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**Beating the Matthew Effect: Head Starts and Catching Up in a Dynamic
All-Pay Auction***

The seal of the University of Oslo, featuring a woman in classical attire playing a harp, surrounded by the Latin text 'UNIVERSITAS OSLOENSIS' and 'MDCCCXXXII'.

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Beating the Matthew Effect: Head Starts and Catching Up in a Dynamic All-Pay Auction*

Derek J. Clark and Tore Nilssen[†]

February 27, 2018

Abstract

We consider an effort-maximizing principal distributing a prize fund over two consecutive all-pay auctions. The two contestants are doubly heterogeneous: one of them has a head start in the first contest; and winning contest one gives an advantage in contest two that varies between players. We show that, with a large head start, the principal chooses a zero prize in contest two, i.e., runs a single contest. Otherwise, the laggard winning contest one may overturn the leader's head start, possibly inciting expected efforts equal to the prize value, avoiding the laggard giving up, and this way mitigating the Matthew effect.

Keywords: contest; all-pay auction; head start, catching up, Matthew effect.

JEL codes: D74, D72

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1 Introduction

Winning breeds winners, making success self-fulfilling. How then should an incentive scheme be organized to ensure that laggards do not simply give up, leaving the spoils to a superior competitor? In this paper, we discuss this question in a contest setting where a principal has a prize fund available and must choose whether to put all of it in a single contest or to spread it out over several contests. There is a synergy between the contests which reflects the fact that success in a previous contest affects the likelihood of success in the current one. We find that the optimal prize division depends upon contestants' heterogeneity, in particular on the interplay of their ex-ante heterogeneity and the difference in how they respond to an early win to affect that ex-ante heterogeneity. When, in a case of two participants, one contestant's ex-ante head start is small relative to the other contestant's ability to use an early win to catch up with and even overtake the rival, the principal should spread the prize fund over more than one contest in order to maximize participants' total expected efforts. Indeed, the principal can in some cases use the prize division to even out the ex-ante heterogeneity between the contestants, capturing all of their surplus.

Contests are common constructs for analyzing competition between agents who compete for a prize. In the all-pay auction, one of the most widely studied contests, the highest bidder wins and all competitors pay the cost of their bid. Numerous applications of this basic model can be found, from promotion or R&D races, to sporting contests and political competition, or indeed military conflict. Equilibrium in an all-pay auction with complete information, where contestants' valuations of the prize is common knowledge, has been characterized by Hillman and Riley (1989) and Baye, *et al.* (1996). Since then, economists have been interested in analyzing how changes to the basic model affect bids, or efforts, in this model. For example, Che and Gale (1998) consider the effect of placing an upper bound on contestants' efforts, Baye, *et al.* (1993) examine how excluding some competitors may increase efforts, and Barut and Kovenock (1998) allow several prizes to be awarded in a single all-pay auction. Clark and Riis (1995) and Konrad (2002) allow one player to have a head-start advantage, implying that he can expend effort at a lower level than the opponent and still win the prize.

Many real-world applications of contests require extending the basic framework to allow for competition over multiple stages (*e.g.*, Sela, 2012). Here, the modeler can choose from many types of contest architecture. An early analysis by Harris and Vickers (1987) considers a race as a best-of- k contest, where there is a single prize, and the winner of that prize is the first to win k stage contests. By winning a stage contest, a competitor gets closer to the finish line so that leader-laggard behavior can be examined.¹ In a contest spanning several rounds, or a series of contests, where the outcomes of previous rounds form the starting point for the current stage contest, it is natural to consider such leader-laggard configurations. This is indeed our point of departure in the current paper: we consider how

¹See Konrad (2012) for a discussion of the “discouragement effect” that often arises in this class of contests, where a laggard will at some stage simply give up.

previous play in a series of contests affects the players' incentives to exert efforts across the series. Here, previous play determines the identity of the player that has a current head start in the stage contest, as well as the size of that head start. Moreover, we solve a design problem in which a contest sponsor or principal wishes to maximize efforts by dividing up a prize fund of fixed size over the stage contests. There are thus two types of synergies between the contests in the series, one that affects the direction and magnitude of the head start, and one due to the fixed prize fund where more granted in the current contest leaves less to fight for in the future.

The situation that we consider has a close relationship to the Matthew effect, coined by Merton (1968) to describe situations in which the rich get richer or where success breeds success. The idea is that an initial advantage can be self-amplifying, and that such mechanisms are prevalent whether we consider "economic wealth, political power, prestige, knowledge, or in fact any other scarce or valued resource" (Perc, 2014; p. 1). The phenomenon can also be used to describe how certain nodes in a network become preferred (as hubs), or how popular products become more popular with social influence (Altszyler, *et al.*, 2017). Faced with a contestant with initial advantage, it may seem likely that the laggard will give up. How might the principal divide his prize to mitigate this effect? We show that the answer depends on the size of the initial advantage, and how this advantage evolves; there is no guarantee that the initial leader will maintain the advantage in the future. In his extensive review of the Matthew effect, Rigney (2010; p. 1) states that "[i]nitial advantage does not always lead to further advantage, and initial disadvantage does not always lead to further disadvantage". By allowing for the possibility that an initially disadvantaged player can catch up and surpass the initial leader, we show that the principal can "beat the Matthew effect" and urge both players to exert high efforts.

To be specific, we consider a two-player model of a series of two complete-information all-pay auctions. At the beginning of the first contest, we allow one player to have a head start in the sense that he may have less effort than the opponent, and still win the stage prize.² The head start is assumed to evolve from one contest to the next according to the identity of the winner of the first contest. Clark and Nilssen (2017) have looked at such dynamics in a long series of contests with a common stage prize, terming the change in head start as the "win advantage". Hence, the winner of the first contest does not only gain the stage prize, but also affects the head start in the second contest. The source of this win advantage is elaborated on below. Should the player with the initial head start win the first contest, his head start is increased at the start of contest two. If the initial underdog should win, then there are two possibilities for the head start: either it will still be in the favour of the original leader, but reduced, or the leader will be overturned so that the underdog now becomes the favorite in contest two.³ In setting the stage prizes, the principal must be mindful of the fact

²This head start may reflect for example an incumbency advantage in political competition, a prior investment made by a player, or some technological superiority.

³Such cases, where yesterday's winner has a head start today, irrespective of whether he was a leader or a laggard yesterday, resemble incumbency contests, as studied by Ofek and Sarvary

that the relationship between the head start and the prize in a stage contest will determine whether the players will exert effort, or if one player gives up, as well as the magnitude of efforts. The principal must also take account of how the head start will evolve for various outcomes of the first contest. We characterize the equilibrium of this game, and use it to demonstrate when the two-contest series can be used to improve on running a single contest, in the sense of yielding more effort in equilibrium. Indeed, we identify situations in which a judicious setting of the contest prizes can neutralize the initial asymmetry between the players, giving an expected level of effort exactly equal to the gross value of the prize; such a full dissipation result is common when players are symmetric, but less so with asymmetry.⁴

The synergy in the head start across rounds of competition has been discussed in many wide-ranging applications, although not directly related to contests. A market leader seeks mechanisms to keep and augment his advantage, so that the lead becomes self-enforcing; while laggards will attempt to break the dominance, catch up, and even surpass the better rival. Merton (1968), in analyzing the sociology of science, purported the Matthew effect at a micro and a macro level: at the micro level, it is suggested that high-status scientists get more credit than their peers for comparable scientific achievements, and at the macro level this process leads to a cumulative advantage to the distinguished scientists. Gallini and Scotchmer (2002; p. 54) suggest that competing for grants is easier for those with high status than others: “[F]uture grants are contingent upon previous success. The linkage between previous success and future funding seems even more specific in the case of the National Science Council”. Those who succeed in obtaining research grants hence experience an increase in status, and win a grant to fund current work; in sum, this gives an advantage in future rounds of competition for scarce research funding.⁵ In the case of innovation contests, Davis and Davis (2004; p. 20) discuss reputational gains associated with winning a contest, stating: “[T]he impact of prizes on reputation, we feel, is a much overlooked phenomenon. They alone can provide the economic justification for a sponsor to design the contest and for contestants to enter.” Winning a contest, and enhancing one’s reputation, can be so lucrative that contestants are willing to invest to gain an incumbency advantage (Konrad, 2002). Having established a lead in a product market, for example, a firm will want to fight to retain the ascendancy, while rivals will want to become the market leader; this mechanism may be particularly important in markets with network economies since these often have the winner-take-all feature of contests. Frank and Cook (1995) discuss in this connection the competition between the VHS and Betamax video recorder platforms, which both vied to become the market standard. Whilst Betamax was widely regarded as the superior

(2003) and Mehlum and Moene (2006, 2008).

⁴In the context of a single all-pay auction with a head start, Li and Yu (2012) and Franke, *et al.* (2016) show how the principal may set the head start to achieve maximum effort.

