MEMORANDUM

No 25/2000

Deriving Belief Operators from Preferences

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ISSN: 0801-1117

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DERIVING BELIEF OPERATORS FROM PREFERENCES

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ABSTRACT. A belief operator derived from preferences is presented. It generalizes 'belief with probability 1' to incomplete preferences and satisfies minimal requirements for belief operators under weak conditions.
JEL Classification no: D81.
Keywords: Decision theory, incomplete preferences, belief operators.

1. Introduction

Morris [6] shows in an interesting contribution how belief operators can be derived from preferences. In particular, he introduces operators called 'Savage-belief' and 'strong belief' and shows that these under given assumptions correspond to KD systems in the sense that

- events which are necessarily true are always believed to be true,
- events which are necessarily false are never believed to be true,
- E is believed and F is believed if and only if $E \cap F$ is believed.

In the case of 'strong belief' one assumption is that utilities are *unbounded* both below and above (cf. Prop. 4 below). When preferences are complete and can thus be presented by a lexicographic probability system (being a hierarchy of subjective probability distributions over states; Blume et al. [5]), then 'strong belief' corresponds to 'belief with probability 1'. However, the 'strong belief' operator can be used also when preferences are not complete.

The purpose of this note is to argue that there is a belief operator (here called simply 'belief') that

- also is applicable for incomplete preferences while corresponding to 'belief with probability 1' when preferences are complete (Prop. 1),
- is in general stronger than 'strong belief' (Props. 2 & 3), and

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Acknowledgment: I thank Martin Dufwenberg for helpful comments.

- corresponds to a KD system even when utilities are not unbounded above and below in constrast to 'strong belief' (Prop. 5). As argued in a concluding remark, the case where utilities are *bounded* both above and below is of particular interest.

The 'belief' operator is used to provide epistemic characterizations of forward and backward induction in Asheim & Dufwenberg [3] and Asheim [2], respectively. In these game-theoretic applications it is of importance to be able to handle preferences that are not complete.

2. Preliminaries

Consider a decision maker under uncertainty. Let Ω be a finite set of states, where the decision maker is uncertain about what state in Ω will be realized. The decision maker has preferences over the set of functions that assign a utility value to any state. Any such function $\mathbf{x} : \Omega \to U$, where $U \subseteq \mathbb{R}$ is a closed and convex set, is called an *act* on Ω. The set U can be bounded above, bounded below, or equal to $\mathbb{R}$. If the true state is ω, then the preferences of the decision maker is a binary relation $\succeq^\omega$ over the set of acts, with $\succ^\omega$ and $\sim^\omega$ denoting the asymmetric and symmetric parts, respectively. For any $\omega \in \Omega$, $\succeq^\omega$ is assumed to be

- *reflexive* and *transitive*, but not necessarily *complete*.
- *nontrivial* in the sense that there exist $\mathbf{x}$ and $\mathbf{y}$ such that $\mathbf{x} \succ^\omega \mathbf{y}$.

If $E \subseteq \Omega$, let $\mathbf{x}_E$ denote the restriction of $\mathbf{x}$ to E. Define the *conditional* binary relation $\succeq^\omega_E$ by $\mathbf{x} \succeq^\omega_E \mathbf{y}$ if, for any $\mathbf{z}$, $(\mathbf{x}_E, \mathbf{z}_{-E}) \succeq^\omega (\mathbf{y}_E, \mathbf{z}_{-E})$, where $-E$ denotes $\Omega \backslash E$. Say that the state $\omega' \in \Omega$ is *Savage-null* given ω if $\mathbf{x} \sim^\omega_{\{\omega'\}} \mathbf{y}$ for all acts $\mathbf{x}$ and $\mathbf{y}$ on Ω. Let κ^ω denote the set of states that are *not* Savage-null given ω. Since $\succeq^\omega$ is nontrivial, $\kappa^\omega \neq \emptyset$. Assume that, for any $\omega \in \Omega$ and $\omega' \in \kappa^\omega$, $\mathbf{x} \succeq^\omega_{\{\omega'\}} \mathbf{y}$ if and only if $\mathbf{x}(\omega') \geq \mathbf{y}(\omega')$.[1]

