

MEMORANDUM

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The Price-Quantity Decomposition of Capital Values Revisited: Framework and Examples

The seal of the University of Oslo, featuring a woman in classical attire playing a lyre, surrounded by the Latin text 'UNIVERSITAS OSLOENSIS' and 'MDCCCXXXIII'.

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THE PRICE–QUANTITY DECOMPOSITION OF CAPITAL VALUES REVISITED:
FRAMEWORK AND EXAMPLES

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ABSTRACT: The often discussed problems of aggregating tangible capital assets across vintages and of decomposing value aggregates into quantity and price aggregates are revisited. For stock values and service flow values, some new results are given, and illustrated by examples, along with reinterpretations of familiar ones. If the definitions and measurement methods for prices and quantities do not ‘match’, a third, ‘quality’, component may be needed. Should this ‘buffer’ component be included in the price or quantity components, or both, or should it be accounted for separately, and in the latter case, how does it depend on the interest rate and the capital’s age? In discussing these issues, five related quantity variables and five related price variables are introduced and discussed. For certain parametric profiles for survival and efficiency loss they are equal. Some variables are observable from market data without large efforts, some are genuinely unobservable, and some can be quantified only if certain (sometimes questionable and often non-testable) assumptions are made. Examples based on three sets of parametric profiles, including exponential decay, are given.

KEYWORDS: Capital accounting. Capital survival. Capital service price. Capital and interest. Arbitrage. Capital quality. Aggregation.

JEL CLASSIFICATION: C43, C82, E22.

1 INTRODUCTION

The aggregation of capital quantities, capital prices, and capital values, *inter alia* in the construction of quantity and price indexes, as well as their relationships, are important and much discussed problems in compiling capital accounts in, *e.g.*, national accounting. A related problem arises when the time profile of second-hand prices of households' consumption capital, say dwellings or automobiles, are intended to be reflected in the consumer price indexes; see *e.g.*, Diewert (2003). In this paper, some new results are given along with reinterpretations of some more familiar ones.¹

Capital goods are less homogeneous than ordinary goods which cannot be stocked and have service lives which are zero or very short. Since capital items which may seem similar and exist alongside each other, belong to different vintages, aggregation across vintages is added to the classical problem of aggregating physically diverse goods. Capital items which appear as homogeneous in generating capital services as inputs in production processes at the moment, may be heterogeneous when considered as wealth objects and traded in a market. This has two implications: First, the problem of measuring capital prices (price indexes) should be considered together with the problem of measuring capital quantities (quantity indexes). A requirement when constructing a price index should be to specify the quantity (index) to which it belongs, and *vice versa*. Claiming for example that the price of a new, or used, capital asset equals the present value of its future (expected) rental prices weighted by the remaining efficiency, may be too vague from this viewpoint since the associated quantity is not specified. Second, inclusion of a third component, called *quality* for lack of a better term, may be needed to properly decompose the observations on capital values or to compensate for definitions of the price and the quantity that do not match.² The third component may act as a buffer to account for improper measurement of the price and quantity components. In its 'manual' on capital measurement, OECD (2009) says:

“While the wealth and the production side of capital are different aspects that help analysing different questions, they are not independent of each other. Quite to the contrary, there is a clear link between the value of an asset and its current and future productive capacity and consistency in capital measures means taking account of this link” (p. 11). “Whereas the introduction of costs of capital services into the accounts has been of interest in itself, they should also be internally consistent with measures of the net capital stock so that the volume and price measures of capital services, depreciation and net income aggregates in the national accounts as well as balance sheets are fully integrated. This allows researchers and statistical offices to produce consistent indicators of multi-factor productivity . . . Thus, capital services are not simply an add-on to measures of the net capital stock they are its analytical counterpart that comes along with the two basic roles of capital a measure of wealth and income and a measure of the contribution of capital to production” (p. 25).

The rest of the paper is organized as follows: In Section 2, the definitions of survival and efficiency functions, and arbitrage conditions connecting vintages are reconsidered. The associated price and quantity concepts, distinguishing between capital in physical units, in efficiency units and in wealth-related units, are presented. Stock related and service-related (flow-related) concepts, as well as their relationships following from arbitrage considerations are discussed, *inter alia*, through examples. It total five quantity

¹See also Biørn (2007) for an overview of some related results.

²Domar (1963, p. 587) says about this: “The recognition of the improving quality of inputs or outputs is equivalent to deflating their money values (if this method is used) by a less rapidly growing price index. As a result, the rates of growth of these inputs or outputs will now increase”.

variables and five price variables are involved. Depending on the parametric profiles for survival and efficiency loss, capital measures that are conceptually different may take the same value, which in practical applications may create confusion. Exponential decay and sudden death/constant efficiency are such boundary cases. In Section 3 the resulting capital stock value and service value measures, age-specific as well as values aggregates across vintages, are considered. Ways of splitting capital values into prices and quantities and the possible need for a third component is the topic of Section 4. Section 5 exemplifies the general results for three sets of parametric survival functions and efficiency functions.³

2 BASIC ASSUMPTIONS, PRICE AND QUANTITY VARIABLES

We first revisit the framework for quantities and capital prices, giving the two variables parallel treatment and considering the capital market from a neo-classical viewpoint. Of specific importance are the malleability of capital as an input and an arbitrage assumption for the prices of capital which is closely attached to this malleability. We first define the stock quantities (Section 2.1), next the related stock prices (Section 2.2), then the capital service prices (Section 2.3) and finally the connection between stock and flow (service) prices which this arbitrage assumption involves (Section 2.4).

2.1 Capital stock quantities

Considering time as continuous, we let $J(t)$ denote the quantity invested at time t . It brings an increase in the capital's capacity for generating capital services at time t , which gradually vanishes as the capital becomes older. This disappearance of service generating capacity is indicated by two functions. The first, the *survival function*, $S(a) \in [0, 1]$, measures the proportion of the initial investment, expressed in *physical units* – say, the number of identical buildings, machines, or automobiles – which survive at age $a (\geq 0)$. The second, the *efficiency function*, $E(a) \in [0, 1]$, measures the efficiency of one *physical* unit, say one machine, existing at age a relative to its efficiency as new. The two functions satisfy

$$(2.1) \quad S'(a) \leq 0, \quad S(0) = 1, \quad S(\infty) = 0,$$

$$(2.2) \quad E'(a) \leq 0, \quad E(0) = 1, \quad E(\infty) = 0.$$

The capital quantity of age a at time t , measured in *physical* units and expressed in *efficiency* units – the latter considered the elementary particles⁴ of capital as a productive input – can then be expressed as, respectively,

$$(2.3) \quad G(t, a) = S(a)J(t-a),$$

$$(2.4) \quad K(t, a) = E(a)G(t, a).$$

It follows that

$$(2.5) \quad K(t, a) = B(a)J(t-a),$$

where

$$(2.6) \quad B(a) = S(a)E(a).$$

³Appendix A gives a full symbol list.

⁴Such 'particles' may certainly be alleged to be metaphysical constructs, yet they are highly useful in formalizing a theory, even if the analyst's intention is to confront it with data.

Obviously,

$$K(t, 0) = G(t, 0) = J(t).$$

Since the service generating efficiency of each unit in $G(t, a)$ in general declines with age, we have $B(a) < S(a) \implies K(t, a) < G(t, a)$ for $a > 0$. An older machine may require more service and repair or be more exposed to unexpected stops etc. than a newer one; the number of elements in $G(t, a)$ exceeds the number of elementary particles in $K(t, a)$.

Capital is measured in efficiency units such that one unit generates one capital service unit per unit of time. This implies that $K(t, a)$ has the interpretation both as a stock and a flow, the latter representing the instantaneous intensity of capital services generated at time t by the stock of capital which is of age a . Correspondingly, $E(a)$ represents the (intensity of the) capital service flow at age a per physical capital unit that remains at this age, and $B(a)$, the combined *survival-efficiency function*, represents the (intensity of the) flow of capital services in efficiency units at age a per unit of capital initially invested. The latter function expresses jointly the disappearance of capital goods and their loss of efficiency and has the same general properties as its two components in (2.1), (2.2) and (2.6), although not necessarily the same curvature.⁵ While $G(t, a)$ is potentially observable, $K(t, s)$ is usually unobservable.⁶

Capital malleability, which in the present context means that *efficiency* units of the same category (say a specific kind of machinery) belonging to different vintages are *perfect substitutes* in production activities, is a core element in *neo-classical* theory. Therefore the relevant capital input at time t can be represented by the sum,

$$(2.7) \quad K(t) = \int_0^\infty K(t, a) da = \int_0^\infty B(a) J(t-a) da,$$

the (*aggregate*) *gross capital stock*, and be given the joint interpretation as the number of capital efficiency units existing at time t and the instantaneous flow of services at time t . The aggregate of capital in physical units, counting for example the number of machines,

$$(2.8) \quad G(t) = \int_0^\infty G(t, a) da = \int_0^\infty S(a) J(t-a) da,$$

usually is of minor interest in describing input.

