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At Last! An Explicit Solution for the Ramsey Saddle Path

Halvor Mehlum*

Abstract

I derive an explicit solution for the saddle path in a Ramsey growth model. The existence of a closed form expression greatly simplifies the analysis of how the parameters of the utility function affects investments and growth.

Keywords: *Ramsey growth model*

JEL: D91O41

1 Introduction

Two of the main contributions to growth theory is the Ramsey growth model (1928) and the Solow model (1956). The novelty of the Solow model was the use of a neo-classical production function with declining returns to capital combined with a fixed savings rate. Later, Ramsey's model got renewed attention as economists, unsatisfied with the savings assumption of the Solow model, wanted to analyze dynamic optimizing savings behavior. By now, both the Ramsey model and the Solow model are central in any exposition of growth theory (see for example Barro and Sala-I-Martin 1995). Central in the solution of the Ramsey model is the saddle path that relates consumption to production and thus determines the rate of investment and the rate of growth. It is generally impossible to find an explicit solution for the Ramsey model. To my knowledge all expositions of the Ramsey model reverts to qualitative assessments, approximate methods or numerical simulations

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when characterizing the saddle path. The contribution of the present paper is to show that an explicit solution is in fact available when the production function is of the fixed coefficient type. The choice of this production function may appear to be restrictive. It nevertheless leaves a lot of room for discussing the main insight of the Ramsey model; the optimizing behavior of a rational consumer with foresight.

The results should be of interest for economist working with growth theory and, not the least, for students struggling with their understanding of what is *really* going on in the Ramsey model.

2 The model

Production is given as a function of capital K by the fixed coefficient production function

$$X = a \min\left(K, \bar{K}\right) \tag{1}$$

where a is the output-capital ratio and $\bar{K}$ is the maximal capital stock that can be used productively. This production function is concave and has the essential properties needed for the Ramsey problem. $\bar{K}$ may be given by factors of production that cannot be accumulated, for example labor. In that case (1) is the limit of a constant elasticity of substitution production function as the elasticity of substitution between labor and capital goes to zero.

When the supply of capital is below $\bar{K}$, the rental price of capital will be equal to the output-capital ratio a. When the capital stock is larger than $\bar{K}$, however, there will be excess supply of capital, and the rental price will be zero.

$$\begin{aligned} K < \bar{K} &\iff r = a \\ K > \bar{K} &\iff r = 0 \end{aligned} \tag{2}$$

I will return to the case of exact equality $K = \bar{K}$ below. The static model (1) and (2) determines X, and r as functions of K.

The dynamics of the economy depends on the capital accumulation. Assume that the initial capital stock K_0 is below its maximum $\bar{K}$ so that there is a scope for further capital accumulation. All income is earned by a representative consumer who is the owner of the firms and the owner of capital. The supply of capital accumulation is then determined by the consumer's savings/investment decision. The consumer maximizes a constant relative risk aversion utility function

$$U = \int_{t=0}^{\infty} \frac{C_t^{1-\frac{1}{\sigma}} - 1}{1 - \frac{1}{\sigma}} e^{-\theta t} \mathrm{d}t \tag{3}$$

where C is consumption, θ the rate of time preferences, and σ the intertemporal elasticity of substitution. When abstracting from depreciation and using (1), investments (the time derivative of capital) is simply

$$\frac{dK}{dt} = aK - C \tag{4}$$

By using standard methods of dynamic optimization[1], maximizing (3) with respect to (4) and inserting from (2), the aggregate consumption path is

$$\frac{dC}{dt} = C\sigma\left[r - \theta\right] = \begin{array}{c} C\sigma\left[a - \theta\right] \text{when } K < \bar{K} \\ -C\sigma\theta \quad \text{when } K > \bar{K} \end{array} \tag{5}$$

When $a > \theta$ consumption grows exponentially, starting out below the level of production aK. In the solution both the capital stock and consumption grow until the steady state is reached where $K = \bar{K}$ and $C = aK$. As the consumption growth is exponential the transition time is finite. At time of termination all capital is employed and the return to capital r drops down to the level where the consumer has no incentive for further savings nor for dissavings. Hence from the point of termination and onwards consumption is stable. From (5) it follows that the rental price of capital that satisfies this condition is $r = \theta$. Hence (2) can be completed by the following condition

$$K = \bar{K} \iff r = \theta \tag{6}$$

The two linear differential equations (4) and (5), in combination with the initial condition $K = K_0$ and the two terminal conditions $C = a\bar{K}$ and $K = \bar{K}$, are sufficient for solving for the two paths and for the transition time.

[1] In order to keep the paper short I deliberately avoid technicalities, as I assume that the readers are familiar with the problem. For a careful presentation of the Ramsey problem see for example Barro and Sala-I-Martin (1995).

2.1 The Saddle Path

In order to display the familiar saddle path in a capital-consumption phase diagram, I transform the two differential equations (4) and (5) to one differential equation between K and C. This is possible since the solutions for both K and C are strictly increasing in t. When dividing (4) by (5) I get the linear differential equation

$$\frac{\partial K(C)}{\partial C} = \frac{aK(C) - C}{C\sigma(a-\theta)} \tag{7}$$

