depend explicitly on age. However, this is not true when we consider optimal passage into E_2 from E_1 followed by a possible later return to E_1.

Theorem 6.2 *Suppose that the optimal migration prescribes moving into the host country E_2 followed by a later return to E_1 at t_{21} determined by (6.11). Denote (for notational simplicity) the optimal transition time t_{21} determined by (6.11) by τ. Then the optimal time t_{12} for transition from E_1 into E_2 is determined by*

$$log\left(\frac{w + B_1}{w - \sigma_{12} + B_{21}(t)}\right) \leq (h_2 - h_1)\frac{\int_t^\tau P(t,s)ds}{\int_t^\infty P(t,s)ds} \tag{6.13}$$

where

$$B_{21}(t) = bvr^{-1} + bd_2 r^{-2} + b(d_1 - d_2)e^{-r(\tau - t)}r^{-2} \tag{6.14}$$

B_1 is given by (6.9) and $B_{21}(t)$ follows from (5.7) since $\Delta b = 0$.

Proof. By assumption it is optimal to move into E_2 at a certain time t. From the value function (3.17) it follows

$$A(t)log[w - \sigma_{12} + B_{21}(t)] + h_2 \int_t^\tau P(t,s)ds + h_1 \int_\tau^\infty P(t,s)ds \geq A(t)\left[log[w + B_1] + h_1\right] \tag{6.15}$$

Since $A(t) = \int_t^\infty P(t,s)ds$, (6.13) follows when we solve inequality (6.15) with respect to the log terms. If inequality (6.15) holds at $t = 0$ then immediate passage into E_2 is recommended. There can be no movement if $w < \sigma_{12} - B_{21}(t)$ $\square$

Condition (6.13) can be seen as determining t in terms of the variable z. Note that τ is a function of z and t. Figure (6.2) illustrates the contents of the two theorems when the optimal transition time $E_1 \to E_2$ is $t = t_{12} = 0$.

Owing to (3.24) this transition will imply a consequent jump in $\lambda(0)$ from $\lambda(0)$ to $\lambda(0^+)$. This jump represents the diminution in w caused by the cost of passage (σ_{12}). The time the migrant returns to E_1 does not occur when $\Delta dk(t)\lambda(t)$ crosses the line Δh (6.4), but is delayed by the necessity to earn the cost of moving back. The switch then take place, not when $k(t)\lambda(t)\Delta d$ first falls to the value Δh, but at the later time t_{21} of Figure (6.2), when it would rise to this value after payment of passage.

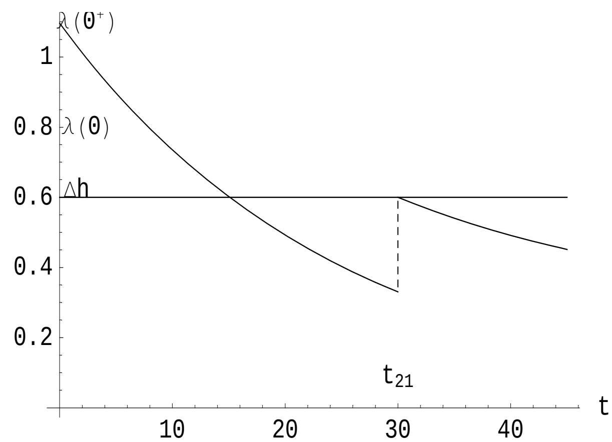

Figure 6.2: Monetary travelling costs

7 The case of several environments

Apart from consideration of return migrants, there seem to have been few empirical studies of multiple movers. However, we know from Da Vanzo's [6] study that a large proportion of those moving in a given period have also moved in the past. For this reason, and also because we believe that our model has other applications in economics, we extend the analyse above to the case of several environments.

