MEMORANDUM

No 24/2002

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By
Atle Seierstad

ISSN: 0801-1117

Department of Economics
University of Oslo

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P. O.Box 1095 Blindern	Gaustadalleén 21
N-0317 OSLO Norway	N-0371 OSLO Norway
Telephone: + 47 22855127	Telephone: +47 22 95 88 20
Fax: + 47 22855035	Fax: +47 22 95 88 25
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Maximum principle for stochastic control in continuous time with hard end constraints.

by

Atle Seierstad, University of Oslo

Abstract. A maximum principle is proved for certain problems of continuous time stochastic control with hard end constraints, (end constraints satisfied a.s.) After establishing a general theorem, the results are applied to problems where the state equation (differential equation) changes at certain stochastic points in time, and to piecewise continuous stochastic problems (including piecewise deterministic problems).

Introduction. Frequently, problems are encountered in which the state at the terminal time has to satisfy a constraint almost surely. An example may be the running of a firm under the constraint that the equity capital at the end of the planning period shall exceed a given level almost surely. The present paper proves necessary conditions, in the form of a maximum principle, for certain types of such problems. First a general theorem is proved, covering the case of a general type of stochastic disturbance in the right hand side of the differential equation and where the state develops continuously in time. From this theorem a result is derived for the case where the right hand side changes at certain stochastic points in time. For the latter type of stochastic disturbances, also results for hard end constrained piecewise continuous stochastic problems are derived. In particular, certain types of hard end constrained piecewise deterministic problems are covered.

In discrete time, maximization problems with hard end constraints are solved, using the dynamic programming equation, by associating a value $-\infty$ to points from which it is not possible to reach the terminal constraint almost surely. Also, maximum principles for such problems have been proved, for example by Arkin and Evstigneev (1982), that involve even more general hard constraints (required "all the time"). In continuous time, for control problems involving diffusions, soft end constraints (constraints satisfied in expectation) have been considered, e.g. Kushner (1972), Haussmann (1986), Peng (1990), Yong and Zhou (1999). For many types of diffusion problems one cannot operate with hard constraints, unless controls are allowed that depend on time as erratically as the diffusion. This conclusion, however, also depends on the manner in which the diffusion enters the problem, (so there are exceptions, see Seierstad (1991)). Below, only smoother controls (controls with smoother effects) are considered, together with smoother systems

(systems having pathwise solutions). The general theorem presented below is closely related to a result in Seierstad (1991), in the present paper the growth conditions are slightly weaker.

The general system. Let T be a fixed positive number, let X and Y be Banach spaces, let x_0 be a given point in X, let π be a bounded linear map from X into Y, and let U be a topological space. Furnish the interval $J = [0, T]$ with the Lebesgue measure. Let $(\Omega, \Phi, \Phi_t, P)$ be a filtered probability space, (i.e. for $t \in [0, T], \Phi_t$ are sub-σ-algebras of the σ-algebra Φ of subsets of $\Omega, \Phi_s \subset \Phi_t$ if $s < t$, P is a probability measure on Φ). The function $f(t, x, u, \omega) : J \times X \times U \times \Omega \to X$ has a Frechet derivative f_x with respect to $x \in X$, f has one-sided limits with respect to t, and f and f_x are, separately, continuous in x and in u, and for any given t, x, u, f is Φ_t-measurable with respect to ω. These conditions are called the Basic Assumptions.

Let U' be the set of functions $u(t, \omega)$ taking values in U, such that $u(., .)$, for each t, when restricted to $[0, t] \times \Omega$, is Lebesgue$\times\Phi_t-$measurable. (I.e. there exists a sequence of Lebesgue$\times\Phi_t-$measurable simple functions converging a.e.$\times$ a.s. to $u(., .)$ on $[0, t] \times \Omega$.) The measurability property is called progressive measurability. Let $u(., .) \in U'$. For each ω, the ("pathwise") solution - continuous in t - of the equation $\partial x(t, \omega)/\partial t :=$

$$\dot{x}(t, \omega) = f(t, x(t, \omega), u(t, \omega), \omega), x(0) = x_0, \tag{1}$$

is denoted $x^{u(.,.)}(t, \omega) = x^u(t, \omega)$, and called a system solution.

The simplest set of assumptions that are used in what follows, called Simple Global Assumptions, are as follows: In addition to the Basic Asssumptions,

f_x is uniformly continuous in x, uniformly in t, u and ω. Furthermore, a constant M^+ exists such that $|f(t, x, u, \omega)| \le M^+$ and $|f_x(t, x, u, \omega)| \le M^+$ for all (t, x, u, ω).

If wanted, the reader may skip all later modifications of these assumptions. These assumptions imply that system solutions always exist on all $[0, T]$, i.e, for any $u(., .) \in U'$, the solution of (1) exists. This property also holds for the so-called Standard Global Assumptions, which are as follows: In addition to the Basic Assumptions,

f_x is uniformly continuous in x, uniformly in t, u and ω. Furthermore, Lebesgue$\times\Phi$ -measurable functions $M^0(t, \omega)$ and $M'(t, \omega)$ are given, both integrable in t, such that $|f(t, 0, u, \omega)| \le M^0(t, \omega)$ and $|f_x(t, x, u, \omega)| \le M'(t, \omega)$ for all $(t, x, u, \omega) \in J \times B(0, \Xi(\omega)) \times U \times \Omega$, where $B(0, \Xi(\omega))$ is an open ball in X around the origin of radius $\Xi(\omega) := 1 + (|x_0| + \int_J M^0(s, \omega)ds)e^{\int_J M'(s,\omega)ds}$. There exists a constant M'_π such that $|\pi f_x(t, x, u, \omega)| \le M'_\pi$ for $(t, x, u, \omega) \in$

$J \times B(0,\Xi(\omega)) \times U \times \Omega$. The inequality $M^{**} := \text{essup}_\omega \int_J M'(t,\omega)dt < \infty$ holds, and $M^0(t,\omega)$ is Lebesgue$\times P$-integrable.

Let $\hat{M}(t,\omega) := M^0(t,\omega)+\Xi(\omega)M'(t,\omega)$, ($\hat{M}(t,\omega)$ is Lebesgue$\times P$-integrable). Then, $|f(t,x,u,\omega)| \leq \hat{M}(t,\omega)$ for $(t,x,u,\omega) \in J \times B(0,\Xi(\omega)) \times U \times \Omega$. In this case, any solution $x^u(t,\omega)$ of (1) belongs to $B(0,\Xi(\omega))$, moreover $f(t,x^u(t,\omega),u(t,\omega),\omega)$ is Lebesgue$\times P$-integrable.

Let $a \in X^*$, the topological dual of X, and consider the problem

$$\max_{u(.,.)\in U'} \mathrm{E}\langle x^{u(.,.)}(T,\omega), a\rangle, \tag{2}$$

subject to the differential equation (1), and

$$\pi x^{u(.,.)}(T,\omega) = \tilde{y} \text{ a.s., } \tilde{y} \text{ fixed in } Y. \tag{3}$$

Let $u^*(.,.) \in U'$ be an optimal control in the problem and write $x^{u^*(.,.)}(.,.) = x^*(.,.)$. Let, for each ω, $C(t,s,\omega)$ be the resolvent of the equation $\dot{q} = f_x(t,x^*(t,\omega),u^*(t,\omega),\omega)q$, ($C(s,s,\omega) = I$, the identity map). In the subsequent necessary conditions, the following local linear controllability condition is needed. Let $B_\alpha = \{\int_0^T z(t,.)dt : z(t,\omega) \in Y, z(.,.)$ is progressively measurable and $|z(.,.)|_\infty < \alpha\} \subset L_\infty(\Omega,Y) := L_\infty(\Omega,\Phi,Y)$, and let co denote convex hull. There exist a number $\alpha > 0$, and a progressively measurable function $\check{z}(t,\omega) : J \times \Omega \to Y$, with $|\check{z}(.,.)|_\infty < \infty$, such that

$$\int_0^T \check{z}(t,.)dt+B_\alpha \subset \text{co}\{\pi \int_0^T C(T,t,.)[f(t,x^*(t,.),\hat{u}(t,.),.)-f(t,x^*(t,.),u^*(t,.),.)]dt : \hat{u}(.,.) \in U'\}. \tag{4}$$

Theorem 1. Assume that $u^*(.,.)$ is optimal, that the Simple Global Assumptions hold and that (4) is satisfied. Then there exist a number $\Lambda_0 \geq 0$ and a linear functional ν on $L_\infty(\Omega,Y)$, bounded on B_α, such that, for all $u(.,.) \in U'$,

$$\langle \int_0^T \pi C(T,t,.)[f(t,x^*(t,.),u(t,.),.) - f(t,x^*(t,.),u^*(t,.),.)]dt, \nu\rangle + \Lambda_0 E\langle \int_0^T C(T,t,.)[f(t,x^*(t,.),u(t,.),.) - f(t,x^*(t,.),u^*(t,.),.)]dt, a\rangle \leq 0. \tag{5}$$

Finally, $(\Lambda_0, \nu|_{B_\alpha}) \neq 0$, where $\nu|_{B_\alpha}$ means ν restricted to B_α .

Remark 1 In this remark, the Standard Global Assumptions are postulated. Let 1_M be the indicator function of the set M, and let $M(t,\omega) = 2\hat{M}(t,\omega)$. Define $U^K := \{u(.,.) \in U'$:essup$_\omega \int_J M(t,\omega) 1_{\{(t,\omega):u(t,\omega)\neq u^*(t,\omega)\}}(t,\omega)dt \leq K, |\pi f(.,x^*(.,.),u(.,.),.) - \pi f(.,x^*(.,.),u^*(.,.),.)|_\infty \leq K\}$. Assume that for some constant $K^* > 0$, (4) holds for U' replaced by U^{K^*}. Then (5) holds for U' replaced by U^K, K any given positive number. (In case of the Simple Global Assumptions, let $\hat{M}(.,.) \equiv M^+$, $K = \max\{2TM^+, 2|\pi|M^+\}$, in which case $U^K = U'$.)

Remark 2 To obtain necessary condition only local well-behaviour conditions on the system are needed. Such conditions are presented below, combined with other modifications. Thus the Simple and Standard Assumptions can be replaced by the following ones. Let $U(t,\omega)$ be a given multifunction, with $U(t,\omega) \subset U$ for all (t,ω), such that $u^*(t,\omega) \in U(t,\omega)$ for all (t,ω). Define an error function $e(d)$ to be a non-negative function on $(0,\infty)$ such that $\lim_{d\to 0} e(d) = 0$. In addition to the Basic Assumptions, assume the following properties: For some $\check{d} > 0$, and some Lebesgue$\times P$-integrable functions $M'(t,\omega)$ and $M(t,\omega)$,

(i) $|f_x(t,x,u,\omega)| \leq M'(t,\omega)$ for all $(t,x,u,\omega) \in J \times B(x^*(t,\omega),\check{d}) \times U(t,\omega) \times \Omega$, and
$|f(t,x^*(t,\omega),u,\omega) - f(t,x^*(t,\omega),u^*(t,\omega),\omega)| \leq M(t,\omega)$ for all $(t,u,\omega) \in J \times U(t,\omega) \times \Omega$.

The inequality $M^{**} :=$ essup$_\omega \int_J M'(t,\omega)dt < \infty$ holds. For some real-valued function $\check{e}_0(d,t)$, being an error function in d, for any t,ω, for any $y \in B(x^*(t,\omega),d) \subset X$, $d \in (0,\check{d}]$,

(ii) $|f_x(t,y,u^*(t,\omega),\omega) - f_x(t,x^*(t,\omega),u^*(t,\omega),\omega)| \leq \check{e}_0(d,t)$, and $\lim_{d\searrow 0} \int_J \check{e}_0(d,t)dt = 0$, $\check{e}_0(d,t)$ assumed to be integrable in t.

The limit condition in (ii) holds if either $M'(t,\omega)$ is independent of ω, or if $\check{e}_0(d,t)$ is independent of t.

Furthermore,

(iii) $\pi f_x(t,x,u^*(t,\omega),\omega)$ is uniformly continuous in $x \in B(x^*(t,\omega),\check{d})$, uniformly in t,ω, and, for some constant M'_π, $|\pi f_x(t,x,u,\omega)| \leq$ M$'_\pi$ for $(t,x,u,\omega) \in J \times B(x^*(t,\omega),\check{d}) \times U(t,\omega) \times \Omega$.

Next,

(iv) $f(t, x^*(t,\omega), u^*(t,\omega), \omega)$ is Lebesgue$\times\Phi$-integrable.

Finally, for U' redefined to equal $\{u(.,.) \in U' : u(t,\omega) \in U(t,\omega)$ for all $(t,\omega)\}$, the following condition hold:

(v) (4) holds in the sense of Remark 1 for some K^*, when this U' and the function $M(t,\omega)$ are the entities appearing in the definition of U^K.

(vi) $u^*(.,.)$ is optimal in the set of controls $u(.,.) \in U'$, for which a solution $x^u(.,.)$ exists on all J satisfying (1) and (3) and for which $f(t, x^u(t,\omega), u(t,\omega), \omega)$ is Lebesgue $\times P$-integrable.

Then (5) holds for U' replaced by U^K as here defined, for any given $K > 0$.

Remark 3 Assume in Remark 2, that $X = X' \times X''$, where X', X'' are Banach spaces. Let π' be the projection on X', π'' be the projection on X'' and assume that $\pi = \pi_Y \pi'$ for some continuous linear map $\pi_Y : X' \to Y$, and that, for $x = (x', x'') \in X = X' \times X''$, $\pi' f(t, x, u, \omega)$ does not depend on x''. Assume that (iii), (iv) and (vi) in Remark 2 still hold, together with the following modifications of (i) and (ii): $M(t,\omega)$ and $M'(t,\omega)$ in (i) in Remark 2 need only pertain to $\pi' f$, i.e. it suffices that $|\pi' f(t, x^*(t,\omega), u, \omega) - \pi' f(t, x^*(t,\omega), u^*(t,\omega), \omega)| \le M(t,\omega)$ for all $(t, u, \omega) \in J \times U(t,\omega) \times \Omega$, and that $|\pi' f_x(t, x, u, \omega)| \le M'(t,\omega)$, for $(t, x, u, \omega) \in J \times B(x^*(t,\omega), \check{d}) \times U(t,\omega) \times \Omega$. Moreover, (ii) need only hold for f replaced by $\pi' f$. Finally, some Lebesgue$\times P$-integrable functions $M_*(t,\omega)$ and $M'_*(t,\omega)$ are assumed to exist, such that $|\pi'' f(t, x^*(t,\omega), u, \omega) - \pi'' f(t, x^*(t,\omega), u^*(t,\omega), \omega)| \le M_*(t)$ for all $(t, u, \omega) \in J \times U(t,\omega) \times \Omega$ and such that $|\pi'' f_x(t, x', x'', u, \omega)| \le M'_*(t,\omega)$ for all $(t, x', x'', u, \omega) \in J \times B(\pi' x^*(t,\omega), \check{d}) \times B(\pi'' x^*(t,\omega), M_{**}(\omega)) \times U(t,\omega) \times \Omega$, where $M_{**}(t,\omega) := 1 + \int_J \max\{M(t,\omega), M_*(t,\omega)\} dt e^{\int_J \max\{M'(t,\omega) M'_*(t,\omega)\} dt}$, with $E[e^{2\int_J M'_*(t,\omega)dt}(1+\int_J M'_*(t,\omega)dt)]^p < \infty$ and $|\int_J \max\{M'_*(t,\omega), M_*(t,\omega)\} dt|_{2q} < \infty, 1/p + 1/q = 1, p \in [1,\infty)$. (If $p = 1$, the last condition can dropped, provided $M(.,.)$ in the definition of U^K is replaced by $\max\{M(.,.), M'_*(.,.), M_*(.,.)\}$.) Then, if (v) in Remark 2 holds for the present $M(t,\omega)$-function, the conclusion of Remark 2 still holds.

Remark 4 For simplicity, assume $X = \mathbb{R}^n, Y = \mathbb{R}^{n'}$. The following results hold even for the assumptions in Remarks 2 and 3. Define $\nu_* := \phi \to \langle \pi\phi, \nu \rangle + \Lambda_0 E\langle \phi, a \rangle, \phi \in L_\infty(\Omega, X)$, and $p(s) := C(T, s, .)^* \nu_*$, where * means taking dual map, (so $p(s, \phi) := \langle \phi, p(s) \rangle = \langle C(T, s, .)\phi, \nu_* \rangle$). For $s < T$, the functions $\nu|_{\Phi_s} := \nu|_{L_\infty(\Omega, \Phi_s, Y)}$ and $p(s)|_{\Phi_s}$ are continuous in ϕ in L_∞-norm, (recall that $\nu|_{B_\alpha}$ is bounded, and that $\pi C(T, s, .)\phi =$

$\pi(I + \int_s^T f_x(\rho, x^*(\rho,.), u^*(\rho,.),.)C(\rho,s,.)d\rho)\phi \in B_\alpha$, for $\phi \in L_\infty(\Omega, \Phi_s, \mathbb{R}^n)$, $|\phi|_\infty$ small). By (5), for any $u(.,.) \in U^K$ and $s < T$, if $\lim_{t \nearrow s}(s-t)^{-1} \int_t^s (f(\rho, x^*(\rho,.), u(\rho,.),.) - f(\rho, x^*(\rho,.), u^*(\rho,.),.))d\rho =$ $f(s, x^*(s,.), u(s,.),.) - f(s, x^*(s,.), u^*(s,.),.)$ in $L_\infty(\Omega, X)$-norm, then

$$\langle f(s, x^*(s,.), u(s,.),.) - f(s, x^*(s,.), u^*(s,.),.), p(s)\rangle \leq 0. \tag{6}$$

The property:

$$\lim_{t\uparrow T} \sup_{|\phi|\leq 1, \phi \in L_\infty(\Omega, \Phi_t, X)} |p(t,\phi) - \langle \phi, \nu_*\rangle| = 0 \tag{7}$$

holds, if either ν is bounded in $|.|_\infty$-norm, or if $M'(t,\omega)$ is independent of ω and for some $\beta > 0$, $[\beta/(T-t)] \int_t^T \pi f_x(s, x^*(s,.), u^*(s,.),.)\phi ds$ belongs to B_α, for all $t \in [0,T)$, all $\phi \in B(0,1) \subset L_\infty(\Omega, \Phi_t, \mathbb{R}^n)$. These conditions, however, hold only in special cases.

Assume in the remaining part of this remark that, for any $s < T$, $\phi \to \langle \pi C(T,s,.)\phi, \nu\rangle$, $\phi \in L_\infty(\Omega, \Phi_s, \mathbb{R}^n)$ is absolutely continuous with respect to P, i.e., for any unit vector e_j, $H \to \langle \pi C(T,s,.)e_j 1_H, \nu\rangle$, $H \in \Phi_s$, is absolutely continuous. (By (5), this property holds in particular, if, in (4), $\check{z}(.,.) = 0$.) Then, $\phi \to p(s,\phi)$, $\phi \in L_\infty(\Omega, \Phi_s, \mathbb{R}^n)$, is also absolutely continuous, and has a Radon-Nikodym derivative $p^+(s,\omega)$, $s < T$, and $p^+(s,\omega)$ satisfies a.s.×a.e.:

$$\partial p^+(s,\omega)/\partial s = -p^+(s,\omega) f_x(s, x^*(s,\omega), u^*(s,\omega), \omega), \tag{8}$$

provided we read $p^+(s,\omega)$ as a row vector (and f_x is the Jacobian matrix).

Furthermore, for all $u(.,.) \in U^K$, a.e.×a.s.,

$$\langle f(s, x^*(s,\omega), u(s,\omega), \omega) - f(s, x^*(s,\omega), u^*(s,.), \omega), p^+(s,\omega)\rangle \leq 0. \tag{9}$$

Even the following inequality evidently holds a.e.×a.s.,

$$\langle f(s, x^*(s,\omega), u(s,\omega), \omega) - f(s, x^*(s,\omega), u^*(s,\omega), \omega), E[p^+(s,\omega)|\Phi_s]\rangle \leq 0. \tag{10}$$

In special cases below, differential equations for multipliers related to $E[p^+(s,\omega)|\Phi_s]$ are given.

Remark 5. Assume (vi) in Remark 2, and that there exist given sets $U_n(t,\omega), n = 1,2,..., U_{n+1}(t,\omega) \supset U_n(t,\omega)$, $u^*(t,\omega) \in U_1(t,\omega)$, such that, for each n, (i)–(iv) in Remark 2 holds for $U(t,\omega)$ replaced by $U_n(t,\omega)$, for functions $M(t,\omega) = M_n(t,\omega), M'(t,\omega) = M'_n(t,\omega)$, (or that the corresponding conditions in Remark 3 hold in the same manner, and with $M_*(t,\omega) = M_{*n}(t,\omega), M'_*(t,\omega) = M'_{*n}(t,\omega)$). Define

$$\begin{aligned}&U_n^K := \{u(.,.) \in U' : u(t,\omega) \in U_n(t,\omega) \text{ for all } (t,\omega),\\ &\text{essup}_\omega \int_J M_n(t,\omega) 1_{\{(t,\omega):u(t,\omega)\neq u^*(t,\omega)\}}(t,\omega)dt \le K,\\ &|\pi f(.,x^*(.,.),u(.,.),.) - \pi f(.,x^*(.,.),u^*(.,.),.)|_\infty \le K\}.\end{aligned}$$

Assume that, for some given $n = n^*, K = K^*$, (4) holds for $U' = U_{n^*}^{K^*}$. Then the necessary condition (5) holds for all $u(.,.) \in \cup_n U_n^n$.