2.2 Capital stock prices

Letting $q(t)$ denote the price of a new capital unit invested at time t , the value of the new investment can be written as

$$V(t, 0) = q(t)J(t) = q(t)G(t, 0) = q(t)K(t, 0).$$

To each of the two capital quantity concepts is, for each vintage, attached a price. Associated with $K(t, a)$ and $G(t, a)$, we define *vintage (stock) prices*: $r(t, a)$ is the price *per*

⁵Since, from (2.6),

$$\frac{B''(a)}{B(a)} = \frac{S''(a)}{S(a)} + \frac{E''(a)}{E(a)} + 2 \frac{B'(a)}{B(a)} \frac{E'(a)}{E(a)},$$

where the last term is non-negative by assumption, $B''(a)$ may be positive [$B(a)$ convex] if $S''(a) < 0$ and $E''(a) < 0$ [$S(a)$ and $E(a)$ concave], while $S''(a) > 0$, $E''(a) > 0 \implies B''(a) > 0$. Section 5 will exemplify cases with $S(a)$ and $E(a)$ concave and $B(a)$ convex.

⁶The distinction between $S(a)$ and $E(a)$ is related to what Solow (1962, p. 77) calls a 'productivity improvement factor' and 'equivalent stock of capital' and Johansen and Sørsveen (1967, Section 4) a 'capital stock corrected for efficiency'; confer: "Here we are exclusively concerned with the change in efficiency due to the assets' increasing age ... we are not considering the fact that a newer asset is better than an older one because the new asset incorporates experience and new technical ideas which were developed in the period between the production of the older and the newer asset." (Johansen and Sørsveen (1967, p. 186).

efficiency unit of age a at time t , and $p(t, a)$ is the price per physical unit. The value of the capital of age a at time t can then be written

$$(2.9) \quad V(t, a) = r(t, a)K(t, a) = p(t, a)G(t, a).$$

While $r(t, a)$ is usually unobservable, corresponding to the non-observability of $K(t, a)$, $p(t, a)$ is potentially observable from transactions in second-hand markets, corresponding to the potential observability of $G(t, a)$ and $V(t, a)$. Hence, the second decomposition in (2.9) is potentially observable, the first not. From (2.4) and (2.9) it follows that

$$(2.10) \quad p(t, a) = E(a) r(t, a).$$

How are $p(t, a)$ and $r(t, a)$ related to $q(t)$? Our answer will revisit answers given in the literature, although differently formulated, and specifically reflects assumptions about the functioning of the market for capital goods, including capital malleability, perfect substitutability and the lack of arbitrage which perfect substitutability (in perfect markets) induces.

2.3 Capital service (flow) prices

Two auxiliary functions will be needed:

$$(2.11) \quad \omega(a) = \int_a^\infty e^{-\rho(z-a)} B(z) dz = \int_a^\infty e^{-\rho(z-a)} S(z) E(z) dz,$$

$$(2.12) \quad \phi(a) = \frac{\omega(a)}{B(a)} = \frac{\int_a^\infty e^{-\rho(z-a)} S(z) E(z) dz}{S(a) E(a)}.$$

The first can be interpreted as the *discounted prospective service flow which one new capital unit generates after age a* when accounting for retirement and efficiency loss, discounted at the rate ρ (≥ 0). The second can be interpreted as the *discounted prospective service flow per efficiency which has attained age a* , since the denominator, $B(a)$, represents the share of the initial investment, measured in efficiency units, which attains age a . Since the entity under discounting, $B(a)$, is a quantity flow, we interpret ρ as a real interest rate; see Biørn (1989, Section 4.5) for details. Using (2.1) and (2.2), it follows that

$$\omega(a) \leq \phi(a) \leq \frac{1}{\rho}, \quad \rho > 0, a > 0.$$

Simple examples (to be substantially expanded later) are:

$$S(a) = e^{-\beta a}, E(a) = e^{-\alpha a}, a \in [0, \infty), \rho > 0 \implies \omega(a) = \frac{e^{-(\beta+\alpha)a}}{\rho+\beta+\alpha} < \phi(a) = \frac{1}{\rho+\beta+\alpha} < \frac{1}{\rho},$$

$$S(a) = E(a) = 1, a \in [0, N], \rho > 0 \implies \omega(a) = \phi(a) = \frac{1}{\rho} [1 - e^{-\rho(N-a)}] < \frac{1}{\rho},$$

$$S(a) = E(a) = 1, a \in [0, N], \rho = 0 \implies \omega(a) = \phi(a) = N - a,$$

while

$$S(a) = E(a) = 1, a \in [0, \infty), \rho > 0 \implies \omega(a) = \phi(a) = \frac{1}{\rho},$$

Two definitions of *the service (flow) price of capital of age a at time t* are obtained by normalizing, respectively, its efficiency unit price and its price per physical unit against the discounted service flow per capital unit of age a :

$$(2.13) \quad c(t, a) = \frac{r(t, a)}{\phi(a)},$$

$$(2.14) \quad z(t, a) = \frac{p(t, a)}{\phi(a)}.$$

To fix ideas $z(t, a)$ could be the price of one physical automobile unit per unit of prospective service hours, while $c(t, a)$ is the price of one efficiency-corrected automobile unit per unit of prospective service hours. Combining these definitions with (2.10) it follows that

$$(2.15) \quad z(t, a) = E(a)c(t, a).$$

2.4 Arbitrage condition for vintage stock and service prices

The missing link in the price structure is a *connection between stock and flow (service) prices*. We specify this link, using (2.10) and (2.13), by assuming that the vintage service prices satisfy

$$(2.16) \quad c(t, a) = c(t, 0) \equiv c(t) \quad \forall t \text{ \& \forall } a \geq 0,$$

which implies

$$(2.17) \quad \frac{r(t, a)}{\phi(a)} = \frac{r(t, 0)}{\phi(0)} = \frac{p(t, 0)}{\phi(0)} = \frac{q(t)}{\phi(0)} \equiv c(t), \quad \forall t \text{ \& \forall } a \geq 0,$$

where \equiv denotes equal by definition, postulating a *law of indifference* to exist between capital of different ages. It is a (*non*)*arbitrage condition* in the following sense: a firm which buys one *efficiency* unit of capital of age a at time t at the price $r(t, a)$, given by (2.17), pays the same sum of money per unit of prospective (discounted) capital services as a firm which buys one new unit at the price $r(t, 0) = p(t, 0) = q(t)$. In other words, *the per unit efficiency service price is postulated to be age invariant at any time*.⁷ This is related to the neo-classical *malleability assumption* since if (2.16) holds and capital can be moved freely, no possibility of arbitrage between vintages exists.

It is essential that this condition applies to efficiency units, not (in general) to physical units. From (2.13)–(2.16) we obtain

$$(2.18) \quad \begin{aligned} r(t, a) &= \phi(a)c(t), \\ z(t, a) &= E(a)c(t), \\ p(t, a) &= E(a)\phi(a)c(t), \end{aligned}$$

which imply that $r(t, a)$, $p(t, a)$ and $z(t, a)$ are linked to $q(t)$ by:⁸

$$(2.19) \quad \begin{aligned} r(t, a) &= \frac{\phi(a)}{\phi(0)}q(t) \iff \\ z(t, a) &= \frac{E(a)}{\phi(0)}q(t) \iff \\ p(t, a) &= E(a)\frac{\phi(a)}{\phi(0)}q(t). \end{aligned}$$

By implication, the age profiles of the three age-dependent prices follow

$$\frac{r(t, a)}{r(t, 0)} = \frac{\phi(a)}{\phi(0)} \iff \frac{z(t, a)}{z(t, 0)} = E(a) \iff \frac{p(t, a)}{p(t, 0)} = E(a)\frac{\phi(a)}{\phi(0)}.$$

Consider two examples:

⁷See Biørn (1989, section 3.5) for a more thorough interpretation.

⁸This equation in combination with observations on $p(t, a)/q(t)$ – or less likely $z(t, a)/q(t)$ or $r(t, a)/q(t)$ – can be utilized in making inference on the form of the survival and efficiency functions, as discussed in, *e.g.*, Biørn (1998, 2005).

