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Pollution Modelling and Multiple-Output Production Theory*

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Last 10 Memoranda

No 09/16	Frikk Nesje and Geir B. Asheim *Intergenerational altruism: A solution to the climate problem?**
No 08/16	Michael Hoel *Optimal control theory with applications to resource and environmental economics*
No 07/16	Geir B. Asheim *Sustainable growth*
No 06/16	Arnaldur Sölvi Kristjánsson *Optimal Taxation with Endogenous Return to Capital*
No 05/16	Øystein Kravdal *Expected and unexpected consequences of childbearing – a methodologically and politically important distinction that is overlooked*
No 04/16	Dag Fjeld Edvardsen, Finn R. Førsund and Sverre A. C. Kittelsen *Productivity Development of Norwegian Institutions of Higher Education 2004 – 2013*
No 03/16	Finn R. Førsund *Multi-equation modelling of Desirable and Undesirable Outputs Satisfying the Material Balance*
No 02/16	Ingrid Hjort *Potential Climate Risks in Financial Markets: Report from a workshop, January 20, 2016*
No 01/16	Ingrid Hjort *Potential Climate Risks in Financial Markets: A Literature Overview*
No 22/15	Geir B. Asheim and Frikk Nesje *Destructive intergenerational altruism*

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Discussion Paper #D-37/1998

Pollution Modelling and Multiple-Output Production Theory[*]

by

Finn R. Førsund

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[*] A preliminary version was presented at the Southern Economic Association Meeting, Baltimore 8-10 November 1998. I have benefited from comments made by Maureen Cropper as the discussant, and from Rolf Färe.

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Pollution Modelling
and
Multiple-Output Production Theory

Abstract

The materials balance principle points to the crucial role of material inputs in generating residuals in production processes. Pollution modelling must be of a multi-output nature. The most flexible transformation function in outputs and inputs used in textbooks is too general to make sense in pollution modelling. Specifying bads *as if* they are inputs, although defendable on a macro level, hides explicit considerations of various modification activities. A complete taxonomy of inputs as to the impact on both residuals and marketed products as joint outputs, is derived, based on factorially determined multi-output production, thus providing information for choice of policy instruments.

Key words: *multiple-output production, factorially determined multi output production, pollution, bads, purification, modification*

1. Introduction

Pollution is generically a problem with joint outputs in economic activities of production and consumption. The first law of thermodynamics tells us that matter cannot disappear. If we weigh the inputs into an activity, including non-paid factors like oxygen from the air, and weigh the products that are the planned purpose of activities, the difference is the residuals that may turn out to be polluting the natural environment. Thus, the general feature of residuals is that they arise from use of inputs in a wide sense. Ayres and Kneese (1969) coined the phrase *materials balance* to underline the inevitability of residuals generation when employing material resources.

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Although multiple outputs is the rule rather than exception at the micro level of production, economists usually specify single output only. This may be interpreted as a reflection of economics being concerned about basic principles of resource allocation, revealing economic mechanisms and incentive structures, devising policy instruments for influencing decision-making units, etc., and the wish to avoid complicating engineering details when pursuing such goals. Operating with a representative firm and an output aggregate at the micro level will usually suffice. To focus e.g. on the optimal size assortment of nails seems too small potatoes for economics. But as pointed out above single output is a difficult position to maintain when dealing with pollution. Therefore, some form of multiple outputs involving "ordinary" or intended outputs, and unintended residuals or pollutants, or generically "bads", have been used at least indirectly in the old externalities literature, and explicitly in the more recent environmental economics literature (see Mishan (1971) for a review of the externalities literature, and Fisher and Peterson (1976), Cropper and Oates (1992) for reviews of the literature covering the 70-ies and 80-ies decades). However, the choice in the literature of specifications of relationships between ordinary outputs and bads vary and is often not based on any explicit consideration of the most suitable model to pick from the field of multiple output production function theory. (But see Whitcomb (1972) for an outstanding exception.) Since policy conclusions that can be drawn on results from environmental economics are basically concerned with choice among instruments for control of externalities, a firm grasp on the modelling of the multi-output nature is essential.

The purpose of the paper is three-fold. Multiple output models are reviewed in Section 2. The most common way of including pollutants is studied in Section 3, and a critique of standard practises is offered. A more suitable multi output structure is discussed in Section 4, and a new and comprehensive taxonomy for inputs based on the materials balance concept, is offered. Section 5 concludes.

2. Multi-output production

When modelling multiple outputs we should be aware of some main types of multiple-output production (see e.g. Frisch (1965), Chapter 1d. for a brief introduction). Inputs may be employed alternatively to produce different outputs, e.g. a piece of agricultural land may be used to produce potatoes or wheat, a wood cutting tool may be used to produce different types of furniture. There is freedom of choice in what outputs to produce. At the other end of the scale we may have multiple outputs due to jointness in production; sheep yield mutton as well as wool, cattle yield beef and hide, growing wheat we get both wheat and straw, and coal can be converted to coke and gas, to use classical examples from Edgeworth and Marshall. As an extreme form of jointness we have that outputs are produced in fixed proportions, as the distillates of crude oil in a refinery. We will regard a firm as the unit, and not discuss issues concerning internal organisation such as parallel production of commodities or process chains of intermediate products, etc. (see e.g. Danø, 1966).

The standard multi-output representation

When employing a functional representation of multi-output production the common practice is to specify a transformation function as a continuously differentiable manifold in the m-dimensional output vector, y, and the n-dimensional input vector, x:

$$F(y,x) \leq 0 \ , \ y \in R_{+}^{m} \ , \ x \in R_{+}^{n} \ , \ \frac{\partial F(y,x)}{\partial y_i} \geq 0 \ , \ \frac{\partial F(y,x)}{\partial x_j} \leq 0 \ , \quad i=1,..,m \ , \ j=1,..,n \tag{1}$$

Efficient utilisation of resources is associated with equality sign and inefficient operations with the inequality sign. As a standard convention outputs have non-negative partial derivatives and inputs non-positive ones. The trade-offs between outputs for given inputs may be termed *factor isoquants*, and substitution possibilities between inputs for given outputs *product isoquants* (Frisch, 1965). The standard concept of marginal productivity of input j in the production of output i is expressed by $dy_i/dx_j = -F_{xj}/ F_{yi}$.