Proof of Theorem 1: A proof is given based on the conditions (i)–(v) of Remark 2, and modifications needed for a proof of the results in Remark 3 are added. If wanted, the reader may assume that, in accordance with the Simple Global Assumptions, the functions $\hat{M}(t,\omega), M'(t,\omega)$, and $M'_*(t,\omega)$ equal the constant M^+, (with $M(t) = 2M^+$), and that the constant M^{**} equals M^+T. Without loss of generality, let $x_0 = 0, T = 1$. In case of Remark 3, we can, and shall, assume that $M'_*(t,\omega) \ge M'(t,\omega)$, and that a number $K \ge \max\{1, K^*\}$ is chosen, so large that $|\int_J M_*(t,\omega)dt|_q < K$. In case of Remark 2, K is any given number $\ge \max\{1, K^*\}$, and in this case, let $M_*(t,\omega) = M(t,\omega)$ and $M'_*(t,\omega) = M'(t,\omega)$. Define $M^*(\omega) := \int_J M'_*(t,\omega)dt$, and, for $u', u \in U^K$ (as defined in Remark 2), define $H_{u,u'} := \{(t,\omega) : u(t,\omega) \neq u'(t,\omega)\}$ and $\hat{\sigma}(u',u,\omega) := \int_J \max\{M(t,\omega), M_*(t,\omega)\} 1_{H_{u,u'}}(t,\omega)dt$. Let $t \to q^{u(.,.)}(t,\omega) := q^u(t,\omega)$ be the solution - continuous in t - of

$$\begin{aligned}&\dot{q}(t,\omega) = f_x(t,x^*(t,\omega),u^*(t,\omega),\omega)q(t,\omega)+\\ &f(t,x^*(t,\omega),u(t,\omega),\omega) - f(t,x^*(t,\omega),u^*(t,\omega),\omega), q(0) = 0,\end{aligned} \tag{11}$$

Define $\check{\sigma}(u,u')$:=essup$_\omega \int_J M(t,\omega)1_{H_{u,u'}}(t,\omega)dt$, let $A^d := \{u \in U^K : \check{\sigma}(u,u^*) < d\}$, and from now on let $u,u' \in A^{\tilde{d}}$, where $\tilde{d}$ is determined by $\tilde{d}e^{M^{**}} = \check{d}$. Note that by (i) in Remark 2 (and the existence of $M'_*(t,\omega)$ in Remark 3) and Gronwall's inequality (see Appendix), $|x^u(t,\omega) - x^*(t,\omega)| \le$

$$\begin{aligned}&(\textstyle\int_0^1 |f(t,x^*(t,\omega),u(t,\omega),\omega) - f(t,x^*(t,\omega),u^*(t,\omega),\omega)|dt)e^{M^*(\omega)} \le\\ &(\textstyle\int_0^1 \max\{M(t,\omega, M_*(t,\omega)\}1_{H_{u,u^*}}(t,\omega)dt)e^{M^*(\omega)} \le \hat{\sigma}(u,u^*,\omega)e^{M^*(\omega)},\\ &|q^u(t,\omega)| \le (\textstyle\int_0^1 \max\{M(t,\omega), M_*(t,\omega)\}1_{H_{u,u^*}}dt)e^{M^*(\omega)} \le \hat{\sigma}(u,u^*,\omega)e^{M^*(\omega)},\end{aligned}$$

$$|q^{u'}(t,\omega)-q^u(t,\omega)|\le 2\hat\sigma(u,u',\omega)e^{M^*(\omega)}, \tag{12}$$

to obtain the last inequality, note that $|f(t,x^*(t,\omega),u'(t,\omega),\omega)-f(t,x^*(t,\omega),u^*(t,\omega),\omega)-[f(t,x^*(t,\omega),u(t,\omega),\omega)-f(t,x^*(t,\omega),u^*(t,\omega),\omega)]|\le 2\max\{M(t,\omega),M_*(t,\omega)\}1_{H_{u',u}}$, to obtain the first inequality we have actually used a continuation argument yielding $|x^u(t,\omega)-x^*(t,\omega)|<\check d$ in case of Remark 2, and $|\pi' x^u(t,\omega)-\pi' x^*(t,\omega)|<\check d$, $|\pi'' x^u(t,\omega)-\pi'' x^*(t,\omega)|<M_{**}(\omega)$ in case of Remark 3.

Some further properties will now be proved.

Proof of (13) below. Consider the expression

$$\delta^*(t):=f(t,x^u(t)+q^{u'}(t)-q^u(t),u'(t))-f(t,x^*(t),u'(t))-f(t,x^u(t),u(t))+f(t,x^*(t),u(t))-f_x(t,x^*(t),u^*(t))[q^{u'}(t)-q^u(t)].$$

Here, and many places below, we have dropped writing ω. Now, on $\complement(H_{u,u^*}\cup H_{u',u^*})$, $|\delta^*(t)|=$

$$|f(t,x^u(t)+q^{u'}(t)-q^u(t),u^*(t))-f(t,x^u(t),u^*(t))-f_x(t,x^*(t),u^*(t))[q^{u'}(t)-q^u(t)]|\le e(t,x^u(t),q^{u'}(t)-q^u(t))|q^{u'}(t)-q^u(t)|$$

where

$$e(t,x,y,\omega):=\sup_{\lambda\in[0,1]}|f_x(t,x+\lambda y,u^*(t,\omega),\omega))-f_x(t,x^*(t,\omega),u^*(t,\omega),\omega)|$$

Then, by (12), for $\hat e(t,u',u):=e(t,x^u(t),q^{u'}(t)-q^u(t))$, on $\complement(H_{u,u^*}\cup H_{u',u^*})$,

$$|\delta^*(t)|\le \hat e(t,u',u)|q^{u'}(t)-q^u(t)|\le 2\hat e(t,u',u)\hat\sigma(u',u)e^{M^*}.$$

Moreover, by (12) and (i) in Remark 2, (and the existence of $M'_*(t,\omega)$ in Remark 3), on $(H_{u,u^*}\cup H_{u',u^*})\cap\complement H_{u',u}$,

$$|\delta^*(t)|:=|f(t,x^u(t)+q^{u'}(t)-q^u(t),u(t))-f(t,x^u(t),u(t))-f_x(t,x^*(t),u^*(t))[q^{u'}(t)-q^u(t)]|\le 2M'_*(t)|q^{u'}(t)-q^u(t)|\le 4M'_*(t)e^{M^*}\hat\sigma(u',u).$$

Finally, using $|x^u(t)+q^{u'}(t)-q^u(t)-x^*(t)|\le|x^u(t)-x^*(t)|+|q^{u'}(t)|+|q^u(t)|\le(2\hat\sigma(u,u^*)+\hat\sigma(u',u^*))e^{M^*}$, see (12), by (i) in Remark 2, (and the existence of $M'_*(t,\omega)$ in Remark 3), by (12), on $(H_{u,u^*}\cup H_{u',u^*})\cap H_{u',u}$,

$$|\delta^*(t)| \le M'_*(t)|x^u(t) + q^{u'}(t) - q^u(t) - x^*(t)| + M'_*(t)|x^u(t) - x^*(t)| + M'_*(t)|q^{u'}(t) - q^u(t)| \le M'_*(t)(4\hat{\sigma}(u,u^*) + 2\hat{\sigma}(u',u^*))e^{M^*},$$

(here $|q^{u'}(t) - q^u(t)| \le |q^{u'}(t)| + |q^u(t)|$ is also used). Hence, $|\delta^*(t)| \le$

$$2\hat{e}(t,u',u)e^{M^*}\hat{\sigma}(u',u)1_{\complement(H_{u,u^*}\cup H_{u',u^*})} + 4M'_*(t)e^{M^*}\hat{\sigma}(u',u)1_{(H_{u,u^*}\cup H_{u',u^*})\cap\complement H_{u',u}} + M'_*(t)(4\hat{\sigma}(u,u^*) + 2\hat{\sigma}(u',u^*))e^{M^*}1_{(H_{u,u^*}\cup H_{u',u^*})\cap H_{u',u}}.$$

Applying Gronwall's inequality (see Appendix) to the equation $\dot{x} = h(t,x) = f(t,x,u'(t))$, for $\check{z}(t) = x^{u'}(t)$ and $\check{y}(t) = x^u(t) + q^{u'}(t) - q^u(t)$, (with $\int_0^t(-d\check{y}(s)/ds)ds + \int_0^t f(s,\check{y}(s),u'(s))ds = \int_0^t \delta^*(s)ds$), gives

$$|x^{u'}(t,\omega) - (x^u(t,\omega) + q^{u'}(t,\omega) - q^u(t,\omega))| \le e^{2M^*(\omega)}\{\int_J 2\hat{e}(t,u',u,\omega)\hat{\sigma}(u',u,\omega)dt + \int_J 4M'_*(t,\omega)\hat{\sigma}(u',u,\omega)1_{(H_{u,u^*}\cup H_{u',u^*})}(t,\omega)dt + \int_J M'_*(t,\omega)(4\hat{\sigma}(u,u^*,\omega) + 2\hat{\sigma}(u',u^*,\omega))1_{H_{u',u}}(t,\omega)dt\}. \tag{13}$$

Until further notice only the case of Remark 2 is now treated. From now on, let $u, u' \in A^d$, $d \le \tilde{d}$. Then

$$|x^{u'}(t,\omega) - (x^u(t,\omega) + q^{u'}(t,\omega) - q^u(t,\omega))| \le e^{2M^*(\omega)}[\int_J 2\hat{e}(t,u',u,\omega)dt + 8d + 6d]\check{\sigma}(u',u).$$

By (ii) in Remark 2, and inequalities obtained above, $\hat{e}(t,u',u,\omega)dt \le \check{e}_0(3de^{M^*(\omega)},t)$, so (by $M^*(\omega) \le M^{**}$), $\int_J 2\hat{e}(t,u',u,\omega)dt \le \hat{e}_1(d)$. Here and below, $\hat{e}(d)$ -functions, with various subscripts, are error functions, independent of ω. Hence,

$$|x^{u'}(t,\omega) - (x^u(t,\omega) + q^{u'}(t,\omega) - q^u(t,\omega))| \le e^{2M^{**}}[\hat{e}_1(d) + 14d]\check{\sigma}(u',u) =: \check{\sigma}(u',u)\hat{e}_2(d). \tag{14}$$

Proof of (16),(17) below. Define $\delta^{**}(t) =$

$$f(t,x^{u'}(t),u'(t)) - f(t,x^*(t),u'(t)) - f(t,x^u(t),u(t)) + f(t,x^*(t),u(t)) - f_x(t,x^*(t),u^*(t))[q^{u'}(t) - q^u(t)].$$

Then, $|\pi\delta^{**}(t) - \pi\delta^*(t)| \le |\pi f(t,x^{u'}(t),u'(t)) - \pi f(t,x^u(t) + q^{u'}(t) - q^u(t),u'(t))| \le M'_\pi|x^{u'}(t) - (x^u(t) + q^{u'}(t) - q^u(t))| \le M'_\pi\hat{e}_2(d)\check{\sigma}(u',u)$. Define $\hat{e}^3(d,t,\omega) =$

$$\sup_{y\in B(x^*(t,\omega),d)} |\pi f_x(t,y,u^*(t,\omega),\omega) - \pi f_x(t,x^*(t,\omega),u^*(t,\omega),\omega)|,$$

(an error function in d, uniformly in (t,ω), due to (iii) in Remark 2). Similar to what was obtained above, (see the last inequality for $|\delta^*(t)|$),

$$|\pi\delta^*(t)| \le \hat{e}_3(d)e^{M^{**}}\check{\sigma}(u',u)1_{\complement(H_{u,u^*}\cup H_{u',u^*})}+4M'_\pi e^{M^{**}}\check{\sigma}(u',u)1_{(H_{u,u^*}\cup H_{u',u^*})\cap\complement H_{u',u}}+ M'_\pi(4\check{\sigma}(u,u^*)+2\check{\sigma}(u',u^*))e^{M^{**}}1_{(H_{u,u^*}\cup H_{u',u^*})\cap H_{u',u}},$$

where $\hat{e}_3(d) := 2\mathrm{essup}_\omega \sup_t \hat{e}^3(3de^{M^{**}},t,\omega)$. Thus,

$$|\pi\delta^{**}(t)| \le M'_\pi\hat{e}_2(d)\check{\sigma}(u',u)+\hat{e}_3(d)e^{M^{**}}\check{\sigma}(u',u)+ 4M'_\pi e^{M^{**}}\check{\sigma}(u',u)1_{H_{u,u^*}\cup H_{u',u^*}}+6M'_\pi e^{M^{**}}d1_{H_{u',u}}. \tag{15}$$

Define $I_i := (1-1/2^i, 1-1/2^{i+1}], i=0,1,...,\tilde{\sigma}(u,u') :=$essup$_\omega \int_J 1_{H_{u',u}}(t,\omega)dt$, $\sigma^*(u',u) := \sup_i 2^{i+1}\tilde{\sigma}(u'1_{I_i},u1_{I_i})$, and $\sigma(u',u) = \max\{\sigma^*(u',u),\check{\sigma}(u',u)\}$. Let $A_d := \{u \in U^K : \sigma(u,u^*) < d\} \subset A^d$. Let $\tilde{x} := (x_0,x_1,...), x_i \in X$, $\pi^\infty(\tilde{x}) := (\pi x_0, \pi x_1, ...)$. Let $\hat{L}_\infty$ be the subspace of $L_\infty(\Omega,\Phi_{1-1/2^1},Y) \times L_\infty(\Omega,\Phi_{1-1/2^2},Y)\times ...$ consisting of elements $z(\omega) = (z_0(\omega), z_1(\omega), ...)$, $z_i(.) \in L_\infty(\Omega,\Phi_{1-1/2^{i+1}},Y)$, for which the norm ${}^\infty|z(.)| := \sup_i 2^i|z_i(.)|_\infty$ is finite. Moreover, let $y_i^u(\omega) := \int_{I_i}\dot{x}^u(t,\omega)dt, y^u(\omega) := (y_0^u(\omega), y_1^u(\omega), ...)$, $q_i^u(\omega) := \int_{I_i}\dot{q}^u(t,\omega)dt$, $\hat{q}^u(\omega) := (q_0^u(\omega), q_1^u(\omega), ...)$.

Assume that $u,u' \in A_d$. Then, by (15) and $2^{n+1}\tilde{\sigma}(u'1_{I_i},u^*1_{I_i}) \le \sigma^*(u',u^*) < d$, $\int_J 1_{I_i}|\pi\delta^{**}(t)|dt \le$

$$M'_\pi\hat{e}_2(d)\check{\sigma}(u',u)/2^{i+1}+\hat{e}_3(d)e^{M^{**}}\check{\sigma}(u',u)/2^{i+1}+4M'_\pi e^{M^{**}}\check{\sigma}(u',u)[\tilde{\sigma}(u'1_{I_i},u^*1_{I_i})+ \tilde{\sigma}(u1_{I_i},u^*1_{I_i})]+M'_\pi 6e^{M^{**}}d\tilde{\sigma}(u'1_{I_i},u1_{I_i}) \le \hat{e}_4(d)\sigma(u',u)/2^{i+1}+ 8M'_\pi e^{M^{**}}\sigma(u',u)d/2^{i+1}+M'_\pi 6e^{M^{**}}d\tilde{\sigma}(u'1_{I_i},u1_{I_i}).$$

Hence, $2^{i+1}\int_J 1_{I_i}|\pi\delta^{**}(t)|dt \le \hat{e}_5(d)\sigma(u',u)+2^{i+1}M'_\pi 6e^{M^{**}}d\tilde{\sigma}(u'1_{I_i},u1_{I_i}) \le$

$$\hat{e}_5(d)\sigma(u',u)+M'_\pi 6e^{M^{**}}d\sigma^*(u',u) \le \hat{e}_6(d)\sigma(u',u).$$

Thus, for any $u,u' \in A_d$,

$${}^\infty|\pi^\infty y^{u'}(.)-\pi^\infty y^u(.)-[\pi^\infty\hat{q}^{u'}(.)-\pi^\infty\hat{q}^u(.)]| \le \hat{e}_6(d)\sigma(u',u)/2 \qquad (16).$$

From (14), we obtain:

$$E|a\cdot\{x^{u'}(1,\omega)-x^u(1,\omega)-[q^{u'}(1,\omega)-q^u(1,\omega)]\}| \le \hat{e}_7(d)\sigma(u',u). \tag{17}$$

Moreover, (12) yields that $u \to (E[a\cdot q^u(1,\omega)], \pi^\infty\hat{q}^u(.))$ is continuous in

$\sigma-$ metric on U^K, to see the continuity of $u \to \pi^\infty \hat{q}^u(.)$ (in $\sigma-$ metric, ${}^\infty|.|$-norm), note that, in a shorthand notation, a.s., $2^i|\int_{I_i}(\pi\dot{q}^{u'}(t) - \pi\dot{q}^u(t))dt| \leq$

$$2^i|\int_{I_i}\{\pi f_x[q^{u'}(t) - q^u(t)] + \pi f(t, x^*(t), u'(t)) - \pi f(t, x^*(t), u(t))\}dt| \leq$$
$$2^i\int_{I_i} M'_\pi|q^{u'}(t) - q^u(t)|dt + 2^i\int_{I_i} K1_{H_{u',u}}dt \leq M'_\pi e^{M^{**}}\check{\sigma}(u', u)+$$
$$2^i K\tilde{\sigma}(u'1_{I_i}, u1_{I_i}) \leq M'_\pi e^{M^{**}}\sigma(u', u) + K\sigma(u', u)/2.$$

In case of Remark 3, define $\sigma(u, u') := \max\{\sigma_*(u, u'), \sigma^*(u, u'), \check{\sigma}(u, u')\}$, where $\sigma_*(u, u') := |\sigma_{**}(u, u', .)|_{2q}|, \sigma_{**}(u, u', .) = \int_J \max\{M'_*(t, \omega)M_*(t, \omega)1_{H_{u,u'}}(t, \omega)dt$, ($A_d$ now corresponds to this σ). In the present case, πf does not depend on x'', so the same arguments work to show (16) and continuity of $u \to \pi^\infty \hat{q}^u(.)$ in σ-metric.

Furthermore, note that when $u, u' \in A_d$ run through sequences u_n, u'_n such that $\hat{\sigma}(u_n, u^*, \omega) \to 0, \hat{\sigma}(u'_n, u^*, \omega) \to 0$, then for each (t, ω), $e(t, x^u(t, \omega), q^{u'}(t, \omega) - q^u(t), \omega) = \hat{e}(t, u', u, \omega) \to 0$. Now, $\vartheta_n(t, \omega) := e^{2M^*(\omega)}e(t, x^{u_n}(t, \omega), q^{u'_n}(t, \omega) - q^{u_n}(t), \omega)$ is bounded by $2e^{2M^*(\omega)}M'_*(t, \omega)$, so by properties stated in Remark 3, $|\vartheta_n(t, \omega)|_p \to 0$. Note that $|\alpha\beta\gamma|_1 \leq |\alpha|_p|\beta\gamma|_q \leq |\alpha|_p|\beta|_{2q}\gamma|_{2q}$, and $\hat{\sigma} \leq \check{\sigma} + \sigma_*$. Using the last inequalities, as well as $\tilde{\sigma}(u', u) \leq \sum \tilde{\sigma}(u'1_{I_i}, u1_{I_i}) \leq \sigma^*(u', u)$, by (13), for some $\hat{e}_8(d)$,

$$|x^{u'}(t, \omega) - (x^u(t, \omega) + q^{u'}(t, \omega) - q^u(t, \omega))|_1 \leq \hat{e}_8(d)|\hat{\sigma}(u', u, .)|_q+$$
$$4|e^{2M^*(.)}|_p|\hat{\sigma}(u', u, .)|_{2q}|\sigma_{**}(u, u^*, .)+\sigma_{**}(u', u^*, .)|_{2q}+|e^{2M^*(.)}|_p|4\hat{\sigma}(u, u^*, .)+$$
$$2\hat{\sigma}(u', u^*, .)|_{2q}|\sigma_{**}(u', u, .)|_{2q} \leq [\hat{e}_8(d) + 14d|e^{2M^*(.)}|_p]2\sigma(u', u). \quad (18)$$

Evidently, (18) implies (17).

Proof of (30),(31) below:

Two lemmas are needed.

