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Last 10 Memoranda

No 07/16	Geir B. Asheim *Sustainable growth*
No 06/16	Arnaldur Sölvi Kristjánsson *Optimal Taxation with Endogenous Return to Capital*
No 05/16	Øystein Kravdal *Expected and unexpected consequences of childbearing – a methodologically and politically important distinction that is overlooked*
No 04/16	Dag Fjeld Edvardsen, Finn R. Førsund and Sverre A. C. Kittelsen *Productivity Development of Norwegian Institutions of Higher Education 2004 – 2013*
No 03/16	Finn R. Førsund *Multi-equation modelling of Desirable and Undesirable Outputs Satisfying the Material Balance*
No 02/16	Ingrid Hjort *Potential Climate Risks in Financial Markets: Report from a workshop, January 20, 2016*
No 01/16	Ingrid Hjort *Potential Climate Risks in Financial Markets: A Literature Overview*
No 22/15	Geir B. Asheim and Frikk Nesje *Destructive intergenerational altruism*
No 21/15	Rolf Golombek, Alfonso A. Irarrazabal, Lin Ma *OPEC's market power: An Empirical Dominant Firm Model for the Oil Market*
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Optimal control theory with applications to resource and environmental economics

Michael Hoel

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Abstract

This note gives a brief, non-rigorous sketch of basic optimal control theory, which is a useful tool in several simple economic problems, such as those in resource and environmental economics. While the mathematical analysis in the note is self-contained, there is not much explanation and intuition on the economic issues. The note should therefore be read together with articles or books that give more discussion of the economics of the problems considered.

Keywords: optimal control theory, exhaustible resources, renewable resources, climate change, water management

JEL classification: C61, Q2, Q3, Q5

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1 Introduction

This note gives a brief, non-rigorous sketch of basic optimal control theory, which is a useful tool in several simple economic problems, such as those in resource and environmental economics. The basic theory is given in section 2, and the subsequent sections are applications to several problems in resource and environmental economics. The note should be read together with articles or books that give more discussion of the economics of the problems considered.

2 The basic theory

Consider the dynamic optimization problem

$$\max \int_0^\infty e^{-rt} f(x(t), S(t), t) dt \tag{1}$$

subject to

$$\dot{S}(t) = g(x(t), S(t), t) \tag{2}$$

$$S(0) = S_0 \text{ historically given} \tag{3}$$

$$S(t) \geq 0 \text{ for all } t \tag{4}$$

where f and g are continuous and differentiable functions (and in many cases concave in (x, S)), and r is an exogenous positive discount rate. The variable $S(t)$ is a *stock* variable, also called a *state* variable, and can only change gradually over time as given by (2). The variable $x(t)$, on the other hand, is a variable that the decision maker chooses at any time. It is often called a *control* variable. In many economic problems the variable $x(t)$ will be constrained to be non-negative.

Remark 1: In the problem above there is only one control variable and one state variable. It is straightforward to generalize to many control and state variables, and the number of control variables need not be equal to the number of state variables.

Remark 2: The constraint (4) is more general than it might seem, as we often can reformulate the problem so we get this type of constraint. Assume e.g. that the constraint was $S(t) \leq \bar{S}$. We can then reformulate the problem by defining $Z(t) = \bar{S} - S(t)$, implying that $Z(t) \geq 0$. In this case the dynamic equation (2) must be replaced by $\dot{Z} = -g(x(t), \bar{S} - Z(t), t)$ and $S(t)$ in (1) must be replaced by $\bar{S} - Z(t)$.

The current value Hamiltonian

The current value Hamiltonian H is defined as

$$H(x, S, \lambda, t) = f(x, S, t) + \lambda g(x, S, t) \tag{5}$$

where $\lambda(t)$ is continuous and differentiable. The variable $\lambda(t)$ is often called a *co-state variable.* This variable will be non-negative in all problems where "more of the state variable" is "good". More precisely: The derivative of the maximized integral in (1) with respect to S_0 is equal to $\lambda(0)$. For this reason $\lambda(t)$ is also often called the *shadow price* of the state variable $S(t)$.

