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Panel data estimators and aggregation

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Panel Data Estimators and Aggregation

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Abstract: For a panel data regression equation with two-way unobserved heterogeneity, individual-specific and period-specific, 'within-individual' and 'within-period' estimators, which can be given Ordinary Least Squares (OLS) or Instrumental Variables (IV) interpretations, are considered. A class of estimators defined as linear aggregates of these estimators, is defined. Nine aggregate estimators, including between, within, and Generalized Least Squares (GLS), are special cases. Other estimators are shown to be more robust to simultaneity and measurement error bias than the standard aggregate estimators and more efficient than the 'disaggregate' estimators. Empirical illustrations relating to manufacturing productivity are given.

Keywords: Panel data. Aggregation. IV estimation. Robustness. Method of moments.
Factor productivity

JEL classification: C13, C23, C43.

1 Introduction

A primary reason for the substantial growth in the use of panel data during the last decades is the opportunity they give for identifying and controlling for *unobserved heterogeneity* which may disturb coefficient estimation. It is well known that (i) the potential nuisance created by *fixed (additive) individual* heterogeneity in OLS estimation can be eliminated by measuring all variables from their individual means or taking individual differences over time, (ii) the potential nuisance created by *fixed (additive) time specific* heterogeneity in OLS estimation can be eliminated by measuring all variables from their time-specific means or taking time-specific differences over individuals, and (iii) efficient estimation in the presence of suitably structured *random individual- or time-specific heterogeneity*, can be performed by (Feasible) Generalized Least Squares.

Less attention has been given to the fact that such aggregate estimators can be constructed from disaggregate building-blocks. Approaching estimation in this way is illuminating because regression coefficients can be estimated consistently from parts of a panel data set in numerous ways and because the disaggregate estimators have different degree of robustness to bias. By combining an increasing number of individual-specific or period-specific estimators, an increasing part of the observations can be included until, at the limit, the full data set is used. Such approaches are interesting both because several familiar estimators (within, between, generalized least squares etc.) for panel data models can be interpreted as linear combinations of elementary estimators, and because we get other suggestions of estimators along the way.

The paper proceeds as follows: After, in Section 2, describing the model and its transformations, we in Section 3 define *disaggregate* within estimators, each having the interpretation as a 'micro' OLS (Ordinary Least Squares) or IV (Instrumental Variables) estimator. Section 4 defines an estimator class by an arbitrary weighting of the latter, while in Section 5, nine estimators, including three 'within', two 'between', three Generalized Least Squares (GLS), and one standard OLS estimator. The general estimator is shown also to contain members which are more robust to violation of the standard assumptions in random coefficient models. Both a standard regression framework and situations with simultaneity (correlation between individual effects, period effects, and/or disturbances on the one hand and the regressor vector on the other) and situations with measurement errors in the regressor vector are considered. Among the latter estimators we select estimators which are more robust to simultaneity and measurement errors and more efficient than the 'disaggregate' estimators. Finally, Section 6 contains an empirical illustration of robustness and efficiency loss, relating to manufacturing productivity.

2 Model, notation, and transformations

A linear regression model relating y to the $(1\times K)$-vector $\boldsymbol{x}$, with observations from N individuals and T periods is

$$\begin{aligned} & y_{it} = k + \boldsymbol{x}_{it}\boldsymbol{\beta} + \epsilon_{it}, \quad \epsilon_{it} = \alpha_i + \gamma_t + u_{it}, \\ & (u_{it}|\boldsymbol{X}) \sim \mathsf{IID}(0,\sigma^2), \quad (\alpha_i|\boldsymbol{X}) \sim \mathsf{IID}(0,\sigma_\alpha^2), \quad (\gamma_t|\boldsymbol{X}) \sim \mathsf{IID}(0,\sigma_\gamma^2), \\ & u_{it} \perp \alpha_j \perp \gamma_s, \qquad i,j=1,\dots,N;\ t,s=1,\dots,T, \end{aligned} \tag{1}$$

where y_{it} and $\boldsymbol{x}_{it} = (x_{1it},\dots,x_{Kit})$ are the values of y and $\boldsymbol{x}$ for individual i in period t, $\boldsymbol{\beta} = (\beta_1,\dots,\beta_K)'$ is the coefficient vector, α_i and γ_t are random individual-specific and period-specific effects (which may alternatively be interpreted as fixed, see Section 5), u_{it} is a disturbance, and k is an intercept. At the moment, we make the above standard assumptions for two-way random effects models, which imply

$$\mathsf{E}(\epsilon_{it}|\boldsymbol{X}) = 0, \qquad \mathsf{E}(\epsilon_{it}\epsilon_{js}|\boldsymbol{X}) = \delta_{ij}\sigma_\alpha^2 + \delta_{ts}\sigma_\gamma^2 + \delta_{ij}\delta_{ts}\sigma^2, \qquad \begin{array}{l} i,j=1,\dots,N, \\ t,s=1,\dots,T, \end{array} \tag{2}$$

where $\delta_{ij}=1$ for $i=j$ and $=0$ for $i\neq j$, and $\delta_{ts}=1$ for $t=s$ and $=0$ for $t\neq s$, and $\boldsymbol{X}$ is the $(NT\times K)$ matrix containing all $\boldsymbol{x}_{it}$s.

Let individual-specific and period-specific vectors and matrices be

$$\boldsymbol{y}_{i\cdot} = \begin{bmatrix} y_{i1} \\ \vdots \\ y_{iT} \end{bmatrix}, \quad \boldsymbol{X}_{i\cdot} = \begin{bmatrix} \boldsymbol{x}_{i1} \\ \vdots \\ \boldsymbol{x}_{iT} \end{bmatrix}, \quad \boldsymbol{y}_{\cdot t} = \begin{bmatrix} y_{1t} \\ \vdots \\ y_{Nt} \end{bmatrix}, \quad \boldsymbol{X}_{\cdot t} = \begin{bmatrix} \boldsymbol{x}_{1t} \\ \vdots \\ \boldsymbol{x}_{Nt} \end{bmatrix},$$

stacked into

$$\boldsymbol{y} = \begin{bmatrix} \boldsymbol{y}_{1\cdot} \\ \vdots \\ \boldsymbol{y}_{N\cdot} \end{bmatrix}, \quad \boldsymbol{X} = \begin{bmatrix} \boldsymbol{X}_{1\cdot} \\ \vdots \\ \boldsymbol{X}_{N\cdot} \end{bmatrix}, \quad \boldsymbol{y}_* = \begin{bmatrix} \boldsymbol{y}_{\cdot 1} \\ \vdots \\ \boldsymbol{y}_{\cdot T} \end{bmatrix}, \quad \boldsymbol{X}_* = \begin{bmatrix} \boldsymbol{X}_{\cdot 1} \\ \vdots \\ \boldsymbol{X}_{\cdot T} \end{bmatrix},$$

and let $\boldsymbol{e}_H$ be the $(H\times 1)$ vector of ones, $\boldsymbol{I}_H$ the H-dimensional identity matrix, $\boldsymbol{A}_H = \boldsymbol{e}_H\boldsymbol{e}_H'/H$, $\boldsymbol{B}_H = \boldsymbol{I}_H - \boldsymbol{A}_H$, $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_N)'$, and $\boldsymbol{\gamma} = (\gamma_1,\dots,\gamma_T)'$. Alternative ways of writing the regression equation are

$$\boldsymbol{y}_{i\cdot} = \boldsymbol{e}_T k + \boldsymbol{X}_{i\cdot}\boldsymbol{\beta} + \boldsymbol{\epsilon}_{i\cdot}, \quad \boldsymbol{\epsilon}_{i\cdot} = \boldsymbol{e}_T\alpha_i + \boldsymbol{\gamma} + \boldsymbol{u}_{i\cdot}, \quad i=1,\dots,N, \tag{3}$$

$$\boldsymbol{y}_{\cdot t} = \boldsymbol{e}_N k + \boldsymbol{X}_{\cdot t}\boldsymbol{\beta} + \boldsymbol{\epsilon}_{\cdot t}, \quad \boldsymbol{\epsilon}_{\cdot t} = \boldsymbol{\alpha} + \boldsymbol{e}_N\gamma_t + \boldsymbol{u}_{\cdot t}, \quad t=1,\dots,T, \tag{4}$$

implying

$$\boldsymbol{y}_{i\cdot} - \bar{\boldsymbol{y}} = (\boldsymbol{X}_{i\cdot} - \bar{\boldsymbol{X}})\boldsymbol{\beta} + \boldsymbol{\epsilon}_{i\cdot} - \bar{\boldsymbol{\epsilon}}, \qquad \boldsymbol{\epsilon}_{i\cdot} - \bar{\boldsymbol{\epsilon}} = \boldsymbol{e}_T(\alpha_i - \bar{\alpha}) + \boldsymbol{B}_T\boldsymbol{\gamma} + \boldsymbol{u}_{i\cdot} - \bar{\boldsymbol{u}}, \tag{5}$$

$$\boldsymbol{y}_{\cdot t} - \bar{\boldsymbol{y}}_* = (\boldsymbol{X}_{\cdot t} - \bar{\boldsymbol{X}}_*)\boldsymbol{\beta} + \boldsymbol{\epsilon}_{\cdot t} - \bar{\boldsymbol{\epsilon}}_*, \qquad \boldsymbol{\epsilon}_{\cdot t} - \bar{\boldsymbol{\epsilon}}_* = \boldsymbol{B}_N\boldsymbol{\alpha} + \boldsymbol{e}_N(\gamma_t - \bar{\gamma}) + \boldsymbol{u}_{\cdot t} - \bar{\boldsymbol{u}}_*, \tag{6}$$

where $\boldsymbol{\epsilon}_{i\cdot}$, $\boldsymbol{u}_{i\cdot}$, $\boldsymbol{\epsilon}_{\cdot t}$, $\boldsymbol{u}_{\cdot t}$ are defined in similar way as $\boldsymbol{y}_{i\cdot}$ and $\boldsymbol{y}_{\cdot t}$, $\bar{\alpha} = \sum_i \alpha_i/N$, $\bar{\gamma} = \sum_t \gamma_t/T$, $\bar{\boldsymbol{X}} = \sum_i \boldsymbol{X}_{i\cdot}/N$, $\bar{\boldsymbol{X}}_* = \sum_t \boldsymbol{X}_{\cdot t}/T$, $\bar{\boldsymbol{y}} = \sum_i \boldsymbol{y}_{i\cdot}/N$, $\bar{\boldsymbol{y}}_* = \sum_t \boldsymbol{y}_{\cdot t}/T$, etc. Premultiplying (3) by $\boldsymbol{B}_T$, (5) by $\boldsymbol{A}_T$, (4) by $\boldsymbol{B}_N$ and (6) by $\boldsymbol{A}_N$, give, respectively,

$$\begin{aligned} \boldsymbol{B}_T\boldsymbol{y}_{i\cdot} &= \boldsymbol{B}_T\boldsymbol{X}_{i\cdot}\boldsymbol{\beta} + \boldsymbol{B}_T\boldsymbol{\epsilon}_{i\cdot}, \\ \boldsymbol{A}_T(\boldsymbol{y}_{i\cdot} - \bar{\boldsymbol{y}}) &= \boldsymbol{A}_T(\boldsymbol{X}_{i\cdot} - \bar{\boldsymbol{X}})\boldsymbol{\beta} + \boldsymbol{A}_T(\boldsymbol{\epsilon}_{i\cdot} - \bar{\boldsymbol{\epsilon}}), \end{aligned} \tag{7}$$

$$\begin{aligned} \boldsymbol{B}_N\boldsymbol{y}_{\cdot t} &= \boldsymbol{B}_N\boldsymbol{X}_{\cdot t}\boldsymbol{\beta} + \boldsymbol{B}_N\boldsymbol{\epsilon}_{\cdot t}, \\ \boldsymbol{A}_N(\boldsymbol{y}_{\cdot t} - \bar{\boldsymbol{y}}_*) &= \boldsymbol{A}_N(\boldsymbol{X}_{\cdot t} - \bar{\boldsymbol{X}}_*)\boldsymbol{\beta} + \boldsymbol{A}_N(\boldsymbol{\epsilon}_{\cdot t} - \bar{\boldsymbol{\epsilon}}_*). \end{aligned} \tag{8}$$

Symbolizing by $\boldsymbol{W}$, $\boldsymbol{V}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ matrices containing *within-individual*, *within-period*, *between-individual*, and *between-period* (co)variation, respectively, individual-specific and period-specific cross-product matrices emerge as

$$\begin{aligned} \boldsymbol{W}_{XXij} &= \boldsymbol{X}_{i\cdot}'\boldsymbol{B}_T\boldsymbol{X}_{j\cdot} = \textstyle\sum_{t=1}^T(\boldsymbol{x}_{it} - \bar{\boldsymbol{x}}_{i\cdot})'(\boldsymbol{x}_{jt} - \bar{\boldsymbol{x}}_{j\cdot}), \\ \boldsymbol{W}_{X\gamma i} &= \boldsymbol{X}_{i\cdot}'\boldsymbol{B}_T\boldsymbol{\gamma} = \textstyle\sum_{t=1}^T(\boldsymbol{x}_{it} - \bar{\boldsymbol{x}}_{i\cdot})'(\gamma_t - \bar{\gamma}), \end{aligned} \qquad i,j=1,\dots,N, \tag{9}$$

$$\begin{aligned} \boldsymbol{V}_{XXts} &= \boldsymbol{X}_{\cdot t}'\boldsymbol{B}_N\boldsymbol{X}_{\cdot s} = \textstyle\sum_{i=1}^N(\boldsymbol{x}_{it} - \bar{\boldsymbol{x}}_{\cdot t})'(\boldsymbol{x}_{is} - \bar{\boldsymbol{x}}_{\cdot s}), \\ \boldsymbol{V}_{X\alpha t} &= \boldsymbol{X}_{\cdot t}'\boldsymbol{B}_N\boldsymbol{\alpha} = \textstyle\sum_{i=1}^N(\boldsymbol{x}_{it} - \bar{\boldsymbol{x}}_{\cdot t})'(\alpha_i - \bar{\alpha}), \end{aligned} \qquad t,s=1,\dots,T, \tag{10}$$

$$\begin{aligned}\boldsymbol{B}_{XXii} &= (\boldsymbol{X}_{i\cdot}-\bar{\boldsymbol{X}})'\boldsymbol{A}_T(\boldsymbol{X}_{i\cdot}-\bar{\boldsymbol{X}}) = T(\bar{\boldsymbol{x}}_{i\cdot}-\bar{\boldsymbol{x}})'(\bar{\boldsymbol{x}}_{i\cdot}-\bar{\boldsymbol{x}}),\\ \boldsymbol{B}_{X\alpha ii} &= (\boldsymbol{X}_{i\cdot}-\bar{\boldsymbol{X}})'\boldsymbol{e}_T(\alpha_i-\bar{\alpha}) = T(\bar{\boldsymbol{x}}_{i\cdot}-\bar{\boldsymbol{x}})'(\alpha_i-\bar{\alpha}),\end{aligned} \quad i=1,\dots,N, \tag{11}$$

$$\begin{aligned}\boldsymbol{C}_{XXtt} &= (\boldsymbol{X}_{\cdot t}-\bar{\boldsymbol{X}}_*)'\boldsymbol{A}_N(\boldsymbol{X}_{\cdot t}-\bar{\boldsymbol{X}}_*) = N(\bar{\boldsymbol{x}}_{\cdot t}-\bar{\boldsymbol{x}})'(\bar{\boldsymbol{x}}_{\cdot t}-\bar{\boldsymbol{x}}),\\ \boldsymbol{C}_{X\gamma tt} &= (\boldsymbol{X}_{\cdot t}-\bar{\boldsymbol{X}}_*)'\boldsymbol{e}_N(\gamma_t-\bar{\gamma}) = N(\bar{\boldsymbol{x}}_{\cdot t}-\bar{\boldsymbol{x}})'(\gamma_t-\bar{\gamma}),\end{aligned} \quad t=1,\dots,T, \tag{12}$$

etc., where $\bar{\boldsymbol{x}}_{i\cdot} = (\boldsymbol{e}'_T/T)\boldsymbol{X}_{i\cdot}$, $\bar{\boldsymbol{x}}_{\cdot t} = (\boldsymbol{e}'_N/N)\boldsymbol{X}_{\cdot t}$, $\bar{\boldsymbol{x}} = (\boldsymbol{e}'_{NT}/(NT))\boldsymbol{X} = (\boldsymbol{e}'_{TN}/(TN))\boldsymbol{X}_*$. We have:

- $\boldsymbol{W}_{XXij}$, of full rank K if $\boldsymbol{x}_{it}$ contains no individual-specific variables, is the $(K\times K)$ matrix of *within-individual covariation* in the xs of individuals i and j, while $\boldsymbol{V}_{XXts}$, of full rank K if $\boldsymbol{x}_{it}$ contains no period-specific variables, is the $(K\times K)$ matrix of *within-period covariation* in the xs of periods t and s.
- $\boldsymbol{B}_{XXii}$ and $\boldsymbol{C}_{XXtt}$, of rank 1, are the matrices of *between-individual cross-products* and *between-period cross-products of* the xs of individual i and period t, respectively.
- $\boldsymbol{W}_{X\gamma i}$ and $\boldsymbol{V}_{X\alpha t}$ are the vectors of, respectively, the *within-covariation* of the xs of individual i and the period-specific effects, and the *within-covariation* of the xs of period t and the individual-specific effects.