Lemma 1. Let $g \in L_1(J\times\Omega, X)$ be progressively measurable. For any $\varepsilon > 0$, there exists a function $h(t, \omega) := \sum_{k=0}^\infty g(t^k, \omega)1_{[\tau^k(\omega), \hat{\tau}^k(\omega))}(t)$, with $t^k \leq \tau^k(\omega) \leq \hat{\tau}^k(\omega) \leq \tau^{k+1}(\omega)$ for all ω, $g(t^k, .) \in L_1(\Omega, \Phi_{t^k}, X)$, $1_{[\tau^k(\omega), \hat{\tau}^k(\omega))}(t)$ progressively measurable, such that $\int_J |g(t, \omega) - h(t, \omega)|dt < \varepsilon$ a.s., with $\lim_{k\to\infty} \tau^k(\omega) = 1$ a.s.

Proof: Let $\varepsilon > 0$. By Dunford and Schwartz, lemma III.11.16, $g(t, .) \in L_1(J, L_1(\Omega, X))$ a.e. For each $\varepsilon' > 0$, there exists a piecewise constant function $a(t, \omega) = \sum_{j=0}^{j^*} a_j(\omega)1_{[t_j, t_{j+1})}(t)$, $t_0 = 0, t_{j^*+1} = 1$, such that $\int_J |g(t, .) - a(t, .)|_1 dt < \varepsilon'^2/2$. Thus, there exists an open set $A \subset J$, such that meas$(A) <$

ε', and A$\supset$ $A_0 := \{t : |g(t,.) - a(t,.)|_1 > \varepsilon'/2\}$, (note that meas$(A_0) < \varepsilon'$, otherwise the inequality involving $\varepsilon'^2/2$ is contradicted). Let $B = \complement A$, and let $s_j := \min B \cap [t_j, t_{j+1})$, if $j \in \Gamma := \{j : B \cap [t_j, t_{j+1}) \neq 0\}$. For $j \in \Gamma$, $|a_j(.) - g(s_j,.)|_1 \leq \varepsilon'/2$, so for $j \in \Gamma, t \in B \cap [t_j, t_{j+1})$, we have $|g(t,.) - g(s_j,.)|_1 \leq |g(t,.) - a_j(.)|_1 + |a_j(.) - g(s_j,.)|_1 \leq \varepsilon'$. Define $b(t,.) := \sum_{j\in\Gamma} g(s_j,.)1_{[s_j,t_{j+1})}$. Then, if $t \in B$, and $t < 1$, then for some j, $t \in [t_j, t_{j+1})$, so for this j, $j \in \Gamma$, $t \in [s_j, t_j)$ and $|g(t,.) - b(t,.)|_1 = |g(t,.) - g(s_j,.)|_1 < \varepsilon'$.

Assume now that ε' is so small that meas$(C) < \varepsilon' \Rightarrow \int_C |g|_1 dt < \varepsilon^2/4$. Then

$$\int_J |g(t,.) - b(t,.)|_1 dt = \int_B |g(t,.) - b(t,.)|_1 dt + \int_A |g(t,.) - b(t,.)|_1 dt \leq$$
$$\varepsilon' + \int_A |g(t,.) - a(t,.)|_1 dt + \int_A |a(t,.) - b(t,.)|_1 dt \leq \varepsilon' + \varepsilon'^2/2 +$$
$$\int_A \sum_{j\in\Gamma} |a_j(.) - g(s_j,.)|_1 1_{[s_j,t_{j+1})}(t) dt + \int_A \sum_{j\in\Gamma} |a_j(.)|_1 1_{[t_j,s_j)}(t) dt +$$
$$\int_A \sum_{j\notin\Gamma} |a_j(.)|_1 1_{[t_j,t_{j+1})}(t) dt \leq \varepsilon' + \varepsilon'^2/2 + \varepsilon'/2 + \int_A |a(t,.)|_1 dt \leq$$
$$\varepsilon' + \varepsilon'^2/2 + \varepsilon'/2 + \int_A |a(t,.) - g(t,.)|_1 dt + \int_A |g(t,.)|_1 dt \leq \varepsilon' + 2\varepsilon'^2/2 + \varepsilon'/2 + \varepsilon^2/4.$$

Hence, there exists a function $h_1(t,\omega) = \sum_i g(t_i^1, \omega) 1_{[t_i^1, r_i^1)}(t)$,(finite sum) , with $t_i^1 \leq r_i^1 \leq t_{i+1}^1$, such that, in a shorthand notation, $\int_\Omega \int_J |g - h_1| dt dP < \varepsilon^2/2$. Then $\Omega_1 := \{\omega : \int_J |g(t,\omega - h_1(t,\omega)| dt > \varepsilon/2\}$ satisfies $P(\Omega_1) < \varepsilon$, (otherwise the inequality involving $\varepsilon^2/2$ is contradicted). Define $S_1 := \{(t,\omega) : \int_0^t |g(s,\omega - h_1(s,\omega)| ds > \varepsilon/2\}$. For some function $\tau_1(\omega), S_1 = \{(t,\omega) : t \in (\tau_1(\omega), 1]\}$. Evidently, S_1 (i.e. $1_{S_1}(t,\omega)$) is progressively measurable, and $\{\omega : \tau_1(\omega) < 1\} = \{\omega : (t,\omega) \in S_1$ for some $t\} = \Omega_1$. Let $\tau_0(\omega) := 0$.

In the above construction, (as a second step), replace ε by $\varepsilon/2$ and g by $g_1 := g 1_{[\tau_1(\omega),1]}(t)$. Then, for some function $h_2(t,\omega) := \sum_i g_1(t_i^2, \omega) 1_{[t_i^2, r_i^2)}(t)$, $t_i^2 \leq r_i^2 \leq t_{i+1}^2$, $\int_\Omega \int_J |g_1 - h_2| dt dP < \varepsilon^2/8$. So, $\Omega_2 := \{\omega : \int_J |g_1(t,\omega) - h_2(t,\omega)| dt > \varepsilon/4\}$ satisfies $P(\Omega_2) < \varepsilon/2$. Write $S_2 := \{(t,\omega) : \int_0^t |g_1(s,\omega) - h_2(s,\omega)| ds > \varepsilon/4\} =: \{(t,\omega) : t \in (\tau_2(\omega), 1]\}$. Evidently, S_2 is progressively measurable, and $\{\omega : \tau_2(\omega) < 1\} = \{\omega : (t,\omega) \in S_2$ for some $t\} = \Omega_2$. Finally, $S_2 \subset S_1$, ($g_1 = h_2 = 0$ for $t \in [0, \tau_1(\omega))$, i.e. $\tau_2(\omega) > \tau_1(\omega)$).

By induction, (replacing ε by $\varepsilon/2^{j-1}$ and g by $g_{j-1} := g 1_{[\tau_{j-1}(\omega),1]}(t)$ at step j), functions $h_j(t,\omega) = \sum_i g_{j-1}(t_i^j, \omega) 1_{[t_i^j, r_i^j)}$, $t_i^j \leq r_i^j \leq t_{i+1}^j$, can be constructed, such that $\int_\Omega \int_J |g_{j-1} - h_j| dt dP < (\varepsilon/2^{j-1})^2/2$, hence such that $\Omega_j := \{\omega : \int_J |g_{j-1}(t,\omega) - h_j(t,\omega)| dt > \varepsilon/2^j\}$ satisfies $P(\Omega_j) < \varepsilon/2^{j-1}$, with $S_j := \{(t,\omega) : \int_0^t |g_{j-1}(s,\omega) - h_j(s,\omega)| ds > \varepsilon/2^j\} =: \{(t,\omega) : t \in (\tau_j(\omega), 1]\}$, ($S_j$ progressively measurable), and $\{\omega : \tau_j(\omega) < 1\} = \{\omega : (t,\omega) \in S_j$ for some $t\} = \Omega_j$. Finally, $S_j \subset S_{j-1}$, ($g_{j-1} = h_j = 0$ on $[0, \tau_{1-1}(\omega))$,

i.e. $\tau_j(\omega) > \tau_{j-1}(\omega)$), so $\Omega_j \subset \Omega_{j-1}$. Evidently, for any j, for any ω, $\int_{\tau_{j-1}(\omega)}^{\tau_j(\omega)} |g(t,\omega) - h_j(t,\omega)|dt = \int_{\tau_{j-1}(\omega)}^{\tau_j(\omega)} |g_{j-1}(t,\omega) - h_j(t,\omega)|dt \leq \varepsilon/2^j$. Let $h = \sum_{j=1}^{\infty} h_j 1_{[\tau_{j-1}(\omega),\tau_j(\omega))}$ and let $\omega \notin \cap_j \Omega_j$, (this intersection being a null set). Then, for some $j = j^*, \omega \in \complement\Omega_{j^*}$, i.e. $\tau_{j^*}(\omega) = 1$. Hence, $\int_J |g(t,\omega) - h(t,\omega)|dt = \sum_{j \leq j^*} \int_{\tau_{j-1}(\omega)}^{\tau_j(\omega)} |g(t,\omega) - h_j(t,\omega)|dt \leq \sum_{1 \leq j \leq j^*} \varepsilon/2^j \leq \varepsilon$, for such an ω.
Note that $h = \sum_j (\sum_i g(t_i^j,\omega) 1_{[\tau_{j-1}(\omega),1]} 1_{[t_i^j, r_i^j)}) 1_{[\tau_{j-1}(\omega),\tau_j(\omega))} =$
$\sum_{j,i} g(t_i^j,\omega) 1_{[\tau^{i,j}(\omega),\hat{\tau}^{i,j}(\omega))}$, where $\tau^{i,j}(\omega) = \max\{t_i^j, \tau_{j-1}(\omega)\}$, $\hat{\tau}^{i,j}(\omega) = \min\{r_i^j, \tau_j(\omega)\}$, so h is of the form described in the lemma.

Lemma 2 Let $g \in L_1(J \times \Omega, X)$ be progressively measurable and let $\tilde{k} \in (0,1)$. Then, for each $\varepsilon > 0$, there exists a progressively measurable set $C \subset J \times \Omega$, such that $|\tilde{k} \int_0^s g(t,\omega)dt - \int_0^s g(t,\omega) 1_C(t,\omega)dt| < \varepsilon$ for all s, a.s.

Proof. Apply Lemma 1 to obtain

$$\begin{aligned} &\text{essup}_\omega \int_J |g(t,\omega) - h(t,\omega)|dt < \varepsilon/4, \text{ for} \\ &h(t,\omega) = \sum_{k=0}^{\infty} a_k(\omega) 1_{[\tau_k(\omega),\tau_{k+1}(\omega))}(t), \end{aligned} \tag{19}$$

where $a_k(\omega) \in L_\infty(\Omega, \Phi_{t_k}, X)$, $t_k \leq \tau_k(\omega) \leq \tau_{k+1}(\omega)$, ($t_k$ independent of ω), $\tau_0(\omega) = 0$, $\lim_k \tau_k(\omega) = 1$ a.s., $1_{[\tau_k(\omega),\tau_{k+1}(\omega))}(t)$ progressively measurable. We need a subdivision finer than $\{\tau_k(\omega)\}$. Now, a.s. $\int_J |g(t,\omega)|dt = K_\omega < \infty$. For each k, by induction on j, define sets $S^{k,j} := \{(t,\omega) : t \in (\tau_k(\omega) + \xi^{k,j}(\omega), \tau_k(\omega) + \xi^{k,j+1}(\omega)]$ such that $S^{k,j} = \{(t,\omega) : \int_{\tau_k+\xi^{k,j}}^{t} |g(s,\omega)|ds \leq \varepsilon/4\}$, with $\xi^{k,0}(\omega) = 0$. After a finite number of steps, say $j(\omega)$ ($\leq 1 + (\varepsilon/4)^{-1} K_\omega$), $\tau_k(\omega) + \xi^{k,j}(\omega)$ becomes $\geq \tau_{k+1}(\omega) \leq 1$ a.s., in which case $\tau_k(\omega) + \xi^{k,j}(\omega)$ is replaced by $\tau_{k+1}(\omega)$ and we stop. Evidently, by induction, $S^{k,j}$ becomes progressively measurable.

Having made this observation, it is evident that we may simply assume that the points $\tau_k(\omega)$ in (19) have the additional property that, a.s.,

$$\int_{\tau_k(\omega)}^{\tau_{k+1}(\omega)} |g(t,\omega)|dt \leq \varepsilon/4. \tag{20}$$

Define $\phi(t,\omega) := \sum_k 1_{[\tau_k(\omega),(1-\tilde{k})\tau_k(\omega)+\tilde{k}\tau_{k+1}(\omega))}(t)$. Now,

$$\int_{\tau_k(\omega)}^{\tau_{k+1}(\omega)} h(t,\omega)\phi(t,\omega)dt = \int_{\tau_k(\omega)}^{\tau_{k+1}(\omega)} a_k(\omega) 1_{[\tau_k(\omega),(1-\tilde{k})\tau_k(\omega)+\tilde{k}\tau_{k+1}(\omega))}(t)dt =$$

$$\tilde{k}a_k(\omega)(\tau_{k+1}(\omega)-\tau_k(\omega)) = \tilde{k}\int_{\tau_k(\omega)}^{\tau_{k+1}(\omega)} a_k(\omega)dt = \tilde{k}\int_{\tau_k(\omega)}^{\tau_{k+1}(\omega)} h(t,\omega)dt. \quad (21)$$

Hence, for any given k^*,

$$\int_0^{\tau_{k^*}(\omega)} h(t,\omega)\phi(t,\omega)dt = \sum_{k<k^*}\int_{\tau_k(\omega)}^{\tau_{k+1}(\omega)} h(t,\omega)\phi(t,\omega)dt = \sum_{k<k^*}\tilde{k}\int_{\tau_k(\omega)}^{\tau_{k+1}(\omega)} h(t,\omega)dt = \tilde{k}\int_0^{\tau_{k^*}(\omega)} h(t,\omega)dt. \quad (22)$$

Moreover, by (22) and (19), a.s.,

$$|\int_0^{\tau_{k^*}(\omega)} g(t,\omega)\phi(t,\omega)dt - \tilde{k}\int_0^{\tau_{k^*}(\omega)} g(t,\omega)dt| \leq |\int_0^{\tau_{k^*}(\omega)} h(t,\omega)\phi(t,\omega)dt - \tilde{k}\int_0^{\tau_{k^*}(\omega)} h(t,\omega)dt| + |\int_0^{\tau_{k^*}(\omega)}(g(t,\omega)-h(t,\omega))\phi(t,\omega)dt| + \tilde{k}|\int_0^{\tau_{k^*}(\omega)}(h(t,\omega)-g(t,\omega))dt| < 2\varepsilon/4 = \varepsilon/2 \quad (23)$$

Finally, for any given t, if $k^* = k^*(\omega)$ is the largest k such that $\tau_k(\omega) \leq t$, then, by (20), a.s.,

$$|\int_{\tau_{k^*}(\omega)}^t g(t,\omega)\phi(t,\omega)dt| \leq \varepsilon/4,\ |\tilde{k}\int_{\tau_{k^*}(\omega)}^t g(t,\omega)dt| \leq \varepsilon/4. \quad (24).$$

The conclusion of Lemma 2 then follows from (23) and (24).

For any $\tilde{u}(.,.)$, write $f(t,\tilde{u}(.,.),.) := f(t,x^*(t,.),\tilde{u}(.,.),.)$. Let $u'', u \in U^K, k \in (0,1)$. Then, for any $\rho > 0$, for some $\hat{u}(t,\omega) \in U'$, for all s, a.s., $b(\hat{u},s,\omega) :=$

$$|\int_{[0,s]}\{f(t,\hat{u}(t,\omega),\omega) - kf(t,u''(t,\omega),\omega) - (1-k)f(t,u(t,\omega),\omega)\}dt| < \rho \quad (25)$$

To prove (25), apply Lemma 2 to obtain, for $\hat{u} = u''1_C + u(1-1_C)$, that, a.s., $b(\hat{u},s,\omega) = |k\int_0^s(f(t,u''(t,\omega),\omega) - f(t,u(t,\omega),\omega))dt - \int_0^s(f(t,u''(t,\omega),\omega) - f(t,u(t,\omega),\omega))1_C dt| < \rho$, from which (25) follows.

Now, replacing $[0,s]$ by $I_m \cap [0,s]$ and ρ by $\rho/2^{m+1}$ in (25), denoting the corresponding subset C by $C^m \subset I_m$, and dropping writing ω, we get, a.s., for all s,

$$|\int_0^s 1_{I_m}\{f(t,\hat{u}(t)) - [kf(t,u''(t)) + (1-k)f(t,u(t))]\}dt|_\infty < \rho/2^{m+1}, \quad (26)$$

where $\hat{u}$ on I_m is defined by $\hat{u} := u''1_{C^m} + u(1-1_{C^m})$. Also, for this $\hat{u}(t,\omega)$, (25) holds.

Now, (25), combined with Gronwall's inequality, give, a.s.,

$$\sup_t |q^{\hat{u}}(t) - [kq^{u''}(t) + (1-k)q^{u'}(t)]| < e^{M^*}\rho. \quad (27)$$

Both in case of Remarks 2 and 3, (27) holds, until further notice, let us now restrict attention to Remark 2. From (27) it follows that, for all t,

$$\text{essup}_\omega |\pi f_x(t, x^*(t), u^*(t))\{q^{\hat{u}}(t) - [kq^{u''}(t) + (1-k)q^u(t)]\}| \le M'_\pi e^{M^{**}}\rho,$$

so

$$\text{essup}_\omega \int_J 1_{I_m} |\pi f_x(t, x^*(t), u^*(t))\{q^{\hat{u}}(t) - [kq^{u''}(t) + (1-k)q^u(t)]\}| dt \le M'_\pi e^{M^{**}}\rho/2^{m+1}.$$

Combining this with (26) gives, a.s., for all s,

$$|\int_0^s 1_{I_m}\pi\{\dot{q}^{\hat{u}}(t) - [k\dot{q}^{u''}(t) + (1-k)\dot{q}^u(t)]\}dt| \le \rho(|\pi| + M'_\pi e^{M^{**}})/2^{m+1}. \quad (28)$$

By (27),

$$E|a \cdot \{q^{\hat{u}}(1) - [kq^{u''}(1) + (1-k)q^u(1)]\}dt| \le |a||e^{M^*}|_1\rho. \quad (29)$$

Now, if $u'' \in A_{d''}$, $u \in A_d$, for $\hat{d}'' := \sigma(u'', u^*) < d''$, $\hat{d} := \sigma(u, u^*) < d$, then, for each m there exist sets D''_m and D_m in $I_m \times \Omega$ such that $D''_m \supset \{(t,\omega) : t \in I_m, u''(t,\omega) \ne u^*(t,\omega)\}, D_m \supset \{(t,\omega) : t \in I_m, u(t,\omega) \ne u^*(t,\omega)\}$, $\text{essup}\int_{I_m} 1_{D''_m} dt \le \hat{d}''/2^{m+1}$, $\text{essup}\int_{I_m} 1_{D_m} dt \le \hat{d}/2^{m+1}$, essup $\int_J M(t)(\sum 1_{D''_m})dt \le \hat{d}'' < d''$, essup $\int_J M(t)(\sum 1_{D_m})dt \le \hat{d} < d$. Let $D = \cup_m D_m$, $D'' = \cup_m D''_m$, $C = \cup C^m$. Assume that we had carried out the construction in (26), for $\rho < k(d'' - \hat{d}'')$ and the functions $f(t, u''(t))$ and $f(t, u(t))$ replaced by $((1 + M(t))1_{D''}, (1 + M(t))(1_{D''} + 1_D), f(t, u''(t)))$ and $((1 + M(t))1_D, 0, f(t, u(t))$, respectively. Then, a.s.,
$\sup_s|\int_0^s (1+M(t))1_{I_m}\{1_{D''}1_{C^m} + 1_D(1-1_{C^m}) - [k1_{D''} + (1-k)1_D]\}dt| \le \rho/2^{m+1}$, so, a.s., both $|\int_J 1_{I_m}\{1_{D''}1_{C^m} + 1_D(1 - 1_{C^m})\}dt| \le \rho/2^{m+1} + [k\hat{d}'' + (1 - k)\hat{d}]/2^{m+1} < [kd'' + (1-k)d]/2^{m+1}$ and (summing over m), $\sup_s|\int_0^s (1 + M(t))\{1_{D''}1_C + 1_D(1 - 1_C) - [k1_{D''} + (1-k)1_D]\}dt| \le \rho$, and hence, a.s., $|\int_J M(t)\{1_{D''}1_C + 1_D(1 - 1_C)\}dt| \le \rho + [k\hat{d}'' + (1-k)\hat{d}] < [kd'' + (1-k)d]$.

Similarly, a.s.,
$\sup_s|\int_0^s 1_{I_m}\{(1+M(t))(1_{D''}+1_D)1_{C^m} - k(1+M(t))(1_{D''}+1_D)\}dt| \le \rho/2^{m+1}$, so, a.s.,$|\int_{I_m}(1_{D''} + 1_D)1_{C^m}dt| \le \rho/2^{m+1} + k(\hat{d}'' + \hat{d})/2^{m+1} < k(d'' + d)/2^{m+1}$. Summing over m, the first inequality gives, a.s., $|\int_J M(t)(1_{D''} + 1_D)1_{C^m}dt| <$

$\rho + k(\hat{d}'' + \hat{d}) < k(d'' + d)$.