Example 1. Exponential decay: $S(a) = e^{-\beta a}$, $E(a) = e^{-\alpha a}$, $a \in [0, \infty)$, $\beta > 0$, $\alpha > 0$.
Using (2.12), this implies $\phi(a) = \frac{1}{\rho + \beta + \alpha}$ and

$$\begin{aligned} c(t) &= (\rho + \beta + \alpha)q(t), \\ r(t, a) &= q(t), \\ z(t, a) &= e^{-\alpha a}(\rho + \beta + \alpha)q(t), \\ p(t, a) &= e^{-\alpha a}q(t), \end{aligned}$$

and hence

$$\frac{r(t, a)}{r(t, 0)} = 1, \quad \frac{z(t, a)}{z(t, 0)} = \frac{p(t, a)}{p(t, 0)} = e^{-\alpha a}.$$

Example 2. Constant efficiency, sudden death: $S(a) = E(a) = 1$, $a \in [0, N]$.
Using (2.12), this implies $\phi(a) = \frac{1}{\rho}[1 - e^{-\rho(N-a)}]$ and

$$\begin{aligned} c(t) = z(t, a) &= \frac{\rho}{1 - e^{-\rho N}}q(t) = \left[\rho + \frac{1}{N} \frac{\rho N}{e^{\rho N} - 1} \right] q(t) \approx \left[\rho + \frac{1}{N} \right] q(t), \\ r(t, a) = p(t, a) &= \frac{1 - e^{-\rho(N-a)}}{1 - e^{-\rho N}}q(t) \approx \left[1 - \frac{a}{N} \right] q(t), \end{aligned}$$

and hence

$$\frac{z(t, a)}{z(t, 0)} = 1, \quad \frac{r(t, a)}{r(t, 0)} = \frac{p(t, a)}{p(t, 0)} = \frac{1 - e^{-\rho(N-a)}}{1 - e^{-\rho N}} \approx \left[1 - \frac{a}{N} \right],$$

where the two approximations rely of the Taylor expansion $e^x = 1 + \sum_{i=1}^{\infty} x^i/i!$.

Equation (2.16) is related to a condition frequently postulated as an *equilibrium condition* in the capital market literature, saying that the acquisition price of an asset (should) equal the (present value of the) future ‘rental prices’ weighted by the remaining efficiency.⁹ This way of expressing the equilibrium condition reads (in our notation)

$$(2.20) \quad q(t) = \int_0^{\infty} e^{-iz} B(z) c(t+z, t) dz,$$

where i is the nominal interest rate and $c(t+z, t)$ is the capital service price at time $t+z$, as expected at time t .¹⁰ Since $B(z)c(t+z, t)$ represents a value flow, a nominal interest rate is the appropriate discount rate. Now, (2.17) expresses a proportionality between $c(t)$ and $q(t)$ for time-invariant ρ and $B(z)$ which can be rewritten as

$$(2.21) \quad q(t) = c(t) \int_0^{\infty} e^{-\rho z} B(z) dz,$$

This expression concurs with (2.20) in the following sense: If, starting from time t , the service price is expected to increase at the rate γ , so that $c(t+z, t) = c(t)e^{\gamma z}$, (2.20) gives (2.21) with $\rho = i - \gamma$.¹¹

3 IMPLIED CAPITAL VALUES

The above results have implications for the capital stock and the capital services in value terms.

3.1 Vintage specific stock and service (flow) values

The expression for the capital value, (2.9), where the stock quantities $K(t, a)$ and $G(t, a)$ are weighted by the respective stock prices $r(t, a)$ and $p(t, a)$, can, by using (2.18), be rewritten as

⁹See Hotelling (1925), Hicks (1973, Chapter II), Jorgenson (1989, section 1.2) and Diewert (2005, Section 2).

¹⁰See Jorgenson (1974, p. 205), Takayama (1985, p. 694), and Diewert (2005, Section 12.2).

¹¹A fuller discussion of this is given in Biørn (1989, section 4.2).

$$(3.1) \quad V(t, a) = q(t)E(a)\frac{\phi(a)}{\phi(0)}G(t, a) = q(t)\frac{\omega(a)}{\omega(0)}J(t-a).$$

Hence, the value of capital of age a at time t equals the replacement value of the initial investment, $q(t)J(t-a)$, times the share of the discounted capital service flow generated by one *new* capital unit from age a onwards, $\omega(a)/\omega(0)$. Equivalently,

$$(3.2) \quad V(t, a) = c(t)E(a)\phi(a)G(t, a) = c(t)\omega(a)J(t-a),$$

i.e., the stock value equals the service value of the initial investment, $c(t)J(t-a)$, times the discounted capital service flow generated by one *new* capital unit from age a onwards.

Consider again the two examples. It follows from (3.1) that:

$$\text{Example 1. Exponential decay: } V(t, a) = e^{-\alpha a}q(t)G(t, a) = e^{-(\beta+\alpha)a}q(t)J(t-a).$$

$$\begin{aligned} \text{Example 2. Constant efficiency, sudden death: } V(t, a) &= \frac{1-e^{-\rho(N-a)}}{1-e^{-\rho N}}q(t)G(t, a) \\ &= \frac{1-e^{-\rho(N-a)}}{1-e^{-\rho N}}q(t)J(t-a) \approx \left[1 - \frac{a}{N}\right]q(t)G(t, a) = \left[1 - \frac{a}{N}\right]q(t)J(t-a). \end{aligned}$$

A concept with a service value dimension is obtained by weighting the two capital stock quantities by the respective service flow prices $c(t)$ and $z(t, a)$, to obtain

$$(3.3) \quad W(t, a) = c(t)K(t, a) = z(t, a)G(t, a),$$

or equivalently,

$$(3.4) \quad W(t, a) = c(t)E(a)G(t, a) = c(t)B(a)J(t-a),$$

i.e., the service value at replacement cost equals the service value of the initial investment, $c(t)J(t-a)$, times the combined survival-efficiency function.

Equations (3.2) and (3.4) imply

$$(3.5) \quad V(t, a) = \phi(a)W(t, a),$$

i.e., for capital of age a the stock value equals the service value times the prospective service flow generated by one efficiency unit of age a from that age onwards.

3.2 The quantity components of $V(t, a)$ and $W(t, a)$

In addition to capital stock in physical and efficiency units, $G(t, a)$ and $K(t, a)$, one may be interested, not least for constructing capital accounts in national accounting, in quantity dimensioned concepts obtained by normalizing the stock and service values against the price of a new capital unit, $q(t)$. Utilizing (3.1) and (3.4), this motivates to defining

$$(3.6) \quad H(t, a) = \frac{V(t, a)}{q(t)} = \frac{\phi(a)}{\phi(0)}K(t, a) = E(a)\frac{\phi(a)}{\phi(0)}G(t, a) = \frac{\omega(a)}{\omega(0)}J(t-a),$$

$$(3.7) \quad L(t, a) = \frac{W(t, a)}{q(t)} = \frac{K(t, a)}{\phi(0)} = \frac{E(a)}{\phi(0)}G(t, a) = \frac{B(a)}{\omega(0)}J(t-a).$$

The former is what we will call the *net capital stock*. Although it has a quantity dimension it does not measure the capital's ability to generate services in physical units at the moment. The latter, also quantity dimensioned, is what we will call the *net capital service (flow)*. It follows from these definitions that the net capital stock at age a equals net capital service quantity times the discounted service flow generated by one efficiency unit of this age, *i.e.*, the quantity counterpart to (3.5):

$$(3.8) \quad H(t, a) = \phi(a)L(t, a).$$

Regarding the net-gross capital distinction, there is by far full conceptual agreement in the literature. Alternative definitions have been suggested, see footnotes 3, 9 and 10 in Biørn (1989, Chapter 3). In their manual on capital measurement, OECD (2009) states:

“The stock of assets surviving from past investment and re-valued at the purchasers prices of new capital goods of a reference period is called the gross capital stock. The gross capital stock is called gross because it has traditionally been thought of as the value of assets before deducting consumption of fixed capital... the gross capital stock has been widely used as a broad indicator of the productive capacity of a country, ... or ... as a measure of capital input in studies of multifactor productivity” (p. 38.) “. . . net capital stock measures the (market) value of capital, and is therefore a measure of wealth. Its evolution over time is governed by flows of investment and depreciation. A more telling terminology for the net stock is the wealth stock” (p. 26). “The net stock is designed to reflect the wealth of the owner of the asset at a particular point in time . . . is the measure that enters balance sheets of institutional sectors” (p. 56).