Premultiplying the two equations in (7) by, respectively, $\boldsymbol{X}'_{i\cdot}\boldsymbol{B}_T$ and $(\boldsymbol{X}_{i\cdot}-\bar{\boldsymbol{X}})'\boldsymbol{A}_T$, and the two equations in (8) by, respectively, $\boldsymbol{X}'_{\cdot t}\boldsymbol{B}_N$ and $(\boldsymbol{X}_{\cdot t}-\bar{\boldsymbol{X}}_*)'\boldsymbol{A}_N$, give

$$\boldsymbol{W}_{XYij} = \boldsymbol{W}_{XXij}\boldsymbol{\beta}+\boldsymbol{W}_{X\epsilon ij}, \quad \boldsymbol{W}_{X\epsilon ij} = \boldsymbol{W}_{X\gamma i}+\boldsymbol{W}_{XUij}, \quad i,j=1,\dots,N, \tag{13}$$
$$\boldsymbol{B}_{XYii} = \boldsymbol{B}_{XXii}\boldsymbol{\beta}+\boldsymbol{B}_{X\epsilon ii}, \quad \boldsymbol{B}_{X\epsilon ii} = \boldsymbol{B}_{X\alpha ii}+\boldsymbol{B}_{XUii}, \quad i=1,\dots,N, \tag{14}$$
$$\boldsymbol{V}_{XYts} = \boldsymbol{V}_{XXts}\boldsymbol{\beta}+\boldsymbol{V}_{X\epsilon ts}, \quad \boldsymbol{V}_{X\epsilon ts} = \boldsymbol{V}_{X\alpha t}+\boldsymbol{V}_{XUts}, \quad t,s=1,\dots,T, \tag{15}$$
$$\boldsymbol{C}_{XYtt} = \boldsymbol{C}_{XXtt}\boldsymbol{\beta}+\boldsymbol{C}_{X\epsilon tt}, \quad \boldsymbol{C}_{X\epsilon tt} = \boldsymbol{C}_{X\gamma tt}+\boldsymbol{C}_{XUtt}, \quad t=1,\dots,T. \tag{16}$$

3 Base estimators

Since $\mathsf{E}(\epsilon_{ij}|\boldsymbol{X}) = 0$ implies $\mathsf{E}(\boldsymbol{W}_{X\epsilon ij}|\boldsymbol{X}) = \mathsf{E}(\boldsymbol{V}_{X\epsilon ts}|\boldsymbol{X}) = \boldsymbol{0}$, (13) and (15) motivate N^2 individual-specific and T^2 period-specific estimators of $\boldsymbol{\beta}$, to be denoted as *base estimators*, or *disaggregate estimators*:

$$\widehat{\boldsymbol{\beta}}_{Wij} = \boldsymbol{W}^{-1}_{XXij}\boldsymbol{W}_{XYij} = (\boldsymbol{X}'_{i\cdot}\boldsymbol{B}_T\boldsymbol{X}_{j\cdot})^{-1}(\boldsymbol{X}'_{i\cdot}\boldsymbol{B}_T\boldsymbol{y}_{j\cdot}), \quad i,j=1,\dots,N, \tag{17}$$
$$\widehat{\boldsymbol{\beta}}_{Vts} = \boldsymbol{V}^{-1}_{XXts}\boldsymbol{V}_{XYts} = (\boldsymbol{X}'_{\cdot t}\boldsymbol{B}_N\boldsymbol{X}_{\cdot s})^{-1}(\boldsymbol{X}'_{\cdot t}\boldsymbol{B}_N\boldsymbol{y}_{\cdot s}), \quad t,s=1,\dots,T, \tag{18}$$

so that $\widehat{\boldsymbol{\beta}}_{Wii}$ is the *OLS* estimator based on the time series from individual i; $\widehat{\boldsymbol{\beta}}_{Wij}$, for $j\neq i$, is the *IV* estimator which instruments the 'within variation' of individual j, $\boldsymbol{B}_T\boldsymbol{X}_{j\cdot}$, by the 'within variation' of individual i, $\boldsymbol{B}_T\boldsymbol{X}_{i\cdot}$; $\widehat{\boldsymbol{\beta}}_{Vtt}$ is the *OLS* estimator based on the cross-section from period t; $\widehat{\boldsymbol{\beta}}_{Vts}$, for $s\neq t$, is the *IV* estimator which instruments the 'within variation' of period s, $\boldsymbol{B}_N\boldsymbol{X}_{\cdot s}$, by the 'within variation' of period t, $\boldsymbol{B}_N\boldsymbol{X}_{\cdot t}$.[1]

If individual-specific variables occur, so that $\boldsymbol{W}_{XXij}$ contains one or more zero rows and columns, their coefficient estimates cannot be obtained from (17), but estimators for the other coefficients can be solved from $\boldsymbol{W}_{XXij}\widehat{\boldsymbol{\beta}}_{Wij} = \boldsymbol{W}_{XYij}$. Likewise, *if period-specific variables occur*, so that $\boldsymbol{V}_{XXts}$ contains one or more zero rows and columns,

[1] One-regressor versions of these estimators, in a measurement error context, are considered in Biørn (2017, Section 7.2.2).

their coefficient estimates cannot be obtained from (18), but estimators for the other coefficients can be solved from $\boldsymbol{V}_{XXts}\widehat{\boldsymbol{\beta}}_{Vts} = \boldsymbol{V}_{XYts}$.

Since inserting for $\boldsymbol{W}_{XYij}$ and $\boldsymbol{V}_{XYts}$ from (13) and (15) in (17) and (18) gives

$$\widehat{\boldsymbol{\beta}}_{Wij} - \boldsymbol{\beta} = \boldsymbol{W}_{XXij}^{-1}\boldsymbol{W}_{X\epsilon ij} = \boldsymbol{W}_{XXij}^{-1}(\boldsymbol{W}_{X\gamma i} + \boldsymbol{W}_{XUij}), \qquad i,j = 1,\dots,N, \tag{19}$$

$$\widehat{\boldsymbol{\beta}}_{Vts} - \boldsymbol{\beta} = \boldsymbol{V}_{XXts}^{-1}\boldsymbol{V}_{X\epsilon ts} = \boldsymbol{V}_{XXts}^{-1}(\boldsymbol{V}_{X\alpha t} + \boldsymbol{V}_{XUts}), \qquad t,s = 1,\dots,T, \tag{20}$$

and (1) implies

$$\mathsf{E}(\boldsymbol{W}_{XUij}|\boldsymbol{X}) = \mathsf{E}(\boldsymbol{W}_{X\gamma i}|\boldsymbol{X}) = \boldsymbol{0}_{K1}, \qquad i,j = 1,\dots,N, \tag{21}$$

$$\mathsf{E}(\boldsymbol{V}_{XUts}|\boldsymbol{X}) = \mathsf{E}(\boldsymbol{V}_{X\alpha t}|\boldsymbol{X}) = \boldsymbol{0}_{K1}, \qquad t,s = 1,\dots,T, \tag{22}$$

$\widehat{\boldsymbol{\beta}}_{Wij}$ and $\widehat{\boldsymbol{\beta}}_{Vts}$ are unbiased. Also, $\widehat{\boldsymbol{\beta}}_{Wij}$ is T-consistent since $\text{plim}(\boldsymbol{W}_{X\epsilon ij}/T) = \boldsymbol{0}_{K,1}$, provided that $\text{plim}(\boldsymbol{W}_{XXij}/T)$ is non-singular, and $\widehat{\boldsymbol{\beta}}_{Vts}$ is N-consistent since $\text{plim}(\boldsymbol{V}_{X\epsilon ts}/N) = \boldsymbol{0}_{K,1}$, provided that $\text{plim}(\boldsymbol{V}_{XXts}/N)$ is non-singular.

Some estimators may be consistent under weaker conditions than (1). The following *robustness results* hold:

- Since (19) does not contain $\boldsymbol{\alpha}$, $\widehat{\boldsymbol{\beta}}_{Wij}$ is T-consistent if α_i is fixed and unstructured or correlated with $\bar{\boldsymbol{x}}_{i\cdot}$. If γ_t is correlated with $\bar{\boldsymbol{x}}_{\cdot t}$, consistency fails. Symmetrically, since (20) does not contain $\boldsymbol{\gamma}$, $\widehat{\boldsymbol{\beta}}_{Vts}$ is N-consistent if γ_t is fixed and unstructured or correlated with $\bar{\boldsymbol{x}}_{\cdot t}$. If α_i is correlated with $\bar{\boldsymbol{x}}_{i\cdot}$, consistency fails.
- Endogeneity of or random measurement error in $\boldsymbol{x}_{it}$ usually violate $\mathsf{E}(u_{it}|\boldsymbol{X}) = 0$ and give $\mathsf{E}(\boldsymbol{x}'_{it}u_{it}) \neq \boldsymbol{0}_{K1}$, $\text{plim}(\boldsymbol{W}_{XUii}/T) \neq \boldsymbol{0}_{K1}$ and $\text{plim}(\boldsymbol{V}_{XUtt}/N) \neq \boldsymbol{0}_{K1}$, making the OLS estimators $\widehat{\boldsymbol{\beta}}_{Wii}$ and $\widehat{\boldsymbol{\beta}}_{Vtt}$ inconsistent, while the IV estimators $\widehat{\boldsymbol{\beta}}_{Wij}$ $(j \neq i)$ and $\widehat{\boldsymbol{\beta}}_{Vts}$ $(s \neq t)$ remain T-consistent and N-consistent. respectively.

In Appendix A it is shown that when (2) holds the matrices of covariances for the base estimators are

$$\mathsf{C}(\widehat{\boldsymbol{\beta}}_{Wij}, \widehat{\boldsymbol{\beta}}_{Wkl}|\boldsymbol{X}) = (\sigma_\gamma^2 + \delta_{jl}\sigma^2)\boldsymbol{W}_{XXij}^{-1}\boldsymbol{W}_{XXik}\boldsymbol{W}_{XXlk}^{-1}, \qquad i,j,k,l = 1,\dots,N, \tag{23}$$

$$\mathsf{C}(\widehat{\boldsymbol{\beta}}_{Vts}, \widehat{\boldsymbol{\beta}}_{Vpq}|\boldsymbol{X}) = (\sigma_\alpha^2 + \delta_{sq}\sigma^2)\boldsymbol{V}_{XXts}^{-1}\boldsymbol{V}_{XXtp}\boldsymbol{V}_{XXqp}^{-1}, \qquad t,s,p,q = 1,\dots,T. \tag{24}$$

For $(k,l) = (i,j)$ and $(p,q) = (t,s)$, the variance-covariance matrices emerge as

$$\mathsf{V}(\widehat{\boldsymbol{\beta}}_{Wij}|\boldsymbol{X}) = (\sigma_\gamma^2 + \sigma^2)\boldsymbol{W}_{XXij}^{-1}\boldsymbol{W}_{XXii}\boldsymbol{W}_{XXji}^{-1}, \qquad i,j = 1,\dots,N, \tag{25}$$

$$\mathsf{V}(\widehat{\boldsymbol{\beta}}_{Vts}|\boldsymbol{X}) = (\sigma_\alpha^2 + \sigma^2)\boldsymbol{V}_{XXts}^{-1}\boldsymbol{V}_{XXtt}\boldsymbol{V}_{XXst}^{-1}, \qquad t,s = 1,\dots,T, \tag{26}$$

from which it follows that $\widehat{\boldsymbol{\beta}}_{Wjj}$ and $\widehat{\boldsymbol{\beta}}_{Vss}$ are always superior to $\widehat{\boldsymbol{\beta}}_{Wij}$ $(j \neq i)$ and $\widehat{\boldsymbol{\beta}}_{Vts}$ $(s \neq t)$, respectively, as $\mathsf{V}(\widehat{\boldsymbol{\beta}}_{Wij}|\boldsymbol{X}) - \mathsf{V}(\widehat{\boldsymbol{\beta}}_{Wjj}|\boldsymbol{X})$ $(i \neq j)$ and $\mathsf{V}(\widehat{\boldsymbol{\beta}}_{Vts}|\boldsymbol{X}) - \mathsf{V}(\widehat{\boldsymbol{\beta}}_{Vss}|\boldsymbol{X})$ $(t \neq s)$ are positive definite. We have:

$$\begin{aligned}
\mathsf{V}(\widehat{\boldsymbol{\beta}}_{Wij}|\boldsymbol{X}) - \mathsf{V}(\widehat{\boldsymbol{\beta}}_{Wjj}|\boldsymbol{X}) &= (\sigma_\gamma^2 + \sigma^2)(\boldsymbol{W}_{XXij}^{-1}\boldsymbol{W}_{XXii}\boldsymbol{W}_{XXji}^{-1} - \boldsymbol{W}_{XXjj}^{-1}) \\
&\equiv (\sigma_\gamma^2 + \sigma^2)(\boldsymbol{A}_{WXij}^{-1}\boldsymbol{A}_{WXji}^{-1} - \boldsymbol{I}_K)\boldsymbol{W}_{XXjj}^{-1}, \\
\mathsf{V}(\widehat{\boldsymbol{\beta}}_{Vts}|\boldsymbol{X}) - \mathsf{V}(\widehat{\boldsymbol{\beta}}_{Vss}|\boldsymbol{X}) &= (\sigma_\alpha^2 + \sigma^2)(\boldsymbol{V}_{XXts}^{-1}\boldsymbol{V}_{XXtt}\boldsymbol{V}_{XXst}^{-1} - \boldsymbol{V}_{XXss}^{-1}) \\
&\equiv (\sigma_\alpha^2 + \sigma^2)(\boldsymbol{A}_{VXts}^{-1}\boldsymbol{A}_{VXst}^{-1} - \boldsymbol{I}_K)\boldsymbol{V}_{XXss}^{-1},
\end{aligned}$$

where

$$\begin{aligned}
\boldsymbol{A}_{WXij} &= \boldsymbol{W}_{XXii}^{-1}\boldsymbol{W}_{XXij}, \\
\boldsymbol{A}_{VXts} &= \boldsymbol{V}_{XXtt}^{-1}\boldsymbol{V}_{XXts}.
\end{aligned}$$

The latter are the matrix of regression coefficients when regressing the j-specific block of $\boldsymbol{X}$, $\boldsymbol{X}_{j\cdot}$, on the i-specific block, $\boldsymbol{X}_{i\cdot}$, and when regressing the s-specific block of $\boldsymbol{X}$, $\boldsymbol{X}_{\cdot s}$, on the t-specific block, $\boldsymbol{X}_{\cdot t}$, respectively, and $(\boldsymbol{A}_{WXij}^{-1}\boldsymbol{A}_{WXji}^{-1}-\boldsymbol{I}_K)$, $j \neq i$, and $(\boldsymbol{A}_{VXts}^{-1}\boldsymbol{A}_{VXst}^{-1}-\boldsymbol{I}_K)$, $s\neq t$, are positive definite when all regressors are two-dimensional.