[1] If acts – following Anscombe & Aumann [1] – are defined as functions from states to objective randomizations on a finite set of *consequences*, this assumption is satisfied if the binary relation over the set of such acts, in addition to reflexity, transitivity and nontriviality, satisfies objective independence, conditional completeness, conditional continuity and non-null state independence (cf. [5], Sect. 2, as well as the appendix of [2]). Because then there will, given ω, exist a von Neumann Morgenstern utility function that for any $\omega' \in \kappa^\omega$ represents complete preferences over the set of objective randomization on the set of consequences and thus represents the binary relation conditional on ω'. In the proof of Prop. 1 it is explicitly assumed that utilities are derived in this manner.

Definition 1. A *belief system* consists of *(a)* a finite state space Ω and *(b)* for each $\omega \in \Omega$, a binary relation $\succeq^\omega$ over the set of acts (where each act is a function $\mathbf{x} : \Omega \to U$ and $U \subseteq \mathbb{R}$ is a closed and convex set), being reflexive, transitive and non-trivial, and satisfying for each non-Savage-null state ω' that $\mathbf{x} \succeq^\omega_{\{\omega'\}} \mathbf{y}$ if and only if $\mathbf{x}(\omega') \geq \mathbf{y}(\omega')$.

Say that $\mathbf{x}_E$ *strongly dominates* $\mathbf{y}_E$ if, $\forall \omega' \in E$, $\mathbf{x}_E(\omega') > \mathbf{y}_E(\omega')$. Say that $\succeq^\omega$ *respects strong dominance* on E if E is non-empty and $\mathbf{x} \succ^\omega \mathbf{y}$ whenever $\mathbf{x}_E$ strongly dominates $\mathbf{y}_E$. Say that $\mathbf{x}_E$ *weakly dominates* $\mathbf{y}_E$ if, $\forall \omega' \in E$, $\mathbf{x}_E(\omega') \geq \mathbf{y}_E(\omega')$, with strict inequality for some $\omega'' \in E$. Say that $\succeq^\omega$ *respects weak dominance* on E if E is non-empty and $\mathbf{x} \succ^\omega \mathbf{y}$ whenever $\mathbf{x}_E$ weakly dominates $\mathbf{y}_E$.

Say that ω' is deemed *infinitely more likely* than ω'' given ω ($\omega' \gg^\omega \omega''$) if ω' is not Savage-null and $\mathbf{x} \succ^\omega_{\{\omega'\}} \mathbf{y}$ implies $(\mathbf{x}_{-\{\omega''\}}, \mathbf{x}'_{\{\omega''\}}) \succ^\omega_{\{\omega', \omega''\}} (\mathbf{y}_{-\{\omega''\}}, \mathbf{y}'_{\{\omega''\}})$ for all $\mathbf{x}'$, $\mathbf{y}'$. According to this definition, ω'' may, but need not, be Savage-null if $\omega' \gg^\omega \omega''$.

3. Savage-Belief and Belief

Say that an event $E \subseteq \Omega$ is *Savage-believed* given ω if $\kappa^\omega \subseteq E$. Hence, the event that E is Savage-believed, KE, is given by

$$KE := \{\omega \in \Omega | \kappa^\omega \subseteq E\} .$$

This definition corresponds to the one given by Morris [6] and means that there is Savage-belief of an event if the complement is Savage-null.

Say that an event $E \subseteq \Omega$ is *believed* given ω if there exists $\beta \subseteq E$ such that $\succeq^\omega$ respects weak dominance on β. If $\succeq^\omega$ respects weak dominance on β, then any $\omega' \in \beta$ is deemed infinitely more likely than any $\omega'' \in \Omega \backslash \beta$. Since ω' being infinitely more likely than ω'' implies that ω'' is *not* infinitely more likely than ω', it follows that $\beta' \subseteq \beta''$ or $\beta' \supseteq \beta''$ whenever $\succeq^\omega$ respects weak dominance on both β' and β''. Since, in addition, $\succeq^\omega$ respects weak dominance on κ^ω, there exists a unique smallest (w.r.t. set inclusion) non-empty set on which $\succeq^\omega$ respects preferences; let this set be denoted β^ω:

$$\succeq^\omega \text{ resp. weak dom. on } \beta^\omega, \text{ and } \beta \supseteq \beta^\omega \text{ if } \succeq^\omega \text{ resp. weak dom. on } \beta .$$