⁵In UK academia, Elton (2000) documents the fact that the early rounds of the Research Assessment Exercise created a transfer market for academics as universities attempted to improve their rank by purchasing the publication lists of high-status academics. This may be seen as a "catching up" strategy similar to that considered by us here.

technology, VHS allowed longer recording times and consequently had more users, leading to more available film titles, more service workshops and so on. Hence, VHS overturned the initial lead of the superior technology, and was helped in the competition for later consumers by having won over earlier purchasers. In a similar argument made by Arthur (1989), widely-used technologies will attract a disproportionate amount of R&D effort, leading to improvements and more chances of capturing future customers; he refers to this as "lock-in through learning".

Our purpose is to model how the experience of winning and losing affects behavior in contests, and how a principal should divide the prize mass in response to this. It is the actual fact of winning that creates the advantage here, not the winning margin. A research team that wins a grant has the possibility to conduct its research program, and position itself for future applications.⁶ Losing teams have to use time looking for alternative – presumably less attractive – funding. In sales-force management, it is customary to give awards to the Seller of the Month and the like. And in such sales forces, it is not uncommon for the more successful agents to be given less administrative duties, better access to back-office resources, more training than the less successful, and better territories; see, *e.g.*, Skiera and Albers (1998), Farrell and Hakstian (2001), and Krishnamoorthy, *et al.* (2005). Being a winning salesperson gives a head start in later contests, and this is not necessarily tied to the size of the winning margin. In a best-of-two, all-pay auction model, Krumer (2013) attributes a psychological momentum to the first contest winner, who then has a lower cost of competing in the second compared to the losing opponent. A similar effect is investigated by Clark, *et al.* (2018) in a series of Tullock contests.⁷

A phenomenon related to the dynamic win effect that we consider here are dynamic *effort* effects, where efforts in an early contest, rather than winning it, gives a player benefits later on. Clark and Nilssen (2013) interpret this as learning by doing in a model of two rounds of contests. Relatedly, Kovenock and Roberson (2009) and Joffrion and Parreiras (2013) model a dynamic advantage as a function of a player's past *net* effort, *i.e.*, his effort over and above that of the other player. In Kovenock and Roberson's (2009) two-contest model, a high net effort in the first contest will, with some decay, give a player a head start in the second. Joffrion and Parreiras (2013) model a single all-pay auction carried out in multiple rounds, with players' effort costs in each round being convex and players observing their standings, *i.e.*, the leader's head start, after each round. These papers consider a different strategic setting to us since, in the early contests, the margin of victory is important; our model limits the advantage to what can be gained from winning, and this is independent of previous efforts. A key issue in papers with effort advantages is how transferable effort is between time periods, *i.e.*, the extent to which current effort decays in future rounds. We do not have effort spillovers

⁶It is of course not just the actual written application that is evaluated in deciding the winners of research grants. The quality of the team, and the amount and level of previous research by the team on the topic is also highly relevant. Each research team has its status at the beginning of the application procedure, and winners will enhance theirs in future grant rounds.

⁷Clark, *et al.* (2018) discuss a design issue similar to ours: how to distribute the prize fund over time; but they do it in the context of Tullock contests with symmetric players.

between contests, and the rate of catching up or lead enhancement is exogenously fixed; hence the effect of a pure win advantage can be examined.

Our analysis is based on two premises: a stage game consisting of a complete-information two-player all-pay auction with a head start, and the principal wishing to maximize the total expected level of effort over the series of contests. Head starts in a single contest have been analyzed by several authors, notably by Konrad (2002), Meiwowitz (2008), Kirkegaard (2012), Li and Yu (2012), Hirata (2014), Segev and Sela (2014), Siegel (2014), and Franke, *et al.* (2016). Among these papers, Konrad (2002) and Hirata (2014) discuss two-stage situations where players can take actions in the first stage that create head starts in the second stage, while Li and Yu (2012) and Franke, *et al.* (2016) discuss the principal's optimal choice of head start in a single contest in order to maximize expected effort.

With two players, Li and Yu (2012) show that the optimal head start is given to the player with the lowest valuation at an amount equal to the difference between the valuations of the two players. This shifts the stronger player's equilibrium effort distribution upwards by the amount of the head start, whilst that of the weaker player is unchanged. Hence, the stronger player has an increase in effort that offsets the head start, and this increase accrues to the contest organizer. Franke, *et al.* (2016) extend this to the case of several players, and where a head start may be combined with a bias term that amplifies the efforts of players in the contest success function. In this case, the head start can first be used to exclude players, and then to induce revenue-maximizing behavior among those who are left.

We also consider a principal who wishes to maximize contest effort or revenue, but we do not assume that a head start is something that a contest designer can control; neither can the principal control how this head start evolves with the experience of winning.⁸ However, the logic behind the result of Li and Yu (2012) belies one of the mechanisms at work in our story: while the principal cannot manipulate the head start directly, he can affect the valuations of the players of winning by a judicious division of the prize mass across contests. In one of our cases, it is possible for the principal to appropriate all of the surplus of the players by neutralizing the head start advantage exactly, in the sense of Li and Yu (2012). In other cases, this is not possible, however, and one of the players expects a surplus that then reduces the payoff of the principal. We find that, when the ex-ante difference between the players is large (*i.e.*, there is a large head start in the first contest), the principal's optimal choice is to spend all the money in the first contest, which in practice means running only that first contest. This is because the win advantages play only a small role in this case; the same result occurs if the players have symmetric win advantages. When the ex-ante head start is sufficiently small, however, it is the difference in win advantages that plays the key role. In particular, when the ex-ante laggard has a sufficiently greater win advantage, the principal prefers to run two contests. How much surplus he

⁸Revenue maximization may not be the only goal of the principal. Groh, *et al.* (2012) look at how seeding can be set in an elimination tournament in order to achieve maximum effort, the largest probability that the two top seeds will meet in the final, or the probability that the top seed wins overall.

can extract from the players depends on the extent that the initial laggard can overturn the head start.

This paper is organized as follows. Section 2 presents the stage game for our model, a single all-pay auction in which one player has a net head start, and the players have different valuations of the prize. Results from this are then utilized in setting up and solving our two-contest model in Section 3. Section 4 considers the reference case of symmetry, and Section 5 concludes. All proofs are contained in an appendix.

2 Preliminaries

Before we go on to consider our two-contest model, it is useful to state a preliminary result for a single all-pay auction in which two risk-neutral players, $i = 1, 2$, have asymmetric head starts (S_i), and different valuations (V_i) of winning the prize. In the all-pay auction, a player is given a score which is comprised of his initial head start and his effort to win the prize: $S_i + X_i$. Without loss of generality, we shall assume that $S_1 \geq S_2$. Define $\Delta := S_1 - S_2 \geq 0$. The probability that player 1 wins the prize, $\rho_1(X_1, X_2)$, is given by the following contest success function:

$$\rho_1(X_1, X_2) = \begin{cases} 1 & \text{if } S_1 + X_1 > S_2 + X_2 \\ \frac{1}{2} & \text{if } S_1 + X_1 = S_2 + X_2 \\ 0 & \text{if } S_1 + X_1 < S_2 + X_2 \end{cases}$$

The probability that player 2 wins is $\rho_2(X_1, X_2) = 1 - \rho_1(X_1, X_2)$. The expected utilities of the two players can be written as

$$EU_i = \rho_i(X_1, X_2) V_i - X_i, \quad i = 1, 2$$

Let $F_i(X_i)$ represent the cumulative distribution function of player i 's mixed strategy. Clark and Riis (1995) prove the following result:⁹

Proposition 1 (Clark and Riis 1995).

- (i) If $\Delta \geq V_2$, then $X_1 = X_2 = 0$, $EU_1 = V_1$, and $EU_2 = 0$.
- (ii) If $V_2 - V_1 \leq \Delta < V_2$, then the unique mixed-strategy Nash equilibrium is

$$\begin{aligned} F_1(0) &= \frac{\Delta}{V_2}; & F_1(X_1) &= \frac{\Delta + X_1}{V_2}, X_1 \in (0, V_2 - \Delta]; \\ F_2(0) &= 1 - \frac{V_2 - \Delta}{V_1}; & F_2(X_2) &= 1 - \frac{V_2 - X_2}{V_1}, X_2 \in (\Delta, V_2]. \end{aligned}$$

Expected efforts and win probabilities are

$$\begin{aligned} EX_1 &= \frac{(V_2 - \Delta)^2}{2V_2}; & EX_2 &= \frac{V_2^2 - \Delta^2}{2V_1}; \\ \rho_1 &= 1 - \frac{V_2^2 - \Delta^2}{2V_1V_2}; & \rho_2 &= \frac{V_2^2 - \Delta^2}{2V_1V_2}; \end{aligned} \tag{1}$$

⁹Li and Yu (2012) present a similar result in their Lemma 1.