Note that, for $t \in I_m$, $\hat{u}(t,\omega) \neq u^*(t,\omega) \Rightarrow (t,\omega) \in (D'' \cap C^m) \cup (D \cap \complement C^m)$, so, for $d'', d \leq K$, $\hat{u} \in A_{kd''+(1-k)d}$, moreover, $\hat{u}(t,\omega) \neq u(t,\omega) \Rightarrow (t,\omega) \in (D'' \cup D) \cap C^m$, so $\sigma(\hat{u}, u) \leq k(d'' + d)$.

Similar arguments work in case of Remark 3: Since $\pi' f$ does not depend on x'', (27) holds for q, M^* replaced by $\pi' q, M^{**}$, and this suffices for (28) to hold in this case. Now, (27) also holds as written, which again implies (29). The sets D''_m and D_m are now chosen to satisfy also $|\int_J M_*(t)1_{D''}dt|_q \leq \hat{d}''$,—$\int_J M_*(t)1_D dt|_q \leq \hat{d}$. In this case the sets C^m are so chosen that even, a.s., $\sup_s |\int_0^s M_*(t)1_{I_m}\{1_{D''}1_{C^m} + 1_D(1-1_{C^m}) - [k1_{D''} + (1-k)1_D]\}dt| \leq \rho/2^{m+1}$ and $\sup_s |\int_0^s 1_{I_m}\{M_*(t)(1_{D''} + 1_D)1_{C^m} - kM_*(t)(1_{D''} + 1_D)\}dt| \leq \rho/2^{m+1}$. These inequalities imply $|\int_J M_*(t)\{1_{D''}1_C + 1_D(1-1_C)\}dt|_q \leq \rho + [k\hat{d}'' + (1-k)\hat{d}] < [kd'' + (1-k)d]$ and $|\int_J M_*(t)(1_{D''} + 1_D)1_{C^m}dt|_q \leq \rho + k(\hat{d}'' + \hat{d}) < k(d'' + d)$. So in both cases, (28), (29) can be obtained for a $\hat{u} \in A_{kd''+(1-k)d}$ with $\sigma(\hat{u}, u) \leq k(d'' + d)$.

Thus, for any $\varepsilon > 0$, $u'' \in A_{d''}, u \in A_d$, $(d'', d \leq K)$ some $\hat{u} \in A_{kd''+(1-k)d}$ exists such that $\sigma(\hat{u}, u) \leq k(d'' + d)$, and

$$^{\infty}|\pi^{\infty}\hat{q}^{\hat{u}}(.) - [k\pi^{\infty}\hat{q}^{u''}(.) + (1-k)\pi^{\infty}\hat{q}^{u}(.)]| < \varepsilon, \tag{30}$$

$$E|a \cdot \{q^{\hat{u}}(1,\omega) - [kq^{u''}(1,\omega) + (1-k)q^{u}(1,\omega)]\}| < \varepsilon. \tag{31}$$

Proof of (32) below. Let $\check{d} \in (0,1]$. In the above construction, let $u'' \in U^K$, $u = u^*$, let k be slightly less than $\check{d}$, and let d and ε' be positive numbers, so close to zero that $k(d'' + d) < \check{d}K$, where $d'' = K + \varepsilon'$, ($u^* \in A_d$, $u'' \in A_{d''}$, so $\hat{u} \in A_{\check{d}K}$. Then, (30),(31) give that $|(Ea \cdot q^{\hat{u}}(1,\omega), \pi^{\infty}\hat{q}^{\hat{u}}(.)) - k(Ea \cdot q^{u''}(1,\omega), \pi^{\infty}\hat{q}^{u''}(.))|$ is arbitrarily small, (the norm is $|.| \times^{\infty} |\,|$-norm, $|.| =$ absolute value), hence $k(Ea \cdot q^{u''}(1,\omega), \pi^{\infty}\hat{q}^{u''}(.)) \in \mathrm{cl}\{(Ea \cdot q^{\hat{u}}(1,\omega), \pi^{\infty}\hat{q}^{\hat{u}}(.)) : \hat{u} \in A_{\check{d}K}\}$, (closure in $|.| \times^{\infty} |.|$-norm) and so also, (for any $u'' \in U^K, \check{d} \in (0,1]$),

$$\check{d}(Ea \cdot q^{u''}(1,\omega), \pi^{\infty}\hat{q}^{u''}(.)) \in \mathrm{cl}\{(Ea \cdot q^{\hat{u}}(1,\omega), \pi^{\infty}\hat{q}^{\hat{u}}(.)) : \hat{u} \in A_{\check{d}K}\} \quad (32).$$

Final proof steps. For $z(.) \in L_\infty(\Omega, Y)$, define $\Pi_1(z(.)) := E[z(.)|\Phi_{1-1/2}]$ and $\Pi_i(z(.)) := E[z(.)|\Phi_{1-1/2^i}] - E[z(.)|\Phi_{1-1/2^{i-1}}], i > 1$. Furthermore, define the subset L^∞ of $L_\infty(\Omega, Y)$ to consist of all element $z(.) \in L_\infty(\Omega, Y)$ such that $_{\infty}|z(.)| := \sup_i 2^i|\Pi_i z(.)|_\infty < \infty$, and such that $z(.) = \lim_{k\to\infty} \sum_{1\leq i\leq k} \Pi_i z(.)$

$= \lim_{k\to\infty} E[z(.)|\Phi_{1-1/2^k}]$, (limit in $|.|_\infty$-norm). It is easily seen that elements of the type $\int_J y(t,\omega)dt$, $y(t,\omega)$ progressively measurable, $|y(.,.)|_\infty < \infty$, precisely make up the set L^∞. To see this, note that $|\Pi_1 \int_J y(t,\omega)dt| \le |y(.,.)|_\infty$, and, for $j > 1$, $|\Pi_j \int_J y(t,\omega)dt| = |\Pi_j \sum_{0\le i<\infty} \int_{I_i} y(t,\omega)dt| = |\sum_{j-1\le i<\infty} \Pi_j \int_{I_i} y(t,\omega)dt)| \le \sum_{j-1\le i<\infty} 1/2^i |y(.,.)|_\infty = 1/2^{j-2}|y(.,.)|_\infty$. Moreover,

$$|\int_J y(t,\omega)dt - \sum_{1\le j\le k} \Pi_j \int_J y(t,\omega)dt| = |\int_J y(t,\omega)dt - E[\int_J y(t,\omega)dt|\Phi_{1-1/2^k}]| =$$
$$|\int_J y(t,\omega)dt - \int_0^{1-1/2^k} y(t,\omega)dt - E[\int_{1-1/2^k}^1 y(t,\omega)dt|\Phi_{1-1/2^k}]| \le (2/2^k)|y(.,.)|_\infty,$$

so $\int_J y(t,\omega)dt \in L^\infty$. Finally, if $z(\omega) \in L^\infty$, then $z(\omega) = \int_J \gamma(t,\omega)dt$, for $\gamma(t,\omega) := 2\sum_{m\ge1} 2^m \Pi_m z(\omega) 1_{[1-1/2^m, 1-1/2^{m+1})}(t)$, where $|\gamma(.,.)|_\infty \le 2_\infty|z(.)|$, $\gamma(.,.)$ progressively measurable.

Let Θ be the linear map from $\hat{L}_\infty$ into L^∞ defined by $(z_0(\omega), z_1(\omega), ...) \to \sum_i z_i(\omega)$. Let us prove that Θ has norm ≤ 8 for the norms ${}^\infty|.|$ and ${}_\infty|.|$: Now, $|\Pi_1 \sum_{i=0}^\infty z_i(.)|_\infty \le |\sum_{i=0}^\infty z_i(.)|_\infty \le \sum_{i=0}^\infty (1/2^i)^\infty|z(.)| \le 2\cdot{}^\infty|z(.)|$, while for $j > 1$, $|\Pi_j \sum_{i=0}^\infty z_i(.)|_\infty = |\Pi_j \sum_{i=j-1}^\infty z_i(.)| = |\sum_{i=j-1}^\infty \{E[z_i(.)|\Phi_{1-1/2^i}] - E[z_i(.)|\Phi_{1-1/2^{i-1}}]\}|_\infty \le \sum_{i=j-1}^\infty \{|z_i(.)|_\infty + |z_i(.)|_\infty\} \le \sum_{i=j-1}^\infty 2\cdot{}^\infty|z(.)|/2^i = {}^\infty|z(.)|/2^{j-3}$. Note that $\Theta\pi^\infty y^u(.) = \pi x^u(1,.)$ and $\Theta\pi^\infty \hat{q}^u(.) = \pi q^u(1,.)$. Thus (16) and (17) imply, for any $u, u' \in A_d$,

$$_\infty|\pi x^{u'}(1,.) - \pi x^u(1,.) - [\pi q^{u'}(1,.) - \pi q^u(1,.)]| \le \hat{e}_9(d)\sigma(u',u) \tag{33}$$

$$E|a\cdot\{x^{u'}(1,.) - x^u(1,.) - [q^{u'}(1,.) - q^u(1,.)]\}| \le \hat{e}_9(d)\sigma(u',u) \tag{34}$$

By (30),(31), for any $k \in (0,1)$, $\varepsilon > 0$, if $u'', u \in A_d$, then, for some $\hat{u} \in A_d$,

$$_\infty|\pi q^{\hat{u}}(1,.) - [k\pi q^{u''}(1,.) + (1-k)\pi q^u(1,.)]| < \varepsilon, \tag{35}$$

$$E|a\cdot\{q^{\hat{u}}(1,.) - [kq^{u''}(1,.) + (1-k)q^u(1,.)]\}| < \varepsilon, \tag{36}$$

$(d \le K)$. By (32), for any $u'' \in U^K, \check{d} \in (0,1]$,

$$\check{d}(Ea\cdot q^{u''}(1,\omega), \pi q^{u''}(1,.)) \in \mathrm{cl}\{(Ea\cdot q^{\hat{u}}(1,\omega), \pi q^{\hat{u}}(1,.)) : \hat{u} \in A_{\check{d}K}\} \tag{37}$$

where cl means closure in $|.| \times_\infty |.|$, (the first $|.|$ -sign being absolute value).

Let us now invoke Theorem B in Appendix, for $\tilde{A} = U^K, M' = 1, \partial(.,.) = \sigma(.,.)/K$, (so $\tilde{A}_d = A_{dK}$), $\hat{Y} = L_1([0,1] \times \Omega, X)$, and, for $\phi \in \hat{Y}, \breve{\pi}\phi = Ea \cdot \int_0^T \phi dt \in \mathbb{R} = \check{Z}$, $\hat{\pi}\phi = \pi \int_0^T \phi(t,\omega)dt \in L^\infty = Z$, $y(a) = \dot{x}^u(.,.), (u = a), y^+(a) = \dot{q}^u(.,.), \bar{a} = u^*$, z' the constant function $\tilde{y}$, $\check{z}^* = 1, d_0 = 1$. Evidently, U^K is complete in the metric σ, (see Theorem C in Appendix), and (33)-(37) imply (A)-(C) to be satisfied in the manner required in Theorem B. Moreover, (4) and (35) imply that $\mathrm{cl}\hat{\pi}y^+(A)$ is a convex body. Finally, continuity of $u \to (Ea \cdot q^u(1,.), \pi q^u(1,.))$, (see discussion subsequent to (17)), and (33),(34) give that $u \to (Ea \cdot x^u(1,.), \pi x^u(1,.))$ is continuous. Hence, Theorem B applies, and yields a continuous linear nonzero functional (Λ_0, z^*), $\Lambda_0 \geq 0$, such that $\Lambda_0 Ea \cdot \int_0^T \dot{q}^u(t,.)dt + \langle \pi \int_0^T \dot{q}^u(t,.)dt, z^* \rangle \leq 0$ for all $u(.,.) \in U^*$. Hence the conclusion of Theorem 1, (or more precisely of Remark 2), follows, for $\nu = z^*$.

Remark 6 By Theorem B in Appendix, in Remarks 2 and 3, it evidently suffices to assume that (4) holds for co replaced by clco, where cl means closure in ${}_\infty|.|$. This weakened condition (4) is implied by the following condition:

For some $\delta > 0$, some $T' \in [0,1]$, some bounded progressively measurable function $y(t,\omega) \in L_\infty([T',1] \times \Omega, Y)$, some complete separable metrizable subset $\tilde{U}$ of U, $B(\pi\dot{x}^*(t,\omega) + y(t,\omega), \delta) \subset \pi f(t, x^*(t,\omega), \tilde{U}, \omega) \subset Y$, for all ω and $t \in [T',1]$. Moreover, $M'(t,\omega)$ (see Remarks 2 and 3), is a constant M' and, for some constant K, $\mathrm{essup}_\omega \int_J M(t,\omega)dt \leq K$ and $|f(t, x^*(t,\omega), u, \omega) - f(t, x^*(t,\omega), u^*(t,\omega), \omega)| \leq K$ for all $\omega, t \in [T',1], u \in \tilde{U}$. (In case of Remark 3, the latter condition need only holds for f replaced by $\pi' f$.) Finally, $\tilde{U} \subset U(t,\omega)$, for all (t,ω).

(It may also be shown that the weakened condition (4) implies the ordinary condition (4).)

Proof of Remark 6 Let $T = 1$. Let $T^* \in [0,1)$ be so large that $M' \int_{T^*}^{\rho} |C(\rho, s, \omega)|ds \leq \delta/2K|\pi|$ for all $\omega, \rho \in [T^*, 1]$. Let $T'' = \max\{T', T^*\}$, and let $L_\infty^{prog}([T'',1] \times \Omega, Y)$ be the closed subspace of $L_\infty([T'',1] \times \Omega, Y)$, consisting of progressively measurable functions. Note that for any $z(t,\omega) \in L_\infty^{prog}([T'',1] \times \Omega, Y)$, for which $|z(.,.)|dt < \delta$, by (an easy extension of) the selection theorem of Kuratowskii, there exists a progressively measurable function $v(t,\omega), t \in [T'',1], \omega \in \Omega$, with values in $\tilde{U}$, (the set of such ones is denoted $\tilde{U}'$), such that, a.e.×a.s., $\pi\dot{x}^*(t,\omega) + y(t,\omega) + z(t,\omega) = \pi f(t, x^*(t,\omega), v(t,\omega), \omega), t \in [T'',1]$. (At this point, actually only a suitable approximate equality is needed, which allows for a weakening of the assumptions in the remark. In particular, it suffices to assume $B(\pi\dot{x}^*(t,\omega)+y(t,\omega),\delta)\subset$clco$\pi f(t,x^*(t,\omega),\tilde{U},\omega)\subset Y$.)

Now, for any $\phi(s,\omega)\in L_\infty^{prog}([T'',1]\times\Omega,X)$, $\int_{T''}^1 C(1,s,.)\phi(s,.)ds=$ $\int_{T''}^1[I+\int_s^1 f_x(\rho,x^*(\rho,.),u^*(\rho,.),.)C(\rho,s,.)d\rho]\phi(s,.)ds=$ $\int_{T''}^1[\phi(\rho,.)+\int_{T''}^\rho f_x(\rho,x^*(\rho,.),u^*(\rho,.),.)C(\rho,s,.)\phi(s,\omega)ds]d\rho$. Let H be the map $\phi(\rho,\omega)\to\psi(\rho,\omega):=\int_{T''}^\rho f_x(\rho,x^*(\rho,.),u^*(\rho,.),.)C(\rho,s,.)\phi(s,\omega)ds$. Now, $\psi(.,.)$ is also a progressively measurable function of (ρ,ω), with L_∞-norm $\le|\phi|_\infty\delta/2K|\pi|$. We just showed that $B_\infty^{T''}(y(.,.),\delta)\in\{\pi f(.,x^*(.),v(.),.)-\pi f(.,x^*(.,.),u^*(.,.),.):v(.,.)\in\tilde{U}'\}$, $B_\infty^{T''}(y(.,.),\delta)$ a ball in $L_\infty^{prog}([T'',1]\times\Omega,Y)$. Moreover, $|\pi H[f(.,x^*(.,.),v(.,.),.)-f(.,x^*(.,.),u^*(.,.),.)]|_\infty\le\delta/2$, so, (see Lemma 11.1, Seierstad (1975)), $B_\infty^{T''}(y(.,.),\delta/2)\in$cl$V$, for $V:=$coV', where

$$V':=\{\pi(I+H)[f(.,x^*(.),v(.),.)-f(.,x^*(.,.),u^*(.,.),.)]:v(.,.)\in\tilde{U}'\}\subset$$

$L_\infty^{prog}([T'',1]\times\Omega,Y)$, cl = closure in $|.|_\infty$. Let $z(.,.)\in B_\infty^{T''}(y(.,.),\delta/2)$, and let $\varepsilon>0$ be arbirary. Then there exists a $z'(.,.)\in V$ such that $|z'(.,.)-z(.,.)|_\infty<\varepsilon$. It follows that $|\int_{I_i}z'(t,\omega)1_{[T'',1]}dt-\int_{I_i}z(t,\omega)1_{[T'',1]}dt|\le\varepsilon/2^{i+1}$, so ${}_\infty|\int_{T''}^1(z'(t,.)-z(t,.))dt|<4\varepsilon$, (recall $|\Theta|\le 8$) . Hence, $\int_{T''}^1 y(t,.)dt+B(0,\delta(1-T'')/4)$ $\in$clco$\{\int_{T''}^1\tilde{z}(t,.)dt:\tilde{z}(.,.)\in V'\}\subset L^\infty$, where $B(.,.)$ and cl refer to the norm ${}_\infty|.|$. To see this, note that any element $z(.)$ in $B(0,\delta(1-T'')/4)$ can be written as $z(.)=\int_0^1\gamma(t,.)dt,\gamma(.,.)\le 2_\infty|z(.)|<\delta(1-T'')/2$ and $\int_0^1\gamma(t,.)dt=\int_{T''}^1(1/(1-T''))\gamma'(s,.)ds$, $\gamma'(s,\omega)=\gamma(s/(1-T'')-T''/(1-T''),\omega)$, $(\gamma'(.,.)/(1-T'')\in B_\infty^{T''}(0,\delta/2)\subset L_\infty([T'',1]\times\Omega,Y)$, progressively measurable since $s\ge s/(1-T'')-T''/(1-T''))$.

Proof of (8). Let $T=1$. Assume for the moment that ν is absolutely continuous with respect to P. Then, by the inequality $|C(1,s,.)|\le e^{M^*}$, also $\phi\to p(s,\phi)$ is absolutely continuous, hence, (considering for the moment $Dp(s,\omega):=D_\omega p(s,\omega)$ and $D\nu(\omega)$ to be linear functionals), evidently, $Dp(s,\omega)=C(1,s,\omega)^*D\nu_*(\omega)$ and $(\partial/\partial s)Dp(s,\omega)=(\partial/\partial s)C(1,s,\omega)^*D\nu_*(\omega)=$ $-(f_x(s,x^*(s,\omega),u^*(s,\omega),\omega))^*C(1,s,\omega)^*D\nu_*(\omega)=$ $-\,(f_x(s,x^*(s,\omega),u^*(s,\omega),\omega))^*Dp(s,\omega)$.

If it is only known that $\phi\to\langle C(1,s,.)\phi,\nu\rangle,\phi\in L_\infty(\Omega,\Phi_s,X)$, $s<1$, is absolutely continuous, (this happens more often), then for any $s<1$, choose a $T'\in(s,1)$, apply the above arguments to $p(s,\omega)=C(T',s,\omega)^*\nu_{T'}(\omega),\nu_{T'}:=$

$C(1,T',.)^*\nu_*$, to obtain $(\partial/\partial s)Dp(s,\omega) = (\partial/\partial s)C(T',s,\omega)^* D\nu_{T'}(\omega) = -(f_x(s,x^*(s,\omega),u^*(s,\omega),\omega)^* C(T',s,\omega)^* D\nu_{T'}(\omega) = -(f_x(s,x^*(s,\omega.),u^*(s,\omega),\omega)^* Dp(s,\omega)$.

Proof of Remark 5 Let $T=1$. By the proof of Theorem 1, for $n^{**} := \max\{1,K^*,n^*\}$, multipliers $(\Lambda_0^n, z_n^*), n = n^{**}, n^{**}+1, ...$, with $\Lambda_0^n \geq 0, \max\{\Lambda_0^n, |z_n^*|\} = 1$, exist for which $\Lambda_0^n Ea \cdot \int_0^T \dot{q}^u(t,.)dt + \langle \pi \int_0^T \dot{q}^u(t,.)dt, z_n^*\rangle \leq 0$, $u(.,.) \in U_n^n$. This same inequality is satisfied by any given weak* cluster point (Λ_0, z^*) of (Λ_0^n, z_n^*), for any $u(.,.) \in \cup_n U_n^n$. From this inequality, it also follows that $(\Lambda_0, z^*) \neq 0$: For simplicity, assume $\lim_n \Lambda_0^n = \Lambda_0$. If $\Lambda_0 = 0$, then for n large, both $|z_n^*| = 1$, and $|\Lambda_0^n Ea \cdot \int_0^T \dot{q}^u(t,.)dt| \leq \alpha/4$, for all $u \in U_{n^*}^{K^*}$, so $\langle \pi \int_0^T \dot{q}^u(t,.)dt, z_n^*\rangle \leq \alpha/4$, which yields, in particular, $\langle z+y, z_n^*\rangle \leq \alpha/4$, $z := \int_0^1 \check{z}(t,.)dt$, for $y \in B_\alpha$, see (4). This means that $\langle z, z_n^*\rangle \leq \inf_{y\in B_\alpha}\langle -y, z_n^*\rangle + \alpha/4 \leq \inf_{z''\in B(0,\alpha/2)}\langle z'', z_n^*\rangle + \alpha/4 = -\alpha/2 + \alpha/4 = -\alpha/4$, ($B(0,\alpha/2)$ a ball in L^∞), the last inequality because it was shown in the proof of Theorem 1, that for any z'' in L^∞, $z'' = \int_0^1 \gamma(t,\omega)$ for some $\gamma(.,.) \in L_\infty^{prog}(J \times \Omega, Y)$ with $|\gamma(.,.)|_\infty \leq 2_\infty |z''|$. I.e. even $\langle z, z^*\rangle \leq -\alpha/4$, so $z^* \neq 0$.