This means that an event $E \subseteq \Omega$ is believed given ω if and only if $\beta^\omega \subseteq E$. Hence, the event that E is believed, BE, is given by

$$BE := \{\omega \in \Omega | \beta^\omega \subseteq E\} .$$

It follows that $KE \subseteq BE$ since $\succeq^\omega$ respects weak dominance on κ^ω; i.e. $\beta^\omega \subseteq \kappa^\omega$. If $\beta^\omega \neq \kappa^\omega$, then $\succeq^\omega$ is not continuous. In addition to $KE \subseteq BE$, it is straightforward to establish that the operators K and B satisfy

$$K\Omega = \Omega \qquad\qquad B\emptyset = \emptyset$$

$$KE \cap KF = K(E \cap F) \qquad\qquad BE \cap BF = B(E \cap F)\ .$$

Since $KE \subseteq BE$ implies that $K\emptyset = \emptyset$ and $B\Omega = \Omega$, both operators K and B correspond to KD systems. Since an event can be Savage-believed given ω even though the true state ω is an element of the complement of the event (i.e. $\omega \in \Omega \backslash E \subseteq \Omega \backslash \kappa^\omega$), it follows that neither operator satisfies the truth axiom (i.e. $KE \subseteq E$ and $BE \subseteq E$ need not hold).

4. Morris' [6] Notion of Strong Belief

In Morris [6], an event $E \subseteq \Omega$ is said to be *strongly believed* given ω if $\succeq^\omega$ respects strong dominance on E. Hence, the event that E is strongly believed, B^*E, is given by[2]

$$B^*E := \{\omega \in \Omega | \succeq^\omega \text{ respects strong dominance on } E\}\ .$$

The first result states that 'belief' coincides with 'strong belief' for any belief system with complete preferences.

Proposition 1. *For any belief system having the property that the binary relation $\succeq^\omega$ is complete for each $\omega \in \Omega$ and any event $E \subseteq \Omega$, $BE = B^*E$.*

Proof. Consider any $\omega \in \Omega$. Since $\succeq^\omega$ is complete (and assuming that utilities are derived as explained in footnote 1), it follows from Blume et al. ([5], Thm. 2.1) that there exists a primary subjective probability distribution μ_1^ω with support equal to β^ω such that $\sum_{\omega' \in \Omega} \mu_1^\omega(\omega')\mathbf{x}(\omega') > \sum_{\omega' \in \Omega} \mu_1^\omega(\omega')\mathbf{y}(\omega')$ implies that $\mathbf{x} \succ^\omega \mathbf{y}$. This means that $\succeq^\omega$ respects strong dominance on E if and only if $E \supseteq \beta^\omega$. Hence, $\omega \in B^*E$ if and only if $\omega \in BE$. □

It follows from the proof of Prop. 1 that both 'belief' and 'strong belief' correspond to 'belief with probability 1' when preferences are complete.

The second result states that 'belief' is in general as strong as 'strong belief'.

[2]This definition is equivalent to Morris' [6] Def. 2 when, $\forall \omega \in \Omega$, $\succeq^\omega$ is non-trivial. Note that Battigalli & Siniscalchi [4] also use the term 'strong belief', but for a different belief operator that will not be discussed here.

Proposition 2. *For any belief system and any event $E \subseteq \Omega$, $BE \subseteq B^*E$.*

Proof. Assume that $\omega \in BE$; i.e., $\succeq^\omega$ respects weak dominance on some $\beta \subseteq E$. Since strong dominance on E implies weak dominance on any subset of E, it follows that $\succeq^\omega$ respects strong dominance on E, which in turn implies that $\omega \in B^*E$. □

Furthermore, there exist belief systems for which 'belief' is strictly stronger than 'strong belief'.

