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Conditions implying the vanishing of the Hamiltonian at the infinite horizon in optimal control problems.

by

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Abstract. In an infinite horizon optimal control problem, the Hamiltonian vanishes at the infinite horizon, when the differential equation is autonomous. The integrand in the integral criterion may contain the time explicitly, but it has to satisfy certain integrability conditions. A generalization of Michels (1982) result is obtained.

Introduction. Michel (1982) proved that the Hamiltonian vanishes when the time goes to infinity in problems where the differential equation is autonomous and the criterion is of the form $\int_0^\infty e^{rt} g(x,u)dt$. Below, a generalization of this result to the case where the criterion is of the form $\int_0^\infty f_0(t,x,u)dt$ is obtained, using a slightly modified proof.

Results. Consider the following problem.

$$\max_{u(.)} \int_0^\infty f_0(t,x,u)dt, \tag{1}$$

subject to

$$\dot{x} = f(x,u) \text{ v.e.}, \qquad u \in U, \qquad x(0) = x_0, \tag{2}$$

$$\begin{aligned} &\lim_{t\to\infty} x_i(t) = x^i, \qquad i = 1, ..., m' \\ &\text{liminf}_{t\to\infty}\ x_i(t) \geq x^i, \qquad i = m'+1, ..., m. \end{aligned} \tag{3}$$

Here, U is a given set in $\mathbb{R}^m$, $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 0, ..., n$, and f_i and $\partial f_i/\partial x, i = 0, ..., n$, are continuous, as well as $\partial f_0/\partial t$, and *v.e.* (virtually everywhere) means everywhere, except at a countable munber of points.

Assume that $(x^*(t), u^*(t))$ is optimal among all pairs $(x(.), u(.))$ satisfying (2), (3), and for which the integral in (1) is convergent,. with $u(.)$ piecewise continuous on each finite interval. Assume also that $\int_0^\infty |f_0(t+\delta, x^*(t), u^*(t))|dt$ exists for all δ in an interval $(-\varepsilon, \varepsilon)$, and that, for some piecewise continuous function $\alpha(t)$, $|\partial f_0(t+\delta, x^*(t), u^*(t))/\partial t| \leq \alpha(t)$ for all $\delta \in (-\varepsilon, \varepsilon), t \in [0, \infty)$ and $\int_0^\infty \alpha(t)dt < \infty$. Then the following theorem holds:

Theorem. There exists a pair $(p_0, p(t))$, $p_0 \geq 0, (p_0, p(t)) \neq 0$ for all t, such that $p(t)$ is a continuous solution of the adjointequation

$$\dot{p} = -H_x(t, x^*(t), u^*(t), p(t)) \text{ v.e.,} \tag{4}$$

where $H = p_0 f_0(t, x, u) + p(t) f(x, u)$. Moreover,

$$\max_{u \in U} H(t, x^*(t), u, p(t)) = H(t, x^*(t), u^*(t), p(t)) \text{ v.e.} \tag{5}$$

Finally,

$$\lim_{t \to \infty} H(t, x^*(t), u^*(t), p(t)) = 0. \tag{6}$$

Proof. Let S be a given positive number at which $u^*(.)$ is continuous. Consider the auxiliary problem

$$\max_{u(.), T>0} \{ \int_0^T f_0(t, x, u) dt + g(T) \}, \tag{7}$$

subject to (2) and $x(T) = x^*(S)$, where $g(T) = \int_T^\infty f_0(t, x^*(t - (T - S)), u^*(t - (T - S))) dt$. Let $(x(.), u(.))$, defined on $[0, T]$, be admissible in the auxiliary problem (i.e. satisfying (2) and $x(T) = x^*(S)$). Extend $(x(.), u(.))$ to all $[0, \infty)$, by letting $x(t) = x^*(t - (T - S)), u(.) = u^*(t - (T - S))$ for $t > T$. An admissible pair in the original problem is then obtained (i.e. satisfying (2),(3)), hence $\int_0^S f_0(t, x^*(t), u^*(t)) dt + g(S) = \int_0^S f_0(t, x^*(t), u^*(t)) dt + \int_S^\infty f_0(t, x^*(t), u^*(t)) dt =$
$\int_0^\infty f_0(t, x^*(t), u^*(t)) dt \geq \int_0^\infty f_0(t, x(t), u(t)) dt = \int_0^T f_0(t, x(t), u(t)) dt + \int_T^\infty f_0(t, x(t), u(t)) dt =$ $\int_0^T f_0(t, x(t), u(t)) dt + g(T)$. Thus, $(x^*(.), u^*(.))$, restricted to $[0, S]$, is optimal in the auxiliary problem. Now, $g(T) = \int_S^\infty f_0(s + T - S, x^*(s), u^*(s)) dt$. Hence, by the standard finite horizon maximum principle, a pair $(p_0^S, p^S(.))$ exists, such that $|(p_0^S, p^S(0))| = 1, p_0^S \geq 0$, (4) and (5) are satisfied on $[0, S]$ and $[H(T, x^*(T), u^*(T), p(T)) + p_0^S dG/dT]_{T=S} = 0$. (The continuous differentiability of $G(T)$ near S follows from the existence of $\alpha(.)$.) Let S_k be an increasing sequence converging to ∞, such that $(p_0^{S_k}, p^{S_k}(0))$ is convergent. Then for each t, $p^{S_k}(t)$ is convergent, and $p_0 := \lim_k p_0^{S_k}$ and $p(t) := \lim_k p^{S_k}(t)$ satisfies $(p_0, p(0)) = 1$ and (4) and (5) on $[0, \infty)$. Finally, as $(d/dt)\, H(t, x^*(t), u^*(t), p^S(t)) = [(\partial/\partial t) H(t, x, u, p)]_{(x,u,p)=(x^*(t), u^*(t), p^S(t))} =$ $p_0^S f_{0t}(t, x^*(t), u^*(t))$ and $H(S, x^*(S), u^*(S), p^S(S)) = -p_0^S [dG(T)/dT]_{T=S}$, then
$H(S, x^*(S), u^*(S), p^S(S)) - H(t, x^*(t), u^*(t), p^S(t)) = \int_t^S p_0^S f_{0t}(t, x^*(t), u^*(t))$ and
$H(t, x^*(t), u^*(t), p^S(t)) = -\int_t^\infty p_0^S f_{0t}(t, x^*(t), u^*(t)) dt$. Letting $S = S_k$ and $k \to \infty$, gives, for each t, $H(t, x^*(t), u^*(t), p(t)) =$
$-\int_t^\infty p_0 f_{0t}(t, x^*(t), u^*(t)) dt$. Then the existence of $\alpha(.)$ yields (6).

Note: Statement and/or proofs of the finite horizon control results applied above can be found in the references number 2, 3 and 4 below.

References:

1. Michel, Ph. (1982), On the transversality condition in infinite horizon optimal problems., *Econometrica*, Vol. 50. No 4, 975-985.

2. Neustadt, L.W. (1976), *Optimization, A Theory of Necessary Conditions*, Princeton University Press, New Jersey.

3. Pontryagin, L.S. et. al. (1962) *The mathematical Theory of Optimal Processses*, Interscience, New York.

4. Seierstad, A, and K. Sydsaeter. (1987), *Optimal Control Theory with Economic Applications*, North-Holland, Amsterdam.

5. Ye, J.J, (1993), Nonsmooth maximum principles for infinite-horizon problems, *Journal of Optimization Theory and Applications*, Vol 76, No. 3, 485-500.