Conditions for an optimal solution

A solution to the problem (1)-(4) is a time path of the control variable $x(t)$ and an associated time path for the state variable $S(t)$. For optimal paths, there exist a differentiable function $\lambda(t)$ and a piecewise continuous function $\gamma(t)$ such that the following equations must hold for all t:

$$\frac{\partial H(x(t), S(t), \lambda(t), t)}{\partial x} = 0 \tag{6}$$

$$\dot{\lambda}(t) = r\lambda(t) - \frac{\partial H(x(t), S(t), \lambda(t), t)}{\partial S} - \gamma(t) \tag{7}$$

$$\gamma(t) \geq 0 \text{ and } \gamma(t)S(t) = 0 \tag{8}$$

$$Lim_{t \to \infty} e^{-rt}\lambda(t)S(t) = 0 \tag{9}$$

Remark 3: If $x(t)$ is constrained to be non-negative, (6) must be replaced by $\frac{\partial H}{\partial x} \leq 0$ and $\frac{\partial H}{\partial x}x(t) = 0$.

Remark 4: An alternative and equivalent formulation of our problem would be to include the term $+\gamma S$ in the Hamilton defined by (5). With this modification equation (7) would simply be (ignoring the time references) $\dot{\lambda} = r\lambda - \frac{\partial H}{\partial S}$.

Remark 5: If we know from the problem that $S(t) > 0$ for all t, we can forget about $\gamma(t)$, since it always will be zero.

Remark 6: Condition (9) is a transversality condition. Transversality conditions are simple in problems with finite horizons, but considerably more complicated for problems with an infinite horizon (like our problem). Most often, however, the simple condition (9) can be included in the set of necessary conditions.

Remark 7: If f and g are concave in (x, S) and $\lambda(t) \geq 0$, the conditions (6)-(9) are sufficient for an optimal solution. If we can find a time path for $x(t)$ and for $S(t)$ satisfying (6)-(9) in this case, we thus know that the time paths $(x(t), S(t))$ are optimal.

Remark 8: As mentioned in Remark 1, it is straightforward to generalize to many control and state variables. If there are n state variables, there are also n co-state variables $(\lambda_1, ...\lambda_n)$, n Lagrangian multipliers $(\gamma_1...\gamma_n)$, and n differential equations of each of the types (2) and (7).

3 The optimal use of a non-renewable resource

Consider the dynamic optimization problem

$$\max \int_0^\infty e^{-rt} u((x(t))dt \tag{10}$$

subject to

$$\dot{S}(t) = -x(t) \tag{11}$$

$$S(0) = S_0 \text{ historically given initial resource stock} \tag{12}$$

$$x(t) \geq 0 \text{ for all } t \tag{13}$$

$$S(t) \geq 0 \text{ for all } t \tag{14}$$

where $u(0) = 0$, $u' > 0$, $u'' < 0$ and $u'(0) = b$. The Hamiltonian in this case is

$$H(x, S, \lambda) = u(x) - \lambda x$$

and the conditions (6)-(9) are now

$$\frac{\partial H}{\partial x} = u'(x(t)) - \lambda(t) \leq 0 \text{ and } [u'(x(t)) - \lambda(t)]\, x(t) = 0 \tag{15}$$

$$\dot{\lambda}(t) = r\lambda(t) - \gamma(t) \tag{16}$$

$$\gamma(t) \geq 0 \text{ and } \gamma(t)S(t) = 0 \tag{17}$$

$$Lim_{t\to\infty} e^{-rt}\lambda(t)S(t) = 0 \tag{18}$$

As long as $S(t) > 0$ we have $\gamma(t) = 0$ implying from (16) that

$$\dot{\lambda}(t) = r\lambda(t) \text{ or } \lambda(t) = \lambda(0)e^{rt} \tag{19}$$

It follows from (15) and (19) that $\dot{x}(t) \leq 0$. For $x(t) > 0$ we have

$$u'(x(t) = \lambda(0)e^{rt}$$

giving a declining $x(t)$. At some time T , $\lambda(0)e^{rT} = b$, giving $x(T) = 0$ since $u'(0) = b$. The resource stock must reach 0 at T: $S(T) < 0$ would violate the condition $S(t) \geq 0$ for all t, while $S(T) > 0$ would violate the transversality condition (18).