Again, the two examples are illuminating. We obtain from (3.5)–(3.8):

Example 1. Exponential decay:

$$\begin{aligned} W(t, a) &= (\rho + \beta + \alpha)V(t, a), \\ H(t, a) &= K(t, a) = e^{-\alpha a}G(t, a) = e^{-(\beta + \alpha)a}J(t - a), \\ L(t, a) &= (\rho + \beta + \alpha)K(t, a) = e^{-\alpha a}(\rho + \beta + \alpha)G(t, a) = e^{-(\beta + \alpha)a}(\rho + \beta + \alpha)J(t - a). \end{aligned}$$

Example 2. Constant efficiency, sudden death:

$$\begin{aligned} W(t, a) &= \frac{\rho}{1 - e^{-\rho(N-a)}}V(t, a) \approx \left[\rho + \frac{1}{N-a} \right] V(t, a), \\ H(t, a) &= \frac{1 - e^{-\rho(N-a)}}{1 - e^{-\rho N}}K(t, a) = \frac{1 - e^{-\rho(N-a)}}{1 - e^{-\rho N}}G(t, a) = \frac{1 - e^{-\rho(N-a)}}{1 - e^{-\rho N}}J(t - a) \\ &\approx \left[1 - \frac{a}{N} \right] K(t, a) = \left[1 - \frac{a}{N} \right] G(t, a) = \left[1 - \frac{a}{N} \right] J(t - a), \\ L(t, a) &= \frac{\rho}{1 - e^{-\rho N}}K(t, a) = \frac{\rho}{1 - e^{-\rho N}}G(t, a) = \frac{\rho}{1 - e^{-\rho N}}J(t - a) \\ &\approx \left[\rho + \frac{1}{N} \right] K(t, a) = \left[\rho + \frac{1}{N} \right] G(t, a) = \left[\rho + \frac{1}{N} \right] J(t - a). \end{aligned}$$

In the expression for $W(t, a)$, $\delta = \beta + \alpha$ in Example 1 has its counterpart in $1/(N - a)$ in Example 2 (approximate formula). Likewise, in the expression for $L(t, a)$, $\delta = \beta + \alpha$ in Example 1 has its counterpart in $1/N$ in Example 2 (approximate formula).

Returning to the general case, we can summarize the definitions of the two values and their components in:

$$(3.9) \quad V(t, a) = r(t, a)K(t, a) = p(t, a)G(t, a) = q(t)H(t, a),$$

$$(3.10) \quad W(t, a) = c(t)K(t, a) = z(t, a)G(t, a) = q(t)L(t, a),$$

and the ratios

$$(3.11) \quad \frac{V(t, a)}{W(t, a)} = \frac{H(t, a)}{L(t, a)} = \frac{r(t, a)}{c(t)} = \frac{p(t, a)}{z(t, a)} = \phi(a),$$

$$(3.12) \quad \frac{K(t, a)}{L(t, a)} = \frac{q(t)}{c(t)} = \phi(0).$$

3.3 Aggregate values

In capital accounting, value aggregates obtained by summation across vintages are usually the variables of primary interest. First, to supplement the aggregates (2.7) and (2.8) we define the aggregate net capital stock and the aggregate net capital service flow at time t as, respectively,

$$(3.13) \quad H(t) = \int_0^\infty H(t, a) da,$$

$$(3.14) \quad L(t) = \int_0^\infty L(t, a) da.$$

Since (3.6), (3.7) and (3.10) imply that $H(t, a)$ and $L(t, a)$, *i.e.*, the quantity components of $V(t, a)$ and $W(t, a)$, have price $q(t)$ and that the quantity component $K(t, a)$ of $W(t, a)$ has price $c(t)$, the two aggregate values are

$$(3.15) \quad \begin{aligned} V(t) &= \int_0^\infty V(t, a) da \\ &= \int_0^\infty r(t, a) K(t, a) da = \int_0^\infty p(t, a) G(t, a) da = \int_0^\infty q(t) H(t, a) da \\ &= q(t) H(t), \end{aligned}$$

$$(3.16) \quad \begin{aligned} W(t) &= \int_0^\infty W(t, a) da \\ &= \int_0^\infty c(t) K(t, a) da = \int_0^\infty z(t, a) G(t, a) da = \int_0^\infty q(t) L(t, a) da \\ &= c(t) K(t) = q(t) L(t), \end{aligned}$$

which, by using (3.1)–(3.3), can be rewritten as

$$(3.17) \quad V(t) = q(t) \int_0^\infty E(a) \frac{\phi(a)}{\phi(0)} G(t, a) da = q(t) \int_0^\infty \frac{\omega(a)}{\omega(0)} J(t-a) da,$$

$$(3.18) \quad W(t) = c(t) \int_0^\infty E(a) G(t, a) da = c(t) \int_0^\infty B(a) J(t-a) da.$$

Inserting (3.6) and (3.7) in (3.13) and (3.14) the aggregate net capital stock and the aggregate capital service quantity can be expressed alternatively as

$$(3.19) \quad H(t) = \int_0^\infty \frac{\phi(a)}{\phi(0)} K(t, a) da = \int_0^\infty E(a) \frac{\phi(a)}{\phi(0)} G(t, a) da = \int_0^\infty \frac{\omega(a)}{\omega(0)} J(t-a) da,$$

$$(3.20) \quad L(t) = \int_0^\infty \frac{1}{\phi(0)} K(t, a) da = \int_0^\infty E(a) \frac{1}{\phi(0)} G(t, a) da = \int_0^\infty B(a) \frac{1}{\phi(0)} J(t-a) da.$$

4 DECOMPOSING CAPITAL VALUES. A THIRD, ‘QUALITY’ COMPONENT?

In the previous sections we have introduced two value variables, V and W , five quantity variables, J , G , K , H , and L , and five price variables, q , c , d , r , and p . Ideally, our aim should be to decompose any value as follows:

$$(4.1) \quad \text{VALUE} = \text{PRICE} \times \text{QUANTITY}$$

However, depending on the measures of price and quantities combined and the data source applied in their compilation, there is no guarantee that such a relationship is satisfied, even approximately. We may therefore be led to consider decompositions of the kind

$$(4.2) \quad \text{VALUE} = \text{PRICE} \times \text{QUALITY} \times \text{QUANTITY},$$

where ‘Quality’, or ‘The Third Component’, serves as a buffer. This is illustrated in Table 1 for the stock value $V(t, a)$ and in Table 2 for the service value $W(t, a)$. The alternatives labeled qH, rK and pG have entry one in the quality column of Table 1, while the same holds for alternatives cK, zG and qL in Table 2, signaling that their price and quantity measures ‘match’, *i.e.*, satisfy (4.1). Among these alternatives, however, only qH in Table 1 and cK in Table 2 have prices that are age invariant. In the other alternatives, the third, ‘quality’ component is effective, making (4.2) the appropriate decomposition. For unknown reasons, the OECD (2009) manual does not discuss relationships between capital values and capital quantities, and its chapter on ‘User Costs’ (Chapter 8) is remarkably ‘detached’ from the discussion of quantities in the rest of the text.

Illustrations for the *age-specific* values, obtained from (3.6), (3.9) and (3.10), are:

$$(4.3) \quad \begin{aligned} V(t, a) &= q(t) \frac{\phi(a)}{\phi(0)} K(t, a) = q(t) \frac{E(a)\phi(a)}{\phi(0)} G(t, a) = q(t) \frac{\omega(a)}{\omega(0)} J(t-a) \\ &= c(t)\phi(a)K(t, a) = c(t)E(a)\phi(a)G(t, a) = c(t)\omega(a)J(t-a), \end{aligned}$$

$$(4.4) \quad \begin{aligned} W(t, a) &= c(t)E(a)G(t, a) = c(t)B(a)J(t-a) \\ &= q(t) \frac{E(a)}{\phi(0)} G(t, a) = q(t) \frac{B(a)}{\omega(0)} J(t-a). \end{aligned}$$

The ‘Third component’ is the middle part of these three-factor expressions inserted between the price and the quantity variables. Illustrations for *vintage-aggregated* are the following value/quantity ratios, obtained from (3.15)–(3.18):

Stock value per efficiency unit:

$$(4.5) \quad \frac{V(t)}{K(t)} = \frac{\int_0^\infty q(t)H(t, a)da}{\int_0^\infty K(t, a)da} = q(t) \frac{\int_0^\infty \frac{\phi(a)}{\phi(0)} K(t, a)da}{\int_0^\infty K(t, a)da}.$$

Stock value per physical unit:

$$(4.6) \quad \frac{V(t)}{G(t)} = \frac{\int_0^\infty q(t)H(t, a)da}{\int_0^\infty G(t, a)da} = q(t) \frac{\int_0^\infty E(a) \frac{\phi(a)}{\phi(0)} G(t, a)da}{\int_0^\infty G(t, a)da}.$$

Service value per physical unit:

$$(4.7) \quad \frac{W(t)}{G(t)} = \frac{\int_0^\infty c(t)K(t, a)da}{\int_0^\infty G(t, a)da} = c(t) \frac{\int_0^\infty E(a)G(t, a)da}{\int_0^\infty G(t, a)da}.$$

Service value per net capital unit:

$$(4.8) \quad \frac{W(t)}{H(t)} = \frac{\int_0^\infty c(t)K(t, a)da}{\int_0^\infty H(t, a)da} = c(t) \frac{\int_0^\infty \frac{\phi(0)}{\phi(a)} H(t, a)da}{\int_0^\infty H(t, a)da}.$$

The rightmost fractional parts of these four expressions have interpretations as quantity-weighted averages of the ‘adjustment factors’ $\phi(a)/\phi(0)$, $E(a)\phi(a)/\phi(0)$, $E(a)$ and $\phi(0)/\phi(a)$, respectively. Their role is to compensate for different vintages being heterogeneous with respect to the respective ‘normalizing quantities’. The ‘correct’ normalizations are $H(t)$ for $V(t)$ and $K(t)$ for $W(t)$, as is known from (3.15) and (3.16).