The Frisch multi-output model

The case of freedom in directing inputs into any output is termed *assorted* production in Frisch (1965). The core production function apparatus of Frisch (1965), Part four) is based on these concepts, and may be described by a set of functions[2]:

$$F^i(y_1,..,y_m,x_1,..,x_n) \leq 0 \ , \ \frac{\partial F^i}{\partial y_s} \geq 0 \ , \ \frac{\partial F^i}{\partial x_j} \leq 0 \ , \qquad i=1,..,\mu \ , \ s=1,..,m \ , \ j=1,..,n \tag{2}$$

The sign convention for outputs and inputs is maintained, remembering that all y's are final outputs, intermediate products are excluded by definition. The relation between number of outputs and equations is defined as the *degree of assortment*[3]: $\alpha = m - \mu$. A special case is $\mu = 1$. We are then back to the common textbook specification (1) of multiple outputs. The degree of assortment is then m - 1, or in the Frisch spirit we have m-1 - dimensional freedom of assortment, i.e. the maximal degree. The flexibility in combining outputs and inputs is maximal. A negative number means that there exist one or more *pure factor bands*, i.e. there are relationships as part of (2) between inputs independent of outputs:
$F^b(x_1,..,x_n) = 0$.

In the case of specific technical relations between products Frisch talks about *couplings* between the outputs. The *product couplings* restrict the freedom of combining outputs. The

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[2] Whitcomb (1972) is the only one to my knowledge that explicitly discusses the appropriate form of multiple-outputs model to use. He sets up a system with each firm both receiving and generating externalities as abstract concepts, but uses electricity production based on fossil fuel as an example, thus illustrating the main purpose of doing multiple outputs in the present paper. Although he refers to Frisch in general he does not relate his system to Frisch. In fact, his specification is a special variant of (2).

[3] As Frisch puts it in characteristic style on p.270: "mnemonically α may be thought of as the first letter in the word "assortment" ".

degree of couplings[4], κ, is the number of relations between outputs, in the system (2), independent of inputs:

$F^c(y_1,..,y_m) = 0$.

Note that the sign convention for partial derivatives cannot apply to factor bands or product couplings. Obviously the direction of a coupling, e.g. $dy_1 / dy_2 = -F_{y2}^c / Fy_1^c$ should be unrestricted in sign, as well as the corresponding direction within a factor band. Thus the sign convention only applies to relations in (2) with both outputs and inputs simultaneously present.

A special case of (2) is the case of *no assortment*. We will later argue that this case is of especial relevance for pollution modelling. Frisch calls this case *factorially determined* multi-output production:

$$y_i = f^i(x_1,..,x_n) \ , \ i=1,..,m \qquad \textbf{(3)}$$

The degree of assortment is (m - m) = 0, meaning that for a given set of inputs, all the outputs are determined. However, this is not the same as output couplings. There are product isoquant maps for each function in (3). To the degree that these do not coincide, we will realise outputs in different proportions varying the inputs. In the terminology of Frisch, we have *product separation.* If the isoquant maps coincide completely we then have the case of couplings, i.e. the outputs cannot be separated. The coupling may be proportional at the simplest, or exhibiting more complex variability. (See Figure 1 in Section 3 for illustrations.)

Comparing the special case (1) with the Frisch variants the latter take us a step closer to real engineering specifications, while keeping a level of aggregation typical of economics (see e.g. Førsund (1998) for some simple examples of engineering relations conforming to the Frisch system).

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[4] Frisch, p.270: “mnemonically κ may be thought of as the first letter in the word “coupling” ” .

3. Pollution modelling

As stated in the Introduction, a fundamental observation as to modelling pollution generation by economic activity is that residuals are an inherent part of this activity, following the "materials balance" approach (Ayres and Kneese, 1969). Generation of residuals can therefore be regarded as joint outputs in economic activities.

Since Equation (1) is the standard representation of multi-output production possibilities it is natural first to extend this formulation also represent pollutant generation, z (see Baumol and Oates (1988), p. 37):

$$F(y,z,x) = 0 \ , \tag{4}$$

where z is the residuals vector of k elements. In accordance with the basic nature of the residuals as joint outputs, following the sign convention of the partial derivatives, F_{zs} should be positive. However, we immediately get into problems with such a formulation, as shown below. Our purpose of pollution modelling is to demonstrate the importance of multiple-output modelling. The framework is therefore as simple as possible and of a partial equilibrium nature. The social planning problem is to maximise consumer plus producer surplus, introducing demand functions on price form, $p_i(y_i)$, for each output, using fixed q_j's as social evaluation coefficients for inputs, and evaluating pollutants through the monetised *damage function* $D(z_1,..,z_k)$ (where $D_{zs} \geq 0$). (We may also think of a firm selling products to fixed prices, p_i, and buying inputs to fixed prices, q_j, in competitive markets, and paying a non-linear pollution tax, D(z) .) The maximisation problem for a single (representative) firm, characterised by some multiple-output technology, is then:

$$Max = \sum_{i=1}^{m} \int_{w_i=0}^{y_i} p_i(w_i)dw_i - \sum_{j=1}^{n} q_j x_j - D(z_1,..,z_k) \tag{5}$$
$$s.t.\ a\ multiple\text{-}output\ technology$$

When considering several firms demand functions have to be adjusted according to type of demand interactions, and it must be specified whether the damage functions are unique to each firms, or the nature of interactions if there are any. This simple model makes the fundamental trade-offs between “ordinary” goods, y_i, and “bads”, z_s, transparent, allowing for Pareto-optimal allocation rules both for these two types of outputs and for inputs, and facilitates introduction of policy instruments implementing optimal solutions, etc. The explicit representation of pollutants and environmental damage distinguishes it from the models of the externalities-literature, where the transmission mechanisms of external effects are implicit.