Suppose that there are n environments, the ith being specified its parameters h_i, b_i and d_i ($i = 1, 2, ..., n$). Thus for a given state vector $x = (w, v, t)$ we have to compare the values of the criterion

$$D_i(t) = h_i + (b_i v + d_i k_i(t))\lambda_i(t) \tag{7.1}$$

for the different environments. Calculating the value of (7.1) for the different environments and choosing the one with the highest value appears to be a simple procedure for finding the optimal switching policy. However, the apparent simplicity of this approach is deceptive. The point is that the marginal value of human capital $k_i(t)$ and the co-state variable $\lambda_i(t)$ depends on the future optimal policy that are generally not known at an arbitrary time t. Hitherto our discussion of optimization of migration policy has been based on control arguments that control theorists would call 'open-loop' control. Considering movements between two environments this theory turned out to be simple to use. However, when we wish to extend this analysis to n environments the determination of $k_i(t)$ and $\lambda_i(t)$ are much more difficult.

However, when it comes to optimization of the switching policy with n environments, there is one case for which the results of the previous sections have a simple and elegant analogue. This is that when all environments value human skills equally, so that the b_i are independent of i and equal to b, say. In this case the marginal value of human capital $k_i(t)$ is independent of future switching policy. For a given value of human capital v one is paid equally (bv) in all environments and $k_i(t)$ reduces to b/r when we assume that skills do not deteriorate $(f(t) = 0)$. The criterion $D_i(t)$ simplifies to

$$G_i(\lambda) = h_i + \left[\frac{bd_i}{r}\right]\lambda \qquad i = 1, 2, ..., n \tag{7.2}$$

That is, one is essentially weighing the effect of decreased h (the intrinsic attachment to an environment) against an increased d (rate of skill acquisition). λ is a function of t and also of i but here we simply view λ as a variable taking values on the positive half-axis. As a guidance for the choice of the optimal environment we take the function.

$$G^v(\lambda) = \max_i G_i(\lambda) \tag{7.3}$$

The value function $G^v(\lambda)$ picks out the active environments and has the piece-wise convex form illustrated in figure (7.1). Since the labelling of the environments is arbitrary, label them so that the first, second, third ... of these linear segments as λ increases correspond to $i = 1, 2, 3, ...$ If there are p such linear segments then the values $i = p+1, p+2, ..., n$ can be assigned arbitrarily to the $n-p$ environments which did not yield the maximiser in (7.3) for any non-negative value of λ. The fact that the gradient of the function $G^v(\lambda)$ increases at the switch-points implies that the sequence $\{d_i, 1 \leq i \leq p\}$

is increasing. Let $\bar{\lambda}_i$ be the value of λ at which the gradient changes from bd_i/r to bd_{i+1}/r. Then

$$\bar{\lambda}_i = \frac{r(h_i - h_{i+1})}{b(d_{i+1} - d_i)} = \frac{(h_i - h_{i+1})}{\frac{b}{r}(d_{i+1} - d_i)} \qquad (1 \leq i \leq p) \tag{7.4}$$

which we compliment by $\bar{\lambda}_0 = 0$ and $\bar{\lambda}_p = +\infty$. The interval $[\bar{\lambda}_{i-1}, \bar{\lambda}_i)$ is denoted by Λ_i.

The overall structure of the optimal migration policy is given in the following theorem. The transitions between the optimal environments are illustrated in figure (7.1)

Theorem 7.1 (i) *If $\lambda(t) \in \Lambda_i$ then E_i is the optimal current environment. Environments for $i > p$ is never optimal. The optimal switch-points occur at ages t at which $\lambda(t) = \bar{\lambda}_i$ for some i.*

(ii) *If the interest rate is larger than the mortality rate ($r > m(t)$) for all t, the co-state variable $\lambda(t)$ will be monotonically decreasing, implying that $\lambda(t)$ will progress through sets Λ_i of decreasing i until it comes to rest in Λ_1. In the opposite case the co-state variable will be monotonically increasing, implying that $\lambda(t)$ will progress through sets Λ_i of increasing i until it comes to rest in Λ_p.*

Proof. Assertion (i) follows directly from the value function (7.3). Assertion (ii) follows from the shape of the value function. The break-points $\bar{\lambda}_i$ of the value function divide the value set of λ into disjoint optimal sets Λ_i. The monotonicity of $\lambda(t)$ follows directly from the differential equation (3.25) implying that $\lambda(t)$ is a decreasing function of age t in the first case and increasing in the second. $\square$