Proof of (7) With the conventions in the proof of Theorem 1 and a shorthand notation, $|C(T,s,\omega) - I| = |\int_s^T f_x C dt| \leq e^{M^*(\omega)} \int_s^T M'(t,\omega)dt \leq e^{M^*(\omega)} M^*(\omega)$, $|\pi(C(T,s,\omega) - I)| = |\int_s^T \pi f_x C dt| \leq e^{M^{**}} \int_s^T M'_\pi dt \leq e^{M^{**}}(T-s)M'_\pi$, and $p(t,\phi) - p(T,\phi) = \langle \pi(C(T,s,.)-I)\phi, \nu\rangle + \Lambda_0 E\langle (C(T,s,.)-I)\phi, a\rangle$, from which (7) follows.

Applications

A. Continuous systems Assume that $X = \mathbb{R}^n, Y = \mathbb{R}^m, \Omega = \{\omega = (\tau_1,\tau_2,...) : \tau_i \in [0,\infty)\}$, and that conditional probability densities $\dot{\mu}(\tau_{j+1}|\tau_1,...,\tau_j)$ are given, (for $j=0$, the density is simply $\dot{\mu}(\tau_1)$, sometimes written $\dot{\mu}(\tau_1|\tau_0), \tau_0 = 0$). The conditional density $\dot{\mu}(\tau_{j+1}|\tau_1,...,\tau_j)$ is assumed to be measurable with respect to $\tau_1,...,\tau_{j+1}$, and integrable with respect to τ_{j+1}, with integral 1. We assume $\dot{\mu}(\tau_{j+1}|\tau_1,...,\tau_j) = 0$ if $\tau_{j+1} < \max_{1\leq i\leq j}\tau_i$. This means that we need only consider nondecreasing sequences $\omega = (\tau_1,\tau_2,...)$, making up the set Ω^*, or even the set Ω' of strictly increasing sequences. For $\omega^j = (\tau_0,\tau_1,...,\tau_j)$, these conditional densities define simultaneous conditional densities $\dot{\mu}(\tau_{j+1},...,\tau_m|\omega^j)$, ($\dot{\mu}(\tau_1,...,\tau_m|\tau_0) = \dot{\mu}(\tau_1,...,\tau_m)$), assumed to satisfy: For some $k^* \in (0,1)$, some positive numbers $\Phi^*(t,j), \upsilon(t,j)$, $\upsilon(t,j) \in (0,\kappa^*)$,

$$\Pr[t \in (\tau_m, \tau_{m+1}]|\omega^j] \leq \Phi^*(t,j)(\upsilon(t,j))^{m-j}, \tag{38}$$

for any given $t \in [0, \infty)$. Property (38), used for $j = 0$, means that with probability 1, the sequences $(\tau_1, \tau_2, ...)$ has the property that $\tau_i \to \infty$, (the subset of such sequences in Ω' is denoted Ω'').

Let the term "nonanticipating function" mean a function $y(t,\omega) = y(t, \tau_1, \tau_2, ...)$ that for each given $t \in [0, T]$, depends only on $\tau_i \leq t$. (Formally, $y(t, \tau_1', \tau_2', ...) = y(t, \tau_1, \tau_2, ...)$ if $\{i : \tau_i' \leq t\} = \{i : \tau_i \leq t\}$ and $\tau_i' = \tau_i$ for $i \in \{i : \tau_i \leq t\}$.). This corresponds to letting Φ_t be the σ-algebra generated by sets of the form $A = A_{B,i} := \{\omega := (\tau_0, \tau_1, ...) : \tau_i \in B\}$, where B is a either a Lebesgue measurable set in $[0, t]$, or $B = (t, \infty), i \in \{1, 2, ...\}$. Condition (38) entails that a probability measure P, corresponding to the conditional densities $\dot{\mu}(\tau_{i+1}|\omega^i)$, is defined on (Ω, Φ), $\Phi = \Phi_T$. If $y(.,.)$ takes values in a topological space $\bar{Y}$, let $M^{nonant}(J \times \Omega, \bar{Y})$ be the set of functions being nonanticipating and simultaneous Lebesgue measurable on each set $[0, T] \times \Omega_i, \Omega_i := \{\omega : \tau_{i+1} > T\}$. (These properties are essentially equivalent to progressive measurability.) As a function of (t, ω), f (in (1)) is now assumed to be nonanticipating. Sometimes we write $y(t, \omega) = y(t, \omega^i)$ when $\tau_i \leq t, \tau_{i+1} > t, \omega \in \Omega'$. Now, we define $U' = M^{nonant}(J \times \Omega, U), B_\alpha = \{\int_J z(t,.)dt : z(.,.) \in M^{nonant}(J \times \Omega, Y), |z(.,.)|_\infty < \alpha\}$, and we let $\check{z}(.,.) \in L_\infty^{nonant}(J \times \Omega, Y) := \{z(.,.) \in M^{nonant}(J \times \Omega, Y), |z(.,.)|_\infty < \infty\}$. For these definitions, Theorem 1 holds.

The present type of systems might be termed continuous, piecewise deterministic. In such systems, the right hand side of the differential equation in (1) exhibits sudden changes at stochastic points in time τ_i. In concrete (economic) situations, this may be earthquakes, inventions, sudden devaluations etc.

In the remaining part of this section A, assume that, for any $t < T$, $C(T, t, .)^* \pi^* \nu|_{\Phi_t}$ is absolutely continuous with respect to P. From now on all ω's occurring will belong to Ω'. Let H_i^t be any Lebesgue measurable set in $[0, t]^i$ and define the corresponding set $H_t^i := \{(\tau_1, \tau_2, ...) \in \Omega' : (\tau_1, ..., \tau_i) \in H_i^t, \tau_{i+1} > t\} \in \Phi_t$. The absolute continuity assumption implies both that, (by (38)), for any $t < T$,

$$\lim_{N \to \infty} \sup_{\phi \in B(0,1) \subset L_\infty(\Omega, \Phi_t, X)} \langle \pi C(T, t, .) \phi 1_{(\tau_N \leq t]}, \nu \rangle = 0, \tag{39}$$

and that, for any unit vector e_j, for each i and $t < T$, $H_i^t \to \langle e_j 1_{H_t^i}, C(T, t, .)^* \pi^* \nu \rangle$

is absolutely continuous with respect to P and so also with respect to Lebesgue measure. Defining $p_j^{**}(s, 1_{H_i^s}) = p(s, e_j 1_{H_s^i})$, we get that $H_i^s \to p_j^{**}(s, 1_{H_i^s})$ is absolutely continuous with respect to Lebesgue measure. Moreover, the Radon - Nikodym derivative of $H_i^s \to (p_1^{**}(s, 1_{H_i^s}), ..., p_n^{**}(s, 1_{H_i^s}))$ with respect to Lebesgue measure, denoted $p^*(s, \omega^i) := (p_1^*(s, \omega^i), ..., p_n^*(s, \omega^i))$, is absolutely continuous in s in any interval (τ_i, T), with a right limit at the left end of the interval, and satisfies, for a.e. $s > \tau_i$,

$$\dot{p}^*(s, \omega^i) = -p^*(s, \omega^i) f_x(s, x^*(s, \omega), u^*(s, \omega), \omega) - p^*(s+, \omega^i, s). \tag{40}$$

if $p^*(s, \omega^i)$ is considered to be a row vector. (For $i = 0$, this equation is satisfied by $p^*(s, \omega^0) := (p(s, e_1 1_{(s,\infty)}(\tau_1), ..., p(s, e_n 1_{(s,\infty)}(\tau_1).)$ If at most N jumps can occur, then $p(s+, \omega^N, s) = 0$. We define the nonanticipating function $p^*(t, \omega)$ by $p^*(t, \omega) = p^*(t, \omega^i)$ if $t \in (\tau_i, \tau_{i+1}) \cap J$. Then, for any $u(.,.) \in U^K$, for a.e. t,

$$\langle f(t, x^*(t, \omega), u(t, \omega), \omega) - f(t, x^*(t, \omega), u^*(t, \omega), \omega), p^*(t, \omega) \rangle \le 0. \tag{41}$$

Sometimes it is useful to work with the function $q^*(t, \omega^i) = p^*(t, \omega^i)/\theta(t, \omega^i)$, $\theta(t, \omega^i) := \int_t^\infty \dot{\mu}(\omega^i, \tau_{i+1}) d\tau_{i+1}$. Let $q^*(t, \omega)$ be the corresponding nonanticipating function. For simplicity, assume $\int_t^\infty \dot{\mu}(\omega^i, \tau_{i+1}) d\tau_{i+1} > 0$ for all $t \in J, \omega^i$. The following differential equation is satisfied by $q^*(t, \omega)$: For any i, for a.e. $t \in (\tau_i, \tau_{i+1}) \cap J$,

$$\begin{aligned} &\dot{q}^*(t, \omega) = -q^*(t, \omega) f_x(t, x^*(t, \omega), u^*(t, \omega), \omega) + \\ &(q^*(t, \omega) - q^*(t+, \omega^i, t)) \dot{\mu}(\omega^i, t)/\theta(t, \omega^i). \end{aligned} \tag{42}$$

(If at most N jumps can occur, let $\dot{\mu}(\tau_{N+1}, \omega^N) = 0, \theta(\tau_{N+1}, \omega^N) = 1$.) The function $q^*(t, \omega)$ (as well as $p^*(t, \omega)$), is absolutely continuous in t in any interval $(\tau_i, \min\{T, \tau_{i+1}\})$, with left and right limits at the ends of the interval, (a left limit only if $\tau_{i+1} < T$). Of course, in this case, for all $u(.,.) \in U^K$, a.s., for a.e. t,

$$\langle f(t, x^*(t, \omega), u(t, \omega), \omega) - f(t, x^*(t, \omega), u^*(t, \omega), \omega), q^*(t, \omega) \rangle \le 0. \tag{43}$$

Finally, a relationship between $p^*(t, \omega)$ and ν that is frequently useful, is obtained from the definitions of ν_* and $p^*(t, \omega)$: For any $\tau_{i+1} > T$, a.s. in ω^i, $\tau_i < T$, $j = 1, ..., n$, (provided the two limits exist):

$$\lim_{t \nearrow T} p_j^*(t, \omega^i) = \Lambda_0 a_j \theta(T, \omega^i) + \lim_{t \nearrow T} e_j D_{(\tau_1, ..., \tau_i)} \{C(T, t, .)^* \pi^* \nu\}((0, \tau_1] \times ... \times (0, \tau_i] \times \{\tau_{i+1} : \tau_{i+1} > t\}), \tag{44}$$

(where for $i = 0$, the right hand side reduces to $\Lambda_0 a_j \theta(T, \omega^0) + \lim_{t \nearrow T} e_j C(T, t, .)^* \pi * \nu(\{\tau_1 : \tau_1 > t\})$. Here $D_{(\tau_1, ..., \tau_i)}$ is a derivative with respect to $(\tau_1, ..., \tau_i)$. To obtain a corresponding condition for $q^*(.,.)$, replace $p^*(t, \omega^i)$ in (44) by $q^*(t, \omega^i)\theta(t, \omega^i)$.

When solving a concrete problem one may start by proving the above absolute continuity of $C(T, t, .)^* \pi^* \nu$ with respect to P. Or, one may even start by *assuming* this absolute continuity. If in the problem at hand this assumption is false, the need to relax this assumption will soon express itself.

Proof of (40).

Write $\nu_t := (\pi C(T, t, .))^* \nu$. By the absolute continuity assumption, for some P-integrable function ψ_t, for any $\psi \in L_\infty(\Omega, \Phi_t, \mathbb{R}^n)$, $\langle \psi, \nu_t \rangle = \int_\Omega \langle \psi, \psi_t \rangle dP(\omega)$. Write $\nu^t = \nu_t + E\langle ., \Lambda_0 C(T, t, .)^* a \rangle = C(T, t, .)^* \nu_*$, $\psi^t = \psi_t + \Lambda_0 C(T, t, .)^* a$. Furthermore, write $f_x^{(*)}(s, \omega) := f_x(s, x^*(s, \omega), u^*(s, \omega), \omega)$ and $\omega_i = (\tau_{i+1}, \tau_{i+2}, ...)$. Recall the following facts about $C(.,.,.)$: $C(t, s, \omega) = I + \int_s^t f_x^{(*)}(\rho, \omega) C(\rho, s, \omega) d\rho$, so for a.e. $s, t \to \partial C(t, s, \omega)/\partial s$ is the solution of the equation: $\partial C(t, s, \omega)/\partial s = -f_x^{(*)}(s, \omega) + \int_s^t f_x^{(*)}(\rho, \omega)(\partial C(t, s, \omega)/\partial s) d\rho$. Since $C(.,.,)$ yields the solution of such equations, then $\partial C(t, s, \omega)/\partial s = C(t, s, \omega)(-f_x^{(*)}(s, \omega))$. This formula in fact holds for all regular points s in (τ_i, τ_{i+1}) of $f_x^{(*)}(s, \omega^i)$.

Let $0 < s < t < T$, (t fixed, s will be varied). Note that $\nu_s = C(t, s, .)^* \nu_t, \psi^s = C(t, s, \omega)^* \psi^t$. Let $\psi = \psi^i 1_{[\tau_i \le s]} 1_{[\tau_{i+1} > s]}$, where $\psi^i = \psi^i(\omega^i) \in L_\infty(\Omega^i, \mathbb{R}^n), \Omega^i := \{(\tau_1, ..., \tau_i) : \tau_k < \tau_{k+1}, k = 1, ..., i-1, \tau_i \le T\}$. Now, $p(s, \psi) = \int_\Omega \langle \psi, \psi^s \rangle dP(\omega) = \int_{\Omega^i} \int_s^\infty \langle \psi^i 1_{[\tau_i \le s]}, E[\psi^s | \omega^i, \tau_{i+1}] \rangle \dot{\mu}(\omega^i, \tau_{i+1}) d\tau_{i+1} d\omega^i$, so, for $\tau_i \le s$, a.s., the Radon-Nicodym derivative $p^*(s, \omega^i) := (d/d\omega^i) p(s, \psi)$ equals

$$\int_s^\infty E[\psi^s | \omega^i, \tau_{i+1}] \dot{\mu}(\omega^i, \tau_{i+1}) d\tau_{i+1} = \int_s^\infty E[C(t, s, \omega)^* \psi^t | \omega^i, \tau_{i+1}] \dot{\mu}(\omega^i, \tau_{i+1}) d\tau_{i+1} = C(0, s, \omega^i)^* \int_s^\infty E[C(t, 0, \omega)^* \psi^t | \omega^i, \tau_{i+1}] \dot{\mu}(\omega^i, \tau_{i+1}) d\tau_{i+1}.$$

The last expression shows that a.s., $p^*(s, \omega^i)$ is absolutely continuous in s in any interval (τ_i, t), with a right limit at the left end of the interval. In particular, a.s., $p^*(\tau_i +, \omega^i)$ equals $\int_{\tau_i}^\infty E[C(t, \tau_i, \omega)^* \psi^t | \omega^i, \tau_{i+1}] \dot{\mu}(\omega^i, \tau_{i+1}) d\tau_{i+1} = E[C(t, \tau_i, \omega) \psi^t | \omega^i] \dot{\mu}(\omega^i)$. Now, using this calculation, (for ω^i replaced by

$(\omega^i, s))$, and the last formula for $p^*(s,\omega^i)$, it is seen that $p^*(s,\omega^i)$ is differentiable for a.e. s in (τ_i, t), and letting $p^*(s,\omega^i)$ for the moment be a column vector, then, a.s., $\dot{p}^*(s,\omega^i)$ equals a.e.

$$[C(0,s,\omega^i)(-f_x^{(*)}(s,\omega^i))]^* \int_s^\infty E[C(t,0,\omega)^*\psi^t|\omega^i,\tau_{i+1}]\dot{\mu}(\omega^i,\tau_{i+1})d\tau_{i+1} - \\ E[C(t,0,\omega)^*\psi^t|\omega^i,s]\dot{\mu}(\omega^i,s) = \\ [-f_x^{(*)}(s,\omega^i)]^* \int_s^\infty E[C(t,s,\omega)^*\psi^t|\omega^i,\tau_{i+1}]\dot{\mu}(\omega^i,\tau_{i+1})d\tau_{i+1} - \\ \int_s^\infty E[C(t,s,\omega^i,s,\omega_{i+1})^*\psi^t|\omega^i,s,\tau_{i+2}]\dot{\mu}(\omega^i,s,\tau_{i+2})d\tau_{i+2} = \\ [-f_x^{(*)}(s,\omega^i)]^* p^*(s,\omega^i) - p^*(s+,\omega^i,s).$$

Hence, the present $p^*(t,\omega)$ is a.s. equal to the one defined in (40).

Remark 7 (The relationship between (41) and (5))
Let us take another look at the relationship between (41) and (5), (still the absolute continuity assumption is postulated). Let b be any number $< T$, and let $u(t,\omega) \in U^K$. Write $\alpha_i^t(\omega) := 1_{[\tau_i \le t]} 1_{[\tau_{i+1} > t]}, t \le b$. From (41), it follows that,

$$\int_{\Omega^i} \alpha_i^t(\omega)\langle [f(t,x^*(t,\omega),u(t,\omega),\omega) - f(t,x^*(t,\omega),u^*(t,\omega),\omega), p^*(t,\omega)\rangle d\omega^i = \\ \langle \alpha_i^t(.)[f(t,x^*(t,.),u(t,.),.) - f(t,x^*(t,.),u^*(t,.),.)], C(T,t,.)^*\nu_*\rangle \le 0.$$

Hence, using that (38) implies $\sum \alpha_i^t = 1$ a.s. and P-integrability of $[f(t,x^*(t,.),u(t,.),.) - f(t,x^*(t,.),u^*(t,.),.)]$ for a.e. t, (so sums and integration with respect to $C(T,t,.)^*\nu_*$ can be interchanges), gives, for a.e. t,

$$0 \ge \sum_i \langle \alpha_i^t(.)[f(t,x^*(t,.),u(t,.),.) - f(t,x^*(t,.),u^*(t,.),.)], C(T,t,.)^*\nu_*)\rangle = \\ \langle f(t,x^*(t,.),u(t,.),.) - f(t,x^*(t,.),u^*(t,.),.), C(T,t,.)^*\nu_*)\rangle = \\ \langle C(b,t,.)[f(t,x^*(t,.),u(t,.),.) - f(t,x^*(t,.),u^*(t,.),.)], C(T,b,.)^*\nu_*)\rangle.$$

Then, using that $C(T,b,.)\nu_*|_{\Phi_b}$ is absolutely continuous, it follows that $0 \ge$

$$\int_0^b \langle C(b,t,.)[f(t,x^*(t,.),u(t,.),.) - f(t,x^*(t,.),u^*(t,.),.)], C(T,b,.)^*\nu_*)\rangle dt = \\ \langle \int_0^b C(b,t,.)[f(t,x^*(t,.),u(t,.),.) - f(t,x^*(t,.),u^*(t,.),.)]dt, C(T,b,.)^*\nu_*)\rangle dt = \\ \langle \int_0^b C(T,t,.)[f(t,x^*(t,.),u(t,.),.) - f(t,x^*(t,.),u^*(t,.),.)]dt, \nu_*)\rangle dt.$$

Let us introduce the condition: For all $u(.,.) \in U^K$,

$$\langle \int_b^T C(T,t,.)[f(t,x^*(t,.),u(t,.),.) - f(t,x^*(t,.),u^*(t,.),.)]dt, \nu_*\rangle \le 0. \quad (45)$$

Evidently, (41) and (45) are equivalent to (5). In fact, (5) is equivalent to

(41) and the following condition: For all $u(.,.) \in U^K$,

$$\limsup_{b\uparrow T} \langle \int_b^T C(T,t,.)[f(t,x^*(t,.),u(t,.),.) - f(t,x^*(t,.),u^*(t,.),.)]dt, \nu_* \rangle \leq 0. \tag{46}$$

Example 1 Consider the problem

$\max E \int_0^1 (-u^2/2)[1_{[0,\tau)}(t) + (1-\tau)1_{[\tau,1]}(t)]dt$, $\dot{x} = u, u \in R, x(0) = 0, x(1) = 1$ a.s.,

where τ is exponentially distributed in $[0,\infty)$ with intensity λ.