Pollutants as outputs in the standard formulation

To see the problem with pollutants, z, as outputs let us assume that the production possibilities are characterised by (4). The necessary first order conditions of the social planning problem (5) are:

$$\begin{aligned} p_i - \lambda F'_{y_i} &\begin{pmatrix} = 0 \\ \le 0 \end{pmatrix} \textit{for} \begin{pmatrix} y_i > 0 \\ y_i = 0 \end{pmatrix}, \ i=1,..,m \\ -q_j - \lambda F'_{x_j} &\begin{pmatrix} = 0 \\ \le 0 \end{pmatrix} \textit{for} \begin{pmatrix} x_j > 0 \\ x_j = 0 \end{pmatrix}, \ j=1,..,n \\ -D'_{z_s} - \lambda F'_{z_s} &\begin{pmatrix} = 0 \\ \le 0 \end{pmatrix} \textit{for} \begin{pmatrix} z_s > 0 \\ z_s = 0 \end{pmatrix}, \ s=1,..,k \end{aligned} \qquad \textbf{(6)}$$

where λ is the shadow price on the production possibilities, expressing alternative costs. Considering interior solutions the first two conditions can be combined to give the textbook result that a factor should be employed so that the value of its marginal productivity equals factor cost:

$$q_j = p_i \frac{-F'_{x_j}}{F'_{y_i}} \qquad \textbf{(7)}$$

There is no explicit reflection here of pollutants generation, i.e. no explicit "cost punishment" for neither inputs or outputs possibly associated with generation of pollutants.

PROPOSITION 1: *The socially optimal level of bads is zero when the multi output technology including bads has maximal flexibility, i.e. the degree of assortment,* $\alpha = m - \mu = m - 1$ $(\mu = 1)$ *in relation (4), is maximal.*

Discussion: From the last condition in (4) it follows that the generation of the polluting residual should be set to zero since equality cannot be obtained in the regular case considered here. Since there is no explicit formalisation of the inevitability of residuals in the formulation (4), this is the logical result, pollutants being "bads" in the objective function. Therefore, marginal rates of transformation like (5) will naturally not reflect any pollution generation. The pure flexibility of the multiple-output modelling leads inevitably to intuitively nonsensical results.

It might be objected that residuals can be made necessary, in the sense of assuming that $F(y,0,x) = 0 \Rightarrow y = 0$, i.e. a form of "null jointness". This may be due to a limit form of cross derivatives, either between pollutants and outputs becoming positive and very large as pollutants decrease, or between inputs and pollutants becoming very small as pollutants decrease, but this line of reasoning seems far fetched and leads logically to introducing lower limits for pollutants. Then these limits will replace the zeros in the solution for pollutants, and Proposition 1 correspondingly changed.

Pollutants as inputs

To avoid the problem above one possibility is to treat residuals generation *as if* they are inputs. (See Martin (1986) for an extensive critique of this approach.) This option is followed without any comment or explanation in the influential textbook by Baumol and Oates (1975). When a defence of the procedure is offered, the most satisfactory position is in a macro setting[5]. It is then argued that outputs increase when residuals generation increases because this means that less resources are used on pollution abatement, and these freed resources are then transferred to output production (see e.g. Brock (1977) for, to my knowledge, the first use of this argument, and Cropper and Oates (1992), as well as Tahvonen and Kuuluvainen (1993) for adopting this explanation). In a macro world of only one good the alternative cost of pollution is correctly expressed in terms of this good. But the same argument maintained at the micro level seems more awkward, since we obviously loose information as to the nature of the firm level purification activity. The procedure seems more a convenience due to the econometrics of the multi-output models estimated (see e.g. Pittman, 1981).

Since the sign of F_z now is *negative* in the last necessary condition in (6), the condition is formally identical to the condition for employing an input, leading to a positive level of pollutants generation in the case of an interior solution. The factor price in (7) is replaced with the marginal damage. There are, however, several weaknesses with this formulation.

PROPOSITION 2: *When the multi output technology has a maximal degree of assortment with bads formally as inputs, i.e. relation (4) describes the technology with* $F_z < 0$, *we have:*
i) the socially optimal level of bads is only determined as a trade-off with goods,
ii) no explicit purification activity in the form of inputs can be identified.

Discussion: Generation of pollutants acts as any other input, and thus has a positive impact on every output in general. The pollutants are thus associated with outputs, not with inputs. The trade-off between outputs and residuals generation is expressed by:

$$D'_{z_s} = p_i \frac{-F'_{z_s}}{F'_{y_i}} \Rightarrow p_i = D'_{z_s} \frac{F'_{y_i}}{-F'_{z_s}} \ , \ i=1,..,m \ , \ s=1,..,k \tag{8}$$

---

[5] We will return to a defence of the practice in a micro setting offered by Hoel (1997).

The first expression in (8) says that polluting residuals should be employed up to the point that the value of the resulting output i is equal to the marginal damage caused by the pollutant. The second, equivalent, expression shows that the marginal value of the output in question is equal to the damage evaluation of the marginal unit requirement of pollutants generation. There is thus a formal trade-off mechanism in place linking outputs and pollutants, so if this is the connection we want to model at the micro level, we have at least a *qualitative* representation, but not a correct quantitative one, since an active part of the resource base is left outside the explicit modelling. Most awkwardly, purification activity is swept under the carpet. So if one is interested in optimal abatement per se this is not the appropriate formulation. Abatement by input substitution cannot be captured neither, because there is no functional or physical link between the "real" inputs, x, and the residuals generation, z, specified as input. The model specification allows low sulphur coal to be substituted for high sulphur coal when factor prices change, but this has no impact on generation of sulphur as a residual!