The structure is transparent in the *one regressor* case ($K=1$), (23)–(26) reducing to

$$\begin{aligned}\mathsf{C}(\widehat{\beta}_{Wij},\widehat{\beta}_{Wkl}|\boldsymbol{X}) &= (\sigma_\gamma^2+\delta_{jl}\sigma^2)\frac{W_{XXik}}{W_{XXij}W_{XXkl}},\\ \mathsf{V}(\widehat{\beta}_{Wij}|\boldsymbol{X}) &= (\sigma_\gamma^2+\sigma^2)\frac{W_{XXii}}{W_{XXij}^2},\end{aligned} \tag{27}$$

$$\begin{aligned}\mathsf{C}(\widehat{\beta}_{Vts},\widehat{\beta}_{Vpq}|\boldsymbol{X}) &= (\sigma_\alpha^2+\delta_{sq}\sigma^2)\frac{V_{XXtp}}{V_{XXts}V_{XXpq}},\\ \mathsf{V}(\widehat{\beta}_{Vts}|\boldsymbol{X}) &= (\sigma_\alpha^2+\sigma^2)\frac{V_{XXtt}}{V_{XXts}^2},\end{aligned} \tag{28}$$

where W_{XXik}, $\widehat{\beta}_{Wij}$, etc. are the scalar counterparts to $\boldsymbol{W}_{XXik}$, $\widehat{\boldsymbol{\beta}}_{Wij}$, etc. The coefficient of correlation between two arbitrary individual-specific and two arbitrary period-specific base estimators can therefore be written as, respectively,

$$\begin{aligned}\rho(\widehat{\beta}_{Wij},\widehat{\beta}_{Wkl}|\boldsymbol{X}) &\equiv \frac{\mathsf{C}(\widehat{\beta}_{Wij},\widehat{\beta}_{Wkl}|\boldsymbol{X})}{[\mathsf{V}(\widehat{\beta}_{Wij}|\boldsymbol{X})\mathsf{V}(\widehat{\beta}_{Wkl}|\boldsymbol{X})]^{1/2}}\\ &= \frac{\sigma_\gamma^2+\delta_{jl}\sigma^2}{\sigma_\gamma^2+\sigma^2}\frac{W_{XXik}}{(W_{XXii}W_{XXkk})^{1/2}}\\ &= \rho(\epsilon_{jt},\epsilon_{lt})R_{WXik},\end{aligned} \tag{29}$$

$$\begin{aligned}\rho(\widehat{\beta}_{Vts},\widehat{\beta}_{Vpq}|\boldsymbol{X}) &\equiv \frac{\mathsf{C}(\widehat{\beta}_{Vts},\widehat{\beta}_{Vpq}|\boldsymbol{X})}{[\mathsf{V}(\widehat{\beta}_{Vts}|\boldsymbol{X})\mathsf{V}(\widehat{\beta}_{Vpq}|\boldsymbol{X})]^{1/2}}\\ &= \frac{\sigma_\alpha^2+\delta_{sq}\sigma^2}{\sigma_\alpha^2+\sigma^2}\frac{V_{XXtp}}{(V_{XXtt}V_{XXpp})^{1/2}}\\ &= \rho(\epsilon_{is},\epsilon_{iq})R_{VXtp},\end{aligned} \tag{30}$$

where $R_{WXik} = W_{XXik}/(W_{XXii}W_{XXkk})^{1/2}$ is the empirical coefficient of correlation between the xs of individuals i and k; $R_{VXtp} = V_{XXtp}/(V_{XXtt}V_{XXpp})^{1/2}$ is the coefficient of correlation between the xs in periods t and p; $\rho(\epsilon_{jt},\epsilon_{lt}) = (\sigma_\gamma^2+\delta_{jl}\sigma^2)/(\sigma_\gamma^2+\sigma^2)$; and $\rho(\epsilon_{is},\epsilon_{iq}) = (\sigma_\alpha^2+\delta_{sq}\sigma^2)/(\sigma_\alpha^2+\sigma^2)$.

Therefore, considering (3) *as an N-equation model* with one equation per individual and common coefficient, $\rho(\widehat{\beta}_{Wij},\widehat{\beta}_{Wkl}|\boldsymbol{X})$ emerges as the product of the coefficient of correlation between two ϵs from individuals (equations) j and l in the same period, and the coefficient of correlation between the regressor (instrument) for individuals (equations) i and k. Likewise, considering (4) as a *T-equation model* with one equation per period and common coefficient, $\rho(\widehat{\beta}_{Vts},\widehat{\beta}_{Vpq}|\boldsymbol{X})$ emerges as the product of the coefficient of correlation between two ϵs from periods (equations) s and q for the same individual, and the coefficient of correlation between the values of the regressor (instrument) in periods t and p. Hence, $\rho(\widehat{\beta}_{Wij},\widehat{\beta}_{Wkl}|\boldsymbol{X})$ has one equation-specific component (j vs. l) and one instrument-specific component (i vs. k), while $\rho(\widehat{\beta}_{Vts},\widehat{\beta}_{Vpq}|\boldsymbol{X})$ has one equation-specific component (s vs. q) and one instrument-specific component (t vs. p). For $j = l$ (same equation/individual) and $i=k$ (same IV) (29) gives, respectively,

$$\rho(\widehat{\beta}_{Wij},\widehat{\beta}_{Wkj}|\boldsymbol{X}) = R_{WXik}, \quad i \neq k,$$
$$\rho(\widehat{\beta}_{Wij},\widehat{\beta}_{Wil}|\boldsymbol{X}) = \frac{\sigma_\gamma^2}{\sigma_\gamma^2+\sigma^2}, \quad j \neq l,$$

and for $s=q$ (same equation/period) and $t=p$ (same IV) (30) gives, respectively,

$$\rho(\widehat{\beta}_{Vts},\widehat{\beta}_{Vps}|\boldsymbol{X}) = R_{VXtp}, \quad t \neq p,$$
$$\rho(\widehat{\beta}_{Vts},\widehat{\beta}_{Vtq}|\boldsymbol{X}) = \frac{\sigma_\alpha^2}{\sigma_\alpha^2+\sigma^2}, \quad s \neq q.$$

From (27) and (28) it follows that the *inefficiency* when instrumenting the (within) variation of individual i by the (within) variation of individual j relative to performing OLS on the observations from individual j and when instrumenting the (within) variation of period t by the (within) variation of period s relative to performing OLS on the observations from period s, can be expressed simply as, respectively,

$$\frac{\mathsf{V}(\widehat{\beta}_{Wij}|\boldsymbol{X})}{\mathsf{V}(\widehat{\beta}_{Wjj}|\boldsymbol{X})} = \frac{1}{A_{WXij}A_{WXji}} = \frac{1}{R^2_{WXij}},$$
$$\frac{\mathsf{V}(\widehat{\beta}_{Vts}|\boldsymbol{X})}{\mathsf{V}(\widehat{\beta}_{Vss}|\boldsymbol{X})} = \frac{1}{A_{VXts}A_{VXst}} = \frac{1}{R^2_{VXts}}.$$

Hence, R^{-2}_{WXij} and R^{-2}_{VXts} measure the efficiency loss when using estimators that are robust to inconsistency caused by simultaneity or random measurement error in the regressor, respectively, (i) in a relationship for individual j using as IV observations from another individual, i, relative to using OLS, and (ii) in a relationship for period s by using as IV observations from another period, t, relative to using OLS.

4 A class of moment estimators

Since each base estimator $\widehat{\boldsymbol{\beta}}_{Wij}$ and $\widehat{\boldsymbol{\beta}}_{Vts}$ uses only a minor part of the panel data set, they are rarely real competitors to estimators utilizing the complete data set, when (1) is valid. And even if correlation between $\boldsymbol{x}_{it}$ and u_{it}, between $\bar{\boldsymbol{x}}_{i\cdot}$ and α_i or between $\bar{\boldsymbol{x}}_{\cdot t}$ and γ_t are allowed for, consistent aggregate estimators which are more efficient than any of the IV estimators $\widehat{\boldsymbol{\beta}}_{Wij}$ ($j \neq i$) and $\widehat{\boldsymbol{\beta}}_{Vts}$ ($s \neq t$) may exist. Yet, the insight provided by examining the base estimators is useful when constructing composite estimators of $\boldsymbol{\beta}$, of which they can serve as building-blocks.

In order to explore this, we define *a class of estimators of* $\boldsymbol{\beta}$ by weighting the individual-specific or period-specific (co)variation in $\boldsymbol{X}$ and $\boldsymbol{y}$. Let $\boldsymbol{\phi} = (\phi_{ts})$ be a $(T \times T)$ matrix and $\boldsymbol{\psi} = (\psi_{ij})$ an $(N \times N)$ matrix of (positive, zero or negative) weights and define a *general moment estimator* as

$$\begin{aligned}
\boldsymbol{b} = \boldsymbol{b}(\boldsymbol{\phi},\boldsymbol{\psi}) &= (\textstyle\sum_{t=1}^T \sum_{s=1}^T \phi_{ts}\boldsymbol{V}_{XXts} + \sum_{i=1}^N \sum_{j=1}^N \psi_{ij}\boldsymbol{W}_{XXij})^{-1} \\
&\quad \times (\textstyle\sum_{t=1}^T \sum_{s=1}^T \phi_{ts}\boldsymbol{V}_{XYts} + \sum_{i=1}^N \sum_{j=1}^N \psi_{ij}\boldsymbol{W}_{XYij}) \\
&\equiv (\textstyle\sum_{t=1}^T \sum_{s=1}^T \phi_{ts}\boldsymbol{V}_{XXts} + \sum_{i=1}^N \sum_{j=1}^N \psi_{ij}\boldsymbol{W}_{XXij})^{-1} \\
&\quad \times (\textstyle\sum_{t=1}^T \sum_{s=1}^T \phi_{ts}\boldsymbol{V}_{XXts}\widehat{\boldsymbol{\beta}}_{Vts} + \sum_{i=1}^N \sum_{j=1}^N \psi_{ij}\boldsymbol{W}_{XXij}\widehat{\boldsymbol{\beta}}_{Wij}),
\end{aligned} \quad (31)$$

or, in simplified notation,

$$\boldsymbol{b} = \sum_{t=1}^{T}\sum_{s=1}^{T}\boldsymbol{G}_{Vts}\widehat{\boldsymbol{\beta}}_{Vts} + \sum_{i=1}^{N}\sum_{j=1}^{N}\boldsymbol{G}_{Wij}\widehat{\boldsymbol{\beta}}_{Wij}, \tag{32}$$

involving weighting matrices $\boldsymbol{G}_{Vts}, \boldsymbol{G}_{Wij}$, $\sum_t\sum_s \boldsymbol{G}_{Vts}+\sum_i\sum_j \boldsymbol{G}_{Wij}=\boldsymbol{I}_K$, given by

$$\begin{aligned} &\boldsymbol{G}_{Vts} = \boldsymbol{Q}^{-1}\phi_{ts}\boldsymbol{V}_{XXts}, && t,s = 1,\dots,T,\\ &\boldsymbol{G}_{Wij} = \boldsymbol{Q}^{-1}\psi_{ij}\boldsymbol{W}_{XXij}, && i,j = 1,\dots,N,\\ &\boldsymbol{Q} = \boldsymbol{Q}(\boldsymbol{\phi},\boldsymbol{\psi}) = \sum_{t=1}^{T}\sum_{s=1}^{T}\phi_{ts}\boldsymbol{V}_{XXts} + \sum_{i=1}^{N}\sum_{j=1}^{N}\psi_{ij}\boldsymbol{W}_{XXij}. \end{aligned} \tag{33}$$

When (1) holds, $\boldsymbol{b}$ is unbiased for any $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$. In Appendix B it is shown that its variance-covariance matrix is[2]

$$\mathsf{V}(\boldsymbol{b}|\boldsymbol{X}) = \boldsymbol{Q}^{-1}\,\boldsymbol{P}\,(\boldsymbol{Q}^{-1})' = \boldsymbol{Q}(\boldsymbol{\phi},\boldsymbol{\psi})^{-1}\boldsymbol{P}(\boldsymbol{\phi},\boldsymbol{\psi},\sigma^2,\sigma_\alpha^2,\sigma_\gamma^2)(\boldsymbol{Q}(\boldsymbol{\phi},\boldsymbol{\psi})^{-1})', \tag{34}$$

where

$$\boldsymbol{P} = \boldsymbol{P}(\boldsymbol{\phi},\boldsymbol{\psi},\sigma^2,\sigma_\alpha^2,\sigma_\gamma^2) = \sigma^2(\boldsymbol{S}_V + \boldsymbol{S}_W + \boldsymbol{S}_{VW}) + \sigma_\alpha^2\,\boldsymbol{Z}_V + \sigma_\gamma^2\,\boldsymbol{Z}_W, \tag{35}$$

with

$$\begin{aligned} &\boldsymbol{S}_V = \boldsymbol{S}_V(\boldsymbol{\phi}) = \sum_{t=1}^{T}\sum_{p=1}^{T}\boldsymbol{V}_{XXtp}(\sum_{s=1}^{T}\phi_{ts}\phi_{ps}),\\ &\boldsymbol{S}_W = \boldsymbol{S}_W(\boldsymbol{\psi}) = \sum_{i=1}^{N}\sum_{k=1}^{N}\boldsymbol{W}_{XXik}(\sum_{j=1}^{N}\psi_{ij}\psi_{kj}),\\ &\boldsymbol{S}_{VW} = \boldsymbol{S}_{VW}(\boldsymbol{\phi},\boldsymbol{\psi}) = \sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{i=1}^{N}\sum_{j=1}^{N}\phi_{ts}\psi_{ij}(\boldsymbol{x}_{is}-\bar{\boldsymbol{x}}_{i\cdot})'(\boldsymbol{x}_{jt}-\bar{\boldsymbol{x}}_{\cdot t}),\\ &\boldsymbol{Z}_V = \boldsymbol{Z}_V(\boldsymbol{\phi}) = \sum_{t=1}^{T}\sum_{p=1}^{T}\boldsymbol{V}_{XXtp}(\sum_{s=1}^{T}\phi_{ts})(\sum_{r=1}^{T}\phi_{pr}),\\ &\boldsymbol{Z}_W = \boldsymbol{Z}_W(\boldsymbol{\psi}) = \sum_{i=1}^{N}\sum_{k=1}^{N}\boldsymbol{W}_{XXik}(\sum_{j=1}^{N}\psi_{ij})(\sum_{l=1}^{N}\psi_{kl}). \end{aligned} \tag{36}$$

If either $\phi_{ts}=\phi$ for all t,s or $\psi_{ij}=\psi$ for all i,j, $\boldsymbol{S}_{WV}=\boldsymbol{0}$, while $\boldsymbol{Z}_V=\boldsymbol{0}$ if $\sum_{s=1}^{T}\phi_{ts}=0$ for all t, and $\boldsymbol{Z}_W=\boldsymbol{0}$ if $\sum_{j=1}^{T}\psi_{ij}=0$ for all i. The standard estimators in fixed and random effects models satisfy at least one of these restrictions, which will be shown below.