5 EXAMPLES

So far we have (with a few exceptions) considered general survival and efficiency functions, $S(a)$ and $E(a)$, satisfying (2.1) and (2.2). In this section we illustrate more systematically the price-quantity decompositions obtained for three parametric functions:

Example 1: Exponential decay for both survival and efficiency:

$$(5.1) \quad S(a) = e^{-\beta a}, \quad E(a) = e^{-\alpha a}, \quad B(a) = e^{-\delta a}, \quad a \in (0, \infty), \quad \beta \geq 0, \alpha \geq 0, \delta = \beta + \alpha.$$

Example 2. Simultaneous retirement and constant efficiency:

$$(5.2) \quad S(a) = E(a) = B(a) = 1, \quad a \in (0, N).$$

Example 3. Two-parametric functions for survival and efficiency:

$$(5.3) \quad S(a) = \left[1 - \frac{a}{N}\right]^\mu, \quad E(a) = \left[1 - \frac{a}{N}\right]^\nu, \quad B(a) = \left[1 - \frac{a}{N}\right]^\tau, \quad a \in (0, N), \mu \geq 0, \nu \geq 0, \tau = \mu + \nu$$

From a formal viewpoint we can remark that Example 1 has two parameters, one for each of the two functions, Example 2 has only one parameter, common for the two functions, while Example 3 has three parameters, one specific for the two functions and one common. Basic characteristics of these examples are listed in Table 3, with the zero interest case ($\rho=0$) explicitly shown.

In describing Examples 2 and 3, the following function is needed:

$$(5.4) \quad h(a, \rho, \tau) = \left[1 - \frac{a}{N}\right]^{-\tau} \int_a^N e^{-\rho(z-a)} \left[1 - \frac{z}{N}\right]^\tau dz.$$

Its properties are shown in Appendix B. In particular,

$$(5.5) \quad \lim_{\rho \rightarrow 0} h(a, \rho, \tau) = \left(1 - \frac{a}{N}\right)^{-\tau} \int_a^N \left(1 - \frac{z}{N}\right)^\tau dz = \frac{N-a}{\tau+1},$$

$$(5.6) \quad \lim_{\rho \rightarrow 0} \frac{h(a, \rho, \tau)}{h(0, \rho, \tau)} = \frac{N-a}{N}.$$

Other properties are listed in the following frames:

Example 1:

The curvature of the survival function and of the survival-efficiency function are equal.

Net capital equals capital in efficiency units, $H(t, a) = K(t, a)$.

The efficiency unit price equals the investment price: $r(t, a) = q(t)$.

Capital in efficiency units is smaller than capital in physical units, $K(t, a) < G(t, a)$.

$p(t, a)$ depends on α , but is independent of β .

$G(t, a)$ depends on β , but is independent of α .

$K(t, a)$, $H(t, a)$ and $V(t, a)$ only depend on $\beta + \alpha = \delta$.

$K(t, a)$, $H(t, a)$ and $V(t, a)$ are ρ -invariant.

Specific cases:

1. $\beta > 0$, $\alpha > 0$ (both retirement and declining efficiency) \implies
 $p(t, a) < r(t, a) = q(t)$, $H(t, a) = K(t, a) < G(t, a) < J(t-a)$, $\forall a > 0$.
2. $\beta = 0$, $\alpha > 0$ (no retirement, declining efficiency) \implies
 $p(t, a) < r(t, a) = q(t)$, $H(t, a) = K(t, a) < G(t, a) = J(t-a)$, $\forall a > 0$.
3. $\beta > 0$, $\alpha = 0$ (retirement, constant efficiency) \implies
 $p(t, a) = r(t, a) = q(t)$, $H(t, a) = K(t, a) = G(t, a) < J(t-a)$, $\forall a > 0$.
4. $\beta = 0$, $\alpha = 0$ (no retirement, constant efficiency) \implies
 $p(t, a) = r(t, a) = q(t)$, $H(t, a) = K(t, a) = G(t, a) = J(t-a)$, $\forall a > 0$.

Example 2 [= Example 3 with $\mu = \nu = 0$]:

The efficiency unit stock price equals the stock price per physical unit: $r(t, a) = p(t, a)$.

The corresponding service prices are equal: $c(t) = z(t, a)$.

Capital in efficiency units equals capital in physical units, $K(t, a) = G(t, a) = J(t-a)$.

Capital in efficiency units exceeds net capital, $K(t, a) > H(t, a)$.

Net capital is ρ -dependent.

Example 3:

The survival-efficiency function is convex, linear and concave according as $\tau = \mu + \nu \begin{matrix} > \\ = \\ < \end{matrix} 1$.

The curvature of the survival function and of the survival-efficiency function may differ.

All stock quantities are in general different.

All stock prices and both service prices are in general different.

Net capital is ρ -dependent.

Specific cases:

1. $\mu > 0, \nu > 0$ (gradual retirement and gradually declining efficiency):
 $p(t, a) < r(t, a) < q(t), \quad H(t, a) < K(t, a) < G(t, a) < J(t-a), \quad \forall a > 0.$
2. $\mu = 0, \nu > 0$ (full retirement at age N , gradually declining efficiency):
 $p(t, a) < r(t, a) < q(t), \quad H(t, a) < K(t, a) < G(t, a) = J(t-a), \quad \forall a > 0.$
3. $\mu > 0, \nu = 0$ (gradual retirement, full efficiency until age N):
 $p(t, a) = r(t, a) < q(t), \quad H(t, a) < K(t, a) = G(t, a) < J(t-a), \quad \forall a > 0.$
4. $\mu = 0, \nu = 0$ (full retirement at age N , full efficiency up to age N):
 $p(t, a) = r(t, a) < q(t), \quad H(t, a) < K(t, a) = G(t, a) = J(t-a), \quad \forall a > 0.$
5. $N \rightarrow \infty \implies H(t, a) = K(t, a), \quad r(t, a) = p(t, a) = q(t), \quad z(t, a) = c(t) = \rho q(t).$

Tables 4 and 6 illustrates, for Examples 1–3, the price-quality-quantity decompositions of the *vintage-specific* capital values surveyed for the general case in Tables 1 and 2. Table 5, an addendum to Table 4, summarizes the dependence of the ‘Third component’ (Quality) on the interest rate ρ and the capital’s age a . In a few cases this component is an *increasing* function of age, otherwise it is either constant or decreasing. Alternative pH, which combines net capital with the price per physical capital unit, is notable in this respect.