Introducing technology restrictions

It should be noted that reducing the degree of assortment in (4), i.e. when representing bads as outputs, by introducing more relations of the form (4) as in (2), does not help in general. We get the same Proposition 1 and 2. This can intuitively be realised by inserting (2) with the bads, z, as part of the output vector in the optimisation problem (5). As long as the partial derivatives F_z^i are non-negative zero values of the z's will result. We must introduce explicitly either product couplings between y and z when both are interpreted as outputs, or factor bands between x and z if z is interpreted as inputs.

Product coupling

As a formally more satisfying amendment to the general representation (4) let us introduce a product coupling following Frisch. The production relationship system is then:

$$
\begin{aligned}
F(y,z,x) &= 0 \,,\ F_{y_i} > 0 \,,\ F_{z_s} > 0 \,,\ F_{x_j} < 0 \\
G(y,z) &= 0 \,,\ i)\ G_{y_i} > 0 \,,\ G_{z_s} < 0 \,,\ ii)\ \ G_{y_i} > 0 \,,\ G_{z_s} > 0
\end{aligned}
\tag{9}
$$

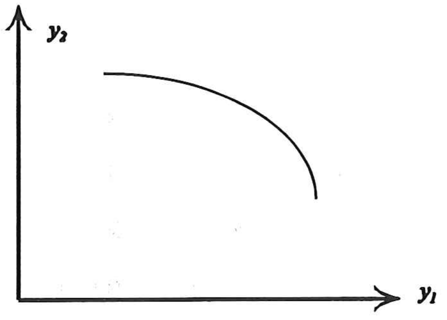
The second relationship is the coupling between outputs and pollution generation. In order to model the inevitability of generating bads together with goods, the signs of the partial derivatives must be opposite (but the choice of which one to be positive and negative is arbitrary), i.e. case i), yielding a positive relationship between a good and a bad. Null jointness may also be assumed, i.e. G(y, 0) = 0 ⇒ y = 0. We will not pursue the issues of special limit conditions. The nature of the relationships between the three sets of variables can be illustrated in Figure 1.

Now we can capture the fact that residuals generation cannot be avoided, and that producing outputs entail an extra cost in terms of pollution generation. If we want to connect residuals generation to a limited number of outputs, this is , of course, straightforward. Necessary conditions of the social planning problem (5) with (9) as technology are:

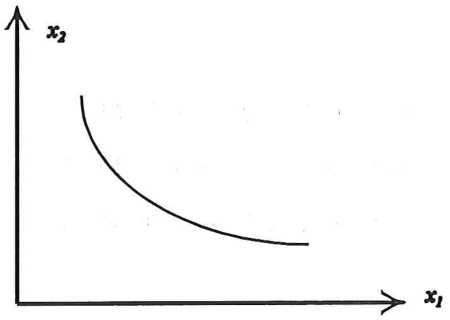
$$
\begin{aligned}
p_i - \lambda F_{y_i} - \gamma G_{y_i} \begin{pmatrix} = 0 \\ \leq 0 \end{pmatrix} &\ for \begin{pmatrix} y_i > 0 \\ y_i = 0 \end{pmatrix} \,,\ i=1,..,m \\
-D_{z_s} - \lambda F_{z_s} - \gamma G_{z_s} \begin{pmatrix} = 0 \\ \leq 0 \end{pmatrix} &\ for \begin{pmatrix} z_s > 0 \\ z_s = 0 \end{pmatrix} \,,\ s=1,..,k \\
-q_j - \lambda F_{x_j} \begin{pmatrix} = 0 \\ \leq 0 \end{pmatrix} &\ for \begin{pmatrix} x_j > 0 \\ x_j = 0 \end{pmatrix} \,,\ j=1,..,n
\end{aligned}
\tag{10}
$$

where λ is the shadow price on the first production relationship and γ on the coupling relationship.

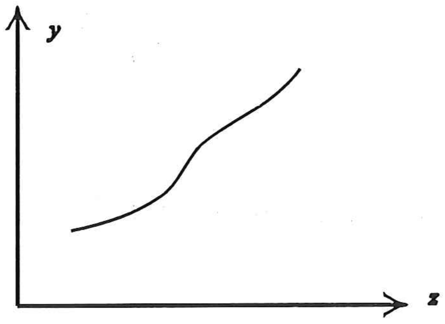
PROPOSITION 3: *When the multi output technology has a maximal degree of assortment with bads formally as outputs, i.e. relation (4) describes the technology, and an output coupling is added, then both resource cost of a good and a bad, and the marginal coupling cost between*



Panel a) The factor isoquant



Panel b) The output isoquant



Panel c) The coupling curve

Figure 1 Multi-output production

them are evaluated determining the socially optimal level of the bad.

Discussion: Assuming interior solutions and eliminating the Lagrangian parameters yield:

$$p_i = q_j \frac{F_{y_i}}{-F_{x_j}} + q_j \frac{F_{z_s}}{-F_{x_j}} \cdot \frac{G_{y_i}}{-G_{z_s}} + D_{z_s} \frac{G_{y_i}}{-G_{z_s}} = q_j\left(\frac{F_{y_i}}{-F_{x_j}} + \frac{F_{z_s}}{-F_{x_j}} \cdot \frac{G_{y_i}}{-G_{z_s}}\right) + D_{z_s} \frac{G_{y_i}}{-G_{z_s}} \,, \qquad \textbf{(11)}$$
$$i=1,..,m \;,\; j=1,..,n \;,\; s=1,..,k$$

In addition to the first standard input cost term on the right hand side of the first equality sign, we have two new types of cost terms due to the coupling. From evaluating changes in the transformation functions in (9) we have that $F_{yi}/(-F_{xj})$ is the marginal unit input requirement of input j producing one unit of output i (or marginal *fabrication coefficient* following Frisch), $F_{zs}/(-F_{xj})$ is the marginal unit requirement of input j producing one unit of z, and $G_{yi}/(-G_{zs})$ is *marginal coupling effect* expressing the marginal increase in z for a unit increase in output i. The interpretation of the two additional cost terms is then for the second term on the right-hand side of (11) that producing output i we must also produce residual z_s, and this has a resource cost expressed in terms of input j; the marginal coupling coefficient is multiplied with the marginal input requirement producing z_s and then costed according to the factor price of input j. The third cost element is expressing the marginal coupling effect of producing output i on z, and then costed at marginal damage of z_s. The last equation shows the two basic cost terms, consisting of a resource cost term encompassing the unit input cost of output i and the corresponding residuals generation input cost.