In this paper we shall always assume that that $r \geq m(t)$ so that $\lambda(t)$ moves through sets Λ_i of decreasing index i. According to (7.4) this means that the environments used in the optimal policy has a natural ordering, in which environments with decreasing utility rate h_i is more than compensated for in early by their increasing growth rates in skill d_i. The ordering of the environments indicates that we can determine the optimal switching policy in 'closed-loop' form. That is, that the optimal decisions are expressed directly in terms of the current value of the state vector $x = (w, v, t)$. In deriving the 'closed-loop' control for optimal migration the equations (3.24) and (3.26) are central, which we state for clarity

$$\lambda(t) = F_w = \frac{A(t)}{w(t) + B(t)} \tag{7.5}$$

$$\lambda(t) = \lambda(0)e^{\int_0^t (m(s)-r)ds} \tag{7.6}$$

When the environments reward skills to the same degree the present value of future earnings $B(t)$ (3.19) splits into the terms bv/r, i.e. the present value of a continuing income of bv, and the present value of increases in future incomes, denoted $I(t,s)$ where s denotes the vector of future switching times. Since the switching times s depend upon the state vector from which one starts $x = (w, v, t)$, we write $I(t,s)$ as $I(x)$ with the x-dependence of the switching times recognized. We can then write the solution for the value function (3.17) as

$$F(x) = A(t)log[z + I(x)] + H(t) + C(t) \tag{7.7}$$

where $z = w + bv/r$. The variable z_t is the sum of current wealth w_t and the present value of a continuing income (bv_t), we call it generalized wealth.

We know that the co-state variable λ has the extremal characterization $\lambda = F_w$, which as a function of the state vector can be expressed

$$\lambda(x) = F_w = \frac{A(t)}{z + I(x)} \tag{7.8}$$

Relation (7.8) is the generalization of (3.24) to the case of optimal switching. We have now to determine expression (7.8) more closely if the switching rule is to become explicit.

We shall prove that if for a given state vector $x = (w, v, t)$ one is on the switching boundary $\bar{\lambda}_i$, i.e. $\lambda(x) = \bar{\lambda}_i$, then the increases in future incomes $I(x)$ is a function of i and t alone.

Theorem 7.2 *On the switching boundary $\lambda(x) = \bar{\lambda}_i$ the function $I(x)$ is a function $I_i(t)$ of i and t alone, where $I_i(t)$ is determined by the recursion*

$$I_i(t) = bd_i \int_t^{s_{i-1}} e^{-r(s-t)}(s-t)ds + e^{-r(s_{i-1}-t)} I_{i-1}(s_{i-1}) \tag{7.9}$$

starting with the initial condition $I_1 = bd_1/r^2$

Proof. We know from theorem 7.1 that if the migrant starts at $\lambda(x) = \bar{\lambda}_i$, then the co-state variable will progress through the sets Λ_i, Λ_{i-1},..,Λ_1 successively. The switches from one environment to the next take place at the boundary points $\bar{\lambda}_i$, $\bar{\lambda}_{i-1}$,..,$\bar{\lambda}_1$, which are determined by (7.4). Assume the migrant enters environment E_i at time t, then by using (7.6) we know that $\lambda(t) = \lambda(0)e^{-\int_0^t (r-m(s))ds} = \bar{\lambda}_i$. We also know that the individual is expected to leave E_i (for E_{i-1}) at the future time s_{i-1} where the value of the co-state variable is given by $\lambda(s_{i-1}) = \lambda(0)e^{-\int_0^{s_{i-1}} (r-m(s))ds} = \bar{\lambda}_{i-1}$. The switch-points $\bar{\lambda}_i$ and $\bar{\lambda}_{i-1}$ are determined by (7.4), so at current time t the optimal switch-point $s_{i-1} > t$ is determined by

$$\bar{\lambda}_i = e^{\int_t^{s_{i-1}} (r-m(s))ds}\bar{\lambda}_{i-1} = M(t, s_{i-1})\bar{\lambda}_{i-1} \tag{7.10}$$

where $M(t, s_{i-1}) = e^{\int_t^{s_{i-1}} (r-m(s))ds}$. The value of s_{i-1} solving (7.10) is a function of current time t. For the given value of s_{i-1} we next determine the switch-point s_{i-2} and successively s_{i-3}, s_{i-4},...,s_1 by the equations

$$\bar{\lambda}_j = M(s_j, s_{j-1})\bar{\lambda}_{j-1} \qquad j = i-1, i-2, ..., 2 \tag{7.11}$$

The set of equations (7.11) determine successively s_{i-2}, s_{i-3},..., and finally s_1.