To conclude: Optimal resource extraction declines gradually over time, making the marginal utility u' rise over time at the rate r. Extraction eventually reaches zero; this occurs simultaneously with the resource stock being completely depleted. In a market economy u' may be interpreted as the price of the good, so that our result states that the optimal resource price should rises at the rate of interest until the resource is completely depleted. This resul twas first given by Hotelling (1931).

4 Non-renewable resource extraction with rising extraction costs

In section 3 it was assumed that resource extraction was limited by a physical limit of available resources. For most resources it is more reasonable to assume that there resource extraction is limited by the costs of extraction, and that extraction costs are higher the lower are the remaining physical resources. With the same notation as in section 2, we can model this as extraction costs being equal to $xc(S)$, with $c(S)$ having the properties that $c' < 0$ and $c(0) > u'(0) = b$. In words, the last inequality says that as the resource stock becomes sufficiently small, the extraction cost will exceed the marginal willingness to pay for even a small amount of the resource. This inequality implies that it will never be optimal to extract all of the resource. Hence, the restriction $S(t) \geq 0$ will be redundant in the present case.

The optimization problem in section 3 is now changed to

$$\max \int_0^\infty e^{-rt} \left[u((x(t)) - x(t)c(S(t)\right] dt \tag{20}$$

subject to

$$\dot{S}(t) = -x(t) \tag{21}$$

$$S(0) = S_0 \text{ historically given initial resource stock} \tag{22}$$

$$x(t) \geq 0 \text{ for all } t \tag{23}$$

where as before $u(0) = 0$, $u' > 0$, $u'' < 0$ and $u'(0) = b$. The Hamiltonian in this case is

$$H(x, S, \lambda) = u(x) - xc(S) - \lambda x \tag{24}$$

and the conditions (6)-(9) are now replaced by (for $x(t) > 0$, see below)[1]

$$\frac{\partial H}{\partial x} = u'(x) - c(S) - \lambda(t) = 0 \tag{25}$$

[1] From now on we skip explicit time references at most places where this cannot cause any misunderstanding.

$$\dot{\lambda} = r\lambda + xc'(S) \tag{26}$$

$$Lim_{t\to\infty} e^{-rt}\lambda(t)S(t) = 0 \tag{27}$$

Whenever extraction is positive it follows from (25) that

$$u'(x) = c(S) + \lambda \tag{28}$$

Differentiating with respect to time gives

$$\dot{u}' = c'(S)\dot{S} + \dot{\lambda}$$

Inserting (21), (26) and (28) into this expression gives

$$\dot{u}' = r\left[u'(x) - c(S)\right] \tag{29}$$

From (26) it is clear that if extraction stops while $\lambda \neq 0$, the transversality condition (27) will be violated. Similarly, if λ becomes negative, it follows from (26) that the transversality condition will be violated (since $c' < 0$). The equilibrium time path of $\lambda(t)$ therefore must reach zero as extraction approaches zero. In other words, the long-run stock of the resource must approach a value S^* given by $c(S^*) = b$.

Note that the resource stock S^* will only be reached asymptotically. To see this assume that S^* is reached at some finite date T. The dynamics of the system imply that all variables remain constant from T and onwards. Moreover, the same dynamics imply that all variables remain constant also when we move *backwards* in time from T. But this can only be a solution to our equations if we already are at the steady state initially.

5 Optimal climate policy for a given carbon budget

Allen et al. (2009) have argued that it is the total amount of carbon emitted to the atmosphere that determines long-run climate change To quote:

the relationship between cumulative emissions and peak warming is remarkably insensitive to the emission pathway (timing of emissions or peak emission rate). Hence policy targets based on limiting cumulative emissions of carbon dioxide are likely to be more robust to scientific uncertainty than emission-rate or concentration targets. Total anthropogenic emissions of one trillion tonnes of carbon (3.67 trillion tonnes of CO2), about half of which has already been emitted since industrialization began, results in a most likely peak carbon-dioxide induced warming of 2°C above pre-industrial temperatures, with a 5–95% confidence interval of 1.3–3.9⁰C.

Based on this, we consider the same problem as in section 3, but now let $u(x)$ be the benefit of emitting carbon (x), i.e. of using fossil fuels. Moreover, let S_0 be the total amount of carbon emissions in the future (from date $t = 0$) that are consistent with a political goal of total temperature increase. The model describes how u' must develop over time. Users of carbon set u' equal to the carbon tax. Hence we can conclude that the optimal carbon tax must rise at interest rate. Moreover, the *level* of this carbon tax is higher the lower is S_0, i.e. the lower temperature increase we accept.