From (34)–(36) $\mathsf{V}(\boldsymbol{b}|\boldsymbol{X})$ can be estimated consistently for any chosen weighting matrices $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ when consistent estimators of the variances σ^2, σ_α^2, and σ_γ^2 are available.

5 Selected aggregate estimators

The estimator $\boldsymbol{b}$ contains several familiar estimators for fixed effects models. We first describe the weighting system $(\boldsymbol{\phi},\boldsymbol{\psi})$ for six such estimators and other, less familiar estimators whose consistency is more robust to violation of the basic assumptions.[3]

Let the matrices of *overall*, *within individual* and *within period* (co)variation be

$$\boldsymbol{W}_{XX} = \sum_{i=1}^{N}\boldsymbol{W}_{XXii} = \sum_{i=1}^{N}\sum_{t=1}^{T}(\boldsymbol{x}_{it}-\bar{\boldsymbol{x}}_{i\cdot})'(\boldsymbol{x}_{it}-\bar{\boldsymbol{x}}_{i\cdot}), \tag{37}$$

$$\boldsymbol{V}_{XX} = \sum_{t=1}^{T}\boldsymbol{V}_{XXtt} = \sum_{t=1}^{T}\sum_{i=1}^{N}(\boldsymbol{x}_{it}-\bar{\boldsymbol{x}}_{\cdot t})'(\boldsymbol{x}_{it}-\bar{\boldsymbol{x}}_{\cdot t}), \tag{38}$$

etc. The corresponding *overall between individual*, and *between period* (co)variation are

$$\begin{aligned} \boldsymbol{B}_{XX} &= \sum_{i=1}^{N}\boldsymbol{B}_{XXii} = T\sum_{i=1}^{N}(\bar{\boldsymbol{x}}_{i\cdot}-\bar{\boldsymbol{x}})'(\bar{\boldsymbol{x}}_{i\cdot}-\bar{\boldsymbol{x}})\\ &= (1/T)\sum_{t=1}^{T}\sum_{s=1}^{T}\boldsymbol{V}_{XXts}, \end{aligned} \tag{39}$$

$$\begin{aligned} \boldsymbol{C}_{XX} &= \sum_{t=1}^{T}\boldsymbol{C}_{XXtt} = N\sum_{t=1}^{T}(\bar{\boldsymbol{x}}_{\cdot t}-\bar{\boldsymbol{x}})'(\bar{\boldsymbol{x}}_{\cdot t}-\bar{\boldsymbol{x}})\\ &= (1/N)\sum_{i=1}^{N}\sum_{j=1}^{N}\boldsymbol{W}_{XXij}, \end{aligned} \tag{40}$$

[2] This specializes to the formula in Biørn (1994, Appendix A) when $K=1$, $\sigma_\gamma^2=0$.

[3] The results below generalize those in Biørn (1994, section 3), where only one regressor is included ($K=1$) and period-specific effects are disregarded ($\gamma_t=0$).

etc., where the last equalities are shown in Appendix C. The matrix of *overall* (co)variation and its decomposition is

$$\begin{aligned}\boldsymbol{G}_{XX} &= \sum_{i=1}^{N}\sum_{t=1}^{T}(\boldsymbol{x}_{it}-\bar{\boldsymbol{x}})'(\boldsymbol{x}_{it}-\bar{\boldsymbol{x}})\\ &= \boldsymbol{W}_{XX}+\boldsymbol{B}_{XX} = \boldsymbol{V}_{XX}+\boldsymbol{C}_{XX}\\ &\equiv \sum_{i=1}^{N}\boldsymbol{W}_{XXii}+(1/T)\sum_{t=1}^{T}\sum_{s=1}^{T}\boldsymbol{V}_{XXts}\\ &\equiv \sum_{t=1}^{T}\boldsymbol{V}_{XXtt}+(1/N)\sum_{i=1}^{N}\sum_{j=1}^{N}\boldsymbol{W}_{XXij}.\end{aligned} \tag{41}$$

Finally, the matrix of *residual* (co)variation, *i.e.*, the (co)variation which remains when all (co)variation between individuals and between periods is eliminated, the *combined within-individual-and-period* (co)variation, is

$$\begin{aligned}\boldsymbol{R}_{XX} &= \sum_{i=1}^{N}\sum_{t=1}^{T}(\boldsymbol{x}_{it}-\bar{\boldsymbol{x}}_{i\cdot}-\bar{\boldsymbol{x}}_{\cdot t}+\bar{\boldsymbol{x}})'(\boldsymbol{x}_{it}-\bar{\boldsymbol{x}}_{i\cdot}-\bar{\boldsymbol{x}}_{\cdot t}+\bar{\boldsymbol{x}})\\ &= \boldsymbol{G}_{XX}-\boldsymbol{B}_{XX}-\boldsymbol{C}_{XX}\\ &\equiv \sum_{i=1}^{N}(\boldsymbol{W}_{XXii}-(1/N)\sum_{j=1}^{N}\boldsymbol{W}_{XXij})\\ &\equiv \sum_{t=1}^{T}(\boldsymbol{V}_{XXtt}-(1/T)\sum_{s=1}^{T}\boldsymbol{V}_{XXts}).\end{aligned} \tag{42}$$

We notice that $\boldsymbol{G}_{XX}$ and $\boldsymbol{R}_{XX}$ can be expressed in terms of the $\boldsymbol{W}_{XXij}$s and the $\boldsymbol{V}_{XXts}$s in two ways.

Combining the decompositions exemplified in (37)–(40) with (17)–(18), we can now, cexpress the familiar within individual, within period, between individual, and between period estimators of $\boldsymbol{\beta}$ as the following aggregates

$$\widehat{\boldsymbol{\beta}}_W = \boldsymbol{W}_{XX}^{-1}\boldsymbol{W}_{XY} = (\sum_{i=1}^{N}\boldsymbol{W}_{XXii})^{-1}(\sum_{i=1}^{N}\boldsymbol{W}_{XXii}\widehat{\boldsymbol{\beta}}_{Wii}), \tag{43}$$

$$\widehat{\boldsymbol{\beta}}_V = \boldsymbol{V}_{XX}^{-1}\boldsymbol{V}_{XY} = (\sum_{t=1}^{T}\boldsymbol{V}_{XXtt})^{-1}(\sum_{t=1}^{T}\boldsymbol{V}_{XXtt}\widehat{\boldsymbol{\beta}}_{Vtt}), \tag{44}$$

$$\widehat{\boldsymbol{\beta}}_B = \boldsymbol{B}_{XX}^{-1}\boldsymbol{B}_{XY} = (\sum_{t=1}^{T}\sum_{s=1}^{T}\boldsymbol{V}_{XXts})^{-1}(\sum_{t=1}^{T}\sum_{s=1}^{T}\boldsymbol{V}_{XXts}\widehat{\boldsymbol{\beta}}_{Vts}), \tag{45}$$

$$\widehat{\boldsymbol{\beta}}_C = \boldsymbol{C}_{XX}^{-1}\boldsymbol{C}_{XY} = (\sum_{i=1}^{N}\sum_{j=1}^{N}\boldsymbol{W}_{XXij})^{-1}(\sum_{i=1}^{N}\sum_{j=1}^{N}\boldsymbol{W}_{XXij}\widehat{\boldsymbol{\beta}}_{Wij}). \tag{46}$$

We know that $\widehat{\boldsymbol{\beta}}_W$ and $\widehat{\boldsymbol{\beta}}_V$ are the MVLUE (Minimum Variance Linear Unbiased Estimator) in the cases with only fixed individual-specific and with only fixed period-specific effects, respectively, and that $\widehat{\boldsymbol{\beta}}_B$ and $\widehat{\boldsymbol{\beta}}_C$ are obtained by OLS estimation of equations in individual-specific and in period-specific means, respectively. Among these, $\widehat{\boldsymbol{\beta}}_W$ and $\widehat{\boldsymbol{\beta}}_C$ utilize the *(co)variation across periods* and disregard the (co)variation across individuals, while $\widehat{\boldsymbol{\beta}}_V$ and $\widehat{\boldsymbol{\beta}}_B$ utilize the *(co)variation across individuals* and disregard the (co)variation across periods. Hence, $\widehat{\boldsymbol{\beta}}_W$ and $\widehat{\boldsymbol{\beta}}_C$ may be said to relate to time-series analysis and $\widehat{\boldsymbol{\beta}}_V$ and $\widehat{\boldsymbol{\beta}}_B$ to cross-section analysis.

Reconsider, with this in mind, the global (standard OLS) (G) and the residual (R) estimators. Both can be written as aggregates, as either

$$\begin{aligned}\widehat{\boldsymbol{\beta}}_G &= \boldsymbol{G}_{XX}^{-1}\boldsymbol{G}_{XY} \equiv (\boldsymbol{B}_{XX}+\boldsymbol{C}_{XX}+\boldsymbol{R}_{XX})^{-1}(\boldsymbol{B}_{XY}+\boldsymbol{C}_{XY}+\boldsymbol{R}_{XY})\\ &= (\sum_{i=1}^{N}\boldsymbol{W}_{XXii}+(1/T)\sum_{t=1}^{T}\sum_{s=1}^{T}\boldsymbol{V}_{XXts})^{-1}\\ &\qquad\times(\sum_{i=1}^{N}\boldsymbol{W}_{XXii}\widehat{\boldsymbol{\beta}}_{Wii}+(1/T)\sum_{t=1}^{T}\sum_{s=1}^{T}\boldsymbol{V}_{XXts}\widehat{\boldsymbol{\beta}}_{Vts}),\end{aligned} \tag{47}$$

$$\begin{aligned}\widehat{\boldsymbol{\beta}}_R &= \boldsymbol{R}_{XX}^{-1}\boldsymbol{R}_{XY}\\ &= [\sum_{i=1}^{N}(\boldsymbol{W}_{XXii}-(1/N)\sum_{j=1}^{N}\boldsymbol{W}_{XXij})]^{-1}\\ &\qquad\times[\sum_{i=1}^{N}(\boldsymbol{W}_{XXii}\widehat{\boldsymbol{\beta}}_{Wii}-(1/N)\sum_{j=1}^{N}\boldsymbol{W}_{XXij}\widehat{\boldsymbol{\beta}}_{Wij})],\end{aligned} \tag{48}$$

or

$$\widehat{\boldsymbol{\beta}}_G = (\textstyle\sum_{t=1}^T \boldsymbol{V}_{XXtt} + (1/N)\sum_{i=1}^N \sum_{j=1}^N \boldsymbol{W}_{XXij})^{-1}$$
$$\times (\textstyle\sum_{t=1}^T \boldsymbol{V}_{XXtt}\widehat{\boldsymbol{\beta}}_{Vtt} + (1/N)\sum_{i=1}^N \sum_{j=1}^N \boldsymbol{W}_{XXij}\widehat{\boldsymbol{\beta}}_{Wij}), \tag{49}$$
$$\widehat{\boldsymbol{\beta}}_R = [\textstyle\sum_{t=1}^T (\boldsymbol{V}_{XXtt} - (1/T)\sum_{s=1}^T \boldsymbol{V}_{XXts})]^{-1}$$
$$\times [\textstyle\sum_{t=1}^T (\boldsymbol{V}_{XXtt}\widehat{\boldsymbol{\beta}}_{Vtt} - (1/T)\sum_{s=1}^T \boldsymbol{V}_{XXts}\widehat{\boldsymbol{\beta}}_{Vts})], \tag{50}$$

which follow from (17)–(18) and (41)–(42). While $\widehat{\boldsymbol{\beta}}_G$ is the MVLUE of $\boldsymbol{\beta}$ in the absence of individual or period-specific heterogeneity, $\widehat{\boldsymbol{\beta}}_R$ has this property when all α_is and γ_ts are interpreted as unknown constants (both fixed individual and period-specific effects).[4]

Briefly, (43)–(50) show that all the six standard aggregate estimators for fixed effects models belong to the class (31) and can be interpreted as follows:

- The *within-individual* estimator $\widehat{\boldsymbol{\beta}}_W$ and the *between-period* estimator $\widehat{\boldsymbol{\beta}}_C$ are matrix weighted averages of the *individual-specific* estimators $\widehat{\boldsymbol{\beta}}_{Wij}$, the former utilizing only the N individual-specific OLS estimators, the latter also the $N(N-1)$ individual-specific IV estimators.
- The *within-period* estimator $\widehat{\boldsymbol{\beta}}_V$ and the *between-individual* estimator $\widehat{\boldsymbol{\beta}}_B$ are matrix weighted averages of the *period-specific* estimators $\widehat{\boldsymbol{\beta}}_{Vts}$, the former utilizing only the T period-specific OLS estimators, the latter also the $T(T-1)$ period-specific IV estimators.
- The *residual* estimator $\widehat{\boldsymbol{\beta}}_R$ is a matrix weighted average of either all the N^2 individual-specific estimators or all the T^2 period-specific estimators.
- The *global OLS* estimator $\widehat{\boldsymbol{\beta}}_G$ is a matrix weighted average of either (a) all the N individual-specific OLS estimators, all the T period-specific OLS estimators, and all the $T(T-1)$ period-specific within period IV estimators, or (b) all the T period-specific OLS estimators, all N individual-specific OLS estimators, and all $N(N-1)$ individual-specific within individual IV estimators.