Having determined the optimal switching times the calculation of increases in future incomes along the expected migration path is readily obtained by recursions. Starting with the initial condition

$$I_1(s_1) = bd_1 \int_{s_1}^{\infty} e^{-r(s-s_1)}(s-s_1)ds = bd_1/r^2 \tag{7.12}$$

and progressing backwards we get

$$I_j(s_j) = bd_j \int_{s_j}^{s_{j-1}} e^{-r(s-s_j)}(s-s_j)ds + e^{-r(s_{j-1}-s_j)}I_{j-1}(s_{j-1}) \quad j > 2 \tag{7.13}$$

At the switch-point $\bar{\lambda}_i$ this amounts to (7.9) $\square$

That is, on the switching boundary $\lambda(x) = \bar{\lambda}_i$, the increases in future incomes $I(x)$ reduces to evaluating $I_i(t)$ by the recursion (7.13). The co-state variable (7.8) depends also upon the wealth variable $z = w + bv/r$ which is easily determined from state vector $x = (w, v, t)$. Thus, the amount of the wealth owned by the individual has a considerable impact on the optimal location. We have the following relation between optimal residence in environments and wealth z.

Theorem 7.3 *The value of i indexing the optimal current environment E_i is a non-increasing function of wealth z.*

Proof. For any index i of environments $1 \leq i \leq p$ let z_i and z_{i-1} be the values of wealth where it is optimal to enter environments E_i and E_{i-1} respectively, i.e. z_i and z_{i-1} satisfy

$$\bar{\lambda}_i = \frac{A(t)}{z_i + I_i(t)} \tag{7.14}$$

$$\bar{\lambda}_{i-1} = \frac{A(t)}{z_{i-1} + I_{i-1}(t)} \tag{7.15}$$

where the increases in future incomes $I_i(t)$ and $I_{i-1}(t)$ are calculated by the recursion (7.13). We know that the parameters d_i increase at the switch-points. $I_i(t)$ is the present value of the future increases bd_i earned in the environments E_i, E_{i-1},..., E_1 where $d_i > d_{i-1} >, .., > d_1$, while I_{i-1} is the present value of the future increases bd_{i-1} earned in the environments E_{i-1}, E_{i-2},...,E_1. Since $I_i(t)$ includes the the extra term bd_i for the time spent in E_i, we realize that $I_i(t) > I_{i-1}(t)$. Since $\bar{\lambda}_i > \bar{\lambda}_{i-1}$ implies $A(t)/(z_i + I_i(t)) > A(t)/(z_{i-1} + I_{i-1}(t))$ which in its turn implies that $(z_{i-1} - z_i) > I_i(t) - I_{i-1}(t)$. From this it follows that $z_{i-1} > z_i$. This means that large values of z corresponds to small of i. This proves the assertion in the theorem. $\square$

In our model migration is a way of developing one's human capital (skills). If one has accumulated sufficient wealth there is less need for further migration and if the wealth z is large enough it is optimal for the individual to reside permanently in the home country.

We can now formulate the optimal migration policy in closed-loop form.

Theorem 7.4 *Define the switching values*

$$z_i(t) = \frac{A(t)}{\bar{\lambda}_i} - I_i(t) \qquad (1 \leq i \leq p) \tag{7.16}$$

Then the optimal switching rule in closed-loop form is: adopt environment E_i if

$$z_i(t) \leq z \leq z_{i-1}(t) \qquad (i = 1, 2..., p) \tag{7.17}$$

Proof. We have already proved in theorem (7.3) that $z_i(t) < z_{i-1}(t)$ for a given value of t. At these values we know there will switch from E_{i+1} to E_i at $z_i(t)$ and from E_i to E_{i-1} at $z_{i-1}(t)$, and so the value of z must lie between these values if E_i is to be optimal. $\square$

Since $c = 1/\lambda$ we can express the bounds (7.17) in terms of consumption levels $\bar{c}_i$, i.e.