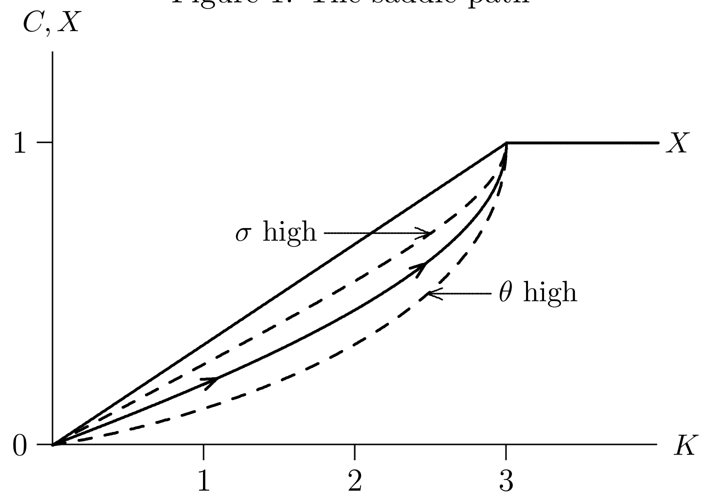
with the end point condition $K(aK) = \bar{K}$. The solution to (7) is found by using standard formulas. It can be confirmed by taking the derivative that the solution is simply

$$K(C) = \frac{\left(a\bar{K}\right)^{1-\beta} C^{\beta} - \beta C}{a(\beta - 1)}, \quad \beta = \frac{a}{\sigma(a-\theta)} \tag{8}$$

This function is the main finding of this paper.[2] It is an explicit function describing the Ramsey saddle path. A saddle path that is discussed extensively in most every advanced course in Macroeconomics and in a countless number of journal articles. The closed form solution provided in (8) gives everyone working with the Ramsey problem an accessible way to investigate the consequences of altering the parameters of the utility function. They can employ standard methods from calculus and need not use numerical or approximate methods.

Figure 1 gives two illustration of shifts in parameter values: When the intertemporal

Figure 1: The saddle path



[2]In the case where $\beta = 1$ the solution becomes $K(C) = C\left[1 + \ln\left(a\bar{K}/C\right)\right]/a$.

elasticity of substitution σ is high, consumption C is low relative to production, hence, savings and investments are high. On the other hand, if the rate of time preferences θ is high savings and investments are low and the growth is low.[3]

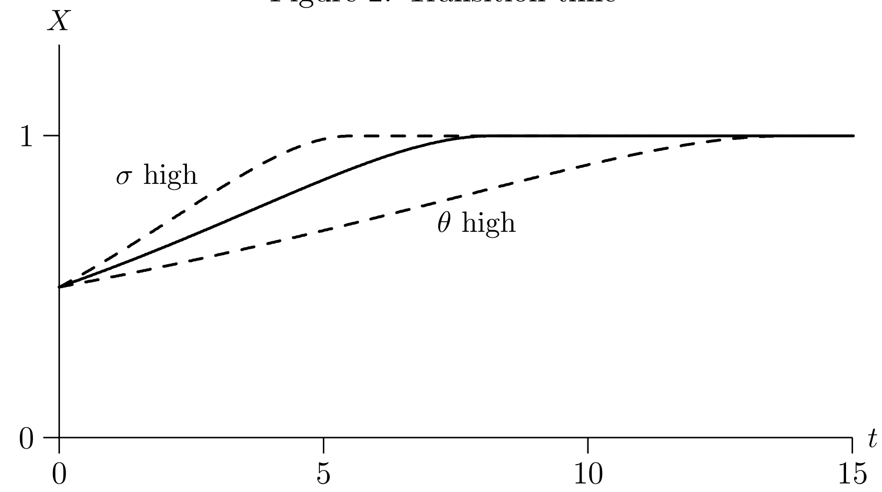
2.2 Transition paths

The saddle path gives the relationship between K and C. In order to find the initial consumption level for a given starting value of the capital stock one needs to solve the equation $K_0 = K(C_0)$. This equation is easily solved numerically. Once C_0 is known (5) gives the exponential growth of C

$$C(t) = C_0 e^{ta/\beta} \tag{9}$$

Once $C(t)$ is known it can be inserted into (8) to also get K as a function of time and finally production X follows. The economic growth over time is illustrated in Figure 2 .

Figure 2: Transition time



When the intertemporal elasticity of substitution σ is high, investments are high and the growth is fast. If the rate of time preferences θ is high, however, investments are low and the growth is slow.

This concludes the main analysis. To summarize: The main result is the existence of the function (8) that gives the closed form solution for the saddle path. Combined with (9) and the initial condition on the capital stock it is simple to calculate the time paths

[3] In the numerical example I use the following parameter values: $a = 1/3$, $\bar{K} = 3$, $\sigma = 0.5$, $\theta = 0.05$, $\sigma\,(\text{high}) = 1$, $\theta\,(\text{high}) = 0.20$, and in addition for the next figure, $K_0 = 1.5$.

for consumption, capital and production.

3 Concluding remarks

I have shown that there exists an explicit function describing the saddle path in a Ramsey problem. This explicit solution makes it a lot easier to get to grips with the essential feature of the Ramsey model: The optimizing consumer's behavior in a specific technological environment. The present model allows the analysis of changes in the important parameters of the utility function: the intertemporal elasticity of substitution and the rate of time preferences.

The explicit function describing the saddle path $K(C)$ in (8) follows from the linear differential equation (7). The essential condition for this result is that the rental price of capital is constant during the transition; a property that follows from the fixed coefficient production function. Given that this condition is satisfied the model's assumptions may be altered in a number of ways. The following alterations can be done without losing the explicit expression for the saddle path. First, the utility function may be modified to include a minimum consumption, a la Stone-Geary, or be changed to the constant absolute risk aversion type. Second, the capital may be subject to depreciation with a fixed coefficient. Third, the production function need not start out in origo but at the constant b. That is

$$X = b + a \min\left(K, \bar{K}\right) \tag{10}$$

Readers who wants to investigate the consequences of shifts in the technology may use (10) with it's degrees of freedom to approximate production functions with more curvature.

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