Table 1, panel A summarizes the weights. Compactly,

$$\begin{aligned}
\widehat{\boldsymbol{\beta}}_R &= \boldsymbol{b}(\boldsymbol{B}_T, \boldsymbol{0}_{N,N}) = \boldsymbol{b}(\boldsymbol{0}_{T,T}, \boldsymbol{B}_N),\\
\widehat{\boldsymbol{\beta}}_B &= \boldsymbol{b}(\boldsymbol{A}_T, \boldsymbol{0}_{N,N}),\\
\widehat{\boldsymbol{\beta}}_C &= \boldsymbol{b}(\boldsymbol{0}_{TT}, \boldsymbol{A}_N),\\
\widehat{\boldsymbol{\beta}}_W &= \boldsymbol{b}(\boldsymbol{B}_T, \boldsymbol{A}_N) = \boldsymbol{b}(\boldsymbol{0}_{T,T}, \boldsymbol{I}_N),\\
\widehat{\boldsymbol{\beta}}_V &= \boldsymbol{b}(\boldsymbol{A}_T, \boldsymbol{B}_N) = \boldsymbol{b}(\boldsymbol{I}_T, \boldsymbol{0}_{N,N}),\\
\widehat{\boldsymbol{\beta}}_G &= \boldsymbol{b}(\boldsymbol{I}_T, \boldsymbol{A}_N) = \boldsymbol{b}(\boldsymbol{A}_T, \boldsymbol{I}_N).
\end{aligned}$$

For the total, residual, and within estimators the weights occur in two versions. We obtain their variance-covariance matrices *when the random effects specification (1) is valid* by inserting the weights in Table 1, panel A, into (34)–(36), using (37)–(42). The

[4]Equations (43)–(46), (48) and (50) generalize one-regressor counterparts in Biørn (2017 Section 7.2.3).

results are summarized in panel B. Compactly,

$$
\begin{aligned}
&\mathsf{V}(\widehat{\boldsymbol{\beta}}_R|\boldsymbol{X}) = \sigma^2\boldsymbol{R}_{XX}^{-1},\\
&\mathsf{V}(\widehat{\boldsymbol{\beta}}_B|\boldsymbol{X}) = (\sigma^2+T\sigma_\alpha^2)\boldsymbol{B}_{XX}^{-1},\\
&\mathsf{V}(\widehat{\boldsymbol{\beta}}_C|\boldsymbol{X}) = (\sigma^2+N\sigma_\gamma^2)\boldsymbol{C}_{XX}^{-1},\\
&\mathsf{V}(\widehat{\boldsymbol{\beta}}_W|\boldsymbol{X}) = (\boldsymbol{R}_{XX}+\boldsymbol{C}_{XX})^{-1}[\sigma^2\boldsymbol{R}_{XX}+(\sigma^2+N\sigma_\gamma^2)\boldsymbol{C}_{XX}](\boldsymbol{R}_{XX}+\boldsymbol{C}_{XX})^{-1},\\
&\mathsf{V}(\widehat{\boldsymbol{\beta}}_V|\boldsymbol{X}) = (\boldsymbol{R}_{XX}+\boldsymbol{B}_{XX})^{-1}[\sigma^2\boldsymbol{R}_{XX}+(\sigma^2+T\sigma_\alpha^2)\boldsymbol{B}_{XX}](\boldsymbol{R}_{XX}+\boldsymbol{B}_{XX})^{-1},\\
&\mathsf{V}(\widehat{\boldsymbol{\beta}}_G|\boldsymbol{X}) = \boldsymbol{G}_{XX}^{-1}[\sigma^2\boldsymbol{R}_{XX}+(\sigma^2+T\sigma_\alpha^2)\boldsymbol{B}_{XX}+(\sigma^2+N\sigma_\gamma^2)\boldsymbol{C}_{XX}]\boldsymbol{G}_{XX}^{-1}.
\end{aligned}
$$

Table 1: The General Moment Estimator (31)

A: Weights ϕ_{ts} and ψ_{ij} for selected aggregate estimators

	ϕ_{tt}	$\phi_{ts}, s\neq t$	ψ_{ii}	$\psi_{ij}, j\neq i$	$\boldsymbol{\phi}$	$\boldsymbol{\psi}$
$\widehat{\boldsymbol{\beta}}_R$	$1-\frac{1}{T}$	$-\frac{1}{T}$	0	0	$\boldsymbol{B}_T$	$\boldsymbol{0}_{N,N}$
$\widehat{\boldsymbol{\beta}}_R$	0	0	$1-\frac{1}{N}$	$-\frac{1}{N}$	$\boldsymbol{0}_{T,T}$	$\boldsymbol{B}_N$
$\widehat{\boldsymbol{\beta}}_B$	$\frac{1}{T}$	$\frac{1}{T}$	0	0	$\boldsymbol{A}_T$	$\boldsymbol{0}_{N,N}$
$\widehat{\boldsymbol{\beta}}_C$	0	0	$\frac{1}{N}$	$\frac{1}{N}$	$\boldsymbol{0}_{T,T}$	$\boldsymbol{A}_N$
$\widehat{\boldsymbol{\beta}}_W$	$1-\frac{1}{T}$	$-\frac{1}{T}$	$\frac{1}{N}$	$\frac{1}{N}$	$\boldsymbol{B}_T$	$\boldsymbol{A}_N$
$\widehat{\boldsymbol{\beta}}_W$	0	0	1	0	$\boldsymbol{0}_{T,T}$	$\boldsymbol{I}_N$
$\widehat{\boldsymbol{\beta}}_V$	$\frac{1}{T}$	$\frac{1}{T}$	$1-\frac{1}{N}$	$-\frac{1}{N}$	$\boldsymbol{A}_T$	$\boldsymbol{B}_N$
$\widehat{\boldsymbol{\beta}}_V$	1	0	0	0	$\boldsymbol{I}_T$	$\boldsymbol{0}_{N,N}$
$\widehat{\boldsymbol{\beta}}_G$	1	0	$\frac{1}{N}$	$\frac{1}{N}$	$\boldsymbol{I}_T$	$\boldsymbol{A}_N$
$\widehat{\boldsymbol{\beta}}_G$	$\frac{1}{T}$	$\frac{1}{T}$	1	0	$\boldsymbol{A}_T$	$\boldsymbol{I}_N$

B: Covariance matrices: values of $\boldsymbol{S}_V+\boldsymbol{S}_W, \boldsymbol{Z}_V, \boldsymbol{Z}_W, \boldsymbol{Q}$ ($\boldsymbol{Z}_{VW}=\boldsymbol{0}$)

	$\boldsymbol{S}_V+\boldsymbol{S}_W$	$\boldsymbol{Z}_V$	$\boldsymbol{Z}_W$	$\boldsymbol{Q}$
$\widehat{\boldsymbol{\beta}}_R$	$\boldsymbol{R}_{XX}$	$\boldsymbol{0}$	$\boldsymbol{0}$	$\boldsymbol{R}_{XX}$
$\widehat{\boldsymbol{\beta}}_B$	$\boldsymbol{B}_{XX}$	$T\boldsymbol{B}_{XX}$	$\boldsymbol{0}$	$\boldsymbol{B}_{XX}$
$\widehat{\boldsymbol{\beta}}_C$	$\boldsymbol{C}_{XX}$	$\boldsymbol{0}$	$N\boldsymbol{C}_{XX}$	$\boldsymbol{C}_{XX}$
$\widehat{\boldsymbol{\beta}}_W$	$\boldsymbol{C}_{XX}+\boldsymbol{R}_{XX}$	$\boldsymbol{0}$	$N\boldsymbol{C}_{XX}$	$\boldsymbol{C}_{XX}+\boldsymbol{R}_{XX}$
$\widehat{\boldsymbol{\beta}}_V$	$\boldsymbol{B}_{XX}+\boldsymbol{R}_{XX}$	$T\boldsymbol{B}_{XX}$	$\boldsymbol{0}$	$\boldsymbol{B}_{XX}+\boldsymbol{R}_{XX}$
$\widehat{\boldsymbol{\beta}}_G$	$\boldsymbol{G}_{XX}$	$T\boldsymbol{B}_{XX}$	$N\boldsymbol{C}_{XX}$	$\boldsymbol{G}_{XX}$

Next reconsider the GLS estimator of $\boldsymbol{\beta}$, which is the MVLUE in (1). Consider first

$$
\begin{aligned}
\widehat{\boldsymbol{\beta}} &= \widehat{\boldsymbol{\beta}}(\mu_B,\mu_C,\mu_R)\\
&= (\mu_B\boldsymbol{B}_{XX}+\mu_C\boldsymbol{C}_{XX}+\mu_R\boldsymbol{R}_{XX})^{-1}(\mu_B\boldsymbol{B}_{XY}+\mu_C\boldsymbol{C}_{XY}+\mu_R\boldsymbol{R}_{XY}),
\end{aligned}
\tag{51}
$$

where (μ_B,μ_C,μ_R) are scalar constants. Using the decompositions exemplified by (39), (40), and (42), it can be expressed in the (31) format as either

$$
\begin{aligned}
\widehat{\boldsymbol{\beta}} = {} & [\mu_B\textstyle\sum_{t=1}^T\sum_{s=1}^T\boldsymbol{V}_{XXts}/T+\mu_R\sum_{i=1}^N\boldsymbol{W}_{XXii}+(\mu_C-\mu_R)\sum_{i=1}^N\sum_{j=1}^N\boldsymbol{W}_{XXij}/N]^{-1}\\
& \times[\mu_B\textstyle\sum_{t=1}^T\sum_{s=1}^T\boldsymbol{V}_{XYts}/T+\mu_R\sum_{i=1}^N\boldsymbol{W}_{XYii}+(\mu_C-\mu_R)\sum_{i=1}^N\sum_{j=1}^N\boldsymbol{W}_{XYij}/N],
\end{aligned}
$$

or

$$\widehat{\boldsymbol{\beta}} = [\mu_C \sum_{i=1}^N \sum_{j=1}^N \boldsymbol{W}_{XXij}/N + \mu_R \sum_{t=1}^T \boldsymbol{V}_{XXtt} + (\mu_B - \mu_R) \sum_{t=1}^T \sum_{s=1}^T \boldsymbol{V}_{XXts}/T]^{-1}$$
$$\times [\mu_C \sum_{i=1}^N \sum_{j=1}^N \boldsymbol{W}_{XYij}/N + \mu_R \sum_{t=1}^T \boldsymbol{V}_{XYtt} + (\mu_B - \mu_R) \sum_{t=1}^T \sum_{s=1}^T \boldsymbol{V}_{XYts}/T];$$

compactly

$$\widehat{\boldsymbol{\beta}} = \boldsymbol{b}(\mu_B \boldsymbol{A}_T, \mu_C \boldsymbol{A}_N + \mu_R \boldsymbol{B}_N) \equiv \boldsymbol{b}(\mu_B \boldsymbol{A}_T + \mu_R \boldsymbol{B}_T, \mu_C \boldsymbol{A}_N). \tag{52}$$

As shown by Fuller and Battese (1973, 1974), the two-way random effects GLS estimator of $\boldsymbol{\beta}$ in Model (1), for known $(\sigma^2, \sigma_\alpha^2, \sigma_\gamma^2)$, its MVLUE, can be written as

$$\begin{aligned}\widehat{\boldsymbol{\beta}}_{GLS} &= \widehat{\boldsymbol{\beta}}(\lambda_B, \lambda_C, 1) = (\lambda_B \boldsymbol{B}_{XX} + \lambda_C \boldsymbol{C}_{XX} + \boldsymbol{R}_{XX})^{-1}(\lambda_B \boldsymbol{B}_{XY} + \lambda_C \boldsymbol{C}_{XY} + \boldsymbol{R}_{XY}) \\ &\equiv \left[\frac{\boldsymbol{R}_{XX}}{\sigma^2} + \frac{\boldsymbol{B}_{XX}}{\sigma^2 + T\sigma_\alpha^2} + \frac{\boldsymbol{C}_{XX}}{\sigma^2 + N\sigma_\gamma^2}\right]^{-1} \left[\frac{\boldsymbol{R}_{XY}}{\sigma^2} + \frac{\boldsymbol{B}_{XY}}{\sigma^2 + T\sigma_\alpha^2} + \frac{\boldsymbol{C}_{XY}}{\sigma^2 + N\sigma_\gamma^2}\right],\end{aligned} \tag{53}$$

where

$$\lambda_B = \frac{\sigma^2}{\sigma^2 + T\sigma_\alpha^2}, \qquad \lambda_C = \frac{\sigma^2}{\sigma^2 + N\sigma_\gamma^2}.$$

The corresponding estimators when, respectively, only random individual effects occur ($\gamma_t = \sigma_\gamma^2 = 0$) and only random period effects occur ($\alpha_i = \sigma_\alpha^2 = 0$) are

$$\begin{aligned}\widehat{\boldsymbol{\beta}}_{GLS(\alpha)} &= \widehat{\boldsymbol{\beta}}(\lambda_B, 1, 1) = (\lambda_B \boldsymbol{B}_{XX} + \boldsymbol{C}_{XX} + \boldsymbol{R}_{XX})^{-1}(\lambda_B \boldsymbol{B}_{XY} + \boldsymbol{C}_{XY} + \boldsymbol{R}_{XY}), \\ \widehat{\boldsymbol{\beta}}_{GLS(\gamma)} &= \widehat{\boldsymbol{\beta}}(1, \lambda_C, 1) = (\boldsymbol{B}_{XX} + \lambda_C \boldsymbol{C}_{XX} + \boldsymbol{R}_{XX})^{-1}(\boldsymbol{B}_{XY} + \lambda_C \boldsymbol{C}_{XY} + \boldsymbol{R}_{XY}).\end{aligned}$$

Their weights, as functions of λ_B or λ_C, are given in Table 2, panel A, compactly:

$$\begin{aligned}\widehat{\boldsymbol{\beta}}_{GLS} &= \boldsymbol{b}(\boldsymbol{B}_T + \lambda_B \boldsymbol{A}_T, \lambda_C \boldsymbol{A}_N) \equiv \boldsymbol{b}(\lambda_B \boldsymbol{A}_T, \boldsymbol{B}_N + \lambda_C \boldsymbol{A}_N), \\ \widehat{\boldsymbol{\beta}}_{GLS(\alpha)} &= \boldsymbol{b}(\boldsymbol{B}_T + \lambda_B \boldsymbol{A}_T, \boldsymbol{A}_N) \equiv \boldsymbol{b}(\lambda_B \boldsymbol{A}_T, \boldsymbol{I}_N), \\ \widehat{\boldsymbol{\beta}}_{GLS(\gamma)} &= \boldsymbol{b}(\boldsymbol{I}_T, \lambda_C \boldsymbol{A}_N) \equiv \boldsymbol{b}(\boldsymbol{A}_T, \boldsymbol{B}_N + \lambda_C \boldsymbol{A}_N),\end{aligned}$$

with variance-covariance matrices, see Appendix D,

$$\begin{aligned}\mathsf{V}(\widehat{\boldsymbol{\beta}}_{GLS}|\boldsymbol{X}) &= \sigma^2[\boldsymbol{R}_{XX} + \lambda_B \boldsymbol{B}_{XX} + \lambda_C \boldsymbol{C}_{XX}]^{-1} \\ &= \left[\frac{\boldsymbol{R}_{XX}}{\sigma^2} + \frac{\boldsymbol{B}_{XX}}{\sigma^2 + T\sigma_\alpha^2} + \frac{\boldsymbol{C}_{XX}}{\sigma^2 + N\sigma_\gamma^2}\right]^{-1},\end{aligned}$$

$$\begin{aligned}\mathsf{V}(\widehat{\boldsymbol{\beta}}_{GLS(\alpha)}|\boldsymbol{X}) &= [\boldsymbol{R}_{XX} + \lambda_B \boldsymbol{B}_{XX} + \boldsymbol{C}_{XX}]^{-1} \\ &\quad \times [\sigma^2 \boldsymbol{R}_{XX} + \lambda_B^2(\sigma^2 + T\sigma_\alpha^2)\boldsymbol{B}_{XX} + (\sigma^2 + N\sigma_\gamma^2)\boldsymbol{C}_{XX}] \\ &\quad \times [\boldsymbol{R}_{XX} + \lambda_B \boldsymbol{B}_{XX} + \boldsymbol{C}_{XX}]^{-1}, \\ \mathsf{V}(\widehat{\boldsymbol{\beta}}_{GLS(\gamma)}|\boldsymbol{X}) &= [\boldsymbol{R}_{XX} + \boldsymbol{B}_{XX} + \lambda_C \boldsymbol{C}_{XX}]^{-1} \\ &\quad \times [\sigma^2 \boldsymbol{R}_{XX} + (\sigma^2 + T\sigma_\alpha^2)\boldsymbol{B}_{XX} + \lambda_C^2(\sigma^2 + N\sigma_\gamma^2)\boldsymbol{C}_{XX}] \\ &\quad \times [\boldsymbol{R}_{XX} + \boldsymbol{B}_{XX} + \lambda_B \boldsymbol{C}_{XX}]^{-1}.\end{aligned}$$