$$z_i(t) = \frac{A(t)}{\bar{\lambda}_i} - I_i(t) = A(t)\bar{c}_i - I_i(t) \qquad (1 \leq i \leq p) \tag{7.18}$$

The inequalities (7.17) determine the optimal switching in 'closed-loop' form. i.e. as a specification in terms of the state vector x. Suppose that for a state vector $x = (w, v, t)$ we know that E_i is the optimal environment for migrant. However, we shall require an explicit determination of $\lambda(x)$ if we are to have a corresponding closed-loop determination of optimal consumption rate $c(x) = \lambda(x)^{-1}$. It is true that, if c has been determined at one value of t, then it is determined for all values, by (3.28), but a closed-loop determination must be determined at some point, and is in any case more robust to disturbances of the path than the open-loop determination of (3.28).

Theorem 7.5 *Suppose that the criterion of theorem 7.4 has indicated that E_i is the optimal environment for a given value of the state vector x, i.e. $\lambda(x) \in \Lambda_i$. Then $\lambda(x)$ is determined by*

$$\lambda(x) = \frac{A(t)}{x + I_i(t, s_{i-1})} = M(t, s_{i-1})\bar{\lambda}_{i-1} \tag{7.19}$$

where the optimal switch-point s_{i-1} is determined by the second equality of (7.19).

Proof. Let $I_i(t, s_{i-1})$ be the present value of future increases in v if one starts from a point x for which $\lambda(x) \in \Lambda_i$ and reaches the switching boundary $\lambda = \bar{\lambda}_{i-1}$ at time s_{i-1} at which the next switch occurs, i.e.

$$I_i(t, s_{i-1}) = bd_i \int_t^{s_{i-1}} e^{-r(s-t)}(s-t)ds + e^{-r(s_{i-1}-t)} I_{i-1}(s_{i-1}) \tag{7.20}$$

The values of the co-state variable $\lambda(t)$ for two points t and s_{i-1} where $\lambda(s_{i-1}) = \bar{\lambda}_{i-1}$ are connected by the relation (7.11))

$$\lambda(t) = e^{\int_t^{s_{i-1}}(r-m(s))ds}\bar{\lambda}_{i-1} = M(t, s_{i-1})\bar{\lambda}_{i-1} \tag{7.21}$$

We also know that $\lambda(x)$ has the characterization (7.8) where $I(x)$ is determined by (7.20) when $\lambda(x) \in \Lambda_i$. That is,

$$\lambda(x) = \frac{A(t)}{z + I_i(t, s_{i-1})} \tag{7.22}$$

Since the two values of the co-state variable λ (7.21) and (7.22) have to agree, we derive equation (7.19) where the optimal switch-point s_{i-1} is determined by the second equality of (7.19). $\square$

If s^* denotes the value of s solving (7.19) the closed-loop determination of optimal consumption with the state vector $x = (z, v, t)$ is

$$c(x) = \frac{z + I(t, s^*)}{A(t)} \tag{7.23}$$

i.e. generalized wealth plus the present value of anticipated incomes accruing from optimal migration divided by residual expected lifetime.

These calculations simplify in the case when the mortality rate $m(t)$ takes the constant value m. Then $A(t) = 1/m$ and the quantities $I_i(t)$ and z_i are independent of t. The set of switch-points $\bar{\lambda}_i$, $\bar{\lambda}_{i-1}$,..., $\bar{\lambda}_1$ are determined by (7.4). If we set $\tau_j = s_j - s_{j-1}$ in (7.11) , then these equations determine τ_j, the time needed to progress from switch-point $\bar{\lambda}_i$ to that at $\bar{\lambda}_{j-1}$ as

$$\tau_j = k^{-1} log[\frac{\bar{\lambda}_j}{\bar{\lambda}_{j-1}}] \tag{7.24}$$

where $k = r - m$

The incomes generated by the future growth in skills v are calculated recursively as explained in (7.13), and with $\tau_j = s_j - s_{j-1}$ now takes the form

$$I_j(\tau_j) = bd_j \int_{s_j}^{s_{j-1}} e^{-r(s-s_j)}(s - s_j)ds + e^{-r\tau_j} I_{j-1}(\tau_{j-1}) \tag{7.25}$$

$$= bd_j(1 - e^{-r\tau_j}(1 + r\tau_j)) + e^{-r\tau_j} I_{j-1}(\tau_{j-1}) \tag{7.26}$$