6 Stock pollution with an environmental damage cost function.

Section 5 described a stock pollution problem. Instead of an absolute limit to the stock of pollution, assume now that there at time t is an environmental cost $D(S(t), t)$ of a stock $S(t)$ of the pollutant in the environment, and that $D_S > 0$ and $D_{SS} \geq 0$. Notice that the relationship between S and D may vary over time; if e.g. income growth implies an increased willingness to pay for avoiding environmental damage we will have $D_t > 0$. The sign and size of D_t is, however, of no importance for the derivations below. (We could also assume that u depends directly on t, this would not affect the analysis below.)

Assume that the development of the stock S depends on the flow x as follows:

$$\dot{S}(t) = x(t) - \delta S(t) \tag{30}$$

where $\delta \geq 0$. This way of modeling the depreciation of carbon in the atmosphere is clearly a drastic simplification, and for $\delta > 0$ is not consistent with the insight from Allen et al. (2009): A less drastic simplification wold be to model the depreciation as in e.g. Farzin and Tahvonen (1996) or Hoel (2011), which is consistent with the recommendations by David Archer (2005) when modeling atmospheric carbon and its decay. He states that *A better approximation of the lifetime of fossil fuel CO2 for public discussion might be "300 years, plus 25% that lasts forever".*

The constraints (11)-(13) remain valid, and it follows from (13) and (30) that $S(t)$ can never become negative, so we need not explicitly include the constraint (14).

The optimization problem is now

$$\max \int_0^\infty e^{-rt} \left[u((x(t)) - D(S(t), t) \right] dt \tag{31}$$

and the Hamiltonian is in this case

$$H(x, S, \lambda, t) = u(x) - D(S, t) + \lambda \left[x - \delta S \right]$$

The conditions (6)-(9) are now

$$\frac{\partial H}{\partial x} = u'(x) + \lambda(t) \leq 0 \text{ and } \left[u'(x) + \lambda(t) \right] x = 0 \tag{32}$$

$$\dot{\lambda} = (r + \delta)\lambda + D_S(S, t) \tag{33}$$

$$Lim_{t \to \infty} e^{-rt} \lambda(t) S(t) = 0 \tag{34}$$

Assume that the problem has properties implying $x(t) > 0$ for all t. Then $S(t)$ must also always be positive, so that the transversality condition implies $Lim_{t\to\infty} e^{-rt}\lambda(t) = 0$.

It is useful to define $q(t) = -\lambda(t)$. Since $x(t)$ is always positive we can rewrite (32) and (33) as

$$u'(x) = q(t) \tag{35}$$

$$\dot{q} = (r + \delta)q - D_S(S, t) \tag{36}$$

We may interpret $q(t)$ as the optimal emission tax. If we know how this tax evolves over time we can deduce from (35) how emissions $x(t)$ will evolve over time. To solve for $q(t)$ we first define

$$\mu(t) = e^{-(r+\delta)t}q(t) \tag{37}$$

implying

$$\dot{\mu}(t) = -(r + \delta)e^{-(r+\delta)t}q(t) + e^{-(r+\delta)t}\dot{q}(t)$$

Inserting from (36) gives

$$\dot{\mu}(t) = -e^{-(r+\delta)t}D_S(S(t), t) \tag{38}$$

By definition, we have for any $T > t$ that

$$\mu(T) - \mu(t) = \int_t^T \dot{\mu}(\tau)d\tau$$

Letting $T \to \infty$ and inserting (38) gives

$$\mu(t) = Lim_{T\to\infty}\mu(T) + \int_t^\infty e^{-(r+\delta)\tau}D_S(S(\tau), \tau)d\tau \tag{39}$$

Since $Lim_{T\to\infty}e^{-rT}\lambda(T) = 0$, it follows that $Lim_{t\to\infty}e^{-rT}\left[-e^{(r+\delta)T}\mu(T)\right] =$