If we compare equation (11) with (8) we see the difference introduction of an explicit coupling has made. Jointness in production of ordinary outputs and residuals generation is now represented formally with the extra resource cost term in (11), absent from (8). We have a formally satisfying exposition of how provision of outputs should be affected by the joint production of "bads". However, the formulation is still too general to allow explicit insight into purification activities proper. The input-specific purification effects are expressed via different ratios of marginal productivities , $F_{zs}/(-F_{xj})$, in bads-generation on marginal productivities in goods production, $F_{yi}/(-F_{xj})$.

Factor bands

Since basically the pollutants are connected to the inputs used, it may also be logical to connect the bads when interpreted as inputs to ordinary inputs through factor bands. The model (4) will then be:

$$\begin{aligned} F(y,z,x) &= 0\ ,\ F_{y_i} > 0\ ,\ F_{z_s} < 0\ ,\ F_{x_j} < 0 \\ f(z,x) &= 0\ ,\ i)\ f_{z_s} > 0\ ,\ f_{x_j} < 0\ ,\ ii)\ f_{z_s} < 0\ ,\ f_{x_j} < 0 \end{aligned} \qquad \textbf{(12)}$$

In the factor band the two types of inputs may have opposite signs of partial derivatives, case i), or equal signs, case ii) (positive or negative do not matter). The former implies that the types runs parallel, while the latter implies a trade-off.

Necessary conditions of the social planning problem (5) with (12) as technology are:

$$\begin{aligned} &p_i - \lambda F_{y_i} \begin{pmatrix} = 0 \\ \le 0 \end{pmatrix} \textit{for} \begin{pmatrix} y_i > 0 \\ y_i = 0 \end{pmatrix},\ i=1,..,m \\ &-D_{z_s} - \lambda F_{z_s} - \gamma f_{z_s} \begin{pmatrix} = 0 \\ \le 0 \end{pmatrix} \textit{for} \begin{pmatrix} z_s > 0 \\ z_s = 0 \end{pmatrix},\ s=1,..,k \\ &-q_j - \lambda F_{x_j} - \gamma f_{x_j} \begin{pmatrix} = 0 \\ \le 0 \end{pmatrix} \textit{for} \begin{pmatrix} x_j > 0 \\ x_j = 0 \end{pmatrix},\ j=1,..,n \end{aligned} \qquad \textbf{(13)}$$

where λ is the shadow price on the first production relationship and γ on the factor band.

PROPOSITION 4: *If factor bands are added to the multiple output relation (4) with the bads as inputs, then it is possible to identify purification activity.*

Discussion: Assuming interior solutions and eliminating the Lagrangian parameters yield:

$$q_j = p_i \frac{-F_{x_j}}{F_{y_i}} + p_i \frac{-F_z}{F'_{y_i}} \cdot \frac{-f_{x_j}}{f_{z_s}} - D_{z_s} \frac{-f_{x_j}}{f_{z_s}} = p_i \left(\frac{-F_{x_j}}{F_{y_i}} + \frac{-F_z}{F_{y_i}} \cdot \frac{-f_{x_j}}{f_{z_s}} \right) - D_{z_s} \frac{-f_{x_j}}{f_{z_s}}, \quad i=1,..,m\ ,\ j=1,..,n\ ,\ s=1,..,k \tag{14}$$

At the margin the costs and benefits of employing a factor should balance if this is to be used at a positive level. The benefit terms after the first equality sign in (14) start with the output price of output i times the marginal productivity of factor j in the production of this good. (The possibility of assortment is expressed by the fact that we do not sum over marginal effects on other outputs.) The next term is the value of good i created by the joint change in the bad as input due to the factor band relation, i.e. $-f_{xj}/f_{zs}$ expresses the amount of the bad z_s generated corresponding to one unit increase in the input x_j, the *marginal band effect*, and $-F_{zs}/F_{yi}$ is the productivity of the bad in the production of good y_i. The third term is the environmental cost term due to the marginal band effect. Note that if a ordinary input and a bad has a positive covariation, then increasing the input yields a positive output effect, but a negative environmental damage effect due to the factor band, while if the covariation is negative, then the two effects change sign, i.e. a negative output effect, but a positive environmental effect due to less damage. It is therefore natural to term the latter case corresponding to ii) in (12), for a case with purification possibility.

The production set representation

When specifying multiple outputs one may use the production set representation (see e.g. Chambers, 1988). This gives a very easy incorporation of multiple outputs. The production possibility set, T is defined by:

$$T = \{\ (y,z,x):\ (y,z)\ \textit{can be produced by}\ x\ \} \tag{15}$$

This representation is equivalent to the functional representation (4) given some regularity conditions on the set T (see e.g. McFadden, 1978) :

$$T = \{\ (y,z,x):\ (y,z)\ \textit{can be produced by}\ x\ \} \equiv \{\ (y,z,x):\ F(y,z,x) \le 0\ \} \tag{16}$$

We note that also a technology set representation gives maximal flexibility in combining inputs into outputs when no more restrictions are imposed. There is, however, a distinction to be made about *disposability* within a production set representation. The ease with which outputs may be disposed of can be classified as *free* or *strong*, and *weak* disposal (see e.g. Färe et al., 1994). The former gives the maximal freedom of flexibility, i.e. outputs may be disposed of without opportunity costs, while the latter introduce a restriction on the ease of disposing of outputs, i.e. there is an opportunity cost of disposal. In addition a condition of *null jointness* can be imposed, i.e. the good output cannot be produced without the bad.[6] This case resembles a product coupling in the Frisch sense.