If the one-way random effects model is valid, *i.e.*, if $\sigma_\gamma^2 = 0$ or $\sigma_\alpha^2 = 0$, respectively, the latter two are simplified to

$$\mathsf{V}(\widehat{\boldsymbol{\beta}}_{GLS(\alpha)}|\boldsymbol{X}) = \left[\frac{\boldsymbol{R}_{XX} + \boldsymbol{C}_{XX}}{\sigma^2} + \frac{\boldsymbol{B}_{XX}}{\sigma^2 + T\sigma_\alpha^2}\right]^{-1},$$
$$\mathsf{V}(\widehat{\boldsymbol{\beta}}_{GLS(\gamma)}|\boldsymbol{X}) = \left[\frac{\boldsymbol{R}_{XX} + \boldsymbol{B}_{XX}}{\sigma^2} + \frac{\boldsymbol{C}_{XX}}{\sigma^2 + N\sigma_\gamma^2}\right]^{-1}.$$

An interesting issue is *robustness* of the members of the class $\boldsymbol{b}(\boldsymbol{\phi}, \boldsymbol{\psi})$ to violation of the assumptions in Model (1). From conclusions in Section 3 it follows that: [1] If $\boldsymbol{x}_{it}$ contains an IID *measurement error* vector, which becomes part of u_{it}, then (i) all estimators satisfying $\phi_{tt} = 0$, $\phi_{ts} \neq 0$ for some $s \neq t$, and all $\psi_{ij} = 0$, are N-consistent, and (ii) all estimators satisfying $\psi_{ii} = 0$, $\psi_{ij} \neq 0$ for some $j \neq i$, and all $\phi_{ts} = 0$, are T-consistent. [2] If *endogeneity* of some variables in $\boldsymbol{x}_{it}$ leads to $\mathsf{E}(\boldsymbol{x}'_{it}u_{it}) \neq \boldsymbol{0}_{K,1}$, while $\mathsf{E}(\boldsymbol{x}'_{it}u_{js}) = \boldsymbol{0}_{K,1}$ for $(j,s) \neq (i,t)$, similar consistency results hold.

Table 2: The General Moment Estimator (31) For Random Effects Models

A: Weights ϕ_{ts} and ψ_{ij}

	ϕ_{tt}	$\phi_{ts},\ s{\neq}t$	ψ_{ii}	$\psi_{ij},\ j{\neq}i$	$\boldsymbol{\phi}$	$\boldsymbol{\psi}$
$\widehat{\boldsymbol{\beta}}_{GLS}$	$1-\frac{1-\lambda_B}{T}$	$-\frac{1-\lambda_B}{T}$	$\frac{\lambda_C}{N}$	$\frac{\lambda_C}{N}$	$\boldsymbol{B}_T + \lambda_B\boldsymbol{A}_T$	$\lambda_C\boldsymbol{A}_N$
$\widehat{\boldsymbol{\beta}}_{GLS}$	$\frac{\lambda_B}{T}$	$\frac{\lambda_B}{T}$	$1-\frac{1-\lambda_C}{N}$	$-\frac{1-\lambda_C}{N}$	$\lambda_B\boldsymbol{A}_T$	$\boldsymbol{B}_N + \lambda_C\boldsymbol{A}_N$
$\widehat{\boldsymbol{\beta}}_{GLS(\alpha)}$	$1-\frac{1-\lambda_B}{T}$	$-\frac{1-\lambda_B}{T}$	$\frac{1}{N}$	$\frac{1}{N}$	$\boldsymbol{B}_T + \lambda_B\boldsymbol{A}_T$	$\boldsymbol{A}_N$
$\widehat{\boldsymbol{\beta}}_{GLS(\alpha)}$	$\frac{\lambda_B}{T}$	$\frac{\lambda_B}{T}$	1	0	$\lambda_B\boldsymbol{A}_T$	$\boldsymbol{I}_N$
$\widehat{\boldsymbol{\beta}}_{GLS(\gamma)}$	1	0	$\frac{\lambda_C}{N}$	$\frac{\lambda_C}{N}$	$\boldsymbol{I}_T$	$\lambda_C\boldsymbol{A}_N$
$\widehat{\boldsymbol{\beta}}_{GLS(\gamma)}$	$\frac{1}{T}$	$\frac{1}{T}$	$1-\frac{1-\lambda_C}{N}$	$-\frac{1-\lambda_C}{N}$	$\boldsymbol{A}_T$	$\boldsymbol{B}_N + \lambda_C\boldsymbol{A}_N$

B: Covariance matrices: values of $\boldsymbol{S}_V{+}\boldsymbol{S}_W, \boldsymbol{Z}_V, \boldsymbol{Z}_W, \boldsymbol{Q}$ $(\boldsymbol{Z}_{VW} = \boldsymbol{0})$

	$\boldsymbol{S}_V + \boldsymbol{S}_W$	$\boldsymbol{Z}_V$	$\boldsymbol{Z}_W$	$\boldsymbol{Q}$
$\widehat{\boldsymbol{\beta}}_{GLS}$	$\lambda_B^2\boldsymbol{B}_{XX}{+}\lambda_C^2\boldsymbol{C}_{XX}{+}\boldsymbol{R}_{XX}$	$\lambda_B^2 T\boldsymbol{B}_{XX}$	$\lambda_C^2 N\boldsymbol{C}_{XX}$	$\lambda_B\boldsymbol{B}_{XX}{+}\lambda_C\boldsymbol{C}_{XX}{+}\boldsymbol{R}_{XX}$
$\widehat{\boldsymbol{\beta}}_{GLS(\alpha)}$	$\lambda_B^2\boldsymbol{B}_{XX}{+}\boldsymbol{C}_{XX}{+}\boldsymbol{R}_{XX}$	$\lambda_B^2 T\boldsymbol{B}_{XX}$	$N\boldsymbol{C}_{XX}$	$\lambda_B\boldsymbol{B}_{XX}{+}\boldsymbol{C}_{XX}{+}\boldsymbol{R}_{XX}$
$\widehat{\boldsymbol{\beta}}_{GLS(\gamma)}$	$\boldsymbol{B}_{XX}{+}\lambda_C^2\boldsymbol{C}_{XX}{+}\boldsymbol{R}_{XX}$	$T\boldsymbol{B}_{XX}$	$\lambda_C^2 N\boldsymbol{C}_{XX}$	$\boldsymbol{B}_{XX}{+}\lambda_C\boldsymbol{C}_{XX}{+}\boldsymbol{R}_{XX}$

6 Illustration: Factor productivity

In this, final section, we illustrate some of the above results for a model with a single regressor ($K = 1$), relating to factor productivity. The data are from successive annual Norwegian manufacturing censuses, collected by Statistics Norway, for the sector *Manufacture of textiles* (ISIC 32), with $N = 215$ firms observed in the years 1983–1990, *i.e.*, $T = 8$. The y_{it}s and x_{it}s are, respectively, the log of the material input and the log of gross production, both measured as values at constant prices, so that the (scalar) coefficient β can be interpreted as the *input elasticity of materials with respect to output.* The OLS estimate of β obtained from the $NT = 1720$ observations is $\widehat{\beta}_G = 1.1450$. From the residuals, $\widehat{\epsilon}_{it}$ and their between-individual, between-period, and residual sum of squares,

$$B_{\widehat{\epsilon}\widehat{\epsilon}} = T\sum_{i=1}^N(\bar{\widehat{\epsilon}}_{i\cdot} - \bar{\widehat{\epsilon}})^2,\ \ C_{\widehat{\epsilon}\widehat{\epsilon}} = N\sum_{t=1}^T(\bar{\widehat{\epsilon}}_{\cdot t} - \bar{\widehat{\epsilon}})^2,\ \ R_{\widehat{\epsilon}\widehat{\epsilon}} = \sum_{i=1}^N\sum_{t=1}^T(\widehat{\epsilon}_{it} - \bar{\widehat{\epsilon}}_{i\cdot} - \bar{\widehat{\epsilon}}_{\cdot t} + \bar{\widehat{\epsilon}})^2,$$

we obtain ANOVA type estimates:

$$\widehat{\sigma}^2_\alpha + \frac{\widehat{\sigma}^2}{T} = \frac{B_{\widehat{\epsilon}\widehat{\epsilon}}}{T(N-1)}, \quad \widehat{\sigma}^2_\gamma + \frac{\widehat{\sigma}^2}{N} = \frac{C_{\widehat{\epsilon}\widehat{\epsilon}}}{N(T-1)}, \quad \widehat{\sigma}^2 = \frac{R_{\widehat{\epsilon}\widehat{\epsilon}}}{(N-1)(T-1)},$$

confer Searle, Casella, and McCulloch (1992, section 4.7.iii), which give

$$\begin{aligned} \widehat{\sigma}^2_\alpha &= \frac{1}{T(N-1)}\left[B_{\widehat{\epsilon}\widehat{\epsilon}} - \frac{R_{\widehat{\epsilon}\widehat{\epsilon}}}{T-1}\right] = 0.14394, \\ \widehat{\sigma}^2_\gamma &= \frac{1}{N(T-1)}\left[C_{\widehat{\epsilon}\widehat{\epsilon}} - \frac{R_{\widehat{\epsilon}\widehat{\epsilon}}}{N-1}\right] = 0.00066, \\ \widehat{\sigma}^2 &= 0.03449, \\ \widehat{\sigma}^2_\epsilon &= \widehat{\sigma}^2_\alpha + \widehat{\sigma}^2_\gamma + \widehat{\sigma}^2 = 0.17909. \end{aligned}$$

The corresponding shares representing individual heterogeneity, period heterogeneity, and residual variation are $\widehat{\sigma}^2_\alpha/\widehat{\sigma}^2_\epsilon = 0.80372$, $\widehat{\sigma}^2_\gamma/\widehat{\sigma}^2_\epsilon = 0.00370$, and $\widehat{\sigma}^2/\widehat{\sigma}^2_\epsilon = 0.19259$, while $B_{YY}/G_{YY} = 0.93992$, $C_{YY}/G_{YY} = 0.00829$, $R_{YY}/G_{YY} = 0.05179$ for log-material-input and $B_{XX}/G_{XX} = 0.83525$, $C_{XX}/G_{XX} = 0.04216$, and $R_{XX}/G_{XX} = 0.12259$ for log-output. Not surprisingly, the between firm variation by far dominates.

We have selected $N=10$ firms randomly from the 215 in the full sample and included the $T=8$ observations from each of them. All results refer to this subsample of $NT=80$ observations, except that the variance components are estimated from the full sample.

The *firm-specific estimates* of the input elasticity of materials $\widehat{\beta}_{Wij}$ are given in Table 3 (upper panel), the OLS estimates on the main diagonal, varying from -0.09 (firm 2) to 1.54 (firm 7), and the IV estimates in the off-diagonal positions, standard errors, obtained from (25), are given in the lower panel. Even the OLS estimates have low precision. The corresponding within-firm coefficients of correlation of log-output, R_{WXij}, given in Table A3, panel A, show considerable variation, are often low, indicating that log-output for other firms are weak IVs for 'own' log-output.

The weights of the firm-specific OLS estimates (Table 3) in the overall within-firm estimate, $\widehat{\beta}_W$, which is 0.9284 (standard error 0.0773), are reported in Table A1, panel A. The estimate for firm 1 by far dominates (weight 38 per cent). The weights of the firm-specific IV/OLS estimates (Table 3) in the overall between-year estimate $\widehat{\beta}_C$, which is 0.7269 (standard error 0.1628), are reported in Table A1, panel B. The estimate for $(i,j) = (1,1)$ by far dominates (weight 15 per cent). Some off-diagonal weights are negative, reflecting negative correlation between the log-output of the relevant firms (Table A3, Panel A).

The *year-specific estimates* $\widehat{\beta}_{Vts}$ for the $T = 8$ years are given in Table 4 (upper panel), with the OLS estimates on the main diagonal, varying between 1.21 (cross section from year 1989) and 1.64 (cross section from year 1985), and the IV estimates in the off-diagonal positions. All of the $T^2 = 64$ estimates exceed one, with standard errors, from (26), given in the lower panel. Overall, the precision is much higher than for the firm-specific estimates. The corresponding across-year correlation of log-output, R_{VXts}, given in Table A3, panel B, show far less variation than the corresponding across-firm correlation. This indicates that log-output for other years are strong instruments for the year's 'own' log-output, cf. (26) and (28).

The weights of the year-specific OLS estimates (Table 4) in the within-year estimate, $\widehat{\beta}_V$, which is 1.4528 (standard error 0.1717), are reported in Table A2, panel A. The weights vary from 20 per cent (for 1984) and 8 per cent (for 1990). The weights of all the period-specific IV/OLS estimates (Table 4) in the overall between-firm estimate $\widehat{\beta}_B$, which is 1.5195 (standard error 0.1965), are reported in Table A2, panel B. Again, the weights vary less than those for the firm-specific estimates and all weights are positive.

The residual estimate, the OLS estimate, and the GLS estimate (with standard error in parenthesis) are, respectively, $\widehat{\beta}_R = 0.9978$ (0.0875), $\widehat{\beta}_G = 1.4222$ (0.1646), and $\widehat{\beta}_{GLS} = 1.0147$ (0.0717). The latter two are known to be weighted averages of $\widehat{\beta}_B$, $\widehat{\beta}_C$, and $\widehat{\beta}_R$, which agrees with the numerical values $\widehat{\beta}_B = 1.5195$, $\widehat{\beta}_C = 0.7269$, and $\widehat{\beta}_R = 0.9978$.

Since all the aggregate estimators considered have either all $\phi_{tt} \neq 0$ or all $\psi_{ii} \neq 0$, they are inconsistent in cases of endogeneity of or measurement errors in the regressor, confer the end of Section 5. *Modifying the between-firm estimator* $\widehat{\beta}_B$ by replacing $\phi_{ts}=1/T$ for all (t,s) by 0 for $s=t$ and $1/T$ for $s\neq t$ (confer Table 1), we get $\widehat{\beta}_{B*} = 1.5307$. This is N-consistent and is slightly larger than the (less robust) between-firm estimate $\widehat{\beta}_B = 1.5195$. Symmetrically, *modifying the between-year estimator* $\widehat{\beta}_C$ by replacing $\psi_{ij} = 1/N$ for all (i,j) by 0 for $j=i$ and $1/N$ for $j\neq i$ (confer Table 1), we get $\widehat{\beta}_{C*} = 0.5976$, which is T-consistent and is substantially smaller than the (less robust) between-year estimate $\widehat{\beta}_C = 0.7279$. On the other hand, if all assumptions of Model (1) hold, $\widehat{\beta}_{B*}$ is somewhat less efficient than $\widehat{\beta}_B$ (standard error 0.2007 against 0.1965), and $\widehat{\beta}_{C*}$ is markedly less efficient than $\widehat{\beta}_C$ (standard errors 0.2442 against 0.1628), *i.e.*, the efficiency loss when eliminating the disaggregate OLS estimates from the aggregate estimator to improve robustness may be substantial.