$-Lim_{t\to\infty}e^{\delta T}\mu(T) = 0$, which can only hold if $Lim_{t\to\infty}\mu(T) = 0$. Hence, it follows from (37) and (39) that

$$q(t) = \int_t^\infty e^{-(r+\delta)(\tau-t)} D_S(S(\tau),\tau)d\tau \tag{40}$$

This equation for the optimal emission tax has an obvious interpretation: The amount of 1 unit of emissions at time t remaining in the atmosphere at $\tau(> t)$ is $e^{-\delta(\tau-t)}$. To get from the additional stock at τ to additional damages at τ we must multiply the additional stock at τ by the marginal damage at τ, which is $D_S(S(\tau),\tau)$, giving a damage equal to $e^{-\delta(\tau-t)}D_S(S(\tau),\tau)$ for 1 unit emissions at t. The total additional damage caused by 1 unit of emissions at time t is the discounted sum of additional damages at all dates from t to infinity caused by the additional stocks from t to infinity. The marginal damage of 1 additional unit of emissions at t is thus given by the expression above.

7 Renewable biological resources

The resource stock of non-renewable resources is always declining for positive extraction (and constant for zero extraction). For renewable biological resources such as e.g. fish or forests the resource dynamics are different. In the absence of any harvesting of the resource the stock will grow according to a biological growth function $G(S(t))$, where $S(t)$ is the resource stock. The exact properties of the function $G(S)$ will of course depend on the resource considered. However, it is typically assumed that $G(S)$ is bell-shaped, i.e. positive for $0 < S < \bar{S}$. Hence, in the absence of any harvest the resource stock will gradually approach its maximal value $\bar{S}$.

When there is positive harvest $x(t)$, the net growth of the resource stock will be the gross biological growth minus the harvest:

$$\dot{S}(t) = G(S(t)) - x(t) \tag{41}$$

Replacing (21) with (41) is the main difference between the present case and the case treated in the previous section. We shall in addition assume a slightly more general cost function $C(x, S)$ where $C_x > 0$ and $C_S < 0$.

The optimization problem in section 6 is now changed to

$$\max \int_0^\infty e^{-rt} \left[u((x(t)) - C(x(t), S(t)\right] dt \tag{42}$$

subject to

$$\dot{S}(t) = G(S(t)) - x(t)$$

$$S(0) = S_0 \text{ historically given initial resource stock} \tag{43}$$

$$x(t) \geq 0 \text{ for all } t \tag{44}$$

$$S(t) \geq 0 \text{ for all } t \tag{45}$$

where as before $u(0) = 0$, $u' > 0$, $u'' < 0$ and $u'(0) = b$. We assume that costs so high for small values of S that the constraint $S(t) \geq 0$ is never binding.

The Hamiltonian in this case is

$$H(x, S, \lambda) = u(x) - C(x, S) + \lambda \left[G(S) - x \right] \tag{46}$$

and the conditions (25)-(27) are now replaced by (for $x(t) > 0$)

$$\frac{\partial H}{\partial x} = u'(x) - C_x(x, S) - \lambda = 0 \tag{47}$$

$$\dot{\lambda} = \lambda \left[r - G'(S) \right] + C_S(x, S) \tag{48}$$

$$Lim_{t \to \infty} e^{-rt} \lambda(t) S(t) = 0 \tag{49}$$

Whenever extraction is positive it follows from (25) that

$$u'(x) = C_x(x, S) + \lambda \tag{50}$$

An obvious question to ask is whether a steady-state solution can be optimal. By a steady state we mean a situation where the harvest and resource stock are both constant. Assume that such a steady state exists, and denote it by (x^*, S^*). It follows from (48) that λ in this case must be constant given by

$$\lambda^* = u'(x^*) - C_x(x^*, S^*) \tag{51}$$

Since λ is constant, we have $\dot{\lambda} = 0$, so that (48) implies

$$\lambda^* \left[r - G'(S^*)\right] = -C_S(x^*, S^*) \tag{52}$$

Finally, a constant resource stock implies from (41) that

$$x^* = G(S^*) \tag{53}$$

Since S and λ are constant, the transversality condition (49) is satisfied. Hence, the three equations (51)-(53) satisfy out optimality conditions. If the initial resource stock is equal to S^*, the optimal harvest is hence constant and given by the equations (51)-(53).