In view of the necessity of introducing additional restriction of couplings and/or factor bands to the flexible model (4) it may be of interest to have a closer look at the use of production set extended to bads. The first utilisation of attaching weak disposability to a bad (to my knowledge) is found in Färe et al. (1986). Figure 2 illustrates the approach. Weak disposability means that the technology set in output space of one desirable good and one bad for given amounts of inputs is portrayed by OABCO, while strong disposability means that the technology set is within the dotted horizontal line from point B to the vertical axis, and the vertical dotted line from point C to the horizontal axis. Now, points on the border OAB correspond to a positive output coupling, case i) in (9) above. Continuing along the border to C the coupling changes sign, we get case ii). Then along CO we are back to case i) again. These switches seem difficult to reconcile with the coupling idea of Frisch. The challenge is to find empirical examples. At this stage suffice to say that it may be not impossible, but that it seems more realistic to portray the weak disposability set by moving down vertically from point B of maximal good output.. The null jointness in Figure 2 is symmetrical. (In the next application of this model in Färe et al. (1989) a non symmetrical null jointness is illustrated , using the vertical line from C to the horizontal axis as the border also of the weak

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[6] "Weak disposability refers to the ability to dispose of an unwanted commodity at positive private cost." (Färe et al. (1994), p.38). Null jointness means that if the bad has zero value, so must the good.

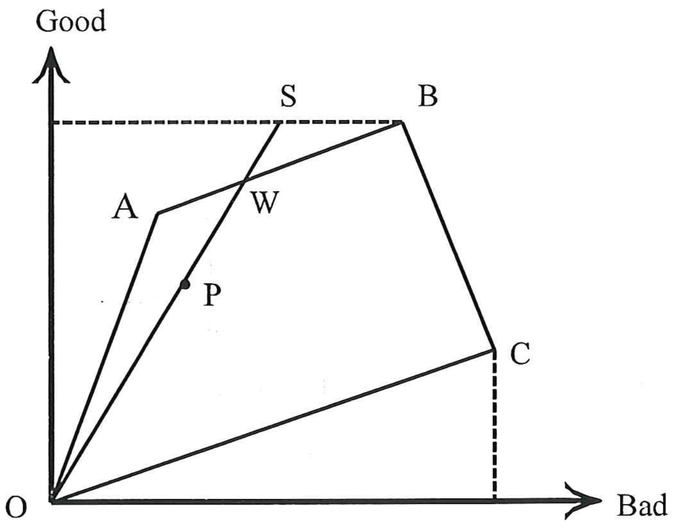

Figure 2. Strong and weak disposability
Source: Färe et al. (1986)

disposability technology.)

We should note some crucial differences with this model compared with our approach. Firstly, it is not the case that reducing the bad is always costly in terms of reducing the good. For efficient observations on the segments BCO the bad can be reduced and the good increased. Indeed, this is the case for any inefficient point inside the region OABCO. It is only for efficient points on the segments OAB that we have costs in this sense. It is only this part of the technology set that corresponds to our model (9), case i). Secondly, purification activities are not shown explicitly in this model. Ideally, purification in real sense should influence the *shape* of the technology set: Bads are reduced for given level of goods when increasing use of inputs. The technology set in Figure 2 should shift North - West. Thirdly, the use of strong and weak disposability to calculate costs of regulations in Färe et al. (1986) and (1989) should be commented upon. The idea there is that strong disposability holds before environmental regulations, and weak after. Considering the inefficient point (observation of a firm) P in Figure 2, the cost of regulation is shown by calculating OW/OS, a radial Farrell output-oriented efficiency calculation, or using 1 - OW/OS as a measure for the loss of goods due to the regulation for observation P. But this is a very peculiar way of seeing regulations. If the technology is of the strong type before environmental concern, certainly the optimal answer for firm P is to move straight to the vertical axis maintaining the same output (or even better to move to efficient goods point on the vertical axis corresponding to B). The same amount of resources is consumed, and no bads produced. A regulator's dream! If strong disposability holds for every production unit before regulation, the only reason for observations to be outside the goods-axes is inefficiency. The realistic situation corresponding to joint production of goods and bads seems to be that the weak disposability is the correct technology description, also before environmental concern. We are still facing the interpretation problems mentioned above. In order to calculate costs of regulation the regulation itself must be introduced, and purification possibilities spelt out. If the weak disposability set in Figure 2 is the complete picture, then another question not addressed is what to assume about the observed inefficiencies after regulation. The rather huge number for regulation costs calculated in Färe et al. (1989) (using a hyperbolic measure instead of a radial one, allowing increase of the goods and decrease of the bads), is therefore of a very doubtful nature. At least in one reference (Tahvonen and Kuuluvainen, 1993) the number is taken at face value as the cost of regulation.

4. Factorially determined multi output production

Residuals arise from use of inputs in a wide sense. Any covariation with outputs is incidental and not expressing any natural law or engineering relationship. Sulphur as a bads output in electricity generation stems from the sulphur content of the fossil fuel used, may it be coal, oil, peat, etc. and is not linked to electricity output as such, but because *both* outputs result from applying certain inputs. Therefore, it would seem desirable to establish a multi-output structure reflecting this fact. Pollutants cannot be inputs in an engineering sense. According to the materials balance principle they are linked to inputs, just as marketed products, but cannot be varied partially in an engineering sense to influence those outputs, as ordinary inputs. One may say that a firm is using "renovation services" of Nature as inputs when discharging residuals, and that the latter serve as measures of such inputs. However, it is still meaningless in an engineering sense to assume that partial increases in the use of these services can increase marketed outputs directly. In addition to modelling the bads as outputs, we would like to see explicitly the purification possibilities. Remember that the main purpose of the exercise is to provide information as to choice of instruments to reduce pollution.