with expected net payoffs

$$EU_1 = V_1 - V_2 + \Delta; EU_2 = 0. \quad (2)$$

(iii) If $0 \leq \Delta \leq V_2 - V_1$, then the unique mixed strategy Nash equilibrium is

$$\begin{aligned} F_1(0) &= 1 - \frac{V_1}{V_2}; & F_1(X_1) &= 1 - \frac{V_1 - X_1}{V_2}, X_1 \in (0, V_1]; \\ F_2(0) &= 0; & F_2(X_2) &= \frac{X_2 - \Delta}{V_1}, X_2 \in (\Delta, V_1 + \Delta]. \end{aligned}$$

Expected efforts and win probabilities are

$$\begin{aligned} EX_1 &= \frac{V_1^2}{2V_2}; & EX_2 &= \frac{V_1}{2} + \Delta; \\ \rho_1 &= \frac{V_1}{2V_2}; & \rho_2 &= 1 - \frac{V_1}{2V_2}; \end{aligned}$$

with expected net payoffs

$$EU_1 = 0; EU_2 = V_2 - V_1 - \Delta.$$

When the net head start of player 1 is sufficiently large (case (i) in Proposition 1), player 2 cannot catch up, and no effort is used. Player 1 wins the prize, gaining a surplus of V_1 . In the more interesting cases (ii) and (iii), the head start is sufficiently small that player 2 may exert positive effort, and this encourages 1 to exert effort also. The two cases differ according to whether the heterogeneity in head starts or the one in valuation is the larger. With player 1 having a net head start, $\Delta \geq 0$, he also has an initial advantage in the contest. However, who is the stronger player of the two is determined by the combination of Δ and the valuations, and two distinct possibilities are delineated by parts (ii) and (iii) of the Proposition. In part (ii), $V_1 + \Delta \geq V_2$. Hence, player 1 is the stronger contestant, since this player has the possibility of making a bid at or below 2's valuation (in the interval $(V_2 - \Delta, V_2]$) and winning with certainty. In part (iii), on the other hand, player 2 is the stronger contestant, since he can make a bid slightly above the maximum score that 1 can attain (in the interval $(V_1 + \Delta, V_2]$) and win with certainty.

When part (ii) of Proposition 1 holds, player 1 gives a positive bid with probability $\frac{V_2 - \Delta}{V_2}$, using a uniform distribution over $[0, V_2 - \Delta]$ conditional on the bid being positive, while player 2 has a probability of $\frac{V_2 - \Delta}{V_1}$ of giving a positive bid, which in this case is generated from a uniform distribution on $[\Delta, V_2]$. Note that the head start affects the equilibrium strategies and expected efforts of both players in this case; in particular, the expected effort of each player is lower, the larger is Δ . Furthermore, by (2), $EU_1 = \Delta - (V_2 - V_1) \geq 0$, and $EU_2 = 0$, reinforcing the idea of player 1 being the strong player in this case. The players' total expected efforts, by (1), can be written as

$$EX^* = EX_1 + EX_2 = V_2 - \Delta + \frac{(V_2^2 - \Delta^2)(V_2 - V_1)}{2V_1V_2} \quad (3)$$

Hence, if players have symmetric valuations and $V_1 = V_2 = V > \Delta$, then $EX^* = V - \Delta$ is the total amount of effort.¹⁰ Starting from this symmetric valuation, increasing V_1 will cause expected effort to fall, as is clear from (3). Hence the asymmetry in valuations reinforces the initial advantage of player 1 yielding lower expected effort. If, on the other hand, it is V_2 that is increased from the symmetric valuation, then expected effort will increase. Li and Yu (2012) show that expected efforts are maximized when $V_2 - V_1 = \Delta > 0$. In this case, parts (ii) and (iii) of Proposition 1 coincide, and $EU_1 = EU_2 = 0$. The asymmetry in the players' valuations (with player 2 having the larger) is neutralized by the head start in favour of the player with the lower valuation. This result will be useful in the next section.

When the player with a head-start disadvantage has sufficiently more to gain from winning than the rival, then part (iii) of Proposition 1 gives expected efforts in equilibrium. Player 1 submits a positive bid with probability $\frac{V_1}{V_2}$, with the positive bid generated by a uniform distribution over the interval $[0, V_1]$. Player 2 submits a positive bid with certainty, generated by a uniform distribution on $[\Delta, V_1 + \Delta]$. Here, it is only the equilibrium strategy of player 2 that is affected by the head start, shifting effort upwards by the amount of this parameter; this is a difference from the result in part (ii) of the Proposition in which both players' expected efforts fall in Δ . When part (iii) applies, $EU_1 = 0$, and $EU_2 = V_2 - V_1 - \Delta \geq 0$; this indicates that player 2 is the stronger in this case.

3 The model and result

A risk-neutral principal has a prize fund of size 1 to distribute among two risk-neutral contestants. He does this by setting up two consecutive all-pay auctions with a prize of $1 - v$ available in the first and a prize of v in the second, where $0 \leq v \leq 1$.¹¹ In the first contest, we assume that one of the players – player 1 without loss of generality – has a head start $h > 0$, which means that this player can win the contest with less effort than the rival. Specifically, we assume the following contest success function for player $i = 1, 2$, in contest 1, given efforts $x_{i,1}$:

$$\begin{aligned} p_{1,1}(x_{1,1}, x_{2,1}) &= \begin{cases} 1 & \text{if } h + x_{1,1} > x_{2,1} \\ \frac{1}{2} & \text{if } h + x_{1,1} = x_{2,1} \\ 0 & \text{if } h + x_{1,1} < x_{2,1} \end{cases} \\ p_{2,1}(x_{1,1}, x_{2,1}) &= 1 - p_{1,1}(x_{1,1}, x_{2,1}) \end{aligned}$$

The winner of the first contest gets not only the stage prize $1 - v$, but also an advantage in the second, a mechanism that has been explored by Clark and Nilssen (2017) in a symmetric version of the model with a general number of contests and a fixed stage prize. The experience of winning gives an advantage in the next contest.

¹⁰With symmetric valuations, player 1 is surely the strong player, and only part (ii) in Proposition 1 is relevant.

¹¹The principal may choose to award the prize in a single contest, in which case $v = 0$.

We call s_i the win advantage of player i . Should player 1 win the first contest, he builds upon his initial head start so that it becomes $h + s_1$ at the start of contest two. In the second contest with effort $x_{1,2}$, player 1 has a contest score of $h + s_1 + x_{1,2}$; the score of player 2 will simply be his effort in contest two, $x_{2,2}$. Should player 2 win the first contest, his score in the second augments his effort $x_{2,2}$ by $s_2 > 0$, giving him a contest score of $s_2 + x_{2,2}$; player 1 in this case has a second-contest score of $h + x_{1,2}$.

Two interesting possibilities can occur: (i) either the net lead of player 1 gets reduced to $h - s_2 \geq 0$, or (ii) player 2 takes over the head start, $s_2 - h > 0$, so that he starts contest two with an initial score of s_2 compared to h for player 1. No efforts are carried over from the first contest to the second. The player with the largest score in contest two is designated the winner of that contest, gaining the stage prize v . The game then ends.

Formally, let $p_{i,2}(x_{1,2}, x_{2,2}; j)$ indicate the win probability for player i in the second contest, given that player j has won the first.

$$\begin{aligned} p_{1,2}(x_{1,2}, x_{2,2}; 1) &= \begin{cases} 1 & \text{if } h + s_1 + x_{1,2} > x_{2,2} \\ \frac{1}{2} & \text{if } h + s_1 + x_{1,2} = x_{2,2} \\ 0 & \text{if } h + s_1 + x_{1,2} < x_{2,2} \end{cases} \\ p_{2,2}(x_{1,2}, x_{2,2}; 1) &= 1 - p_{1,2}(x_{1,2}, x_{2,2}; 1), \end{aligned}$$

and

$$\begin{aligned} p_{1,2}(x_{1,2}, x_{2,2}; 2) &= \begin{cases} 1 & \text{if } h + x_{1,2} > s_2 + x_{2,2} \\ \frac{1}{2} & \text{if } h + x_{1,2} = s_2 + x_{2,2} \\ 0 & \text{if } h + x_{1,2} < s_2 + x_{2,2} \end{cases} \\ p_{2,2}(x_{1,2}, x_{2,2}; 2) &= 1 - p_{1,2}(x_{1,2}, x_{2,2}; 2). \end{aligned}$$

In what follows, we shall suppress the effort arguments in the probability function, writing simply $p_{i,2}(j)$.

Similarly, denote by $\pi_{i,2}(j)$ the expected payoff of player i in the second contest, given that j has won the first. Then

$$\pi_{i,2}(j) = p_{i,2}(j)v - x_{i,2},$$

since the loser gains 0 in this final contest. Denoting the expected equilibrium payoff in contest two with an asterisk, we can write the expected payoff function of player i in contest one as

$$\pi_{i,1} = p_{i,1} [1 - v + \pi_{i,2}^*(i)] + (1 - p_{i,1}) \pi_{i,2}^*(j) - x_{i,1}, \quad i \neq j.$$

Should player i win the first contest, he gains the stage prize $1 - v$ and the payoff that the contest-one winner expects in the second contest, $\pi_{i,2}^*(i)$. If player i loses the first contest, then he gains no stage prize and expects $\pi_{i,2}^*(j)$ in contest two.