Table 3: Firm-specific Estimates of Materials–Output Elasticity: $\widehat{\beta}_{Wij}$

Within deviation of firm i used as IV for within deviation of firm j

$i\downarrow\ j\rightarrow$	1	2	3	4	5	6	7	8	9	10
1	0.92	-0.03	1.29	3.41	0.99	0.92	1.74	1.23	0.12	-0.85
2	0.70	-0.09	1.92	1.80	1.15	4.94	3.20	1.42	0.69	4.67
3	0.95	-0.09	0.55	3.17	1.01	1.02	1.46	1.16	0.54	0.26
4	1.02	-0.43	14.42	1.22	0.78	-0.06	0.77	2.53	0.91	-2.77
5	0.94	-0.04	0.08	-3.46	0.99	0.94	1.62	1.16	0.36	-0.11
6	1.08	0.55	-0.64	0.67	1.05	0.90	1.21	1.13	0.92	0.74
7	1.11	-0.81	0.68	2.06	1.02	0.88	1.54	1.02	2.01	0.85
8	0.97	-0.02	0.32	-11.62	1.04	0.90	1.63	1.16	0.61	0.27
9	0.93	-0.05	2.91	1.39	1.14	1.14	0.91	1.30	0.53	-1.67
10	1.24	0.25	-2.19	0.38	1.07	0.79	1.58	0.91	-2.78	0.78

Standard errors

$i\downarrow\ j\rightarrow$	1	2	3	4	5	6	7	8	9	10
1	0.28	1.73	1.05	6.87	0.49	1.27	1.19	0.52	2.21	2.07
2	0.64	0.75	2.19	3.01	1.28	12.69	8.85	1.09	2.30	8.49
3	0.31	1.73	0.95	6.95	0.47	0.86	1.02	0.46	2.02	1.22
4	1.90	2.26	6.62	1.00	6.44	2.79	2.33	20.73	2.84	3.83
5	0.30	2.16	1.00	14.49	0.45	0.93	1.03	0.48	3.22	1.08
6	0.54	14.53	1.25	4.26	0.63	0.66	1.43	0.53	2.52	1.05
7	0.46	9.26	1.36	3.25	0.64	1.30	0.72	0.64	6.47	0.85
8	0.32	1.84	0.99	46.51	0.47	0.79	1.03	0.45	2.13	1.07
9	0.49	1.37	1.52	2.25	1.13	1.31	3.67	0.75	1.27	7.60
10	0.84	9.36	1.70	5.63	0.71	1.01	0.89	0.70	14.13	0.68

Table 4: Year-specific Estimates of Materials–Output Elasticity: $\widehat{\beta}_{Vts}$

Within deviation of year t used as IV for within deviation of year s

$t\downarrow$ $s\rightarrow$	1983	1984	1985	1986	1987	1988	1989	1990
1983	1.267	1.433	1.572	1.383	1.514	1.567	1.407	1.613
1984	1.232	1.375	1.483	1.290	1.390	1.483	1.302	1.526
1985	1.374	1.508	1.642	1.465	1.589	1.576	1.468	1.663
1986	1.414	1.529	1.660	1.483	1.586	1.604	1.499	1.669
1987	1.441	1.595	1.751	1.588	1.606	1.652	1.435	1.618
1988	1.519	1.668	1.803	1.671	1.712	1.625	1.394	1.623
1989	1.454	1.589	1.676	1.584	1.570	1.477	1.212	1.487
1990	1.502	1.665	1.809	1.683	1.626	1.614	1.330	1.551

Standard errors

$t\downarrow$ $s\rightarrow$	1983	1984	1985	1986	1987	1988	1989	1990
1983	0.080	0.073	0.099	0.105	0.113	0.118	0.142	0.158
1984	0.083	0.071	0.097	0.099	0.109	0.116	0.133	0.152
1985	0.086	0.074	0.093	0.093	0.103	0.105	0.123	0.140
1986	0.092	0.077	0.095	0.091	0.101	0.105	0.121	0.136
1987	0.094	0.080	0.100	0.096	0.096	0.097	0.106	0.118
1988	0.102	0.088	0.105	0.103	0.100	0.093	0.101	0.116
1989	0.117	0.097	0.117	0.113	0.105	0.097	0.097	0.115
1990	0.113	0.096	0.116	0.110	0.101	0.096	0.100	0.112

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Appendices and Appendix Tables

A: The covariance matrices of the base estimators: In order to derive the variance-covariance matrices of $\widehat{\boldsymbol{\beta}}_{Wij}$ and $\widehat{\boldsymbol{\beta}}_{Vts}$ in Model (1) is valid, we first need expressions for the variance-covariance matrices of $\boldsymbol{W}_{XUij}$, $\boldsymbol{V}_{XUts}$, $\boldsymbol{W}_{X\gamma i}$, and $\boldsymbol{V}_{X\alpha t}$. Since

$$\begin{aligned}
&\mathsf{E}(\boldsymbol{\alpha}\boldsymbol{\alpha}'|\boldsymbol{X}) = \sigma_\alpha^2 \boldsymbol{I}_N, \qquad \mathsf{E}(\boldsymbol{\gamma}\boldsymbol{\gamma}'|\boldsymbol{X}) = \sigma_\gamma^2 \boldsymbol{I}_T,\\
&\mathsf{E}(\boldsymbol{u}_{j\cdot}\boldsymbol{u}'_{l\cdot}|\boldsymbol{X}) = \delta_{jl}\sigma^2 \boldsymbol{I}_T, \qquad \mathsf{E}(\boldsymbol{u}_{\cdot s}\boldsymbol{u}'_{\cdot q}|\boldsymbol{X}) = \delta_{sq}\sigma^2 \boldsymbol{I}_N,\\
&\mathsf{E}(\boldsymbol{u}_{j\cdot}\boldsymbol{u}'_{\cdot q}|\boldsymbol{X}) = \sigma^2 \boldsymbol{i}_{Tq}\boldsymbol{i}'_{Nj}, \qquad j,l = 1,\dots,N,\ s,q = 1,\dots,T,
\end{aligned}$$

where $\boldsymbol{i}_{Hj}$ denotes the j'th column of $\boldsymbol{I}_H$, we get, after some algebra,

$$\begin{aligned}
&\mathsf{E}(\boldsymbol{W}_{XUij}\, \boldsymbol{W}'_{XUkl}|\boldsymbol{X}) = \delta_{jl}\, \sigma^2\, \boldsymbol{W}_{XXik},\\
&\mathsf{E}(\boldsymbol{W}_{X\gamma i}\, \boldsymbol{W}'_{X\gamma k}|\boldsymbol{X}) = \sigma_\gamma^2\, \boldsymbol{W}_{XXik},\\
&\mathsf{E}(\boldsymbol{W}_{X\epsilon ij}\, \boldsymbol{W}'_{X\epsilon kl}|\boldsymbol{X}) = (\sigma_\gamma^2 + \delta_{jl}\sigma^2)\boldsymbol{W}_{XXik},
\end{aligned} \tag{a.1}$$

$$\begin{aligned}
&\mathsf{E}(\boldsymbol{V}_{XUts}\, \boldsymbol{V}'_{XUpq}|\boldsymbol{X}) = \delta_{sq}\, \sigma^2\, \boldsymbol{V}_{XXtp},\\
&\mathsf{E}(\boldsymbol{V}_{X\alpha t}\, \boldsymbol{V}'_{X\alpha p}|\boldsymbol{X}) = \sigma_\alpha^2\, \boldsymbol{V}_{XXtp},\\
&\mathsf{E}(\boldsymbol{V}_{X\epsilon ts}\, \boldsymbol{V}'_{X\epsilon pq}|\boldsymbol{X}) = (\sigma_\alpha^2 + \delta_{sq}\sigma^2)\boldsymbol{V}_{XXtp},
\end{aligned} \tag{a.2}$$

$$\left.\begin{aligned}
&\mathsf{E}(\boldsymbol{W}_{XUij}\, \boldsymbol{V}'_{XUpq}|\boldsymbol{X})\\
&\mathsf{E}(\boldsymbol{W}_{X\epsilon ij}\, \boldsymbol{V}'_{X\epsilon pq}|\boldsymbol{X})
\end{aligned}\right\} = \sigma^2(\boldsymbol{x}_{iq} - \bar{\boldsymbol{x}}_{i\cdot})'(\boldsymbol{x}_{jp} - \bar{\boldsymbol{x}}_{\cdot p}), \qquad \begin{aligned}&i,j,k,l = 1,\dots,N,\\ &t,s,p,q = 1,\dots,T.\end{aligned} \tag{a.3}$$

Combining (a.1)–(a.3) with (19)–(20), it follows that the matrices of covariances between the individual-specific and between the period-specific base estimators, respectively, can be expressed as

$$\mathsf{C}(\widehat{\boldsymbol{\beta}}_{Wij}, \widehat{\boldsymbol{\beta}}_{Wkl}|\boldsymbol{X}) = (\sigma_\gamma^2 + \delta_{jl}\sigma^2)\boldsymbol{W}_{XXij}^{-1}\boldsymbol{W}_{XXik}\boldsymbol{W}_{XXlk}^{-1}, \qquad i,j,k,l = 1,\dots,N, \tag{a.4}$$

$$\mathsf{C}(\widehat{\boldsymbol{\beta}}_{Vts}, \widehat{\boldsymbol{\beta}}_{Vpq}|\boldsymbol{X}) = (\sigma_\alpha^2 + \delta_{sq}\sigma^2)\boldsymbol{V}_{XXts}^{-1}\boldsymbol{V}_{XXtp}\boldsymbol{V}_{XXqp}^{-1}, \qquad t,s,p,q = 1,\dots,T. \tag{a.5}$$

B: The covariance matrix of $\boldsymbol{b}$: Inserting for $\boldsymbol{W}_{XYij}$ and $\boldsymbol{V}_{XYts}$ from (13) and (15) in (31), using (33), we find

$$\begin{aligned}
\boldsymbol{b} - \boldsymbol{\beta} &= \boldsymbol{Q}^{-1}\left[\textstyle\sum_{t=1}^T \sum_{s=1}^T \phi_{ts}\boldsymbol{V}_{X\epsilon ts} + \sum_{i=1}^N \sum_{j=1}^N \psi_{ij}\boldsymbol{W}_{X\epsilon ij}\right]\\
&= \boldsymbol{Q}^{-1}\Big[\textstyle\sum_{t=1}^T \sum_{s=1}^T \phi_{ts}\boldsymbol{V}_{XUts} + \sum_{t=1}^T \left(\sum_{s=1}^T \phi_{ts}\right)\boldsymbol{V}_{X\alpha t}\\
&\qquad + \textstyle\sum_{i=1}^N \sum_{j=1}^N \psi_{ij}\boldsymbol{W}_{XUij} + \sum_{i=1}^N \left(\sum_{j=1}^N \psi_{ij}\right)\boldsymbol{W}_{X\gamma i}\Big].
\end{aligned}$$

Combining this equation with (19), (20), and (a.1)–(a.3), we find that $\boldsymbol{b}$ is an unbiased estimator of $\boldsymbol{\beta}$ for any $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ and has variance-covariance matrix

$$\mathsf{V}(\boldsymbol{b}|\boldsymbol{X}) = \boldsymbol{Q}^{-1}\boldsymbol{P}\,(\boldsymbol{Q}^{-1})' = \boldsymbol{Q}(\boldsymbol{\phi},\boldsymbol{\psi})^{-1}\boldsymbol{P}(\boldsymbol{\phi},\boldsymbol{\psi},\sigma^2,\sigma_\alpha^2,\sigma_\gamma^2)(\boldsymbol{Q}(\boldsymbol{\phi},\boldsymbol{\psi})^{-1})', \tag{b.1}$$

where

$$\boldsymbol{P} = \boldsymbol{P}(\boldsymbol{\phi},\boldsymbol{\psi},\sigma^2,\sigma_\alpha^2,\sigma_\gamma^2) = \sigma^2(\boldsymbol{S}_V + \boldsymbol{S}_W + \boldsymbol{S}_{VW}) + \sigma_\alpha^2\, \boldsymbol{Z}_V + \sigma_\gamma^2\, \boldsymbol{Z}_W, \tag{b.2}$$

$$\begin{aligned}
\boldsymbol{S}_V &= \boldsymbol{S}_V(\boldsymbol{\phi}) = \textstyle\sum_{t=1}^T \sum_{p=1}^T \boldsymbol{V}_{XXtp}\left(\sum_{s=1}^T \phi_{ts}\phi_{ps}\right),\\
\boldsymbol{S}_W &= \boldsymbol{S}_W(\boldsymbol{\psi}) = \textstyle\sum_{i=1}^N \sum_{k=1}^N \boldsymbol{W}_{XXik}\left(\sum_{j=1}^N \psi_{ij}\psi_{kj}\right),\\
\boldsymbol{S}_{VW} &= \boldsymbol{S}_{VW}(\boldsymbol{\phi},\boldsymbol{\psi}) = \textstyle\sum_{t=1}^T \sum_{s=1}^T \sum_{i=1}^N \sum_{j=1}^N \phi_{ts}\psi_{ij}(\boldsymbol{x}_{is} - \bar{\boldsymbol{x}}_{i\cdot})'(\boldsymbol{x}_{jt} - \bar{\boldsymbol{x}}_{\cdot t}),\\
\boldsymbol{Z}_V &= \boldsymbol{Z}_V(\boldsymbol{\phi}) = \textstyle\sum_{t=1}^T \sum_{p=1}^T \boldsymbol{V}_{XXtp}\left(\sum_{s=1}^T \phi_{ts}\right)\left(\sum_{r=1}^T \phi_{pr}\right),\\
\boldsymbol{Z}_W &= \boldsymbol{Z}_W(\boldsymbol{\psi}) = \textstyle\sum_{i=1}^N \sum_{k=1}^N \boldsymbol{W}_{XXik}\left(\sum_{j=1}^N \psi_{ij}\right)\left(\sum_{l=1}^N \psi_{kl}\right).
\end{aligned} \tag{b.3}$$

C: Proof of (39)–(40): Since $\bar{\boldsymbol{x}}_{i\cdot} - \bar{\boldsymbol{x}} = \sum_{t=1}^T(\boldsymbol{x}_{it} - \bar{\boldsymbol{x}}_{\cdot t})/T$, $\bar{\boldsymbol{x}}_{\cdot t} - \bar{\boldsymbol{x}} = \sum_{i=1}^N(\boldsymbol{x}_{it} - \bar{\boldsymbol{x}}_{i\cdot})/N$, etc., and

$$\begin{aligned}
&\textstyle\sum_{i=1}^N(\boldsymbol{X}_{i\cdot} - \bar{\boldsymbol{X}})'\boldsymbol{A}_T(\boldsymbol{X}_{i\cdot} - \bar{\boldsymbol{X}}) = \frac{1}{T}\sum_{t=1}^T \sum_{s=1}^T \boldsymbol{X}'_{\cdot t}\, \boldsymbol{B}_N\, \boldsymbol{X}_{\cdot s},\\
&\textstyle\sum_{t=1}^T(\boldsymbol{X}_{\cdot t} - \bar{\boldsymbol{X}})'\boldsymbol{A}_N(\boldsymbol{X}_{\cdot t} - \bar{\boldsymbol{X}}) = \frac{1}{N}\sum_{i=1}^N \sum_{j=1}^N \boldsymbol{X}'_{i\cdot}\, \boldsymbol{B}_T\, \boldsymbol{X}_{j\cdot}
\end{aligned}$$

hold identically, (11) and (12) can be rewritten as

$$\begin{aligned}
\boldsymbol{B}_{XXii} &= \tfrac{1}{T}\textstyle\sum_{t=1}^T \sum_{s=1}^T (\boldsymbol{x}_{it} - \bar{\boldsymbol{x}}_{\cdot t})'(\boldsymbol{x}_{is} - \bar{\boldsymbol{x}}_{\cdot s}),\\
\boldsymbol{B}_{X\alpha ii} &= \textstyle\sum_{t=1}^T (\boldsymbol{x}_{it} - \bar{\boldsymbol{x}}_{\cdot t})'(\alpha_i - \bar{\alpha}),
\end{aligned} \qquad i = 1,\dots,N, \tag{c.1}$$

$$\begin{aligned}
\boldsymbol{C}_{XXtt} &= \tfrac{1}{N}\textstyle\sum_{i=1}^N \sum_{j=1}^N (\boldsymbol{x}_{it} - \bar{\boldsymbol{x}}_{i\cdot})'(\boldsymbol{x}_{jt} - \bar{\boldsymbol{x}}_{j\cdot}),\\
\boldsymbol{C}_{X\gamma tt} &= \textstyle\sum_{i=1}^N (\boldsymbol{x}_{it} - \bar{\boldsymbol{x}}_{i\cdot})'(\gamma_t - \bar{\gamma}),
\end{aligned} \qquad t = 1,\dots,T, \tag{c.2}$$

and the following identities hold

$$\sum_{i=1}^{N} \boldsymbol{B}_{XXii} = \frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T} \boldsymbol{V}_{XXts}, \qquad \sum_{t=1}^{T} \boldsymbol{C}_{XXtt} = \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} \boldsymbol{W}_{XXij}. \tag{c.3}$$