Usually the initial resource stock will not be equal to S^*. However, it is possible to show that for reasonable function forms the optimal solution will gradually approach the steady-state solution.[2]The analysis required for showing this will also reveal that $\lambda^* > 0$. From the interpretation of λ this is as expected: An increase in the stock of the resource will always give an increase in social welfare. Since $C_S < 0$, it follows from (52) that the long-run resource stock S^* must satisfy $G'(S^*) < r$.

8 Water management

Freshwater is used by households, agriculture, industry and for power generation. Water stored in reservoirs, lakes and aquifers is a stock, and its management is hence a problem that can be analyzed using optimal control theory. We first consider (section 8.1) a very simple case of an exogenous and time-independent gross inflow of water into the stock. In section 8.2 we turn

[2] As in the case studied in the previous section, the steady state will only be reached asymptotically.

to the management of a water stock that has a yearly cycle of inflow and a value (utility u) that also varies over the year. An obvious interpretation of this section is how one should manage water reservoirs through a year for the production of hydropower.

8.1 Constant water inflow and a stationary utility function

The problem is identical to the problem (41)-(45) except that we now assume that G is exogenous and constant and that the costs C are independent of the amount of water in the reservoir, i.e. the stock S. With the exception of (48), the conditions (46)-(50) are valid also for the present case. Since $G' = C_S = 0$ by assumption in the present case, the condition (48) is replaced by

$$\dot{\lambda}(t) = r\lambda(t) - \gamma(t) \tag{54}$$

with $\gamma(t) \geq 0$ and $\gamma(t)S(t) = 0$, since the constraint $S(t) \geq 0$ may now be binding for some t.

There are two possible outcomes. First, if x^0 defined by $u'(x^0) = 0$ does not exceed G, the solution is to have $x(t) = x^0$ for all t. In this case $\lambda(t) = \gamma(t) = 0$ for all t, and we immediately see that all our optimality conditions are satisfied.

The more interesting outcome is when $x^0 > G$. In this case $x(t)$ is determined by $u'(x(t)) = \lambda(t)$. Moreover, from (54) it is clear that $\lambda(t)$ rises at the rate r as long as $S(t) > 0$. This implies that the water use $x(t)$ will gradually decline over time, until at some date T we have $u'(G) = \lambda(T)$. The initial value $\lambda(0)$ is determined such that $S(t)$ reaches zero at the date T. After T we have $\gamma = r\lambda(T)$, so that λ remains constant after this date, making the water use x stay constant equal to the inflow G after this date.

From the discussion above it is clear that this problem is very similar to the problem considered in section 3. In both case the initial resource stock is completely depleted. The only difference is that while x was zero after depletion in section 3, we now have x equal to the exogenous and positive water inflow after depletion has occurred.

8.2 Optimal management of water for hydropower over a yearly cycle

For most issues related to water management, the inflow is not constant over time. There will typically be a yearly cycle with variations of inflow over the year. The demand for water, expressed by the utility function u, typically also varies through a year. Water used for the production of hydropower will in most countries have such properties. In Norway, the inflow is high from early May, when the snow melting in the mountains is high. The inflow remains relatively high til about October, after which the precipitation in the mountains comes in the form of snow. As for electricity demand, measured by marginal utility, it is higher in winter than in the summer, and also vary over the daily 24 hour cycle.

To model the yearly production of hydropower, we use a finite horizon version of our optimal control problem.[3] The time unit is now one hour, and the time horizon is one year, i.e. 8760 hours. The utility of electricity for a particular hour t of the year is $u(x(t), \sigma(t))$ where the development of $\sigma(t)$ over the year is exogenous and $u_{x\sigma} > 0$, so that demand is higher the higher is σ.

Discounting within the time frame of one year is of no practical importance; we therefore set $r = 0$. Our optimization problem for the period for period $[0, T]$ is hence

$$max \int_0^T u(x(t), \sigma(t))dt$$

subject to

$$\dot{S}(t) = G(t) - x(t) \tag{55}$$

$$S(0) = 0 \text{ initial (and final) water in reservoir}^4 \tag{56}$$

[3] See Førsund (2013) for a more thorough treatment of hydropower.