Given the structure of the game, the principal wishes to distribute the fixed budget between the two contests in order to obtain the most effort in aggregate. Effort in the first contest is affected by the amount of the asymmetry (h), the

stage prize, and the value of the continuation payoff. Intuitively, saving more of the prize for the second contest will increase the efforts in that contest. However, the efforts in that contest will also be affected by the net head start that evolves after the first contest. The asymmetry may be larger in magnitude than in the first contest (if player 1 wins contest one), less in favour of player 1 (if player 2 wins and $h - s_2 > 0$), or in favour of player 2; player 2 surpasses the head start of player 1 from the first contest if $s_2 - h > h > 0$.

Note that the contest analyzed in the previous section forms a stage game for our model. When deciding how to distribute the prize mass, the principal must be mindful of the fact that the choice of prize in the first contest will affect the type of behavior in contest two. Specifically, the principal knows that equilibrium play in the second contest will progress according to either part (ii) or part (iii) of Proposition 1, or there will be no effort induced (part i), depending on the underlying parameters h, s_1 , and s_2 , and the prize division.

The following proposition, proved in the Appendix, solves for the optimal division of the principal's budget between the two contests.

Proposition 2 *Consider a sequence of two all-pay auctions with two players and a total prize budget equal to 1, so that the prize in the second contest is $v \in [0, 1]$ and the prize in the first contest is $1 - v$. Let player 1 have a head start $h \in (0, 1)$, and let there be a win advantage to each player of $s_i \in (0, 1), i = 1, 2$. A principal who wants to maximize total expected effort over the two contests should set v as follows:*

- (i) *If $s_2 \leq h < 1$, then $v = 0$ is optimal and gives rise to total expected efforts of $1 - h$.*
- (ii) *If $\frac{s_2 - s_1}{2} < h < s_2$, then any $v \in [0, s_2 - h]$ is optimal, giving total expected efforts of $1 - h$.*
- (iii) *If $\frac{s_2 - s_1}{3} < h \leq \frac{s_2 - s_1}{2}$, then any $v \in [s_2 - h, 1]$ is optimal, giving total expected efforts of $1 - 3h + s_2 - s_1$.*
- (iv) *If $0 < h \leq \frac{s_2 - s_1}{3}$, then $v = 2h + s_1$ is optimal, giving total expected efforts of 1.*

In case (i), player 1 has such a large initial head start that his net advantage in contest two will still be positive, even after a loss in contest one. In the other cases, it is the winner of contest one that has the head start in contest two.

Figure 1 depicts the parameter space, delineating the cases addressed in Proposition 2, assuming that all four cases exist. Note that case (iv) cannot exist if $h \geq \frac{1}{3}$, and case (iii) disappears if $h \geq \frac{1}{2}$.

Note, from Proposition 1, that, if the principal puts $v = 0$ and a single contest is run – with a prize of 1, and a head start of h for one of the players – then the amount of expected effort from the players is $1 - h$. More effort than this can only be achieved when the parameters are such that cases (iii) and (iv) in Proposition 2 exist. Hence, the scope for using this design to induce extra effort depends upon the initial head start (h) not being too large, and that the degree of potential catching up by the initial laggard (s_2) is sufficiently large compared to the win advantage of the initial leader (s_1). How much s_2 must be larger than s_1 is discussed below.

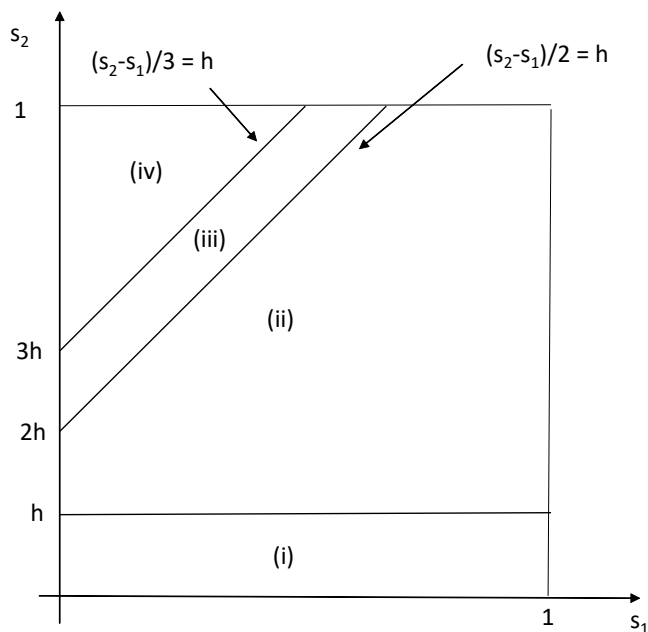


Figure 1: Delineation of cases

Figure 2 depicts the optimal prize v^* in the second contest, as well as the types of contest behavior that one may see there. The areas here depict parameter combinations that lead to effort in contest two no matter who wins contest one (area A), no effort in contest two (area B), and effort in contest two dependent on who has won the first contest (area C only if player 2 has won, and area D only if player 1 has won).¹² The figure is drawn for the case of $s_2 > s_1$; when $s_1 \geq s_2$, we are left with only cases (i) and (ii) of Proposition 2 and area D disappears. In Figure 2, v^* can be seen for each of the cases of Proposition 2. When the initial head start is large, $h \in [s_2, 1)$, then $v^* = 0$, which is case (i). The optimal second contest prize for cases (ii) and (iii) are a range of prizes, represented by the checkered areas in Figure 2. Finally, for low values of the initial head start, $h \in (0, \frac{s_2-s_1}{3})$, the optimal second round prize follows the line $v^* = 2h + s_1$; this is case (iv) of the Proposition.

For high values of the head start, $h > \frac{s_2-s_1}{2}$, the principal can do no better than run a single contest. In the other cases, it is interesting to consider how much of the prize is “saved” for contest two. In case (iii), a range of possibilities occurs since the principal can choose the second-contest prize from an interval. In the extreme, he can distribute the whole prize mass in the second contest, so that the first is simply a contest for position. In case (iv), the prize must be set at a certain level in order to gain the maximum level. Given the restriction on h in this case, one can consider parameter values for which more than half of the prize is awarded in the second contest (*i.e.*, where $2h + s_1 > \frac{1}{2}$). This can occur only if $s_2 > \max \left\{ s_1, \frac{3}{4} - \frac{s_1}{2} \right\}$.

A final illustration of the results in Proposition 2 is provided in Figure 3, which

¹²We shall return to the intuition behind these areas below.

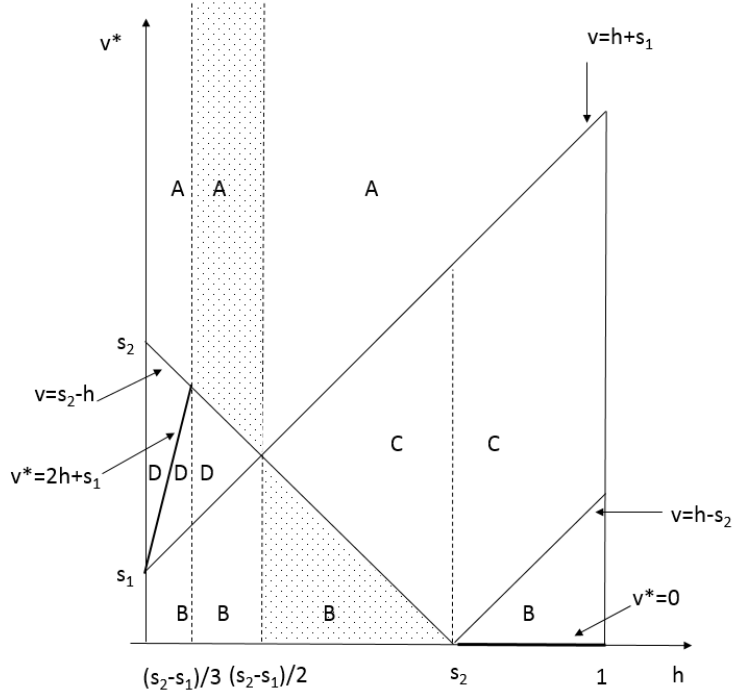


Figure 2: Optimal second-contest prize, for $s_2 > s_1$

depicts the connection between the initial head start and the total expected effort over the two contests. Whilst we have set up the possibility that the principal may use two contests in order to solicit the most effort, it is fully possible to award the whole prize in the first one. As noted above, total expected effort here is $1 - h$, which forms a minimum effort that the principal seeks to increase by judicious setting of the prize instrument. This is only possible when a win for player 2 gives a sufficient leap past player 1 in terms of head start, should player 2 win the first contest. From Figure 3, it is clear that the initial head start for player 1 must be below $\frac{s_2 - s_1}{2}$ in order to achieve more expected effort than $1 - h$. The minimum head start for player 2 in contest two that is needed to achieve higher effort is $s_2 - \frac{s_2 - s_1}{2} = \frac{s_1 + s_2}{2}$, and expected effort of magnitude 1 can be reached as long as the head start of player 2 is at least $s_2 - \frac{s_2 - s_1}{3} = \frac{1}{3}(2s_2 + s_1)$.