The decomposition equations for aggregate capital, (3.15) and (3.16), are illustrated in the frames below. Notice, in particular, how the factors $e^{-\alpha a}$ and $e^{-\beta a}$ in Example 1, $h(a, \rho, \tau)/h(0, \rho, \tau)$ in Example 2, and $h(a, \rho, \tau)/h(0, \rho, \tau), [\frac{N-a}{N}]^\nu$ and $[\frac{N-a}{N}]^\mu$ in Example 3 are ‘transmitted’ between the price and quantity variables:

Example 1:

$$\begin{aligned}
 V(t) &= \int_0^\infty \underbrace{q(t)e^{-\alpha a}}_{p(t, a)} \underbrace{e^{-\beta a} J(t-a)}_{G(t, a)} da = \int_0^\infty \underbrace{q(t)}_{r(t, a)} \underbrace{e^{-\alpha a} e^{-\beta a} J(t-a)}_{H(t, a) = K(t, a)} da \\
 W(t) &= \int_0^\infty \underbrace{c(t)}_{c(t)} \underbrace{e^{-\alpha a} e^{-\beta a} J(t-a)}_{K(t, a) = H(t, a)} da = \int_0^\infty \underbrace{c(t)e^{-\alpha a}}_{z(t, a)} \underbrace{e^{-\beta a} J(t-a)}_{G(t, a)} da
 \end{aligned}$$

Example 2:

$$\begin{aligned}
 V(t) &= \int_0^N \underbrace{q(t) \frac{h(a, \rho, 0)}{h(0, \rho, 0)}}_{p(t, a) = r(t, a)} \underbrace{J(t-a)}_{G(t, a) = K(t, a)} da = \int_0^N \underbrace{q(t)}_{q(t)} \underbrace{\frac{h(a, \rho, 0)}{h(0, \rho, 0)} J(t-a)}_{H(t, a)} da \\
 W(t) &= \int_0^N \underbrace{c(t)}_{c(t)} \underbrace{J(t-a)}_{K(t, a) = G(t, a)} da = \int_0^N \underbrace{c(t)}_{z(t, a)} \underbrace{J(t-a)}_{G(t, a) = K(t, a)} da
 \end{aligned}$$

Example 3:

$$\begin{aligned}
 V(t) &= \int_0^N \underbrace{q(t) \frac{h(a, \rho, \tau)}{h(0, \rho, \tau)} \left[\frac{N-a}{N} \right]^\nu}_{p(t, a)} \underbrace{\left[\frac{N-a}{N} \right]^\mu J(t-a)}_{G(t, a)} da \\
 &= \int_0^N \underbrace{q(t) \frac{h(a, \rho, \tau)}{h(0, \rho, \tau)}}_{r(t, a)} \underbrace{\left[\frac{N-a}{N} \right]^\nu \left[\frac{N-a}{N} \right]^\mu J(t-a)}_{K(t, a)} da \\
 &= \int_0^N \underbrace{q(t) \frac{h(a, \rho, \tau)}{h(0, \rho, \tau)} \left(\frac{N-a}{N} \right)^\nu}_{r(t, a)} \underbrace{\left[\frac{N-a}{N} \right]^\mu J(t-a)}_{K(t, a)} da \\
 W(t) &= \int_0^N \underbrace{q(t)}_{c(t)} \underbrace{\left[\frac{N-a}{N} \right]^\nu \left[\frac{N-a}{N} \right]^\mu J(t-a)}_{H(t, a)} da \\
 &= \int_0^N \underbrace{c(t) \left[\frac{N-a}{N} \right]^\nu}_{z(t, a)} \underbrace{\left[\frac{N-a}{N} \right]^\mu J(t-a)}_{G(t, a)} da
 \end{aligned}$$

TABLE 1: DECOMPOSITIONS OF AGE-SPECIFIC CAPITAL STOCK VALUE $V(t, a)$.
General case

<i>Label</i>	<i>Price</i>	<i>Quality</i>	<i>Quantity</i>
qK	$q(t)$	$\frac{\phi(a)}{\phi(0)}$	$K(t, a)$
qG	$q(t)$	$\frac{E(a)\phi(a)}{\phi(0)}$	$G(t, a)$
qH	$q(t)$	1	$H(t, a)$
qJ	$q(t)$	$\frac{\omega(a)}{\omega(0)}$	$J(t-a)$
rK	$r(t, a)$	1	$K(t, a)$
rG	$r(t, a)$	$E(a)$	$G(t, a)$
rH	$r(t, a)$	$\frac{\phi(0)}{\phi(a)}$	$H(t, a)$
rJ	$r(t, a)$	$B(a)$	$J(t-a)$
pK	$p(t, a)$	$\frac{1}{E(a)}$	$K(t, a)$
pG	$p(t, a)$	1	$G(t, a)$
pH	$p(t, a)$	$\frac{\phi(0)}{E(a)\phi(a)}$	$H(t, a)$
pJ	$p(t, a)$	$S(a)$	$J(t-a)$
cK	$c(t)$	$\phi(a)$	$K(t, a)$
cG	$c(t)$	$E(a)\phi(a)$	$G(t, a)$
cH	$c(t)$	$\phi(0)$	$H(t, a)$
cJ	$c(t)$	$\omega(a)$	$J(t-a)$

TABLE 2: DECOMPOSITIONS OF AGE-SPECIFIC CAPITAL SERVICE VALUE $W(t, a)$.
General case

<i>Label</i>	<i>Price</i>	<i>Quality</i>	<i>Quantity</i>
cK	$c(t)$	1	$K(t, a)$
cG	$c(t)$	$E(a)$	$G(t, a)$
cJ	$c(t)$	$B(a)$	$J(t-a)$
cL	$c(t)$	$\phi(0)$	$L(t, a)$
zK	$z(t, a)$	$\frac{1}{E(a)}$	$K(t, a)$
zG	$z(t, a)$	1	$G(t, a)$
zJ	$z(t, a)$	$S(a)$	$J(t-a)$
zL	$z(t, a)$	$\frac{\phi(0)}{E(a)}$	$L(t, a)$
qK	$q(t)$	$\frac{1}{\phi(0)}$	$K(t, a)$
qG	$q(t)$	$\frac{E(a)}{\phi(0)}$	$G(t, a)$
qJ	$q(t)$	$\frac{B(a)}{\phi(0)}$	$J(t-a)$
qL	$q(t)$	1	$L(t, a)$