Starting with initial condition $I_1 = bd_1/r^2$ and τ_j determined by (7.24), increases in future incomes are readily calculated by the recursion (7.26). If we start in E_i, i.e. $\lambda(x) \in \Lambda_i$ the quantity (7.20) is calculated by the equation

$$I_i(t, s_{i-1}) = \int_t^{s_{i-1}} e^{-r(s-t)}(s - t)ds + e^{-r(s_{i-1}-t)} I_{i-1}(\tau_{i-1}) \tag{7.27}$$

$$= bd_i r^{-2}[1 - e^{-r\tau}(1 + r\tau)] + e^{-r\tau} I_{i-1}(\tau_{i-1}) \tag{7.28}$$

where we have set $\tau = (s_{i-1} - t)$

Setting $I_i(\tau) = I_i(t, s_{i-1})$ the equation (7.19) now becomes

$$\lambda(x) = \frac{1}{m(z + I_i(\tau))} = e^{k\tau}\bar{\lambda}_{i-1} \tag{7.29}$$

where the optimal value of τ is determined by the second equality of (7.29).

The extension of these results to the case when b is i-dependent seems a great deal more difficult. In the case of constant b one is essentially weighing the effect of decreased h (the intrinsic attractiveness of an environment) against an increased d (rate of skill acquisition). If one also brings in differing skill valuations then a great variety of qualitatively different cases can occur .

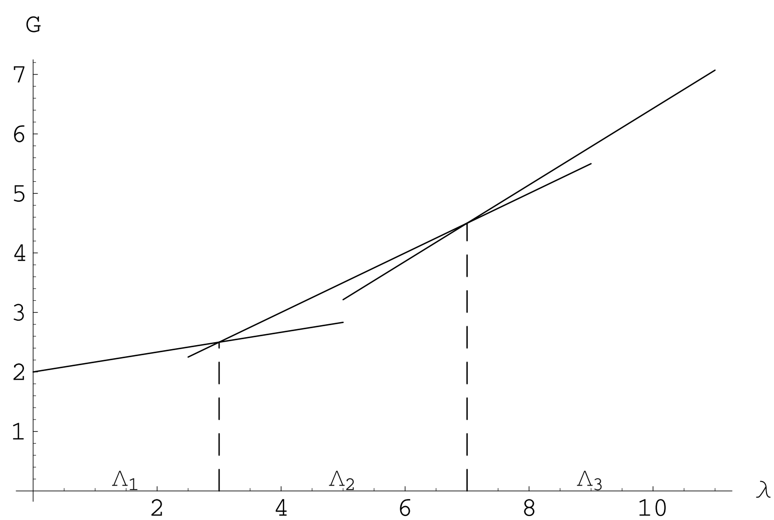

Figure 7.1: Passages between several environments

8 Conclusion

We have developed and analysed a life-cycle models for individuals that have the option of moving between several environments of differing characteristics. The individuals have a utility depending upon the rate of consumption and, in addition, are allowed to have locational preferences. The option of switching between different environments or categories introduces discontinuities in options that is a characteristic of a number of interesting problems in micro-economics. International migration is an immediate specimen of this class. We show how control theory can be applied to deduce the optimal policy when people are free to move between a finite set discrete locations. We also work out a number of implications that can be confronted with relevant empirical data. We find that predictions that can be drawn from the optimal policy agree quite well with the observable data.

We should also note that the multi-environment formulation can be given other interpretation. Labour market participation is another field of application. Suppose, for example, that the individual values leisure and does not always want to be working at a standard rate. Our model allow for this by passage to an 'environment' or position in which the b and d are lower, reflecting slower rates of earning and learning, but the h value is higher, reflecting the appreciation of increased leisure. Typical categories for the supply of labour are: (*i*) working full time, (*ii*) working part-time, and (*iii*) not participating in the labour market. Married couples may have different preferences for these categories, because for instance of the up-bringing of children. The three categories correspond to different environments. The present model extends the usual life-cycle model in that it explicitly recognizes that an individual's choice of labour market career my progress through definite phases whose length, nature and number are determined by optimization.

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Appendix

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