In considering different distributions of the prize mass, the principal must take account of the way in which contest two may develop. If player 1 wins the first contest, he has a net head start in contest two of $h + s_1$. Should player 2 win the first contest, then either player 1 will get a reduced head start in contest two ($h - s_2$), or player 2 will have a net head start of $s_2 - h$, depending on which of these expressions is positive. If the prize in contest two, v , is set below the net head start, then it will not be profitable for the disadvantaged player to make any positive effort, and the advantaged player wins this contest with no effort. In Figure 2, the lines $v = h + s_1$, $v = h - s_2$, and $v = s_2 - h$ distinguish cases where there is effort in the second contest. In area B, we have $v < h + s_1$ and $v < h - s_2$, for $h > s_2$; and $v < s_2 - h$, for $h < s_2$. Hence, whoever wins the first contest, the loser will not fight at contest two. In area A, the second contest prize is above

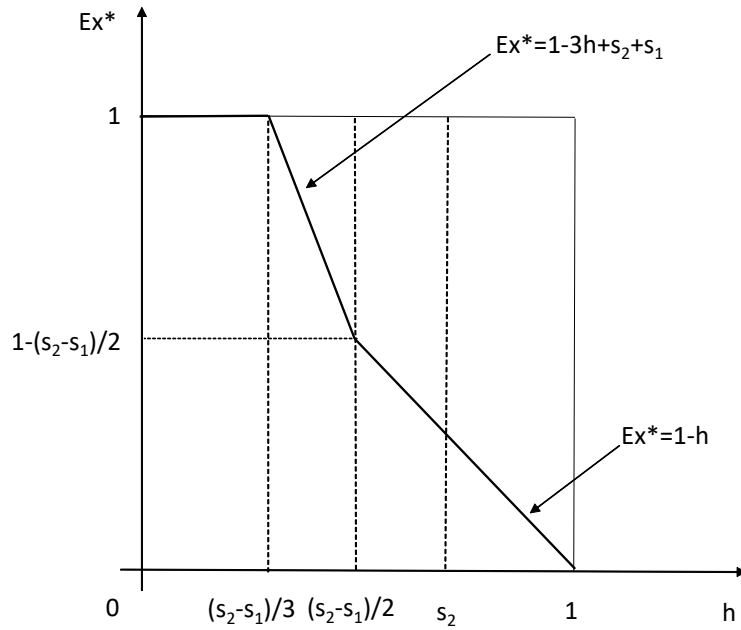


Figure 3: Total expected effort

all of the possible net head starts, and thus this prize is fought over, no matter who wins the first contest. For (v, h) combinations in area C, the prize on offer is so low that player 2 cannot profitably make effort in the second contest if he loses the first; this follows since the prize is below the head start of 1 in this case. On the other hand, the prize is above both of the potential head starts that can occur if player 1 wins the first contest. Hence, in area C, there will be effort in contest two if and only if player 1 wins the first. Area D is characterized by the fact that the second-contest prize is below the net head start of player 2 if that player won contest one, so that player 1 will not exert effort in this case and 2 wins effortlessly; also, the prize is above the head start of player 1 in contest two, if he has won the first. Thus, there is effort in contest two in this case if and only if player 1 has won the first.

When the second-contest prize is determined, the players and the principal can calculate the value of winning the first contest for each player, given the amount of the prize mass that is left for the second contest. In terms of Proposition 1, let us denote by V_i the value to player i of winning the first contest and set $\Delta = h$. When the V_i and h are such that part (ii) of Proposition 1 applies, then total expected effort in equilibrium is given by (3). It turns out that, for optimal choices made by the principal, equilibrium behavior is always determined by part (ii) of Proposition 1 (see the Appendix).¹³ Hence, (3) gives the total expected amount of effort for each of the cases in Proposition 2. We now explain the intuition underlying each case in Proposition 2.

Consider case (i), in which player 1 has a large head start that will still be

¹³In case (iv) of Proposition 2, we have $h = V_2 - V_1$, so that parts (ii) and (iii) of Proposition 1 give identical results.

positive in the second contest even if the rival wins the first. Suppose that player 1 wins the first contest so that the head start in contest two is $h + s_1$. With a second-contest prize $v < h + s_1$, player 2 will not find it worthwhile to fight, and player 1 wins v with no effort. Hence, the total effort would come simply from the first contest. But with $v > h + s_1$, there will be effort in the second contest. If, in this case, player 2 should win the first contest, then player 1's head start in the second one is reduced to $h - s_2$. If $v < h - s_2$, then player 2 will not exert effort in contest two, independent of who wins the first contest, and player 1 has a payoff in the second contest of v whilst 2 gets 0. In contest one, the players fight over the same prize $1 - v$, since the payoff in contest two is not affected by who wins the first. Given that player 1 has a head start of h , total efforts are $1 - v - h$, which is maximized for $v = 0$, the optimal second-contest prize in case (i). Most effort is gained by inducing no effort in contest two, and as much effort as possible in contest one. In terms of the discussion above, each player fights for the same prize in contest one, $V_1 = V_2 = 1$, so that $x^* = 1 - h$, following from (3).

The intuition for case (ii) is broadly similar, although now it is the winner of contest one that will have the head start in contest two. It is player 2 that has the lower positive head start in contest two, since $s_2 - h < h + s_1$ in this case. By the same reasoning as above, setting a prize of $v < s_2 - h$ in contest two will induce no effort in that contest, and the winner of the first contest wins also the second-contest prize v . Seen from the first contest, the value of winning is $(1 - v) + v = 1$ for each player, and hence total efforts are $1 - h$ given the initial head start; again, $V_1 = V_2 = 1$ in (3). Any second-contest prize up to $s_2 - h$ will achieve this result, since no prize in this range induces effort in contest two.

In case (iii), it is possible to induce the same result – expected effort of $1 - h$ – by setting $0 \leq v < h + s_1$. This is since, in this region, $s_2 - h > h + s_1$, and such a second-contest prize induces no effort there, irrespective of who wins the first. However, it is possible to improve upon this result, as indicated in Figure 3. The second-contest prize is optimally set sufficiently high that there will be effort in contest two, irrespective of who wins the first, $v \in [s_2 - h, 1]$. The winner of the first contest is thus the player with a positive expected payoff in contest two. The value of winning the first contest is hence, for each player, the stage prize $1 - v$ plus the expected payoff in contest two: $V_1 = 1 - v + h + s_1$, and $V_2 = 1 - v + s_2 - h$. Player 2 has more to gain, since $V_2 - V_1 = (s_2 - h) - (h + s_1) > 0$. Equation (3) gives a total expected effort of $1 - 3h + s_2 - s_1 > 1 - h$; the inequality holds for the parameter space of case (iii).

In case (iv), the optimal prize structure induces effort in contest two only if player 1 wins the first. Indeed, in this case, the optimal prize distribution induces a total expected effort that is equal to the total prize fund, in spite of the players being asymmetric at the outset. In the proof of Proposition 2 in the Appendix, we show, for this case, that the initial head start of player 1 is completely cancelled out by the induced difference in the players' valuations in the first contest: $h = V_2 - V_1$, and hence neither player expects a positive surplus.¹⁴ With a prize of $h + s_1 \leq v \leq s_2 - h$ in contest two, the players compete in that

¹⁴See the discussion around Proposition 1.

contest only if player 1 wins the first; in which case the players contest the same prize (v) in contest two, player 1 expects a second-contest payoff of $h + s_1$, and player 2 expects zero. Hence, total effort in the second contest, by (3), equals $v - h - s_1$. Seen from the first contest, player 1 has a value of winning of $1 - v$ (the stage prize in the first contest) plus $h + s_1$ from contest two: $V_1 = 1 - v + h + s_1$. Player 2 wins the second contest with certainty if he wins contest one, gaining prize v , and winning contest one gives a stage prize of $1 - v$: $V_2 = 1 - v + v = 1$. Hence $V_2 - V_1 = v - h - s_1$. Furthermore, part (ii) of Proposition 1 determines equilibrium behavior for $h + s_1 \leq v \leq 2h + s_1$. From (3), with $\Delta = h$, we see that V_2 is independent of v , and the second term in this expression is decreasing in V_1 ; hence, V_1 should be as low as possible, which is achieved by fixing v at the top of its range: $v = 2h + s_1$. Hence $V_2 - V_1 = h$, and total expected effort in the first contest is given, by (3), as $1 - \frac{h(1-h)}{2}$. The probability of positive effort in the second contest is the probability that player 1 wins the first, which from part (ii) of Proposition 1 is $\frac{1-h}{2}$, and with v at $2h + s_1$, we have an expected effort in the second contest of h . Hence, total expected effort is $1 - \frac{h(1-h)}{2} + \frac{h(1-h)}{2} = 1$. The principal captures all of the surplus in this case.