[4] Zero is simply a normalization; the minimum acceptable water in a reservoir is for environmental and other reasons actually positive. Replacing zero wtih an exogenous and positive term $S^{\min}$ would not change our analysis.

$$x(t) \geq 0 \text{ for all } t \tag{57}$$

$$x(t) \leq \bar{x} \text{ for all } t \tag{58}$$

$$S(t) \geq 0 \text{ for all } t \tag{59}$$

$$S(t) \leq \bar{S} \text{ for all } t \tag{60}$$

As in section 8.1, $G(t)$ is exogenous. However unlike in the previous case, $G(t)$ varies over the year.

The interpretation of the constraint (58) is that there is a maximal capacity limit to how much electricity can be produced in any hour.

We assume that the constraint $S(t) \geq 0$ is only binding at T. The interpretation of this is that during the first part of the whole period $[0, T]$, i.e. from May onwards in Norway, the exogenous inflow $G(t)$ is larger than the optimal production $x(t)$, so that the amount of water in the reservoir $S(t)$ is increasing. Later in the period, from about October in Norway, the opposite is true, and $S(t)$ declines towards zero[5] as we approach T.

The interpretation of the constraint (60) is that when $S(t) = \bar{S}$, the reservoir is full. Any additional inflow must be either be used for electricity production so the outflow is equal to the inflow, or additional precipitation simply overflows (so $\dot{S}(t) = 0$ even if $G(t) > x(t)$ in this case; however, as long as $u_x > 0$ it will not be optimal to let the water overflow).

The Hamiltonian is

$$H = u(x, \sigma) + \lambda(G - x) + \alpha(\bar{S} - S)$$

The term $+\alpha(\bar{S} - S)$ represents the constraint $S(t) \leq \bar{S}$,with $\alpha(t) \geq 0$ and $\alpha(t)(\bar{S} - S(t)) = 0$. See also Remark 4 in Section 2, and remember that, due to our assumption that in the optimal solution we have $S(t) > 0$ for $t \in (0, T)$, we need not explicitly include any term for the constraint $S(t) \geq 0$.[6]

[5]Zero is simply a normalization, the minimum acceptable water in a reservoir is for environmental and other reasons actually positive. Replacing zero wtih and exogenous and positive term $S^{\min}$ would not change our analysis.

[6]By introducing an additional state variable $Z(t) = \bar{S} - S(t)$ we could reformulate our problem so that it becomes formally identical to a generalized version (to two state variables) of the problem described in Section 2.

The conditions for the optimum are (for $x(t) > 0$)

$$\frac{\partial H}{\partial x} = u_x(x, \sigma) - \lambda \geq 0, \ = 0 \text{ for } x(t) < \bar{x} \tag{61}$$

$$\dot{\lambda} = -\frac{\partial H}{\partial S} = \alpha \geq 0, \ = 0 \text{ for } S(t) < \bar{S} \tag{62}$$

$$\lambda(T)S(T) = 0 \tag{63}$$

Consider first the simplest case when $x(t) < \bar{x}$ and $S(t) < \bar{S}$ for all t in the optimal solution. Then $u_x = \lambda$ for all t, and λ is constant. The level of λ is determined so that $S(T) = 0$. In other words, electricity production through the year is varied in a manner that makes the marginal utility (i.e. price) constant throughout the year.

Assume next that there is a time period $[t_1, t_2]$ during which the constraint (60) is binding. In Norway, such a period would typically occur in the fall after a summer with a lot of rain. Assuming that the constraint (60) is strictly binding in the sense that $\alpha > 0$, it is clear from (62) that $\lambda(t)$ will have a lower value before t_1 than after t_2. Hence, the electricity price u_x will be lower before t_1 (summer) than after t_2 (winter).

Notice also that when $\alpha > 0$ so that the constraint $S(t) = \bar{S}$ is binding, λ and hence the electricity price u_x is rising. This has an obvious interpretation: If u_x were declining, social welfare would increase by using more water at an early date and less at a later date. An adjustment of this type would not be prevented by the constraint $S(t) \leq \bar{S}$. A period of $S(t) = \bar{S}$ and u_x declining can therefore not be optimal.

Assume next that there is a time period $[t_3, t_4]$ during which the constraint (58) is binding. In Norway, such a period would typically occur in the winter when there is high demand. If this constraint is strictly binding, it follows from (61) that $u_x > \lambda$ during this period. Hence, the electricity price u_x will be higher during such a period of high demand than it is in other times of the year.

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