This problem is rewritten by introducing an auxiliary state variable x^0 , governed by $\dot{x}^0 = (-u^2/2)(1_{[0,\tau)} + (1-\tau)1_{[\tau,1]}), x^0(0) = 0$, with $\pi = (x^0, x) \to x$. Then the criterion is $Ex^0(1)$. Let us agree that we only look for candidates $u^*(.,)$, for which $\mathrm{E}\int_J u^*(t,.)^2 dt < \infty$. We shall apply Remark 5, (combined with Remark 3), and we let $U_n(t,\omega) = [-n,n] + u^*(t,\omega)$. We take $M'_{*,n}(t,\omega) := 0, M_{*n}(t,\omega) = n|u^*(t,\omega)| + n^2/2, M(t,\omega) = n, M'(t,\omega) = 0, K^* = 1$. Evidently, $C(1,t,\omega) = I$, so, obviously, (4) holds for $U_1^1, \alpha = 1$, $\check{z}(t,\omega) = 0$. The adjoint variable $q^*_{x^0}(.,.)$ corresponding to x^0 equals Λ_0, (by (44), $\lim_{t\uparrow 1} q^*_{x^0}(t,\tau) = \Lambda_0$, both for $\tau > 1$, and $\tau \leq 1$, and the adjoint equation for $q^*_{x^0}(.,.)$ is evidently satisfied by $q^*_{x^0}(.,.) \equiv \Lambda_0$.)The adjoint variable q^*_x corresponding to x is, for simplicity, written q^*. Let us show that Λ_0 is nonzero. In general, if in Theorem 1, $\check{z}(t,\omega) = 0$, Λ_0 is always nonzero. To see this in the present example, inserting $u = u^* \pm v$, v an arbitrary bounded nonanticipating measurable function, in (5) gives, when $\Lambda_0 = 0$, that $\langle \int_J \pm v dt, \nu \rangle \leq 0$, i.e. $\langle \int_J v dt, \nu \rangle = 0$, contradicting $\nu|_{B_\alpha} \neq 0$. So let $\Lambda_0 = 1$. Next, let us show that $\nu|_{\Phi_s}, s < 1$, is absolutely continuous. Let $z(t,\tau)$ be an arbitrary bounded nonanticipating measurable function, let $b < 1$ be arbitrarily chosen and let I be a Lebesgue measurable set in $[0,b]$. Inserting the nonanticipating function $u = 1_{[b,1]}(t)1_I(\tau)(\pm z(t,\tau)) + u^*$ in the maximum condition (5) gives

$\langle 1_I \int_b^1 \pm z(t,.)dt, \nu\rangle + E[1_I \int_b^1 \{((-(u^*(t,.) \pm z(t,.))^2/2) + (-u^{*2}(t,.)/2)\}(1_{[0,\tau)} +$ (1-τ)$1_{[\tau,1]} dt] \leq 0$.

Letting $z(t,\tau) \equiv \pm 1$, gives $\pm(1-b)\langle 1_I, \nu\rangle \leq E[1_I \int_b^1 (|u^*(t,.)| + 1/2)dt]$, which yields absolute continuity. Moreover, letting $I = [0, \infty)$, the next to last inequality also gives $\lim_{b\uparrow 1}\langle \int_b^1 \pm z(t,\tau), \nu\rangle \leq 0$, as the second integrand is bounded by $|u^*||z| + |z|^2/2$. For $t > \tau$, by (42), $\dot{q}^*(t,\tau) = 0$, so write $q^*(t,\tau) := q_*(\tau)$. By (43), $-u^*(t,\tau)(1-\tau) + q_*(\tau) = 0$, so $u^*(t,\tau)$ is independent of t, and we write it $u_*(\tau)$. Then $x^*(t,\tau) = x^*(\tau,\tau) + u_*(\tau)(t-\tau)$. Evidently, we must have (at least a.s.), $x^*(\tau,\tau) + u_*(\tau)(1-\tau) = 1$, in order to reach the point (1,1) from $(\tau, x^*(\tau,\tau))$. Hence, $u_*(\tau) = (1-(x^*(\tau,\tau))/(1-\tau)$, which gives $q_*(\tau) = u^*(t,\tau)(1-\tau) = u_*(\tau)(1-\tau) = 1 - x^*(\tau,\tau)$. Before the jump, by (43), $-u^*(t) + q^*(t) = 0$, (the "before jump" entities are written without τ appearing), so $\dot{x}^*(t) = q^*(t)$. Moreover, $\dot{q}^* = [q^*(t) - q^*(t+,t)]\lambda = [q^*(t) + x^*(t) - 1]\lambda$. In fact, we have to solve, simultaneously, the two equations, $\dot{x} = q^*$ and $\dot{q}^* = [q^* + x - 1]\lambda$, which yields the second order equation $\ddot{x} = [\dot{x} + x - 1]\lambda$. The latter equation has the solution $x^*(t) := Ce^{r_+ t} + De^{r_- t} + 1$, $r_\pm = (1/2)[\lambda \pm (\lambda^2 + 4\lambda)^{1/2}]$, where C and D are determined by $C + D + 1 = 0$ and $Ce^{r_+} + De^{r_-} + 1 = 1$. As $x^*(1) = 1$, and $\dot{x}^*(t)$ is bounded, then, $|x^*(\tau) - 1| = |x^*(\tau,\tau) - 1| \leq K(1-\tau)$, for some K, so $u_*(\tau)$ is bounded.

The sufficient conditions of Remark 11 below give optimality of $u^*(t)$ and $u_*(\tau)$ in the set of controls taking values in $U_n(t,\omega), n = 1, 2, ...$ (The condition (5) is satisfied, because (46) holds.)

B. Piecewise continuous systems Let us now consider piecewise continuous systems, where the state jumps at the times τ_i introduced in A above. Hence, to the set-up in A, add the feature that

$$x(\tau_i+, \omega) = \hat{g}(\tau_i, x(\tau_i-, \omega), i). \tag{47}$$

So now, $t \to x(t,\omega)$ is only absolutely continuous (and governed by (1)) between the points τ_i, with left and right limits at each $\tau_i, i = 1, 2, ...$ satisfying (47). We take $t \to x(t,\omega)$ to be left continuous, (yet we often, "unnecessarily", write $x(t-,\omega)$). The function f satisfies the conditions in Remark 2. It is assumed that $\hat{g}(.,.,i)$ and $\hat{g}_x(.,.,i)$ (exist and) are continuous and that, for some $\check{d} > 0$, $\hat{g}_x(.,.,i)$ is uniformly continuous in $x \in B(x^*(t,\omega), \check{d})$, uniformly in t, i. Hence, in this case, for any $\varepsilon > 0$, for some δ, $|x - x'| < \delta, x, x' \in B(x^*(t,\omega), \check{d}) \Rightarrow |\hat{g}_x(t,x,i) - \hat{g}_x(t,x',i)| < \varepsilon$ for all t, i. This property automatically holds, if at most N jumps can occur, N some positive natural number, and the Simple Global Assumptions hold, (then $x^*(t,\omega)$

and so also $B(x^*(t,\omega),\check{d})$ are contained in a compact set, independent of ω). It is assumed that for some positive numbers M_i, $|\hat{g}_x(t,x,i)-I| \leq M_i$ for all $(t,x) \in B(x^*(t,\omega),\check{d})$, for all ω, and $\sum_i M_i < \infty$. Finally, it is assumed that $E[\int_0^T |f(t,x^*(t,\omega),u^*(t,\omega),\omega)|dt + \sum_{\tau_i<T} |\hat{g}(\tau_i,x^*(\tau_i-,\omega),i) - x^*(\tau_i-,\omega)|] < \infty$.

In this section, define $C(t,s,\omega)$ as follows: In each interval $(\tau_i,\tau_{i+1}) \cap J$, $t \to C(t,s,\omega)$ is absolutely continuous, with left and right limits at the ends of the interval, and with $t \to C(t,s,\omega)$ governed by

$$\dot{C}(t,s,\omega) = f_x(t,x^*(t,\omega),u^*(t,\omega),\omega)C(t,s,\omega), C(s+,s,\omega) = I, \tag{48}$$

while $t \to C(t,s,\omega)$ has jumps at each τ_i given by

$$C(\tau_i+,s,\omega) = \hat{g}_x(\tau_i,x^*(\tau_i-,\omega),i)C(\tau_i-,s,\omega). \tag{49}$$

The function $C(t,s,\omega)$ is right, and piecewise, continuous as a function of s, jumping only at the τ_i's, and is (taken to be) left continuous in t. The following theorem holds:

Theorem 2 Assume that (4) holds when the present definition of $C(T,t,.)$ is used in (4), (for B_α, see Part A). Then the necessary conditions of Theorem 1 also hold for the present piecewise continuous system, when the present definition of $C(T,t,.)$ is used also in (5).

Again, if, for $t < T$, $C(T,t,.)^*\pi^*\nu|_{\Phi_t}$ is absolutely continuous with respect to P, then an adjoint nonanticipating function $p^*(t,\omega)$ is defined, satisfying the adjoint equation (50) below: For any i, for a.e. $t \in (\tau_i,\tau_{i+1}) \cap J$,

$$\begin{aligned}&\dot{p}^*(t,\omega) = \\ &-p^*(t,\omega)f_x(t,x^*(t,\omega),u^*(t,\omega),\omega) - p^*(t+,\omega^i,t)\hat{g}_x(t,x^*(t,\omega),i+1),\end{aligned} \tag{50}$$

$p^*(t,\omega)$ being absolutely continuous in t in any interval $(\tau_i,\min\{T,\tau_{i+1}\}) \subset [0,T]$, with left and right limits at the end of the interval, (a left limit at τ_{i+1} only if $\tau_{i+1} < T$). Moreover, the maximum condition (41) is satisfied. The end condition (44) holds in the same way as before. Finally, (5) is still equivalent to (41),(46).

Proof: Let $T = 1, x_0 = 0$, and write $g = \hat{g} - x$. The above jumping system can be rewritten as a nonjumping system, as follow:

Let $M = \sum_{i=1}^{\infty} M_i$, $M_0 = 0$, and write $a_j = \sum_{i=0}^{j} M_i$. For $i = 1, 2, ...$, let $\tau^i := \tau^i(\tau_i) := \tau_i + a_{i-1}$, if $\tau_i < 1$, and $\tau^i := \tau^i(\tau_i) := M + \tau_i + a_{i-1}$ if $\tau_i \geq 1$, $\tau_0 = \tau^0 = 0$. There is an one-one correspondence between the τ^i's and the τ_i's. Define also $\check{\tau}^{i+1} := \min\{\tau^{i+1}, \tau^i + M_i + 1 - \tau_i\}$, and note that $\tau^{i+1} < 1+M \Leftrightarrow \tau_{i+1} < 1$. Hence, $\tau^{i+1} < 1+M \Leftrightarrow \tau^{i+1} = \check{\tau}^{i+1}$. To see the latter equivalence, observe that $\tau^i+M_i+1-\tau_i = \tau_i+a_{i-1}+M_i+1-\tau_i = 1+a_i$ and $1+a_i > \tau_{i+1}+a_i =: \tau^{i+1}$ if and only if $\tau_{i+1} < 1$, so $\tau^{i+1} < 1+M \Rightarrow \check{\tau}^{i+1} = \tau^{i+1}$ and $\tau^{i+1} \geq 1 + M \Rightarrow \check{\tau}^{i+1} = 1 + a_i$. For $t \in [0, 1 + M]$, define

$h(t, z(.), v, \tau^1, \tau^2, ...) = \sum_{i=0}^{\infty} f(t - a_i, z(t), v, \tau_1, \tau_2, ...)1_{(\tau^i+M_i,\check{\tau}^{i+1}]}(t) + \sum_{i=0}^{\infty} g(\tau_{i+1}, z(\tau^{i+1}), i + 1)1_{(\tau^{i+1},\tau^{i+1}+M_{i+1}]}(t)/M_{i+1}$.

Let $\omega' = (\tau^1, \tau^2, ...)$ and let $v(t, \tau^1, \tau^2, ...)$ take values in U, be nonanticipating and simultaneously measurable on each set $[0, 1 + M] \times \Omega'_i, \Omega'_i := \{(\tau^1, \tau^2, ...) : \tau^{i+1} > 1 + M\}$. (The set of such controls is denoted U''.) The probability measure of the ω''s is denoted P', it is generated by P, (the density $\dot{\mu}'(\tau^1, ..., \tau^i)$ of $(\tau^1, ..., \tau^i)$ generated by $\dot{\mu}(\tau_1, ..., \tau_i)$). Let, for any given $v(t, \omega')$, $z^v(t, \omega') := z(t, \omega')$, for $t \in [0, 1 + M]$, be the solution - continuous in t - of

$$\dot{z}(t, \omega') = h(t, z(.), v(t, \omega'), \omega') \quad (51)$$

Define, for $s \in [0, 1]$, $x(s, \omega) = \sum_{i=0}^{\infty} z(s + a_i, \omega')1_{(\tau^i+M_i,\check{\tau}^{i+1}]}(s + a_i)$, and

$$u(s, \omega) = \sum_{i=0}^{\infty} v(s + a_i, \omega')1_{(\tau^i+M_i,\check{\tau}^{i+1}]}(s + a_i) \quad (52),$$

Now, $z(t, \omega')$ satisfies $\dot{z}(t, \omega') = f(t - a_i, z(t, \omega'), v(t, \omega'), \tau_1, \tau_2, ...)$ for $t \in (\tau^i + M_i, \check{\tau}^{i+1}]$. If $\tau^{i+1} < 1 + M$, then $x(\tau_{i+1}-, \omega) - x(\tau_i+, \omega) =$

$z(\tau^{i+1}, \omega') - z(\tau^i + M_i, \omega') = \int_{\tau^i+M_i}^{\tau^{i+1}} f(t - a_i, z(t, \omega'), v(t, \omega'), \omega)dt = \int_{\tau_i}^{\tau_{i+1}} f(s, z(s + a_i, \omega'), v(s + a_i, \omega'), \omega)ds = \int_{\tau_i}^{\tau_{i+1}} f(s, x(s, \omega), u(s, \omega), \omega)ds.$

Similarly, for $t' \in [\tau_i, \tau_{i+1}], \tau_{i+1} < 1$, $\int_{\tau_i}^{t'} f(s, x(s, \omega), u(s, \omega), \omega)ds = x(t', \omega) - x(\tau_i+, \omega)$. Also, for $\tau_{i+1} \geq 1 > \tau_i$, for $t \in (\tau_i, 1)$ $(\Rightarrow t + a_i \in (\tau^i + M_i, \check{\tau}^{i+1}) = (\tau^i + M_i, \tau^i + M_i + 1 - \tau_i))$, $x(t, \omega) - x(\tau_i+, \omega) = z(t + a_i, \omega') - z(\tau^i + M_i, \omega') =$

$\int_{\tau^i+M_i}^{t+a_i} f(\rho - a_i, z(\rho, \omega'), v(\rho, \omega'), \omega)d\rho = \int_{\tau_i}^{t} f(s, x(s, \omega), u(s, \omega), \omega)ds.$

Moreover, for $\tau_{i+1} < 1$,

$$x(\tau_{i+1}+,\omega) - x(\tau_{i+1}-,\omega) = z(\tau^{i+1} + M_{i+1},\omega') - z(\tau^{i+1},\omega') =$$
$$\int_{\tau^{i+1}}^{\tau^{i+1}+M_{i+1}} (1/M_{i+1}) g(\tau_{i+1}, z(\tau^{i+1},\omega'), i+1) =$$
$$g(\tau_{i+1}, z(\tau^{i+1},\omega'), i+1) = g(\tau_{i+1}, x(\tau_{i+1}-,\omega), i+1).$$

Hence, $(x(.,\omega), u(.,\omega))$ satisfies (1) and (47). Symmetrically, if $(x(.,.), u(.,.))$ satisfies (1) and (47), there is a pair $(z(.,.), v(.,.))$ satisfying (51), ($u(.,\omega)$ and $v(.,\omega')$ again related as in (52)). For $t \in [0, 1+M]$, define $u_h^*(t,\omega')$ by $u_h^*(t, \tau^1, \tau^2, ...) = u^*(t - a_i, \tau_1, \tau_2, ...)$ for t in $(\tau^i + M_i, \check{\tau}^{i+1}]$, $u_h^*(.,.)$ arbitrary elsewhere, and let $x_h^*(.,.)$ be the solution of (51), for $v(.,.) = u_h^*(.,.)$.

Now, (51) is a retarded differential equation. So let us instead consider the following ordinary differential equation system:

$$\begin{aligned}
&\dot{y}_0 = h_0(t, y_0(t), v(t,\omega'), \omega') := h(t, y_0(.), v(t,\omega'), \omega') 1_{[0,\check{\tau}^1]}(t), y_0(0) = 0,\\
&\dot{y}_i = h_i(t, y_0(t), ..., y_i(t), v(t,\omega'), \omega') :=\\
&h(t, \textstyle\sum_{0 \le j \le i-1} y_j(.), v(t,\omega'), \omega') 1_{(\tau^i, \tau^i + M_i]}(t) +\\
&h(t, \textstyle\sum_{0 \le j \le i} y_j(.), v(t,\omega'), \omega') 1_{(\tau^i + M_i, \check{\tau}^{i+1}]}(t),\\
&y_i(0) = 0, i > 0.
\end{aligned} \tag{53}$$

(The system does become non-retarded, as y_j is constant on $(\check{\tau}^{j+1}, 1+M]$.) Write $y = (y_0, y_1, ...)$, $|y| = \sum_i |y_i|$ and

$$\begin{aligned}
&\dot{y} = (\dot{y}_0, \dot{y}_1, ...) = F(t, y, v(t,\omega'), \omega'), \text{ where } F(t, y, v, \omega') =\\
&(h_0(t, y_0, v, \omega'), h_1(t, y_0, y_1, v, \omega'), h_2(t, y_0, y_1, y_2, v, \omega'), ...).
\end{aligned} \tag{54}$$

Let $y^*(t,\omega)$ be the solution of (54) corresponding to $u_h^*(.,.)$. and let $\Pi(y) = \sum_i y_i$. Then, if $y(t,\omega')$ is a solution of (54), $\Pi y(t,\omega')$ is a solution of (51). Evidently, for the pair $(y^*(.,.), u_h^*(.,.))$, the system defined by F satisfies all conditions in Remark 2, for π replaced by $\pi\Pi$, a replaced by $\Pi^* a$, $T = 1$ replaced by $T = 1 + M$, M'_π replaced by $\max\{1, M'_\pi\}$, and for $M(t,\omega)$ and $M'(t,\omega)$ replaced by $M_F(t,\omega)$ and $M'_F(t,\omega)$, respectively, where $M_F(t,\omega)$ and $M'_F(t,\omega)$ are defined as follows: Let $M_F(t,\omega) := \sum_i M(t - a_i, \omega) 1_{(\tau^i + M_i, \check{\tau}^{i+1}]}(t)$ and $M'_F(t,\omega) := \sum_{i=0}^{\infty} \{M'(t - a_i, \omega) 1_{(\tau^i + M_i, \check{\tau}^{i+1}]}(t) + 1_{(\tau^{i+1}, \tau^{i+1} + M_{i+1}]}(t)\}$. Then $E\int_0^{1+M} M_F(t,\omega) dt = E\int_0^1 M(t,\omega) dt$ and $\text{essup} \int_0^{1+M} M'_F(t,\omega) dt =$ $\text{essup} \int_0^1 M'(s,\omega) ds + \sum M_i < \infty$.

Hence, (33) - (37) are satisfied by the system defined by the F of (54), for $T = 1$ replaced by $1 + M$, π replaced by $\pi\Pi$, a replaced by $\Pi^* a$, $x^u(.,\omega)$

replaced by $y^v(.,\omega')$, and q^u replaced by q_F^v, the linear perturbation pertaining to the system F, $u, u', u'', \hat{u}$ are then renamed $v, v', v'', \hat{v}$, they belong to the set U_K'' corresponding to U^K.

Now, $\Pi q_F^v(1+M,\omega') = q_z^v(1+M,\omega')$, where $q_z^v(0,\omega') = 0$, and

$$\begin{aligned}\dot{q}_z^v(t,\omega') = \textstyle\sum_{i=0}^{\infty}\{& f_x(t-a_i, x_h^*(t,\omega'), u_h^*(t,\omega'),\omega)1_{(\tau^i+M_i,\check{\tau}^{i+1}]}(t)q^z(t,\omega')+\\ &(1/M_{i+1})g_x(\tau_{i+1}, x_h^*(\tau^{i+1},\omega'), i+1)1_{(\tau^{i+1},\tau^{i+1}+M_{i+1}]}(t)q^z(\tau^{i+1},\omega')\}\\ &+h(t, x_h^*(.,\omega'), v(t,\omega'),\omega') - h(t, x_h^*(.,\omega'), u_h^*(t,\omega'),\omega').\end{aligned} \tag{55}$$

Define the piecewise continuous function $t \to q^u(t,\omega)$ by $\dot{q}^u(t,\omega) =$

$$f_x(t,x^*(t,\omega),u^*(t,\omega),\omega)q^u(t,\omega)+f(t,x^*(t,\omega),u(t,\omega),\omega)-f(t,x^*(t,\omega),u^*(t,\omega),\omega),$$
$q^u(0,\omega) = 0$, $q^u(\tau_i+,\omega) = \hat{g}(\tau_i, x^*(\tau_i-,\omega), i)q^u(\tau_i-,\omega)$.