It is clear from Figure 3 that total expected efforts decrease as the initial head start increases. With a small head start, it is possible to design the contests in such a way that the whole value of the prize is dissipated in expectation. The larger the head start is at the outset, the larger payoff the advantaged player achieves, reducing the amount of effort in equilibrium; for these larger values of h , it is optimal to just induce effort in the first contest, *i.e.*, to run a single contest.

4 Symmetric win advantages

An interesting special case of Proposition 2 occurs when there is symmetry in the win advantages: $s_1 = s_2 = s$. Indeed, with such symmetry, the principal can achieve no better than the minimum level of expected effort. Considering $s_1 = s_2$ in Figure 1, it is immediate that the only cases that exist in Proposition 2 are (i) and (ii). Hence, the cases that allow the principal to adjust the prize mass and increase effort above the minimum (single contest) level disappear.

Corollary 1 *Suppose that $s_1 = s_2 = s$. A principal who wants to maximize total expected effort over the two contests should set v as follows, depending on the relation between the ex-ante asymmetry, measured by h , and the (symmetric) win advantage measured by s :*

(i) *If $\frac{h}{s} \in [1, \infty)$, then $v = 0$ is optimal and gives rise to a total expected effort of $1 - h$.*

(ii) *If $\frac{h}{s} \in (0, 1)$, then any $v \in [0, s - h]$ is optimal and gives rise to a total expected effort of $1 - h$.*

These results follow from Proposition 2, parts (i) and (ii). The optimal prize distribution for symmetric win advantages depends upon the relative magnitudes of the initial head start and the win advantage. In case (i) of Corollary 1, the

initial head start is large in relation to the win advantage, so that player 1's head start is little affected should 2 win the first contest. Both contests are asymmetric, and the principal chooses to induce all effort in the first contest. Due to the initial asymmetry, the whole of the prize is not dissipated, leaving an expected payoff for player 1 of h .

In case (ii) of Corollary 1, the head start is small in relation to the win advantage. Should player 2 win contest 1, then he takes over the head start in contest two. The principal maximizes aggregate efforts by setting a sufficiently low value of v that there is effort in contest one only. The exception here is that, with v exactly at $s - h$, there will be effort in contest one, and in contest two if and only if player 2 wins the first.

Hence, in the symmetric case, the principal prefers to leave so little of the prize for the second contest that there will be no effort there. This leaves an expected payoff for player 1 of h so that the principal cannot capture efforts equal to the whole value of the prize. Comparing with the general case above, it is interesting to note the limitations of the contest design through prize distribution when players have the same win advantage. The principal can use the difference in the win advantages to design a prize scheme that can achieve more effort in expectation than he can in the symmetric case. In the symmetric case, the principal cannot improve upon the situation in which he runs a single contest with the whole prize available at once.

5 Conclusion

A head start in a contest may reflect a previous investment made by a competitor, some technological superiority, or an incumbency advantage. In some situations, the initial advantage can be self-amplifying, leading to the Matthew effect where "success breeds success". In a contest setting, this may lead to discouragement of the laggard, and a fall in contest efforts. With the same set of contestants and a series of contests, we suggest that the head start can adjust over time to reflect previous contest outcomes. A win by the initial leader will augment his head start, whereas a win for the underdog will reduce the advantage of the leader, or even turn it in favor of the initially disadvantaged player. In this setting, we have considered a design question in which a principal may divide up the prize mass in order to maximize the expected effort of the players. When the ex-ante head start is large and the degree to which the underdog can catch up small, then the principal can do no better than to run a single contest with the whole prize mass on offer. If the amount that the underdog can overturn the initial disadvantage is sufficiently large, then the principal prefers to run two contests, dividing the prize mass between the two. When the win advantage is the same for each player, then again the principal chooses a single contest. The design problem would seem to be quite complex since the synergy between contests makes the principal mindful of the fact that the prize division can lead to contests in which the contestants prefer not to compete, and he also needs to take into account the development of the head start from one contest to the next. We have shown that a quite simple design can

increase the amount of expected effort over the single contest, and that it can even lead to full dissipation of the prize, hence “beating” the Matthew effect. Elsewhere (Clark and Nilssen, 2018), we have shown that this analysis carries over to a case in which a competitor does not have a head start, but has a bias which multiplies his effort by a factor greater than one in the contest success function. This is a somewhat simpler design problem since, with payoffs there being continuous, one does not have to take into account the fact that some prize divisions will lead to one opponent giving up, as we do in the present analysis. Nevertheless, we show also for that case, that for sufficiently small initial heterogeneity, the principal can capture all of the surplus of the competitors.

Extending the analysis to more than two contests is an obvious, but challenging, extension of our analysis. One may also like to consider a combination of a win advantage and an effort advantage where some part of previous efforts may affect the net head start; we have simply assumed exogenous win advantages here. Also, the win advantage does not necessarily modify the head start in a linear way, as we have assumed. Nevertheless, we believe that our analysis opens up an interesting design issue in a setting with synergy in the head start between contests in a series.

A Appendix

This Appendix contains the proof of Proposition 2. For the purpose of this proof, let player i 's expected equilibrium effort in contest t be x_{it}^* , total expected equilibrium effort across the two players in contest t be $x_t^* := x_{1t}^* + x_{2t}^*$, and total expected equilibrium efforts across contests be x^* , for $i, t \in \{1, 2\}$. Furthermore, denote the equilibrium win probability of player i in contest 1 by ρ_i^* .

Proof. Part (i). Let $s_2 < h$. Consider contest two, and suppose first that player 1 wins contest one, so that, in terms of Section 2, contest two has $\Delta = h + s_1$ and $V_1 = V_2 = v$. If $0 \leq v \leq h + s_1$, then, by part (i) of Proposition 1, there are no efforts, and so $x_2^* = 0$, $\pi_{1,2}(1) = v$, and $\pi_{2,2}(1) = 0$. If $h + s_1 < v \leq 1$, then, by part (ii) of Proposition 1, we have

$$\begin{aligned} x_{12}^* &= \frac{(v - h - s_1)^2}{2v}; \quad x_{22}^* = \frac{v^2 - (h + s_1)^2}{2v}; \quad x_2^* = v - h - s_1; \quad (\text{A1}) \\ \pi_{1,2}^*(1) &= h + s_1; \quad \pi_{2,2}^*(1) = 0. \end{aligned}$$

Suppose next that player 2 wins contest one. This reduces player 1's head start by s_2 , so that it becomes $h - s_2$ in contest two, and thus also $\Delta = h - s_2$. If $0 \leq v \leq h - s_2$, then, by part (i) of Proposition 1, there are no efforts: $x_2^* = 0$, $\pi_{1,2}(1) = v$, and $\pi_{2,2}(1) = 0$. If $h - s_2 < v \leq 1$, then, by Proposition 1 part (ii), we have

$$\begin{aligned} x_{12}^* &= \frac{(v - h + s_2)^2}{2v}; \quad x_{22}^* = \frac{v^2 - (h - s_2)^2}{2v}; \quad x_2^* = v - h + s_2; \quad (\text{A2}) \\ \pi_{1,2}^*(1) &= h - s_2; \quad \pi_{2,2}^*(1) = 0. \end{aligned}$$

Consider next contest one. Suppose that $h + s_1 < v \leq 1$, so that we will have efforts in contest two, regardless of who wins contest one. In terms of Section 2, we have $\Delta = h$. Even if player 2 wins contest one, he will have zero expected payoff in contest two, as we saw above, implying that $V_2 = 1 - v$. Player 1 will have $1 - v + h + s_1$ if he wins contest one and $h - s_2$ if he loses it, so that his value of winning that contest is $V_1 = (1 - v + h + s_1) - (h - s_2) = 1 - v + s_1 + s_2$, and $V_1 - V_2 = s_1 + s_2$.

There are three subcases that can occur:

- (a) $1 - h \geq h + s_2$.
- (b) $h + s_1 \geq 1 - h \geq h - s_2$.
- (c) $h - s_2 \geq 1 - h > 0$.

We shall give a detailed analysis of subcase (a), and more cursory ones of (b) and (c) since they follow closely.

For case (a), if $1 - h \leq v \leq 1$, then $h \geq 1 - v$, *i.e.*, $\Delta \geq V_2$, and so player 1 wins contest one without effort, by part (i) of Proposition 1. Total expected efforts are therefore the ones in contest two following a contest-one win by player 1, which we found above to be $v - h - s_1$, and this is maximized at $v = 1$, with total expected efforts at $1 - h - s_1$.