Similarly,

$$\sum_{i=1}^{N} \boldsymbol{B}_{X\alpha ii} = \sum_{t=1}^{T} \boldsymbol{V}_{X\alpha t}, \qquad \sum_{t=1}^{T} \boldsymbol{C}_{X\gamma tt} = \sum_{i=1}^{N} \boldsymbol{W}_{X\gamma i}.$$

The overall between individual and overall between period (co)variation can then be written as

$$\boldsymbol{B}_{XX} = \sum_{i=1}^{N} \boldsymbol{B}_{XXii} = T\sum_{i=1}^{N}(\bar{\boldsymbol{x}}_{i\cdot} - \bar{\boldsymbol{x}})'(\bar{\boldsymbol{x}}_{i\cdot} - \bar{\boldsymbol{x}}) = (1/T)\sum_{t=1}^{T}\sum_{s=1}^{T} \boldsymbol{V}_{XXts}, \tag{c.4}$$

$$\boldsymbol{C}_{XX} = \sum_{t=1}^{T} \boldsymbol{C}_{XXtt} = N\sum_{t=1}^{T}(\bar{\boldsymbol{x}}_{\cdot t} - \bar{\boldsymbol{x}})'(\bar{\boldsymbol{x}}_{\cdot t} - \bar{\boldsymbol{x}}) = (1/N)\sum_{i=1}^{N}\sum_{j=1}^{N} \boldsymbol{W}_{XXij}. \tag{c.5}$$

D: The covariance matrix of $\widehat{\boldsymbol{\beta}}_{GLS}$: Recalling (45), (46), (48), and (53), the GLS weights in the variance-covariance matrix can be obtained from Table 2, panel A, by adding λ_B times the weights in row 1, λ_C times the weights in row 2, and the weights in row 3 (or 4). Expressions for the variance-covariance matrix of $\widehat{\boldsymbol{\beta}}_{GLS}$ can be derived by inserting the weights in Table 2, panel A, rows 1 or 2, into (34)–(36). The result is given in Table 2, panel B, row 1. In deriving $\mathsf{V}(\widehat{\boldsymbol{\beta}}_{GLS}|\boldsymbol{X})$, we use

$$\sum_{s=1}^{T} \phi_{ts} = \lambda_B, \quad \sum_{s=1}^{T} \phi_{ts}\phi_{ps} = \delta_{tp} - \frac{1-\lambda_B^2}{T} \qquad t,p = 1,\dots,T,$$

$$\sum_{j=1}^{N} \psi_{ij} = \lambda_C, \quad \sum_{j=1}^{N} \psi_{ij}\psi_{kj} = \delta_{ik} - \frac{1-\lambda_C^2}{N} \qquad i,k = 1,\dots,N,$$

so that, using (36), we have

$$\boldsymbol{Z}_V = \lambda_B^2 \sum_{t=1}^{T}\sum_{p=1}^{T} \boldsymbol{V}_{XXtp} = \lambda_B^2 T \boldsymbol{B}_{XX},$$

$$\boldsymbol{Z}_W = \lambda_C^2 \sum_{i=1}^{N}\sum_{k=1}^{N} \boldsymbol{W}_{XXik} = \lambda_C^2 N \boldsymbol{C}_{XX},$$

which are the expressions given in Table 2, panel B, columns 2 and 3. Obviously, $\boldsymbol{S}_{VW} = \boldsymbol{0}$. From (36), in combination with the weights in Table 2, rows 1 and 2, we get

$$\boldsymbol{S}_V + \boldsymbol{S}_W = \boldsymbol{V}_{XX} - (1-\lambda_B^2)\boldsymbol{B}_{XX} + \lambda_C^2\boldsymbol{C}_{XX} = \lambda_B^2\boldsymbol{B}_{XX} + \boldsymbol{W}_{XX} - (1-\lambda_C^2)\boldsymbol{C}_{XX},$$

$$\boldsymbol{Q} = \boldsymbol{V}_{XX} - (1-\lambda_B)\boldsymbol{B}_{XX} + \lambda_C\boldsymbol{C}_{XX} = \lambda_B\boldsymbol{B}_{XX} + \boldsymbol{W}_{XX} - (1-\lambda_C)\boldsymbol{C}_{XX},$$

which, since $\boldsymbol{V}_{XX} - \boldsymbol{B}_{XX} = \boldsymbol{W}_{XX} - \boldsymbol{C}_{XX} = \boldsymbol{R}_{XX}$, can be simplified to

$$\boldsymbol{S}_V + \boldsymbol{S}_W = \boldsymbol{R}_{XX} + \lambda_B^2\boldsymbol{B}_{XX} + \lambda_C^2\boldsymbol{C}_{XX},$$

$$\boldsymbol{Q} = \boldsymbol{R}_{XX} + \lambda_B\boldsymbol{B}_{XX} + \lambda_C\boldsymbol{C}_{XX}.$$

These are the expressions given in Table 2, panel B, columns 1 and 4. Finally, since

$$\sigma^2(\boldsymbol{S}_V + \boldsymbol{S}_W) + \sigma_\alpha^2\boldsymbol{Z}_V + \sigma_\gamma^2\boldsymbol{Z}_W = \sigma^2[\boldsymbol{R}_{XX} + \lambda_B\boldsymbol{B}_{XX} + \lambda_C\boldsymbol{C}_{XX}],$$

the covariance matrix of $\widehat{\boldsymbol{\beta}}_{GLS}$ can be written as

$$\mathsf{V}(\widehat{\boldsymbol{\beta}}_{GLS}|\boldsymbol{X}) = \sigma^2[\boldsymbol{R}_{XX} + \lambda_B\boldsymbol{B}_{XX} + \lambda_C\boldsymbol{C}_{XX}]^{-1} = \left[\frac{\boldsymbol{R}_{XX}}{\sigma^2} + \frac{\boldsymbol{B}_{XX}}{\sigma^2 + T\sigma_\alpha^2} + \frac{\boldsymbol{C}_{XX}}{\sigma^2 + N\sigma_\gamma^2}\right]^{-1}. \tag{d.1}$$

The covariance matrices of the *one-way* GLS estimators $\widehat{\boldsymbol{\beta}}_{GLS(\alpha)}$ and $\widehat{\boldsymbol{\beta}}_{GLS(\gamma)}$ when the *two-way* effects model is valid, obtained from Table 2, panel B, rows 2 and 3, are

$$\mathsf{V}(\widehat{\boldsymbol{\beta}}_{GLS(\alpha)}|\boldsymbol{X}) = [\boldsymbol{R}_{XX} + \lambda_B\boldsymbol{B}_{XX} + \boldsymbol{C}_{XX}]^{-1}[\sigma^2\boldsymbol{R}_{XX} + \lambda_B^2(\sigma^2 + T\sigma_\alpha^2)\boldsymbol{B}_{XX} + (\sigma^2 + N\sigma_\gamma^2)\boldsymbol{C}_{XX}] \times [\boldsymbol{R}_{XX} + \lambda_B\boldsymbol{B}_{XX} + \boldsymbol{C}_{XX}]^{-1}, \tag{d.2}$$

$$\mathsf{V}(\widehat{\boldsymbol{\beta}}_{GLS(\gamma)}|\boldsymbol{X}) = [\boldsymbol{R}_{XX} + \boldsymbol{B}_{XX} + \lambda_C\boldsymbol{C}_{XX}]^{-1}[\sigma^2\boldsymbol{R}_{XX} + (\sigma^2 + T\sigma_\alpha^2)\boldsymbol{B}_{XX} + \lambda_C^2(\sigma^2 + N\sigma_\gamma^2)\boldsymbol{C}_{XX}] \times [\boldsymbol{R}_{XX} + \boldsymbol{B}_{XX} + \lambda_C\boldsymbol{C}_{XX}]^{-1}, \tag{d.3}$$

which for the one-way random effects models ($\sigma_\gamma^2 = 0$ and $\sigma_\alpha^2 = 0$, respectively) are simplified to

$$\mathsf{V}(\widehat{\boldsymbol{\beta}}_{GLS(\alpha)}|\boldsymbol{X}) = \left[\frac{\boldsymbol{R}_{XX} + \boldsymbol{C}_{XX}}{\sigma^2} + \frac{\boldsymbol{B}_{XX}}{\sigma^2 + T\sigma_\alpha^2}\right]^{-1}, \tag{d.4}$$

$$\mathsf{V}(\widehat{\boldsymbol{\beta}}_{GLS(\gamma)}|\boldsymbol{X}) = \left[\frac{\boldsymbol{R}_{XX} + \boldsymbol{B}_{XX}}{\sigma^2} + \frac{\boldsymbol{C}_{XX}}{\sigma^2 + N\sigma_\gamma^2}\right]^{-1}. \tag{d.5}$$

Table A1: Weights of $\widehat{\beta}_{Wij}$ in aggregate estimates. $N = 10,\ T = 8$.

A. Weights of $\widehat{\beta}_{Wii}$ in $\widehat{\beta}_W$, per cent. Average = 10 per cent

$i \rightarrow$	1	2	3	4	5	6	7	8	9	10
	38.25	5.22	3.24	2.93	14.86	6.84	5.71	14.76	1.84	6.35

B. Weights of $\widehat{\beta}_{Wij}$ in $\widehat{\beta}_C$, per cent. Average = 1 per cent

$i \downarrow\ j \rightarrow$	1	2	3	4	5	6	7	8	9	10
1	14.95	-2.40	3.95	-0.60	8.50	3.28	-3.47	8.00	-1.87	2.00
2	-2.40	2.04	-0.70	0.51	-1.20	-0.12	0.17	-1.40	0.67	0.18
3	3.95	-0.70	1.27	-0.17	2.58	1.41	-1.18	2.61	-0.60	0.99
4	-0.60	0.51	-0.17	1.15	0.18	0.41	0.49	0.06	0.40	0.30
5	8.50	-1.20	2.58	0.18	5.81	2.78	-2.51	5.43	-0.80	2.38
6	3.28	-0.12	1.41	0.41	2.78	2.67	-1.23	3.28	-0.69	1.67
7	-3.47	0.17	-1.18	0.49	-2.51	-1.23	2.23	-2.50	0.25	-1.89
8	8.00	-1.40	2.61	0.06	5.43	3.28	-2.50	5.77	-1.21	2.41
9	-1.87	0.67	-0.60	0.40	-0.80	-0.69	0.25	-1.21	0.72	0.12
10	2.00	0.18	0.99	0.30	2.38	1.67	-1.89	2.41	0.12	2.48

Table A2: Weights of $\widehat{\beta}_{Vts}$ in aggregate estimates. $N = 10,\ T = 8$.

A. Weights of $\widehat{\beta}_{Vtt}$ in $\widehat{\beta}_V$, per cent. Average = 12.5 per cent

$t \rightarrow$	1983	1984	1985	1986	1987	1988	1989	1990
	15.503	20.005	11.536	12.091	10.844	11.551	10.556	7.915

B. Weights of $\widehat{\beta}_{Vts}$ in $\widehat{\beta}_B$, per cent. Average = 1.56 per cent

$t \downarrow\ s \rightarrow$	1983	1984	1985	1986	1987	1988	1989	1990
1983	2.222	2.437	1.793	1.705	1.579	1.516	1.260	1.129
1984	2.437	2.868	2.084	2.043	1.857	1.749	1.517	1.330
1985	1.793	2.084	1.654	1.662	1.497	1.468	1.255	1.101
1986	1.705	2.043	1.662	1.733	1.555	1.499	1.303	1.154
1987	1.579	1.857	1.497	1.555	1.554	1.543	1.403	1.266
1988	1.516	1.749	1.468	1.499	1.543	1.656	1.524	1.328
1989	1.260	1.517	1.255	1.303	1.403	1.524	1.513	1.276
1990	1.129	1.330	1.101	1.154	1.266	1.328	1.276	1.135

Table A3: Coefficients of Correlation, Log-Output. $N = 10, T = 8$.

A. Within Firm, R_{WXij}

$i \downarrow\ j \rightarrow$	1	2	3	4	5	6	7	8	9	10
1	1.000	-0.435	0.909	-0.146	0.912	0.518	-0.601	0.861	-0.572	0.329
2	-0.435	1.000	-0.435	0.333	-0.347	-0.052	0.081	-0.408	0.550	0.080
3	0.909	-0.435	1.000	-0.144	0.952	0.765	-0.701	0.964	-0.626	0.559
4	-0.146	0.333	-0.144	1.000	0.069	0.235	0.309	0.022	0.446	0.178
5	0.912	-0.347	0.952	0.069	1.000	0.706	-0.696	0.938	-0.393	0.628
6	0.518	-0.052	0.765	0.235	0.706	1.000	-0.503	0.835	-0.501	0.647
7	-0.601	0.081	-0.701	0.309	-0.696	-0.503	1.000	-0.695	0.196	-0.803
8	0.861	-0.408	0.964	0.022	0.938	0.835	-0.695	1.000	-0.594	0.637
9	-0.572	0.550	-0.626	0.446	-0.393	-0.501	0.196	-0.594	1.000	0.090
10	0.329	0.080	0.559	0.178	0.628	0.647	-0.803	0.637	0.090	1.000

B. Within Year, R_{VXts}

$t \downarrow\ s \rightarrow$	1983	1984	1985	1986	1987	1988	1989	1990
1983	1.000	0.965	0.936	0.869	0.850	0.790	0.687	0.711
1984	0.965	1.000	0.957	0.916	0.879	0.803	0.728	0.737
1985	0.936	0.957	1.000	0.982	0.934	0.887	0.794	0.804
1986	0.869	0.916	0.982	1.000	0.947	0.885	0.805	0.823
1987	0.850	0.879	0.934	0.947	1.000	0.962	0.915	0.954
1988	0.790	0.803	0.887	0.885	0.962	1.000	0.963	0.969
1989	0.687	0.728	0.794	0.805	0.915	0.963	1.000	0.974
1990	0.711	0.737	0.804	0.823	0.954	0.969	0.974	1.000