TABLE 3. EXAMPLES 1–3. BASIC CHARACTERISTICS

	$\rho = 0$	$\rho > 0$
<i>Example 1: Exponential decay</i>		
$B(a)$	$e^{-\delta a}$	$e^{-\delta a}$
$\phi(a)$	$1/\delta$	$1/(\rho+\delta)$
$\omega(a)$	$e^{-\delta a}/\delta$	$e^{-\delta a}/(\rho+\delta)$
$c(t)$	$\delta q(t)$	$(\rho+\delta)q(t)$
$z(t, a)$	$e^{-\alpha a}c(t) = \delta e^{-\alpha a}q(t)$	$e^{-\alpha a}c(t) = (\rho+\delta)e^{-\alpha a}q(t)$
$r(t, a)$	$q(t)$	$q(t)$
$p(t, a)$	$e^{-\alpha a}r(t, a) = e^{-\alpha a}q(t)$	$e^{-\alpha a}r(t, a) = e^{-\alpha a}q(t)$
$G(t, a)$	$e^{-\beta a}J(t-a)$	$e^{-\beta a}J(t-a)$
$K(t, a)$	$e^{-\alpha a}G(t, a) = e^{-\delta a}J(t-a)$	$e^{-\alpha a}G(t, a) = e^{-\delta a}J(t-a)$
$H(t, a)$	$K(t, a) = e^{-\alpha a}G(t, a) = e^{-\delta a}J(t-a)$	$K(t, a) = e^{-\alpha a}G(t, a) = e^{-\delta a}J(t-a)$
$L(t, a)$	$\delta K(t, a) = \delta H(t, a) = \delta e^{-\delta a}J(t-a)$	$(\rho+\delta)K(t, a) = (\rho+\delta)H(t, a) = (\rho+\delta)e^{-\delta a}J(t-a)$
$V(t, a)$	$q(t)H(t, a) = q(t)K(t, a)$	$q(t)H(t, a) = q(t)K(t, a)$
$W(t, a)$	$c(t)K(t, a) = c(t)H(t, a)$	$c(t)K(t, a) = c(t)H(t, a)$
<i>Example 2: Simultaneous retirement and constant efficiency</i>		
$B(a)$	1	1
$\phi(a)$	$N - a$	$h(a, \rho, 0)$
$\omega(a)$	$N - a$	$h(a, \rho, 0)$
$c(t)$	$\frac{1}{N}q(t)$	$\frac{1}{h(0, \rho, 0)}q(t)$
$z(t, a)$	$c(t) = \frac{1}{N}q(t)$	$c(t) = \frac{1}{h(0, \rho, \tau)}q(t)$
$r(t, a)$	$\frac{N-a}{N}q(t)$	$\frac{h(a, \rho, 0)}{h(0, \rho, 0)}q(t)$
$p(t, a)$	$r(t, a) = \frac{N-a}{N}q(t)$	$r(t, a) = \frac{h(a, \rho, \tau)}{h(0, \rho, \tau)}q(t)$
$G(t, a)$	$J(t-a)$	$J(t-a)$
$K(t, a)$	$G(t, a) = J(t-a)$	$G(t, a) = J(t-a)$
$H(t, a)$	$\left[\frac{N-a}{N}\right] K(t, a) = \left(\frac{N-a}{N}\right) J(t-a)$	$\frac{h(a, \rho, 0)}{h(0, \rho, 0)} K(t, a) = \frac{h(a, \rho, 0)}{h(0, \rho, 0)} J(t-a)$
$L(t, a)$	$\frac{1}{N} K(t, a) = \frac{1}{N-a} H(t, a) = \frac{1}{N} J(t-a)$	$\frac{1}{h(0, \rho, 0)} K(t, a) = \frac{1}{h(a, \rho, 0)} H(t, a) = \frac{1}{h(0, \rho, \tau)} J(t-a)$
$V(t, a)$	$q(t)H(t, a) = q(t) \left[\frac{N-a}{N}\right] K(t, a)$	$q(t)H(t, a) = q(t) \frac{h(a, \rho, 0)}{h(0, \rho, 0)} K(t, a)$
$W(t, a)$	$c(t)K(t, a) = c(t) \frac{N}{N-a} H(t, a)$	$c(t)K(t, a) = c(t) \frac{h(0, \rho, \tau)}{h(a, \rho, \tau)} H(t, a)$
<i>Example 3: Two-parametric survival and efficiency functions</i>		
$B(a)$	$\left[\frac{N-a}{N}\right]^\tau$	$\left[\frac{N-a}{N}\right]^\tau$
$\phi(a)$	$\frac{N-a}{\tau+1}$	$h(a, \rho, \tau)$
$\omega(a)$	$\frac{N-a}{\tau+1} \left[\frac{N-a}{N}\right]^\tau$	$h(a, \rho, \tau) \left[\frac{N-a}{N}\right]^\tau$
$c(t)$	$\frac{\tau+1}{N} q(t)$	$\frac{1}{h(0, \rho, \tau)} q(t)$
$z(t, a)$	$\left[\frac{N-a}{N}\right]^\nu c(t) = \frac{\tau+1}{N} \left[\frac{N-a}{N}\right]^\nu q(t)$	$\left[\frac{N-a}{N}\right]^\nu c(t) = \frac{1}{h(0, \rho, \tau)} \left[\frac{N-a}{N}\right]^\nu q(t)$
$r(t, a)$	$\frac{N-a}{N} q(t)$	$\frac{h(a, \rho, \tau)}{h(0, \rho, \tau)} q(t)$
$p(t, a)$	$\left[\frac{N-a}{N}\right]^\nu r(t, a) = \left[\frac{N-a}{N}\right]^{\nu+1} q(t)$	$\left[\frac{N-a}{N}\right]^\nu r(t, a) = \frac{h(a, \rho, \tau)}{h(0, \rho, \tau)} \left[\frac{N-a}{N}\right]^\nu q(t)$
$G(t, a)$	$\left[\frac{N-a}{N}\right]^\mu J(t-a)$	$\left[\frac{N-a}{N}\right]^\mu J(t-a)$
$K(t, a)$	$\left[\frac{N-a}{N}\right]^\nu G(t, a) = \left[\frac{N-a}{N}\right]^\tau J(t-a)$	$\left[\frac{N-a}{N}\right]^\nu G(t, a) = \left[\frac{N-a}{N}\right]^\tau J(t-a)$
$H(t, a)$	$\left[\frac{N-a}{N}\right] K(t, a) = \left[\frac{N-a}{N}\right]^{\tau+1} J(t-a)$	$\frac{h(a, \rho, \tau)}{h(0, \rho, \tau)} K(t, a) = \frac{h(a, \rho, \tau)}{h(0, \rho, \tau)} \left[\frac{N-a}{N}\right]^\tau J(t-a)$
$L(t, a)$	$\frac{\tau+1}{N} K(t, a) = \frac{\tau+1}{N-a} H(t, a) = \frac{\tau+1}{N} \left[\frac{N-a}{N}\right]^\tau J(t-a)$	$\frac{1}{h(0, \rho, \tau)} K(t, a) = \frac{1}{h(a, \rho, \tau)} H(t, a) = \frac{1}{h(0, \rho, \tau)} \left[\frac{N-a}{N}\right]^\tau J(t-a)$
$V(t, a)$	$q(t)H(t, a) = q(t) \left[\frac{N-a}{N}\right] K(t, a)$	$q(t)H(t, a) = q(t) \frac{h(a, \rho, \tau)}{h(0, \rho, \tau)} K(t, a)$
$W(t, a)$	$c(t)K(t, a) = c(t) \frac{N}{N-a} H(t, a)$	$c(t)K(t, a) = c(t) \frac{h(0, \rho, \tau)}{h(a, \rho, \tau)} H(t, a)$

TABLE 4:
DECOMPOSITIONS OF AGE-SPECIFIC CAPITAL STOCK VALUE $V(t, a)$ IN EXAMPLES 1–3

Label	Price	Quant.	Ex. 1: Qual.		Ex. 2: Qual.		Ex. 3: Qual.	
			$\rho = 0$	$\rho > 0$	$\rho = 0$	$\rho > 0$	$\rho = 0$	$\rho > 0$
qK	$q(t)$	$K(t, a)$	1	1	$\frac{N-a}{N}$	$\frac{h(a, \rho, 0)}{h(0, \rho, 0)}$	$\frac{N-a}{N}$	$\frac{h(a, \rho, \tau)}{h(0, \rho, \tau)}$
qG	$q(t)$	$G(t, a)$	$e^{-\alpha a}$	$e^{-\alpha a}$	$\frac{N-a}{N}$	$\frac{h(a, \rho, 0)}{h(0, \rho, 0)}$	$\left[\frac{N-a}{N}\right]^{\nu+1}$	$\frac{h(a, \rho, \tau)}{h(0, \rho, \tau)} \left[\frac{N-a}{N}\right]^{\nu}$
qH	$q(t)$	$H(t, a)$	1	1	1	1	1	1
qJ	$q(t)$	$J(t-a)$	$e^{-\delta a}$	$e^{-\delta a}$	$\frac{N-a}{N}$	$\frac{h(a, \rho, 0)}{h(0, \rho, 0)}$	$\left[\frac{N-a}{N}\right]^{\tau+1}$	$\frac{h(a, \rho, \tau)}{h(0, \rho, \tau)} \left[\frac{N-a}{N}\right]^{\tau}$
rK	$r(t, a)$	$K(t, a)$	1	1	1	1	1	1
rG	$r(t, a)$	$G(t, a)$	$e^{-\alpha a}$	$e^{-\alpha a}$	1	1	$\left[\frac{N-a}{N}\right]^{\nu}$	$\left[\frac{N-a}{N}\right]^{\nu}$
rH	$r(t, a)$	$H(t, a)$	1	1	$\frac{N}{N-a}$	$\frac{h(0, \rho, 0)}{h(a, \rho, 0)}$	$\frac{N}{N-a}$	$\frac{h(0, \rho, \tau)}{h(a, \rho, \tau)}$
rJ	$r(t, a)$	$J(t-a)$	$e^{-\delta a}$	$e^{-\delta a}$	1	1	$\left[\frac{N-a}{N}\right]^{\tau}$	$\left[\frac{N-a}{N}\right]^{\tau}$
pK	$p(t, a)$	$K(t, a)$	$e^{\alpha a}$	$e^{\alpha a}$	1	1	$\left[\frac{N}{N-a}\right]^{\nu}$	$\left[\frac{N}{N-a}\right]^{\nu}$
pG	$p(t, a)$	$G(t, a)$	1	1	1	1	1	1
pH	$p(t, a)$	$H(t, a)$	$e^{\alpha a}$	$e^{\alpha a}$	$\frac{N}{N-a}$	$\frac{h(0, \rho, 0)}{h(a, \rho, 0)}$	$\left[\frac{N}{N-a}\right]^{\nu+1}$	$\frac{h(0, \rho, \tau)}{h(a, \rho, \tau)} \left[\frac{N}{N-a}\right]^{\nu}$
pJ	$p(t, a)$	$J(t-a)$	$e^{-\beta a}$	$e^{-\beta a}$	1	1	$\left[\frac{N-a}{N}\right]^{\mu}$	$\left[\frac{N-a}{N}\right]^{\mu}$
cK	$c(t)$	$K(t, a)$	$\frac{1}{\delta}$	$\frac{1}{\rho+\delta}$	$N-a$	$h(a, \rho, 0)$	$\frac{N-a}{\tau+1}$	$h(a, \rho, \tau)$
cG	$c(t)$	$G(t, a)$	$\frac{e^{-\alpha a}}{\delta}$	$\frac{e^{-\alpha a}}{\rho+\delta}$	$N-a$	$h(a, \rho, 0)$	$\frac{N-a}{\tau+1} \left[\frac{N-a}{N}\right]^{\nu}$	$h(a, \rho, \tau) \left[\frac{N-a}{N}\right]^{\nu}$
cH	$c(t)$	$H(t, a)$	$\frac{1}{\delta}$	$\frac{1}{\rho+\delta}$	N	$h(0, \rho, 0)$	$\frac{N}{\tau+1}$	$h(0, \rho, \tau)$
cJ	$c(t)$	$J(t-a)$	$\frac{e^{-\delta a}}{\delta}$	$\frac{e^{-\delta a}}{\rho+\delta}$	$N-a$	$h(a, \rho, 0)$	$\frac{N-a}{\tau+1} \left[\frac{N-a}{N}\right]^{\tau}$	$h(a, \rho, \tau) \left[\frac{N-a}{N}\right]^{\tau}$

TABLE 5:
CAPITAL STOCK THIRD COMPONENT DEPENDING ON DISCOUNT RATE AND AGE?