If instead $h + s_1 \leq v \leq 1 - h$, then there will be efforts also in contest one. By using the expressions for expected efforts and win probabilities in Proposition 1, part (ii), as well as those for contest-two expected efforts in equations (A1) and (A2), we find that total expected efforts over the two contests in this case equal

$$\begin{aligned}
x^* &= x_{11}^* + x_{21}^* + \rho_1^*(v - h - s_1) + \rho_2^*(v - h + s_2) \\
&= \frac{(1 - v - h)^2}{2(1 - v)} + \frac{(1 - v)^2 - h^2}{2(1 - v + s_1 + s_2)} \\
&\quad + \left(1 - \frac{(1 - v)^2 - h^2}{2(1 - v + s_1 + s_2)(1 - v)}\right) (v - h - s_1) \\
&\quad + \frac{(1 - v)^2 - h^2}{2(1 - v + s_1 + s_2)(1 - v)} (v - h + s_2) \\
&= 1 - 2h - s_1.
\end{aligned}$$

Suppose next that $h - s_2 \leq v \leq h + s_1$, so that we will have efforts in contest two if and only if player 2 wins contest one. We still have $\Delta = h$ and $V_2 = 1 - v$ in contest one, but now, since player 1 will win contest two effortlessly if he wins contest one, $V_1 = (1 - v + v) - (h - s_2) = 1 - h + s_2$, and $V_1 - V_2 = v - h + s_2$. By using Proposition 1, part (ii), and equation (A2) and recalling that there will be efforts in contest two only if player 2 wins contest one, we have

$$\begin{aligned}
x^* &= x_{11}^* + x_{21}^* + \rho_2^*(v - h + s_2) \\
&= \frac{(1 - v - h)^2}{2(1 - v)} + \frac{(1 - v)^2 - h^2}{2(1 - h + s_2)} \\
&\quad + \frac{(1 - v)^2 - h^2}{2(1 - h + s_2)(1 - v)} (v - h + s_2) \\
&= 1 - v - h.
\end{aligned}$$

Thus, x^* is maximized in the low end of this range, at $v = h - s_2$, for which total expected efforts equal $1 - 2h + s_2$.

Finally, suppose that $0 \leq v < h - s_2$, so that player 1 wins contest two without any effort, regardless of who wins contest one. In this case, the only efforts exerted are in contest one. Now, $V_1 = V_2 = 1 - v$, so that $V_1 - V_2 = 0$, while still $\Delta = h$. This is a well-known static contest with head start h and a symmetric prize valuation at $1 - v$, where $x^* = 1 - v - h$. To maximize this, the principal sets $v = 0$, so that total expected efforts become $1 - h$.

For case (b) above, a similar analysis shows that, for $1 \geq v \geq h + s_1$, player 1 wins contest one with no effort. Hence, total effort is that exerted in the second contest: $v - h - s_1$, which is maximized at $v = 1$. For $h + s_1 > v \geq 1 - h$, player 1 wins the first contest with no effort, while there would be effort in contest two only if player 2 won the first; hence, $x^* = 0$. When $1 - h > v \geq h - s_2$, there is a fight in the first contest, but the second-contest prize is contested only if player 2 wins. As above, this yields expected effort of $1 - v - h$, which is maximized at $v = h - s_2$, yielding $x^* = 1 - 2h + s_2$. Finally, for case (b), when $h - s_2 > v \geq 0$, there is effort only in the first contest, in the magnitude $1 - v - h$, which is maximal for $v = 0$. Since $1 - h > 1 - 2h + s_2$, and $1 - h > 1 - h - s_1$, expected effort is maximized for case (b) by setting $v = 0$.

For case (c), the following values of expected effort can be determined: if $1 \geq v \geq h + s_1$, then $x^* = 1 - h - s_1$; if $h + s_1 > v \geq 1 - h$, then $x^* = 0$; if $1 - h > v \geq 0$, then $x^* = 1 - v - h$, which is maximized at $v = 0$. Again $1 - h$ is the maximum level of expected effort.

To conclude the analysis of part (i), we find that the principal maximizes total expected effort by choosing $v = 0$, giving rise to a total expected effort of $1 - h$.

For parts (ii) through (iv) of the Proposition, we have $s_2 > h$, implying the winner of contest one has a head start in contest two. Suppose, first, that player 1 has won contest one, so that, in terms of Section 2, contest two has $\Delta = h + s_1$ and $V_1 = V_2 = v$. If $0 \leq v \leq h + s_1$, then, by part (i) of Proposition 1, there are no efforts, and so $x_2^* = 0$, $\pi_{1,2}(1) = v$, and $\pi_{2,2}(1) = 0$. If $h + s_1 < v \leq 1$, then, by part (ii) of Proposition 1, we have

$$\begin{aligned} x_{12}^* &= \frac{(v - h - s_1)^2}{2v}; \quad x_{22}^* = \frac{v^2 - (h + s_1)^2}{2v}; \quad x_2^* = v - h - s_1; \quad (\text{A3}) \\ \pi_{1,2}^*(1) &= h + s_1; \quad \pi_{2,2}^*(1) = 0. \end{aligned}$$

Suppose next that player 2 wins contest one, so that player 2 has a net head start in contest two of $\Delta = s_2 - h$, with $V_1 = V_2 = v$. If $0 \leq v \leq s_2 - h$, then there are no efforts, and so $x_2^* = 0$, $\pi_{1,2}(2) = 0$, and $\pi_{2,2}(2) = v$. If $s_2 - h < v \leq 1$, then

$$\begin{aligned} x_{12}^* &= \frac{v^2 - (s_2 - h)^2}{2v}; \quad x_{22}^* = \frac{(v + h - s_2)^2}{2v}; \quad x_2^* = v + h - s_2 \quad (\text{A4}) \\ \pi_{1,2}^*(1) &= 0; \quad \pi_{2,2}^*(1) = s_2 - h. \end{aligned}$$

The three cases differ in contest-one outcomes. We take them in turn.

Part (ii). Let $\frac{s_2 - s_1}{2} < h < s_2$, implying $h + s_1 > s_2 - h > 0$. Consider contest one, and suppose first that $h + s_1 < v \leq 1$, so that we have efforts in contest

two, regardless of who wins contest one: there will be efforts in contest two if $v > \max\{h + s_1, s_2 - h\}$, which here equals $h + s_1$. In terms of Section 2, we have $\Delta = h$; moreover, $V_1 = 1 - v + h + s_1$, and $V_2 = 1 - v + s_2 - h$, so that $V_1 - V_2 = 2h - (s_2 - s_1) > 0$.

If $v \geq 1 - 2h + s_2$, then $\Delta \geq V_2$ in contest one, and so, by part (i) of Proposition 1, there are no efforts there, with player 1 therefore winning and moving to contest two with a head start of $h + s_1$. Total expected effort will be $x^* = v - h - s_1$. This is maximized at $v = 1$, giving a total expected effort of $1 - h - s_1$.

If instead $h + s_1 < v < 1 - 2h + s_2$, then there will be positive efforts also in contest one. By using the expressions for expected efforts and win probabilities in Proposition 1, part (ii), as well as those for contest-two expected efforts in equations (A3) and (A4), we find that total expected efforts over the two contests in this case equal

$$\begin{aligned}
x^* &= x_{11}^* + x_{21}^* + \rho_1^*(v - h - s_1) + \rho_2^*(v + h - s_2) \\
&= \frac{(1 - v + s_2 - 2h)^2}{2(1 - v + s_2 - h)} + \frac{(1 - v + s_2 - h)^2 - h^2}{2(1 - v + s_1 + h)} \\
&\quad + \left(1 - \frac{(1 - v + s_2 - h)^2 - h^2}{2(1 - v + s_1 + h)(1 - v + s_2 - h)}\right) (v - h - s_1) \\
&\quad + \frac{(1 - v + s_2 - h)^2 - h^2}{2(1 - v + s_1 + h)(1 - v + s_2 - h)} (v + h - s_2) \\
&= 1 - 3h + s_2 - s_1. \tag{A5}
\end{aligned}$$

Suppose next that $s_2 - h \leq v \leq h + s_1$, so that we will have efforts in contest two if and only if player 2 wins contest one. If player 1 wins contest one, then he will win contest two also, without efforts, so that $\pi_{1,2}^*(1) = v$ and $\pi_{2,2}^*(1) = 0$. If player 2 wins contest one, then our findings in (A4) apply, so that $x_2^* = v + h - s_2$, $\pi_{1,2}^*(1) = 0$, and $\pi_{2,2}^*(1) = s_2 - h$. In terms of Section 2, we still have $\Delta = h$ and $V_2 = 1 - v + s_2 - h$ in contest one, but now $V_1 = 1 - v + v = 1$, so that $V_1 - V_2 = v - s_2 + h$. By using Proposition 1, part (iii), and equation (A4), and recalling that there will be efforts in contest two only if player 2 wins contest one, we have

$$\begin{aligned}
x^* &= x_{11}^* + x_{21}^* + \rho_2^*(v + h - s_2) \\
&= \frac{(1 - v + s_2 - 2h)^2}{2(1 - v + s_2 - h)} + \frac{(1 - v + s_2 - h)^2 - h^2}{2} \\
&\quad + \frac{(1 - v + s_2 - h)^2 - h^2}{2(1 - v + s_2 - h)} (v + h - s_2) \\
&= 1 - v - 2h + s_2.
\end{aligned}$$

This is maximized when v is at the low end of the range, *i.e.*, at $v = s_2 - h$, giving total expected efforts at $1 - h > 1 - 3h + s_2 - s_1$, where the inequality follows from the supposition that $h > \frac{s_2 - s_1}{2}$.