Proposition 3. *There exist a belief system and an event $E \subseteq \Omega$ such that $BE \neq B^*E$.*

Proof. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and $E = \{\omega_1, \omega_2\}$. Assume that $\succeq^{\omega_1}$ is determined by $\mathbf{x} \sim^{\omega_1} \mathbf{y}$ if and only if $\mathbf{x} = \mathbf{y}$, and $\mathbf{x} \succ^{\omega_1} \mathbf{y}$ if and only if $\mathbf{x}$ strongly dominates $\mathbf{y}$ on E or $\mathbf{x}$ weakly dominates $\mathbf{y}$ on Ω. Then $\omega_1 \in B^*E$, while $\omega_1 \in \Omega \backslash BE$. □

In addition to $KE \subseteq BE \subseteq B^*E$, it is straightforward to establish that the operator B^* satisfies

$$B^*\Omega = \Omega \qquad\qquad B^*\emptyset = \emptyset$$

$$B^*E \cap B^*F \supseteq B^*(E \cap F)\ .$$

However, to show that $B^*E \cap B^*F \subseteq B^*(E \cap F)$ one need to assume that the range of acts, U, is unbounded; i.e. $U = \mathbb{R}$.

Proposition 4. *For any belief system having the property that $U = \mathbb{R}$ and any events $E, F \subseteq \Omega$, $B^*E \cap B^*F \subseteq B^*(E \cap F)$.*

Proof. See Morris ([6], Thm. 4(iii)). □

Proposition 5. *There exist a belief system and events $E, F \subseteq \Omega$ such that $B^*E \cap B^*F \neq B^*(E \cap F)$ whenever the closed and convex set U is bounded above or below.*

Proof. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $E = \{\omega_1, \omega_2\}$, and $F = \{\omega_1, \omega_3\}$. Assume that $\succeq^{\omega_1}$ is determined by $\mathbf{x} \sim^{\omega_1} \mathbf{y}$ if and only if $\mathbf{x} = \mathbf{y}$, and $\mathbf{x} \succ^{\omega_1} \mathbf{y}$ if and only if *(a)* $\mathbf{x}$ strongly dominates $\mathbf{y}$ on E or *(b)* $\mathbf{x}$ strongly dominates $\mathbf{y}$ on F or *(c)* $\mathbf{x}$ weakly dominates $\mathbf{y}$ on Ω or *(d)* combining (a) and (b) through the use of transitivity. Then $\omega_1 \in B^*E$ and $\omega_1 \in B^*F$, while we need to show that $\omega_1 \in \Omega \backslash B^*\{\omega_1\}$ whenever U is bounded above or below.

Suppose that the closed and convex set U is bounded above and let w.l.o.g. 1 be the maximal element of U. Consider $\mathbf{x} = (1, 0, 0)$ and $\mathbf{y} = (0, 1, 1)$. If $\omega_1 \in B^*\{\omega_1\}$, then $\mathbf{x} \succ^{\omega_1} \mathbf{y}$. However, $\mathbf{z} \succ^{\omega_1} \mathbf{y}$ only if $\mathbf{z} = (u, 1, 1)$ with $u > 0$, implying that $\mathbf{x} \succ^{\omega_1} \mathbf{y}$ does not hold. Likewise, suppose that the closed and convex set U is bounded below and let w.l.o.g. 0 be the minimal element of U. Consider again $\mathbf{x} = (1, 0, 0)$ and $\mathbf{y} = (0, 1, 1)$. If $\omega_1 \in B^*\{\omega_1\}$, then $\mathbf{x} \succ^{\omega_1} \mathbf{y}$. However, $\mathbf{x} \succ^{\omega_1} \mathbf{z}$ only if $\mathbf{z} = (u, 0, 0)$ with $u < 1$, implying that $\mathbf{x} \succ^{\omega_1} \mathbf{y}$ does not hold. □

5. Concluding Remark

If – as will be the case under the assumptions mentioned in footnote 1 – the utilities arise from a von Neumann Morgenstern utility function representing complete preferences over the set of objective randomizations on a set of consequences and, in addition, this set of consequences is finite, then the set U will be closed and convex and bounded both above and below. Since this case often applies (e.g. in the analysis of games with a finite outcome space), it seems of particular interest to consider belief operators derived from preferences that correspond to KD systems also when utilities are bounded above and below. The 'belief' operator presented here has this property, while Morris' [6] notion of 'strong belief' has not.

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