As for the nonlinear system (51), so also for the linear system (55) there is a one to one correspondence between solutions $q_z^v(.,\omega')$ and $q^u(.,\omega)$, where again $u(.,)$ and $v(.,.)$ are related as in (52). Hence, for $u(.,)$ and $v(.,.)$ thus related, $x^u(1,\omega) = \Pi y^v(1+M,\omega')$, $q^u(1,\omega) = \Pi q_F^v(1+M,\omega')$, where $x^u(t,\omega)$ satisfies (1) and (47). Thus even the jumping system (1),(47) satisfies (33)-(37), and continuity of $u(.,.) \to (Ea\cdot x^u(1,.), \pi x^u(1,.))$ continues to hold. As (4) holds, Theorem B in Appendix again applies. Thus, for some $\Lambda_0 \geq 0, \nu$, ν bounded on B_α, $(\Lambda_0, \nu|_{B_\alpha}) \neq 0$, for all $u(.,.) \in U^K$, $\langle q^u(1,.), \nu_* \rangle \leq 0$, where $\nu_* = \pi^*\nu + \Lambda_0 E\langle ., a\rangle$. For $C(t,s,.)$ as defined in Theorem 2, from the last inequality, then (5) follows.

The inequality $\langle q^u(1,.), \nu_*\rangle \leq 0$ means that $\langle q_z^v(1+M,.), \nu_*\rangle \leq 0$, or $\langle \int_0^{1+M} C_z(1+M,s,.)(h(s,x_h^*(s,.),u(s,,.),.)-h(s,x_h^*(s,.),u_h^*(s,.),.))ds, \nu_*\rangle \leq 0$, where $C_z(.,.,.)$ is the resolvent of equation (55). Let $p_z(s,.) = C_z(1+M,s,.)^*\nu_*$. If, for $t<1$, $C(1,t,.)^*\nu|_{\Phi_t}$ is absolutely continuous with respect to P, then the Radon-Nikodym derivative $p_z^*(t,\omega')$, (here for the moment a row vector), satisfies, for a.e. $s \in (\tau^k + M_k, \check{\tau}^{k+1}) \cap (1, 1+M)$, $\dot{p}_z^*(s,\omega') =$

$$\begin{aligned}&-p_z^*(s,\omega')f_x(s-a_i, x_h^*(s,\omega'), u_h^*(s,\omega'),\omega)1_{(\tau^k+M_k,\check{\tau}^{i+1}]}(s) - p_z^*(s+,\omega'^k,s) =\\ &-p_z^*(s,\omega')f_x(s-a_i, x_h^*(s,\omega'), u_h^*(s,\omega'),\omega)1_{(\tau^k+M_k,\check{\tau}^{i+1}]}(s) -\\ &p_z^*(s+M_{k+1}+,\omega'^k,s)(I+g_x(s,x^*(s,\omega'),i+1)),\end{aligned}$$

the last equality having the following explanation: Evidently, $C_z(\tau^k+M_k,\tau^k,\omega') = I+\int_{\tau^k}^{\tau^k+M_k}(1/M_k)g_x(\tau_k, x_h^*(\tau^k-,\omega'),k)C_z(\tau^k,\tau^k,\omega')d\rho = I+g_x(\tau_k, x^*(\tau_k-,\omega),k)$. Let $t' \in (\tau^k+M_k, 1+M)$. Multiplying by $C_z(t',\tau^k+M_k,.)$ gives $C_z(t',\tau^k,.) =$

$C_z(t', \tau^k + M_k, .)(I + g_x(\tau_k, x_h^*(\tau^k -, \omega'), k)$. Now, recall from part A. that there $p^*(\tau_i +, \omega^i)$ equals $E[C(t, \tau_i, \omega)^* \psi^t | \omega^i] \dot{\mu}(\omega^i) = \int_{\tau_i}^{\infty} E[C(t, \tau_i, \omega)^* \psi^t | \omega^i, \tau_{i+1}] \dot{\mu}(\omega^i, \tau_{i+1}) d\tau_{i+1}$. Applying the corresponding formula for $p_z(\tau^k +, \omega'^k)$ on both sides of the next to last equality yields $p_z^*(\tau^k +, \omega'^k) = (I + g_x(\tau^k, x_h^*(\tau^k), k))^* p_z^*(\tau^k + M_k +, \omega'^k)$, ($\dot{\mu}'(\tau^{k+1} | \omega'^k)$ vanishes on $(\tau^k, \tau^k + M_k)$).

Define $p^*(s', \omega^k) = p_z^*(s' + a_k, \omega'^k)$ when $s' \in (\tau_k, \min\{1, \tau_{k+1}\})$. Then, for $s' + a_k = s, p^*(s' +, \omega^k, s') = p_z^*(s' + a_{k+1} +, \omega'^k, s) = p_z^*(s + M_{k+1} +, \omega'^k, s)$. Hence, the equation for $p_z^*(., .)$ yields (50).

Remark 8. The conclusion of Theorem 2 also holds for the following modifications of the assumptions. Assume that $\pi = \pi' \pi_Y$ as in Remark 3, that f satisfies the conditions in Remark 3 for $M_{**}(., .)$ as defined below, that $\hat{g}$ and $\hat{g}_x$ are continuous, that $\pi' \hat{g}(t, (x', x''), i)$ does not depend on x'', and that for any $\varepsilon > 0$, for some δ, $|x - \hat{x}| < \delta, x, \hat{x} \in B(x^*(t, \omega), \check{d}) \Rightarrow |\pi' \hat{g}_x(t, x, i) - \pi' \hat{g}_x(t, \hat{x}, i)| < \varepsilon$ for all t, i, ω. Moreover, for some numbers M_i and M_i'', $|\pi'(\hat{g}_{x'}(t, x', i) - I)| \leq M_i$ for all $(t, x') \in J \times B(\pi' x^*(t, \omega), \check{d})$ for all ω, $\sum M_i < \infty$, $|\pi''(\hat{g}_x(t, x, i) - I)| \leq M_i''$ for all $(t, x) \in J \times B(\pi' x^*(t, \omega), \check{d}) \times B(\pi'' x^*(t, \omega), M_{**}(t, \omega))$ for all ω, and $E[\phi(1, .)^2 \{1 + \int_J M_*'(t, \omega) + \sum_{\tau_i \leq T} M_i'')\}]^p < \infty$, ($p$ as in Remark 3), where $\phi(t, \omega)$ is the piecewise continuous solution of $\dot{\phi} = \max\{M'(t, \omega), M_*'(t, \omega)\}\phi(t), \phi(0) = 1, \phi(\tau_i +) - \phi(\tau_i -) = \max\{M_i, M_i''\}$ and $M_{**}(t, \omega) = \phi(T, \omega) \int_J \max\{M(t, \omega), M_*(t, \omega)\} dt$. Finally,
$E[\int_0^T |f(t, x^*(t, \omega), u^*(t, \omega), \omega)| dt + \sum_{\tau_i < T} |\hat{g}(\tau_i, x^*(\tau_i -, \omega), i) - x^*(\tau_i -, \omega)|] < \infty$.

The proof is a simple modification of the above proof of Theorem 2. On the rewritten system the same arguments as used in case of Remark 3 will work.

Remark 9 Frequently, π is a projection, and let us consider this case. Often for an admissible solution (and then even more for an optimal solution) to exist, we must have $\pi g = 0$. When $|f^i| \leq K$ for all (t, x, u, ω), we cannot have true jumps in a component x^i of x on which there is a terminal bound of the form $x^i(T, .) = \tilde{x}^i$ a.s. Jumps can occur arbitrarily close to T, and $x^i(t)$ cannot then be steered a.s. to $\tilde{x}^i$. The situation may be different if f^i can be chosen arbitrarily large or small by suitable choices of u. Note also that, sometimes, the body condition may be more difficult to obtain if $\pi g \neq 0$.

Let us now turn back to the general system (1).

Modification of the general set-up

Remark 10 Assume the conditions in Remark 3. Let r be a continuously differentiable map from Y into a Banach space V. Assume that the derivative of r is uniformly bounded and uniformly continuous. Replace (3) by $r(\pi x) = 0$ a.s. and assume, instead of (4), that for some $y^V(t,\omega) \in L_\infty(J \times \Omega, V), y^V(.,.)$ progressively measurable, and for some $K^* > 0, \alpha > 0$, with $B_\alpha^V := \{\int_J v(.,.)dt : v(.,.) \in L_\infty(J \times \Omega, V) : v$ progressively measurable $, |v(.,.)|_\infty < \alpha\}$, the inclusion $\int_J y^V(t,\omega)dt + B_\alpha^V \subset$ $\text{co}\{r_x(\pi x^*(T,.)) \int_J \pi C(T.,t,.)(f(t,x^*(t,.),u(t.,.),.)-f(t,x^*(t,.),u^*(t,.)))dt : u(.,.) \in U^{K^*}\}$ holds. Then, for some $\Lambda_0 \geq 0$, some linear functional ν^+ on $L_\infty(\Omega,\Phi,V)$, bounded on $B_\alpha^V, (\Lambda_0, \nu^+|_{B_\alpha^V}) \neq 0$, the inequality (57) below holds for ν_{**} replaced by $\pi^*(r_x(\pi x^*(T,.))^*\nu^+ + \Lambda_0 E\langle ., a\rangle$.

Proof: Extend the control interval to $[0, T+1]$. Let an auxiliary state y be governed by $\dot{y} = r(\pi x)$ on $(T, T+1]$, $\dot{y} = 0$ on [0,T]. With $f = 0$ on $(T, T+1]$, x is governed by $\dot{x} = f$, $x(0) = x_0$, moreover $y(0) = 0$, $x(T+1,.)$ is free, $y(T+1,.) = 0$ a.s.. Then $r(\pi x(T,.)) = y(T+1,.)$, so applying the above results to this system yields the results in this remark.

Remark 11 Let $\tilde{Y}$ be a Banach space, let π_* be a bounded linear map from X' into $\tilde{Y}$, and write $\tilde{\pi} = \pi_*\pi'$. To the terminal condition (3) (i.e. $\pi x(T,.) = \tilde{y}$ a.s.), add the condition :

$$\tilde{\pi}x(T,.) \in W \text{ a.s., } W \text{ a fixed closed convex body in } \tilde{Y}. \tag{56}$$

Assume the conditions in Remark 3. Then, for some $\Lambda_0 \geq 0$, some ν as before, some $\hat{\nu} \in L_\infty(\Omega, \Phi, \tilde{Y})^*, (\Lambda_0, \nu|_{B_a}, \hat{\nu}) \neq 0$, for $\nu_{**} := \pi^*\nu + \tilde{\pi}^*\hat{\nu} + \Lambda_0 E\langle ., a\rangle$, for all $u(.,.) \in U^K$, (K any given positive number),

$$\langle \int_J C(T,t,.)(f(t,x^*(t,.),u(t,.),.) - f(t,x^*(t,.),u^*(t,.),.))dt, \nu_{**}\rangle \leq 0. \tag{57}$$

Moreover,

$$\langle w(.) - \tilde{\pi}x^*(T,.), \hat{\nu}\rangle \geq 0, \text{ for all } w(.) \in \tilde{W} := \{w(\omega) \in L_\infty(\Omega, \Phi, \tilde{Y}) : w(\omega) \in W \text{ a.s.}\} \tag{58}$$

If, for $t < T$, $C(T,t,.)^*(\pi^*\nu + \tilde{\pi}^*\hat{\nu})|_{\Phi_t}$ is absolutely continuous with respect to P, then in the cases A and B in the Applications, the adjoint equations (40) and (50), respectively, hold. Moreover, the maximum condition (41) is satisfied. Finally, the end condition (44) on $p^*(., \omega^i)$ holds as before, provided ν_* is replaced by ν_{**}.

Proof: Let $T = 1, K \geq \max\{1, K^*\}$, and $W' = (\tilde{W} - \tilde{\pi}x^*(1,.)) \cap \mathrm{cl}B(0,1) \subset L_\infty(\Omega, \Phi, \tilde{Y})$. Then W' is a closed bounded convex body. The condition $\tilde{\pi}x(1,.) - \tilde{\pi}x^*(1,.) - w(.) = 0$ a.s. for some $w(.) \in W'$ implies (56). We now apply Theorem B, to the system $y(a) = (\dot{x}^u(.,.), \tilde{\pi}x^u(1,.) - \tilde{\pi}x^*(1,.) - w(.))$, $y^+(a) = (\dot{q}^u(.,.), \tilde{\pi}q^u(1,.) - w(.))$, $a = (u(.,.), w(.)) \in \tilde{A} := U^K \times W'$, furnished with the product metric of σ/K and $|.|_\infty$. The functions $y(a)$ and $y^+(a)$ take values in $\hat{Y} = L_\infty(J \times \Omega, X) \times L_\infty(\Omega, \Phi, \tilde{Y})$, moreover, $Z = L^\infty \times L_\infty(\Omega, \Phi, \tilde{Y})$, $\check{Z} = L_\infty(\Omega, \Phi, X)$, for $\psi = (\phi, \phi') \in \hat{Y}$, $\hat{\pi}(\psi) = (\pi \int_J \phi dt, \phi'), \check{\pi}(\psi) = \int_J \phi dt, \check{z}^* = E\langle ., a\rangle, z' = (\tilde{x}, 0)$, a constant function in $L^\infty \times L_\infty(\Omega, \Phi, \tilde{Y})$. By (14) (applied to $x' \to \pi' f$, such that x, q are replaced by $\pi' x, \pi' q$), (33), and (34), (B) is satisfied, by (27) (applied to $\pi' q$ instead of q), (35), and (36), and convexity of W', (A) and (C) are satisfied. Finally, $\mathrm{cl}\hat{\pi}y^+(\tilde{A})$ is a convex body, by Lemma 11.2 in Seierstad (1975), (4) and the fact that W' is a convex body in $L_\infty(\Omega, \Phi, \tilde{Y})$. Hence, Theorem B applies and (57) and (58) follow from the conclusion in Theorem B.

Remark 12 Suppose, for simplicity, that X is a Euclidean space, that π and $\tilde{\pi}$ are projections, that $W = \{y \in \tilde{Y} : y_i \geq \tilde{y}^i \text{ for all } i\}$ for given numbers $\tilde{y}^i$, and that the Simple Global Assumptions are satisfied. Assume that $(x^*(.,.), u^*(.,.))$ satisfies the necessary conditions (57) and (58) of Remark 11, with $U' = U^K$, for some $\nu, \hat{\nu}$ and $\Lambda_0 = 1$. Assume, finally, that the function

$$\hat{H}(x(.,.)) := \sup_{u(.,.) \in U'} \langle \int_J C(T,t,.) f(x(.,.), u(t,.),.), \nu_{**}\rangle$$

is concave in $x(.,.) \in \Delta :=$
$\{x(.,.) : x(t,\omega) = x_0 + \int_0^t y(t,\omega)dt \text{ for some } y(.,.) \in L_\infty^{prog}(J \times \Omega, X)\}$, and is bounded from above by a fixed constant on a $|.|^\infty \times |.|_\infty$-neighborhood of $x^*(.,.)$ in Δ. Then $u^*(.,.)$ is optimal in U^K .

A proof is presented in Seierstad (1991). (Provided also $\hat{g}$ satisfies the assumptions subsequent to (47), and the conditions $|\hat{g}(t,x,i) - I| \leq M_i$ for all (t,x), and $x(.) \to \langle h(x(.,.),\omega), \nu_{**}\rangle$ is concave for $x(.,.) \in \Delta$, and is bounded from above by a fixed constant on a $|.|^\infty \times |.|_\infty$-neighborhood of $x^*(.,.)$ in

Δ, where $h(x(.,.),\omega) = \sum_{i\in\{i:\tau_i<T\}} C(T,\tau_i,\omega)\hat{g}(\tau_i, x(\tau_i-,\omega), i)$, then the sufficiency result above also holds in the case of piecewise continuous systems, $C(.,.,.)$ now defined by (48),(49).)

Let us now consider the following example of a (jumping) piecewise deterministic system.

Example 2. $\max E\{\int_0^3(3-s)(1-u)ds\}$, subject to $dx/dt = u \in [0,1]$, $x(0) = 0$, $x(3) \geq 1$ with probability 1. There is a probability with intensity $\lambda > 0$ that a single downwards jump occurs with size $\tau/3$, i.e., $x(\tau+) - x(\tau-) = -\tau/3$, (thus τ is exponentially distributed with intensity $\lambda > 0$) .

Solution. An auxiliary state variable y is used to rewrite the system. Let $y(0) = 0, \dot{y} = (3-s)(1-u)$, and let us maximize $Ey(3)$. The adjoint variable corresponding to x is denoted q^* and the one corresponding to y equals $\Lambda_0 = 1$, ($\Lambda_0 = 0$ is considered later on). The pointwise maximum condition (43) gives, in a shorthand notation,

$$[q^* + t - 3](u - u^*) \leq 0 \tag{59}$$

After a jump at time τ, the solution is $u^*(t;\tau) = 0$ if $x^*(\tau+,\tau) \geq 1$, and if $x^*(\tau+,\tau) < 1$, $u^*(t;\tau) = 0$ for $t < x^*(\tau+,\tau) + 2$, while $u^*(t;\tau) = 1$ for $t > x^*(\tau+,\tau) + 2$. To show the latter assertion, note first that $q^*(t;\tau)$ is independent of t, so we write it $q_*(\tau)$. Evidently $u = 0$ to begin with, here $3 - t > q_*(\tau)$, (if at all), and $u = 1$ at the end. Let $\sigma := \sigma(\tau)$ be the point at which we switch from 0 to 1, $\sigma \in [\tau, 3]$. Then $x^*(t;\tau) = x^*(\tau+,\tau)$ for $t \in (\tau,\sigma)$, $x^*(t;\tau) = x^*(\tau+,\tau) + t - \sigma$, for $t > \sigma$, with σ determined by $x^*(3;\tau) = 1$, i.e. $x^*(\tau+,\tau) + 3 - \sigma = 1$, or $\sigma = x^*(\tau+,\tau) + 2$. At $s = \sigma$, if $\sigma \in (\tau, 3)$, by the pointwise maximum condition (59), $3 - \sigma = q_*$, so $q_*(t) = 1 - x^*(\tau+,\tau)$. Thus, $x^*(t,\tau) = \max\{x^*(\tau+,\tau), t-2\}$ (also if $\sigma = 3$, i.e. $x^*(\tau+,\tau) \geq 1$, or $\sigma = \tau$, i.e. $x^*(\tau+,\tau) = \tau - 2$).

Let us find the control function before a jump τ, written simply $u^*(t)$, (similarly we also write $x^*(t), q^*(t)$). We must have $x^*(3) \geq 2$, a jump downwards can occur arbitrarily close to 3, its size being roughly 1. We guess that $x^*(3) = 2$, (or perhaps a glance at the original problem tells us that this must be so). From (59) it is obtained that $u = 0$ when $3 - t > q^*(t)$, while $u = 1$ when $3 - t < q^*(t)$.

Now, the lowest path possible is $x^*(t) = \max\{0, t-1\}$, (which below will be shown to be the optimal one). Even for this path (and so for any path), if a jump down of size $\tau/3$ occurs, to some value $x^*(\tau+, \tau)$, $\tau < 3$, then $x^*(\tau+, \tau) \geq \max\{0, \tau - 1\} - \tau/3 > \tau - 2$, so σ must be $> \tau$. Thus, $dq^*/dt = \lambda q^*(t) - \lambda q^*(t+, t) = \lambda q^* - \lambda + \lambda(x^*(t) - t/3), (q^*(t+, t) = 1 - x^*(t+, t) = 1 - (x^*(t) - t/3), t < 3)$. Let us prove that $q^*(1) = 2$: Consider first the possibility $q^*(1) \geq 2$. Then, $\dot{q}^*(t) = \lambda(q^{*(}t) + x^*(t) - 1 - t/3) \geq \lambda(q^*(t) + (t-1) - 1 - t/3) = \lambda(q^* - 2 + 2t/3)$, which is > 0 for $t > 0$, if $q^*(t) \geq 2$. So if $q^*(1) \geq 2$, $q^*(t)$ increases when t increases, all the way to $t = 3$. Now, $q^*(1) > 2$ means that $u = 1$ on an interval greater than $[1, 3]$, which gives $u = 1$ here and $x^*(3) > 2$, a contradiction. On the other hand, on $(0, 1)$, the above inequality for $\dot{q}^*(t)$ gives that if $q^*(1) < 2$, then $q^*(t)$ stays below 2 when t decreases, (if $q^*(t)$ moves close to 2, then it gets a positive derivative, and moves away from 2). Moreover, if $q^*(1) = 2$, then $q^*(t) < 2$, for $t < 2$, close to 2, and stays below 2 when t decreases. Now, $q^*(1) < 2$ means that $u = 0$ in an interval somewhat larger than [0,1], making it impossible to obtain $x^*(3) = 2$. Hence, $q^*(1) = 2$.