Finally, consider the case of $0 \leq v < s_2 - h$. Now, there will be no efforts in contest two, irrespective of who wins the first contest: the winner of contest

one will win contest two without efforts. In terms of Section 2, we now have $V_1 = V_2 = 1$, so that $V_1 - V_2 = 0$, while still $\Delta = h$. All the effort takes place in contest one, and so this case is of a well-known static contest with a head start of h and a symmetric prize valuation at 1. Total expected effort is $1 - h$.

To conclude the analysis of part (ii), we find that the principal maximizes total expected effort by choosing any $v \in [0, s_2 - h]$, giving rise to a total expected effort of $1 - h$.

Part (iii). Let $\frac{s_2 - s_1}{3} \leq h \leq \frac{s_2 - s_1}{2}$, implying $h \geq s_2 - s_1 - 2h \geq 0$. Consider contest one. Suppose first that $s_2 - h < v \leq 1$, so that we will have efforts in contest two, regardless of who wins contest one; now $s_2 - h > h + s_1$, so this is the relevant restriction. In terms of Section 2, we have $\Delta = h$; moreover, $V_1 = 1 - v + h + s_1$, and $V_2 = 1 - v + s_2 - h$, so that $\Delta > V_2 - V_1 > 0$, with $V_2 - V_1 = s_2 - s_1 - 2h$.

We can now disregard the possibility of player 1 winning in contest one without effort, since this only happens when $\Delta \geq V_2$, which here amounts to $v \geq 1 - 2h + s_2$. Given the restriction $v \leq 1$, this would require $h \geq \frac{s_2}{2}$, which breaches with the supposition, here in part (iii), that $h \leq \frac{s_2 - s_1}{2}$.

Since $\Delta > V_2 - V_1$, Proposition 1, part (ii), applies, with an analysis identical to that in the proof of part (ii), so that, following equation (A5), we have $x^* = 1 - 3h - s_1 + s_2$.

Suppose next that $h + s_1 \leq v \leq s_2 - h$, so that we will have efforts in contest two if and only if player 1 wins contest one. If player 2 wins contest one, then he will in this case win contest two also, without any efforts, with $\pi_{1,2}^*(1) = 0$ and $\pi_{2,2}^*(1) = v$. If player 1 wins contest one, then he will have a head start in contest two of $h + s_1$, and our analysis in (A3) applies, with $x_2^* = v - h - s_1$, $\pi_{1,2}^*(1) = h + s_1$, and $\pi_{2,2}^*(1) = 0$. In terms of Section 2, we now have, in contest one, $\Delta = h$, $V_1 = 1 - v + h + s_1$, and $V_2 = 1 - v + v = 1$. This implies that $\Delta > V_2 - V_1 \geq 0$, with $V_2 - V_1 = v - h - s_1$. The inequality $V_2 - V_1 \geq 0$ follows from the restriction that $v \geq h + s_1$. The inequality $V_2 - V_1 < h$ is equivalent to $v < 2h + s_1$, which holds under the restriction that $v \leq s_2 - h$, since here in part (iii) we have $3h > s_2 - s_1$. By using Proposition 1, part (ii), and equation (A3), and recalling that there will be efforts in contest two only if player 1 wins contest one, we have

$$\begin{aligned}
x^* &= x_{11}^* + x_{21}^* + \rho_1^*(v - h - s_1) \\
&= \frac{(1 - h)^2}{2} + \frac{1 - h^2}{2(1 - v + h + s_1)} \\
&\quad + \left(1 - \frac{1 - h^2}{2(1 - v + h + s_1)}\right)(v - h - s_1) \\
&= 1 + v - 2h - s_1,
\end{aligned} \tag{A6}$$

so that x^* is maximized when v is in the upper end of its range, at $v = s_2 - h$, giving total expected efforts at $1 - 3h + s_2 - s_1$.

Finally, suppose that $0 \leq v < h + s_1$. Now, there will be no efforts in contest two, irrespective of who wins contest one: the winner of contest one will win also contest two without efforts. In this case, contest one has $V_1 = V_2 = 1$, so that

$V_2 - V_1 = 0$, while still $\Delta = h$. Total expected efforts equal $1 - h < 1 - 3h + s_2 - s_1$, where the inequality follows from the supposition that $h \leq \frac{s_2 - s_1}{2}$.

To conclude the analysis of part (iii), we find that the principal maximizes total expected efforts by choosing any $v \in [s_2 - h, 1]$, giving rise to total expected profits equal to $1 - 3h + s_2 - s_1$.

Part (iv). Let $0 < h < \frac{s_2 - s_1}{3}$, implying $s_2 - s_1 - 2h > h > 0$. Consider contest one. Suppose first that $s_2 - h < v \leq 1$, so that we will have efforts in contest two, regardless of who wins contest one. As in part (ii), we can disregard the case of player 1 winning contest one without effort. Also, as in part (iii), we have $\Delta = h$, $V_1 = 1 - v + h + s_1$, $V_2 = 1 - v + s_2 - h$, and $V_2 - V_1 = s_2 - s_1 - 2h$, but now this implies $\Delta < V_2 - V_1$, since $h \leq \frac{s_2 - s_1}{3}$. We therefore use Proposition 1, part (iii), together with equations (A3) and (A4), to obtain

$$\begin{aligned}
x^* &= x_{11}^* + x_{21}^* + \rho_1^*(v - h - s_1) + \rho_2^*(v + h - s_2) \\
&= \frac{(1 - v + h + s_1)^2}{2(1 - v + s_2 - h)} + \frac{1 - v + h + s_1}{2} + h \\
&\quad + \frac{1 - v + h + s_1}{2(1 - v + s_2 - h)}(v - h - s_1) \\
&\quad + \left(1 - \frac{1 - v + h + s_1}{2(1 - v + s_2 - h)}\right)(v + h - s_2) \\
&= 1 + 3h - (s_2 - s_1).
\end{aligned}$$

Suppose next that $h + s_1 \leq v \leq s_2 - h$, so that we will have efforts in contest two if and only if player 1 wins contest one. If player 2 wins contest one, he will in this case win contest two also, without any efforts, with $\pi_{1,2}^*(1) = 0$ and $\pi_{2,2}^*(1) = v$. If player 1 wins contest one, then he will have a head start in contest two of $h + s_1$, and our analysis in (A3) applies, with $x_2^* = v - h - s_1$, $\pi_{1,2}^*(1) = h + s_1$, and $\pi_{2,2}^*(1) = 0$. In terms of Section 2, we now have, in contest one, $\Delta = h$, $V_1 = 1 - v + h + s_1$, $V_2 = 1 - v + v = 1$, and $V_2 - V_1 = v - h - s_1$. $\Delta < V_2 - V_1$ holds if and only if $v > s_1 + 2h$. Since $s_2 - h > 2h + s_1$ when $h < \frac{s_2 - s_1}{3}$, we have that Proposition 1, part (ii), applies for $h + s_1 \leq v \leq 2h + s_1$, and Proposition 1, part (iii), for $2h + s_1 < v \leq s_2 - h$. Using Proposition 1, part (iii), in this latter case together with equation (A3), and recalling that there are efforts in contest two only if player 1 wins contest one, we have

$$\begin{aligned}
x^* &= x_{11}^* + x_{21}^* + \rho_1^*(v - h - s_1) \\
&= \frac{(1 - v + h + s_1)^2}{2} + \frac{1 - v + h + s_1}{2} + h \\
&\quad + \frac{1 - v + h + s_1}{2}(v - h - s_1) \\
&= 1 - v + h + s_1,
\end{aligned}$$

so that x^* is maximized when v is at the lower end of its range, at $v = 2h + s_1$, giving rise to total expected efforts equal to $1 - h$. For $2h + s_1 \geq v \geq h + s_1$, Proposition 1, part (ii), applies, giving expected effort as in (A6): $x^* = 1 + v - 2h - s_1$, which is maximized at the higher end of this range, at $v = 2h + s_1$, giving $x^* = 1$.

Finally, suppose that $0 \leq v < h + s_1$. Now, there will be no efforts in contest two, irrespective of who wins contest one: the winner of contest one will win also contest two without efforts. In this case, contest one has $V_1 = V_2 = 1$, so that $V_2 - V_1 = 0$, while still $\Delta = h$. Total expected efforts equal $1 - h$.

To conclude the analysis of part (iv), we find that the principal maximizes total expected efforts by choosing $v = 2h + s_1$ to obtain total expected profits equal to 1. ■

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