The special system Frisch called *factorially determined multi-output production* seems best suited to capture both joint production of bads and how to use inputs to reduce generation of them. This model has been used by several authors on environmental pollution (see e.g. Mäler (1974), Martin, 1986) without recognising the generic character of the approach (or Frisch as a reference) (as well as in papers recognising Frisch, see e.g. Førsund, 1972). The general model (4) is replaced with:

$$\begin{aligned} y_i &= f^i(x_1,..,x_n) \ , \ f^i_{x_j} \geq 0 \ ,i=1,..,m \ , \ j=1,..,n \\ z_s &= g^s(x_1,..,x_n) \ , \ g^s_{x_j} \begin{bmatrix} > \\ = \\ < \end{bmatrix} 0 \ , \ j=1,..,n \ , \ s=1,..,k \end{aligned} \tag{17}$$

Both the outputs and residuals generation are functions of the same set of inputs. Choosing input levels, the production of bads follow. The formulation is best suited for grouping the normal goods on one hand, and the pollutants on the other. (The first relations in (17) may be written more generally as in (2) (without bads) for increased realism, thus mixing the models (2) and (17).) The crucial feature for pollution modelling is the second set of relations.

Note that the marginal productivity of some inputs may be zero in the first relationship, given that the list of n inputs is exhaustive. Such inputs will be unique to purification activities. The marginal productivities of ordinary production inputs may be positive, zero or negative in the residuals generation function. Positive productivities in the latter correspond to ordinary inputs with positive productivities also in the output relations. These inputs will typically be materials and energy. Inputs with zero productivities in the residuals generation function may have positive productivities in the output relations, but do not generate residuals (e.g. capital equipment, labour). Inputs with negative productivities in the residuals generation function are either purification inputs proper, i.e. they then have zero productivities in the output relations, or they may be both purification inputs and production inputs with positive productivities in the output relations, e.g. labour that operates processes and the more labour the less waste of materials and the more output, and using "more" capital to make the process more resource efficient. Such effects may also be connected with recycling or materials recovery activities. These efforts cannot then be identified as a separate purification activity, but such inputs may be applied with also the residuals generation effect in mind. It is, in fact, quite realistic that it is impossible to identify a purification activity within a plant. In the case of "end of pipe" purification or a separate "purification division" we have the case of inputs with zero productivities in the production relations, and negative productivities in the residuals generation function.

In order to cover as many options as possible the list of n inputs should be general and also cover inputs not in current use. Consider the example of input substitution that did not work within the framework of the general formulations (4) or (9). The use of various sources of energy may be chosen to be only heavy oil in the production relations in (17) without any concern for the environment. If the price of this input is increased, sooner or later the firm will switch to light oil, and then the generation of sulphur will, quite correctly, decrease according to the residuals generation function (keeping the same calory value).[7]

We have above identified five types of inputs. We will term them *dirty*, *clean* , *green* , *pure purification*, and *integrated purification*. The classification of the first three is according to the total impact on environmental damages. Corresponding to the materials balance principle we have opened for keeping track of the complete picture of residuals generation, i.e. when

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[7] In Hoel (1997) a simple version of the model (17) with one good output, y, and one bad output, z, is used to demonstrate that solving for an x-input in the second relation and substituting for this variable in the first relation, we have a production function where z *appears* as an input. One ordinary input is then suppressed. However, it seems rather odd to sacrifice the explicit insight of the complete picture.

employing an input, residuals may increase as well as decrease. The total environmental impact is measured by the damage function. In addition, we will add the characterisation strong and weak to the dirty, clean and green terminology, according to the nature of the residuals generation.

The general social planning problem (5) will now take the following form:

$$\begin{aligned} &Max = \sum_{i=1}^{m} \int_{w_i=0}^{y_i} p_i(w_i)dw_i - \sum_{j=1}^{n} q_j x_j - D(z_1,..,z_k) \\ &s.t. \\ &\quad y_i = f^i(x)\ ,\ i=1,..,m\ ,\ z_s = g^s(x)\ ,\ s=1,..,k \end{aligned} \tag{18}$$

Since the level of outputs and bads follow from a vector of inputs the problem may be simplified by inserting the production relation into the objective function. The necessary first order conditions are then:

$$\sum_{i=1}^{m} p_i f^i_{x_j} - q_j - \sum_{s=1}^{k} D_{z_s} g^s_{x_j} \begin{pmatrix} = 0 \\ \le 0 \end{pmatrix} \ for \begin{pmatrix} x_j > 0 \\ x_j = 0 \end{pmatrix}\ , j=1,..,n \tag{19}$$

The rule for whether or not an input should be used at all follows from the first order conditions in the standard way of the Kuhn -Tucker problem, e.g. it may be economic only to use one type of energy source, etc., depending on whether the value of the marginal productivities can meet resource costs and environmental costs generated by the simultaneous generation of residuals, z, at a positive level of the input in question.

PROPOSITION 5: *Specifying the multi output production to be factorially determined as in (17), allows:*

i) an exhaustive classification of the effects of inputs

ii) reveal all technical possibilities of changing residuals generation.

Discussion: Assuming interior solutions and rearranging yield:

$$q_j + \sum_{s=1}^{k} D_{z_s} g^s_{x_j} = \sum_{i=1}^{m} p_i f^i_{x_j} \text{ , } j=1,..,n \tag{20}$$

Using these relations the different cases of inputs are set out in Table 1. The horizontal line in the middle distinguishes between application of inputs without any environmental concern (top part) and environmentally conscious applications (bottom part), based on the damages impact.