Let us sum up what we have obtained: Before a jump, $u = 0$ before $t = 1$, and $u = 1$ afterwards. The state $x^*(t)$ equals $\max\{0, t-1\}$ before a jump, so just after a jump $x^*(\tau+, \tau) = \max\{0, \tau - 1\} - \tau/3$. After a jump at τ, $u = 0$ until $t = \sigma = 2 + \max\{0, \tau - 1\} - \tau/3 > \tau$ is reached, from then on $u = 1$ is used. Moreover, $x^*(3, \tau) = 1$, and $q^*(3, \tau) > 0$ for $\tau < 3$. If a jump does not occur, $x^*(3) = 2 > 1$. Now, as (59), i.e (41) holds, then (5) follows, once (46) is proved to hold for ν_* replaced by ν_{**}: Let $\hat{u}^*(t, \tau)$ be any bounded nonanticipating function. Then (45) reduces to $\int_0^\infty \int_b^3 (3 - s)(u^*(s, \tau) - \hat{u}^*(s, \tau)) ds \lambda e^{-\lambda\tau} d\tau + \langle \int_b^3 \hat{u}^*(s, .) - u^*(s, .) ds, \hat{\nu} \rangle$. When $b \uparrow 3$, both terms become equal to zero, the second term, since $\hat{\nu}$ is continuous in $|.|_\infty$ - norm. Sufficient conditions, Remark 11, give optimality. Note that the solution in the problem is the same for all $\lambda > 0$.

Though we don't need the complete specification of $\hat{\nu}$, let us nevertheless write it down: By (58), $\hat{\nu}$ is nonnegative, and $\hat{\nu}((3, \infty))$ $(= \hat{\nu}(1_{(3,\infty)})(\tau)) = 0$, because, using (58), for $\tau \in (3, \infty)$, $x^*(3) = x^*(3, \tau) = 2 > 1$, so choosing $w(.)$ equal to $1_{(3,\infty)}(\tau) + x^*(3, \tau)$ and to $x^*(3, \tau) - 1_{(3,\infty)}(\tau)$, gives $\hat{\nu}(1_{(3,\infty)})(\tau)) \leq 0$, $\hat{\nu}(-1_{(3,\infty)})(\tau)) \leq 0$. Now, by (44), used for $i = 1$, in $(0, 3)$, $\hat{\nu}$ is given by a density, namely

$$\lim_{t \uparrow 3} q^*(t, \tau) = q^*(3, \tau) = q_*(\tau) = 1 - \max\{0, \tau - 1\} - \tau/3.$$

Next, in $(1,3)$, before any jump, $\dot{q}^*(t) = \lambda q^* - \lambda + \lambda(t-1-t/3)$, with $q^*(1) = 2$, this implies $q^*(t) = 2 + (2/3 + 2/3\lambda)e^{\lambda(t-1)} - 2t/3 - 2/3\lambda > 0$. Then, using (44) for $i = 0$, gives $0 < q^*(3)e^{-\lambda 3} = \lim_{t\uparrow 3} q^*(t)e^{-\lambda t} = \lim_{t\uparrow 3} \hat{\nu}((t,\infty)) = \lim_{t\uparrow 3} \hat{\nu}((t,3])$. As a bounded linear functional on L_∞-space, $\hat{\nu}$ vanishes on $P-$null sets (here then also on Lebesgue null sets). Still, informally speaking, (or considering a representation of $\hat{\nu}$ restricted to the space of functions continuous near 3), we may say that $\hat{\nu}$ has an atom at $\tau = 3$. This corresponds to the fact that $\lim_{\tau\nearrow 3} x^*(3,\tau) = 1$.

When using sufficient conditions, it is not necessary to prove $\Lambda_0 \neq 0$, and the absolute continuity of $\hat{\nu}$ on $[0,s]$, s a given point < 3. Both properties, however follow easily from (5), i.e.

$$\langle \int_0^3 (u(s,.) - u^*(s,.))ds, \hat{\nu}\rangle \leq \Lambda_0 E[\int_0^3 (3-s)(u(s,.) - u^*(s,.))ds]. \quad (60)$$

as we shall see. First, let us show that $\Lambda_0 \neq 0$. For the moment, we know nothing about $u^*(.,.), x^*(.,.)$, except of course that $x^*(.,.)$ has to satisfy the conditions $x^*(3,\tau) \geq 1$. Assume that $\Lambda_0 = 0$, and let $A := \{\tau : \int_0^3 u^*(t,\tau)dt \geq 2.5\}$. Then, a.s., $x^*(3,\tau) \geq 1.5$, so, by (58), $\hat{\nu}(A) = 0$ and $\hat{\nu}(\complement A) \neq 0$. Inserting $u = 1$ in (60), we get $\langle 1_{\complement A}(\tau)(3-2.5), \hat{\nu}\rangle \leq \langle 1_{\complement A}(\tau)\int_0^3 (1-u^*(t,\tau))dt, \hat{\nu}\rangle = \langle \int_0^3 (1-u^*(t,\tau))dt, \hat{\nu}\rangle \leq 0$, which gives $\hat{\nu}(\complement A) = 0$, a contradiction.

So $\Lambda_0 = 1$. Next, let $s \in [0,3)$. Choose $b \in (0,1)$, b so small that $(3 - 3b - s) =: 3k > 0$. Let $D : \{\tau : x^*(3,\tau) \geq 1+b\}$. By (58), $\hat{\nu}(D) = 0$. Moreover, for any $r \in [0,3)$, for $\tau \in (r,3)$, τ close to 3, $x^*(r,\tau) + 3 - r = x^*(r,\tau) + \int_r^3 dt \geq x^*(r,\tau) + \int_r^3 u^*(t,\tau)dt \geq 2$, (otherwise 1 cannot be reached at $t = 3$, as a jump down of size roughly 1 can occur). By non-anticipation, from these inequalities it follows that for all $\tau \in (r,3)$, $r - 1 \leq x^*(r,\tau) = \int_0^r u^*(t,\tau)dt$. Let $r \uparrow \tau$. Then the last inequality also holds for $r = \tau-$. If $\tau \notin D$, $\tau \in (0,3)$, then $x^*(3,\tau) \leq 1+b$, so $2+b-\tau = 1+b-(\tau-1) \geq x^*(3,\tau) - x^*(\tau-,\tau) = \int_\tau^3 u^*(t,\tau)dt - \tau/3$, and $\int_\tau^3 u^*(t,\tau)dt \leq 2+b-2\tau/3$. Thus, for $\tau \notin D$, $\tau \leq s$, $\int_\tau^3 (1 - u^*(t,\tau))dt \geq 3 - \tau - [2 + b - 2\tau/3] = 1 - b - \tau/3 \geq k > 0$. Now, let H be a measurable set in $[0,s]$. Then, $k\langle 1_H(\tau), \hat{\nu}\rangle = k\langle 1_H(\tau)1_{\complement D}(\tau), \hat{\nu}\rangle \leq \langle 1_H(\tau)1_{\complement D}(\tau)\int_\tau^3 (1 - u^*(t,\tau))dt, \hat{\nu}\rangle \leq \langle 1_H(\tau)\int_\tau^3 (1 - u^*(t,\tau))dt, \hat{\nu}\rangle \leq E[1_H(\tau)\int_\tau^3 (3-t)(1 - u^*(t,\tau))dt]$. Note that $1_H(\tau)1_{(\tau,3]}(t)$ is nonanticipating, so the last inequality follows from (60). This sequence of inequalities gives the absolute continuity property of $\hat{\nu}$.

Let us consider for a moment a problem where the state is a scalar, and where there is an inequality restriction of the form $x(T,\omega) \geq \check{x}$ a.s., and where a maximum number N of jumps can occur, all with nonzero probabilities in all intervals, and where the jumps have equal and constant size c. Moreover, assume $|f| \leq K$ for all (t,x,ω). Then if c is positive, for any given number $j < N$ of jumps having occurred at a certain time t, so long as no further jumps occurs, it suffices to steer the state in such a manner that $x(T,\omega^j) \geq \check{x}$: If $x(T,\omega^j) \geq \check{x}$ then if one or more jumps occur in $(T - c/2K, T]$, we automatically have $x(T,\tau_1,...,\tau_k) \geq \check{x}$. (For one jump, for $\tau_{j+1} \in (T - c/2K] : x(\tau_{j+1}-,\omega^j) \geq \check{x} - c/2$, $x(\tau_{j+1}+,\omega^j) \geq \check{x} + c/2$ and $x(T,\omega^j) \geq \check{x} + c/2 - c/2 \geq \check{x}$.) If c is negative, we have to steer the state in such a manner that the even the most demanding terminal restriction in this case is satisfied, namely $x(T,\omega^j) \geq \check{x} - (N-j)c$, (as many as $N-j$ downwards jumps can occur arbitrarily close to T).

For a general jump function g, in case j jumps have occurred before T, one has to steer the state in such a manner that $x(T,\tau_1,...,\tau_j) \geq \check{x}$,
$g(T,x(T,\tau_1,...,\tau_j)) \geq \check{x}$, $g(T,g(T,x(T,\tau_1,...,\tau_j))) \geq \check{x}$,...
$(g^{(N-j)}(T,x(T,\tau_1,...,\tau_j)) \geq \check{x}$ are all satisfied, $g^{(N-j)}(T,.)$ meaning the composition of $x \to g(T,x)$, $N-j$ times).

Remark 13 Consider the problem of Remark 11. Let $\bar{Y}$ be a Euclidean space, and let $\bar{\pi}$ be a bounded linear map from $L_1(\Omega,\Phi,X)$ into $\bar{Y}$. To the terminal conditions in Remark 11, add yet another terminal condition, namely $\bar{\pi}x(T,.) = \bar{y}$, where $\bar{y}$ is a fixed point in $\bar{Y}$. Then (58) and (57) hold, (57) for ν replaced by $\pi^*\nu + \tilde{\pi}^*\tilde{\nu} + \bar{\pi}^*\bar{\nu} + \Lambda_0 E\langle .,a\rangle$, where $\bar{\nu} \in \bar{Y}^* = \bar{Y}$, $\Lambda_0 \geq 0, (\Lambda_0, \nu|_{B_\alpha}, \tilde{\nu}, \bar{\nu}) \neq 0$.

The proof of this condition (for $T = 1$) in the case where $\bar{C} :=$
$\text{clco}\{(\pi q^u(1,.), \tilde{\pi}q^u(1,.) - w(.), \bar{\pi}q^u(1,.)) : u \in U^*, w(.) \in W'\}$ is a convex body in $L^\infty \times L_\infty(\Omega,\Phi,\tilde{Y}) \times \bar{Y}$ is closely parallel to the proof of Remark 11: Just add the components $\bar{\pi}x^u(1,.))$ and $\bar{\pi}q^u(1,.))$ to the components of $y(a)$ and $y^+(a)$ as defined in Remark 11. Now, it is easily seen that $\bar{C}$ is either a convex body or contained in a closed hyperplane, (if necessary, see Seierstad (1975), Remark 11.1), and in the latter case the existence of $(\Lambda_0, \nu, \tilde{\nu}, \bar{\nu})$, $(\Lambda_0, \nu|_{B_\alpha}, \tilde{\nu}, \bar{\nu}) \neq 0$ is trivial.

Of course, the terminal conditions introduced here encompasse, when X is Euclidean, terminal conditions of the "soft" type: $Ex^i(T,.) = \bar{x}^i$.

Appendix.

Gronwall's lemma Assume that $h(t,x)$ is measurable in t, and Lipschitz continuous in $x \in X$ with constant $M(t)$. Let $\check{y}(t)$ and $\check{z}(t)$ be two continuous functions such that $|\check{y}(0) - \check{z}(0)| = \varepsilon, |\check{y}(t) - \check{y}(0) - \int_0^t h(s, \check{y}(s))ds| \leq \alpha(t), |\check{z}(t) - \check{z}(0) - \int_0^t h(s, \check{z}(s))ds| \leq \beta(t)$. Then $|\check{z}(t) - \check{y}(t)| \leq [\varepsilon + \sup_{s \leq t}(\alpha(s) + \beta(s))]e^{\int_0^t M(s)ds}$.

For example, if $|f_x(t,x)| \leq M(t)$, and $|f(0)| \leq a(t)$, then for $\check{z}(.) = 0, \check{y}(.)$ the solution $x(.)$ of $\dot{x} = f(t,x), x(0) = x_0$, it follows that $|x(t)| \leq (|x_0| + \int_s^t a(s)ds)e^{\int_0^t M(s)ds}$.

Theorem A. (Seierstad (1970). Let Y be a normed space, and let $\tilde{A}$ be a complete pseudometric space with metric $\partial(.,.)$. There are given numbers $M' > 0, \bar{e} > 0$, $d_0 \in (0,1]$ and elements $\bar{a} \in \tilde{A}$ and $\bar{p} \in Y$. Let $\tilde{A}_d = \{a \in \tilde{A} : \partial(a, \bar{a}) < M'd\}$. Let $y(.) : \tilde{A} \to Y$ and $y^+(.) : \tilde{A} \to Y$ be given functions with $y^+(\bar{a}) = 0$, and $y(.)$ continuous. Assume $M' \geq \sup_{a \in \tilde{A}} \partial(a, \bar{a})$. Assume furthermore, for all $d \in (0,1], \varepsilon > 0, a'', a \in \tilde{A}_d, k \in [0,1]$, that there exists an element $a' \in \tilde{A}_d$ such that

$$|ky^+(a'') + (1-k)y^+(a) - y^+(a')| \leq \varepsilon, \; \partial(a, a') \leq 2M'kd. \qquad (A)$$

Moreover, let $\check{e}(d) \geq 0$ be a given error function, (i.e. here an extended real-valued function on $(0, \infty)$ such that $\lim_{d \searrow 0} \check{e}(d) = 0$), and assume that for all $d \in (0, d_0]$,

$$|y(a') - y^+(a') - (y(a) - y^+(a))| \leq \check{e}(d)\partial(a', a) \text{ for all } a, a' \in \tilde{A}_d, \qquad (B)$$

$$dy^+(\tilde{A}) \subset cly^+(\tilde{A}_d). \qquad (C)$$

Finally, assume that

$$B(\bar{p}, 2\bar{e}) \subset cly^+(\tilde{A}). \qquad \text{(D)}$$

Then, for some $d' \in (0, d_0]$, $B(d\bar{p}, d\bar{e}) + y(\bar{a}) \subset y(\tilde{A})$ for all $d \in (0, d']$.

Theorem B. Let $\hat{Y}$ be a normed space, and let $\tilde{A}$ be a complete pseudometric space with metric $\partial(.,.)$. There are given numbers $M' > 0, d_0 \in (0,1]$ and an element $\bar{a} \in \tilde{A}$. Define $\tilde{A}_d = \{a \in \tilde{A} : \partial(a, \bar{a}) < M'd\}$. Let $y(.) : \tilde{A} \to \hat{Y}$ and $y^+(.) : \tilde{A} \to \hat{Y}$ be given functions with $y^+(\bar{a}) = 0$. Assume

$M' \geq \sup_{a \in \tilde{A}} \partial(a, \bar{a})$. Let Z and $\check{Z}$ be Banach spaces, $\hat{\pi}$ a linear map from $\hat{Y}$ into Z, $\check{\pi}$ a linear map from $\hat{Y}$ into $\check{Z}$. Assume that $\text{cl}\hat{\pi}\text{y}^+(\tilde{A})$ is a convex body. Assume also that $\bar{a}$ is optimal in the problem $\max_{a \in \tilde{A}} \langle \check{\pi} y(a), \check{z}^* \rangle$, subject to $\hat{\pi} y(a) = z'$, where $z' \in Z, \check{z}^* \in \check{Z}^*$ are fixed. Assume that (A)-(C) in Theorem A are satisfied by y, y^+ replaced by $(\hat{\pi} y(a), \langle \check{\pi} y(a), \check{z}^* \rangle)$,$(\hat{\pi} y^+(a), \langle \check{\pi} y^+(a), \check{z}^* \rangle)$ with $a \to (\hat{\pi} y(a), \langle \check{\pi} y(a), \check{z}^* \rangle)$ continuous. Then, for some $\Lambda_0 \geq 0, z^* \in Z^*, (\Lambda_0, z^*) \neq 0$, $\Lambda_0 \langle \check{\pi} y^+(a), \check{z}^* \rangle + \langle \hat{\pi} y^+(a), z^* \rangle \leq 0$ for all $a \in \tilde{A}$.

Proof Note that in Theorem A, by continuity of $y(.)$, (B), and (C), in fact $y^+(\tilde{A})$ is bounded. It is easily seen that $C := \text{cl}\{(\hat{\pi} y^+(a), \langle \check{\pi} y^+(a), \check{z}^* \rangle + \gamma) : a \in \tilde{A}, \gamma \in [-4M'', 0]\}$ is a convex body, $M'' = \max_{a \in A} |\langle \check{\pi} y^+(a), \check{z}^* \rangle|$, (if necessary use Lemma 11.2 in Seierstad (1975)). Let $(a, \gamma), (\bar{a}, 0), \tilde{A} \times [-4M'', 0]$, $(\hat{\pi} y(a), \langle \check{\pi} y(a), \check{z}^* \rangle + \gamma), (\hat{\pi} y^+(a), \langle \check{\pi} y^+(a), \check{z}^* \rangle + \gamma), \max\{\partial(a', a''), M'/4M''|\gamma' - \gamma''|\}$, play the roles of $a, \bar{a}, \tilde{A}, y(a), y^+(a), \partial(a', a'')$ in Theorem A. Then, for no $\alpha > 0$ can $(0, \alpha)$ be interior in C, for if so, $d'(0, \alpha) + (\hat{\pi} y(\bar{a}), \langle \check{\pi} y(\bar{a}), \check{z}^* \rangle) \in (\hat{\pi} y(a), \langle \check{\pi} y(a), \check{z}^* \rangle + \gamma : a \in \tilde{A}, \gamma \in [-4M'', 0]\}$, contradicting optimality. As $\text{int}C$ and $\{0\} \times (0, \infty)$ are disjoint, they are separated by a nonzero continuous linear functional represented by (Λ_0, z^*), which yields the conclusion in the Theorem.

Theorem C. U^K is complete in σ.

Proof: First $\tilde{\sigma}$-completeness is shown: Let $u_n(.,.)$ be a Cauchy-sequence in $\tilde{\sigma}$. Choose a subsequence u_{n_j} such that $\text{essup} \int_J 1_{H_{u_{n_j}, u_{n_{j+1}}}} dt \leq 1/2^j$. Let $B_i = \cup_{j \geq i} H_{u_{n_j}, u_{n_{j+1}}}$. Then, for any i, for $k \geq i$, a.s., $u_{n_i} \neq u_{n_k}$ only for $(t, \omega) \in B_i$. Define $u = u_{n_i}$, for $(t, \omega) \in C_i \cap B_{i-1}$, $i = 1, 2, ...,$, $C_i = \complement B_i, C_0 = \emptyset$. Evidently, $\text{essup} \int_J 1_{B_i} dt \leq \sum_{j \geq i} \int 1_{H_{u_{n_j}, u_{n_{j+1}}}} dt \leq 1/2^{i-1}$. Then $\tilde{\sigma}(u, u_{n_i}) \leq 1/2^{i-1}$, which suffices to conclude that $\tilde{\sigma}(u, u_n) \to 0$ when $n \to \infty$. Next, let u_n be a Cauchy-sequence in σ^*. Let $u(.)$ be the $\tilde{\sigma}$-limit of u_n. For each ϵ, for $k, n \geq$ some N, $\epsilon \geq \sigma^*(u_k, u_n) \geq 2^{i+1} \tilde{\sigma}(u_k 1_{I_i}, u_n 1_{I_i})$. Letting $n \to \infty$ gives $\epsilon \geq 2^{i+1} \tilde{\sigma}(u_k 1_{I_i}, u 1_{I_i})$, for all i, hence $\sigma^*(u_k, u) \to 0$, when $k \to \infty$. Furthermore, given any $\epsilon > 0$, for some N, for $k, n \geq N$, $\epsilon \geq \check{\sigma}(u_k, u_n) = \text{essup} \int_J M(t, \omega) 1_{H_{u_k, u_n}} dt$. Now, a.s., by dominated convergence, $\lim_n \int_J M(t, \omega) 1_{H_{u_k, u_n}} dt = \int_J M(t, \omega) 1_{H_{u_k, u}} dt$, so also, a.s., $\epsilon \geq \int_J M(t, \omega) 1_{H_{u_k, u}} dt, k \geq N$. (Finally, in case of Remark 3, $\sigma_*(u, u_n) \to 0$, because, by Lebesgue's dominated convergence theorem, both
$\int_J \max\{M(t, \omega), M_*(t, \omega)\} 1_{H_{u, u_n}} dt \to 0$ a.s. and
$| \int_J \max\{M(t, \omega), M_*(t, \omega)\} 1_{H_{u, u_n}} dt|_{2q} \to 0$.)

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