Label	Example 1		Example 2		Example 3	
	ρ -dep.?	a -dep.?	ρ -dep.?	a -dep.?	ρ -dep.?	a -dep.?
qK	No	Invar.	Yes	Decr.	Yes	Decr.
qG	No	Decr.	Yes	Decr.	Yes	Decr.
qH	No	Invar.	No	Invar.	No	Invar.
qJ	No	Decr.	Yes	Decr.	Yes	Decr.
rK	No	Invar.	No	Invar.	No	Invar.
rG	No	Decr.	No	Invar.	No	Decr.
rH	No	Invar.	Yes	Incr.	Yes	Incr.
rJ	No	Decr.	No	Invar.	No	Decr.
pK	No	Incr.	No	Invar.	No	Incr.
pG	No	Invar.	No	Invar.	No	Invar.
pH	No	Incr.	Yes	Incr.	Yes	Incr.
pJ	No	Decr.	No	Invar.	No	Decr.
cK	Yes	Invar.	Yes	Decr.	Yes	Decr.
cG	Yes	Decr.	Yes	Decr.	Yes	Decr.
cH	Yes	Invar.	Yes	Invar.	Yes	Invar.
cJ	Yes	Decr.	Yes	Decr.	Yes	Decr.

TABLE 6:
DECOMPOSITIONS OF AGE-SPECIFIC CAPITAL SERVICE VALUE $W(t, a)$

Price	Quant.	Qual., Ex. 1		Qual., Ex. 2		Qual., Ex. 3	
		$\rho = 0$	$\rho > 0$	$\rho = 0$	$\rho > 0$	$\rho = 0$	$\rho > 0$
$c(t)$	$K(t, a)$	1	1	1	1	1	1
$c(t)$	$G(t, a)$	$e^{-\alpha a}$	$e^{-\alpha a}$	1	1	$\left[\frac{N-a}{N}\right]^\nu$	$\left[\frac{N-a}{N}\right]^\nu$
$c(t)$	$J(t-a)$	$e^{-\delta a}$	$e^{-\delta a}$	1	1	$\left[\frac{N-a}{N}\right]^\tau$	$\left[\frac{N-a}{N}\right]^\tau$
$c(t)$	$L(t, a)$	$\frac{1}{\delta}$	$\frac{1}{\rho+\delta}$	N	$h(0, \rho, 0)$	$\frac{N}{\tau+1}$	$h(0, \rho, \tau)$
$z(t, a)$	$K(t, a)$	$e^{\gamma a}$	$e^{\gamma a}$	1	1	$\left[\frac{N}{N-a}\right]^\nu$	$\left[\frac{N}{N-a}\right]^\nu$
$z(t, a)$	$G(t, a)$	1	1	1	1	1	1
$z(t, a)$	$J(t-a)$	$e^{-\beta a}$	$e^{-\beta a}$	1	1	$\left[\frac{N-a}{N}\right]^\mu$	$\left[\frac{N-a}{N}\right]^\mu$
$z(t, a)$	$L(t, a)$	$\frac{e^{\gamma a}}{\delta}$	$\frac{e^{\gamma a}}{\rho+\delta}$	N	$h(0, \rho, 0)$	$\frac{N}{\tau+1} \left[\frac{N}{N-a}\right]^\nu$	$h(0, \rho, \tau) \left[\frac{N}{N-a}\right]^\nu$
$q(t)$	$K(t, a)$	δ	$\rho+\delta$	$\frac{1}{N}$	$\frac{1}{h(0, \rho, 0)}$	$\frac{\tau+1}{N}$	$\frac{1}{h(0, \rho, \tau)}$
$q(t)$	$G(t, a)$	$\delta e^{-\alpha a}$	$(\rho+\delta)e^{-\alpha a}$	$\frac{1}{N}$	$\frac{1}{h(0, \rho, 0)}$	$\frac{\tau+1}{N} \left[\frac{N-a}{N}\right]^\nu$	$\frac{1}{h(0, \rho, \tau)} \left[\frac{N-a}{N}\right]^\nu$
$q(t)$	$J(t-a)$	$\delta e^{-\delta a}$	$(\rho+\delta)e^{-\delta a}$	$\frac{1}{N}$	$\frac{1}{h(0, \rho, 0)}$	$\frac{\tau+1}{N} \left[\frac{N-a}{N}\right]^\tau$	$\frac{1}{h(0, \rho, \tau)} \left[\frac{N-a}{N}\right]^\tau$
$q(t)$	$L(t, a)$	1	1	1	1	1	1

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APPENDIX A: SYMBOL LIST

t	: Time
a	: Age
$J(t)$: Gross investment
$K(t, a)$: Gross capital stock = Capital quantity in efficiency units
$G(t, a)$: Capital quantity stock in physical units
$H(t, a)$: Net capital stock = Capital stock value deflated by investment price
$L(t, a)$: Net capital service flow = Capital service value deflated by investment price
$V(t, a)$: Capital stock value
$W(t, a)$: Capital service value
$q(t)$: Investment price
$p(t, a)$: Price per physical capital stock unit
$r(t, a)$: Price per efficiency capital stock unit
$c(t, a)$: Efficiency unit price per capital service unit
$z(t, a)$: Physical unit price per capital service unit
$S(a)$: Capital survival function
$E(a)$: Capital efficiency function
$B(a)$: Combined capital survival-efficiency function = $S(a)E(a)$
$\phi(a)$: Prospective (discounted) service flow per efficiency capital unit which has attained age a
$\omega(a)$: Service flow from one new capital unit generated after age a

APPENDIX B: THE AUXILIARY FUNCTION $h(a, \rho, \tau)$. PROPERTIES AND PROOFS

The function is defined as

$$(B.1) \quad h(a, \rho, \tau) = \left(1 - \frac{a}{N}\right)^{-\tau} \int_a^N e^{-\rho(z-a)} \left(1 - \frac{z}{N}\right)^{\tau} dz$$

Substituting $s = z/N$, $b = a/N$, $\theta = \rho N$, it can be written as

$$(B.2) \quad h(a, \rho, \tau) = N(1-b)^{-\tau} \int_b^1 e^{-\theta(s-b)} (1-s)^{\tau} ds$$

Letting

$$(B.3) \quad f(b, \theta, \tau) = \int_b^1 e^{-\theta(s-b)} (1-s)^{\tau} ds,$$

we can write (B.2) as

$$(B.4) \quad h(a, \rho, \tau) = N(1-b)^{-\tau} f(b, \theta, \tau) = N \left(1 - \frac{a}{N}\right)^{-\tau} f\left(\frac{a}{N}, \rho N, \tau\right).$$

Using integration by parts, (B.3) can be shown to satisfy

$$(B.5) \quad f(b, \theta, \tau) = \frac{1}{\theta} [(1-b)^{\tau} - \tau f(b, \theta, \tau-1)], \quad \theta \neq 0, \tau \geq 1.$$

Multiplying this expression by $N(1-b)^{-\tau}$, substituting $s = z/N$, $b = a/N$, $\theta = \rho N$, and rearranging, we find that the recursion expressed in terms of $h(a, \rho, \tau)$ is

$$(B.6) \quad h(a, \rho, \tau) = \frac{1}{\rho} \left[1 - \frac{\tau}{N-a} h(a, \rho, \tau-1)\right], \quad \rho > 0, \tau \geq 1.$$

When the conditions $\tau \geq 1$ and $\rho \neq 0$ are not satisfied, it follows from (B.1) that

$$(B.7) \quad h(a, \rho, 0) = \int_a^N e^{-\rho(z-a)} dz = \frac{1}{\rho} [1 - e^{-\rho(N-a)}], \quad \rho > 0,$$

$$(B.8) \quad h(a, 0, \tau) = \left(1 - \frac{a}{N}\right)^{-\tau} \int_a^N \left(1 - \frac{z}{N}\right)^{\tau} dz = \frac{N-a}{\tau+1}, \quad \tau \geq 0.$$

Equation (B.7) provides the initial value when the recursion (B.6) is applied for $\tau = 1, 2, \dots$