Table 1. Classification of inputs

Input type	Production impact	Residuals impact	Damage impact	Optimality condition
Strongly dirty	$f_{xj}^i > 0, \forall i$	$g_{xj}^s \geq 0, \forall s$	$\sum_s D_{zs} g_{xj}^s > 0$	$q_j + \sum D_{zs} g_{xj}^s = \sum p_i f_{xj}^i$
Weakly dirty	$f_{xj}^i > 0, \forall i$	$g_{xj}^s \gtrless 0, \forall s$	$\sum_s D_{zs} g_{xj}^s > 0$	$q_j + \sum D_{zs} g_{xj}^s = \sum p_i f_{xj}^i$
Strongly clean	$f_{xj}^i > 0, \forall i$	$g_{xj}^s = 0, \forall s$	$\sum_s D_{zs} g_{xj}^s = 0$	$q_j = \sum p_i f_{xj}^i$
Weakly clean	$f_{xj}^i > 0, \forall i$	$g_{xj}^s \gtrless 0, \forall s$	$\sum_s D_{zs} g_{xj}^s = 0$	$q_j = \sum p_i f_{xj}^i$
Strongly green	$f_{xj}^i > 0, \forall i$	$g_{xj}^s \leq 0, \forall s$	$\sum_s D_{zs} g_{xj}^s < 0$	$q_j = \sum p_i f_{xj}^i + \sum D_{zs}(-g_{xj}^s)$
Weakly green	$f_{xj}^i > 0, \forall i$	$g_{xj}^s \gtrless 0, \forall s$	$\sum_s D_{zs} g_{xj}^s < 0$	$q_j = \sum p_i f_{xj}^i + \sum D_{zs}(-g_{xj}^s)$
Separable purification	$f_{xj}^i = 0, \forall i$	$g_{xj}^s \gtrless 0, \forall s$	$\sum_s D_{zs} g_{xj}^s < 0$	$q_j = \sum_s D_{zs}(-g_{xj}^s)$
Integrated purification	$f_{xj}^i \gtrless 0, \forall i$	$g_{xj}^s \gtrless 0, \forall s$	$\sum_s D_{zs} g_{xj}^s < 0$	$q_j + \sum p_i(-f_{xj}^i) = \sum D_{zs}(-g_{xj}^s)$

Let us try to give examples of the classification above. A strongly dirty input means that it has a positive productivity in the production of marketed goods, but that the use of this input also increases the generation of a subset of residuals, and no component is reduced, therefore the term strong. The input may be fossil fuel in electricity production. Residuals such as SO_2, NO_x, CO_2, soot, particles, etc. may all increase by increased use of the fuel. The total marginal damage increases, therefore the term dirty. Weakly dirty means that while some residuals are increased, one or more residuals are also decreased, but the total marginal damage is still increasing. One example may be from cement production; increasing the input of calcium materials increases the amount of particles, but reduces the amount of sulphur due to chemical reactions.

A clean input means that there are no environmental damage according to the damage function. Strongly clean means that no residuals are generated. For this to be true we must have service inputs from stock variables like capital and labour. If the use of capital is increased, e.g. substituting labour at no increased use of energy, no extra residuals are generated. Weakly clean means that some residuals may increase, other decrease, but the total marginal damage remains constant. This is a limiting case of weakly dirty. A material input must be involved. In the corresponding cement example above it may be the case that the increased damage from particles is exactly balanced by the reduction in sulphur emissions.

An input is strongly green if all residuals that are generated decrease, so that the total marginal damage is reduced, and weakly green if some increase, some decrease, but still reducing total marginal damage. An example of the former may be labour engaged in better supervision of the production process and succeeding in reducing various forms of waste of raw materials. A weakly green input may be capital, increasing the use of energy and then generating associated residuals, but making better use of e.g. a fibre inputs in a pulp and paper process.
The reduction of fibres is weighing more in damages reduction than the increase in energy-related residuals.

The separable purification is the standard end-of-pipe purification activity, e.g. a waste water treatment plant connected to a firm using process water. The inputs used in the end-of-pipe plant have no impacts on the production of marketed goods, and this plant may increase generation of some residuals, and necessarily reducing the generation of others. To capture this materials balance principle, *modification* is used as the terminology instead of purification (see e.g. Russell and Spofford, 1972). Organic waste may be removed from the waste water, but a sludge is created instead as a solid waste. In addition some chemicals may be used to deal with bacteria, so we have a plus and minus situation here to. But the total activity may be termed purification because total marginal damage is reduced.

Finally, some technological ways of dealing with residuals are integrated within the production process of the marketed goods. Increasing inputs to reduce damage impact may then divert resources from production proper, resulting in partial decrease of marketable products, but production may also be increased due to recycling of waste as inputs. Although some residuals obviously are decreased, some may also increase. An example may be that some key process machinery is improved in order to contain residuals, but this feature reduces the production capacity, another example may be redesign of equipment to reduce some emissions like particles, but needing more energy, with accompanying residuals increases. An example of both increasing production of marketed goods and reducing residuals may be recovering of waste heat integrated in the production equipment, e.g. a closed aluminium oven. This will increase production for the same input of primary energy, and reduce energy residuals as well as other gases and particles.

5. Conclusions

The materials balance principle points to the crucial role of material inputs in generating residuals in production processes. Thus, material-based production must also produce "bads" as joint outputs to marketed goods. Pollution modelling therefore must be of a multi-output nature. It has been demonstrated that the most flexible transformation function in outputs and inputs used in textbooks is too general to make sense in pollution modelling. The standard approach has therefore been to specify the bads *as if* they are inputs. It is demonstrated that this practice, although defendable on a macro level, hides explicit considerations of various modification activities. This is most awkward on a micro level, since a main purpose of environmental economics is to come up with results helping making choices as to the most efficient policy instruments to apply in order to reduce pollution. The main result of the paper is to come up with a complete taxonomy of inputs as to the impact on both residuals and marketed products as joint outputs, thus providing the information for choice of instruments. The multi-output model making such a taxonomy possible, is what Frisch (1965) termed *factorially determined multi output production.*

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