MEMORANDUM

Taxation, uncertainty, and the cost of equity for a multinational firm

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Taxation, uncertainty, and the cost of equity for a multinational firm[*]

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Abstract

From a CAPM-type model the cost of equity is derived for a firm operating under various foreign tax systems. The firm's shares are traded in a market which is unaffected by these systems. The cost of capital depends on the foreign tax system, even for fully equity financed projects. This is neglected in much of the literature. For a corporate income tax the main factor which reduces the cost of equity is the depreciation deductions. Compared with a neutral cash flow tax, this reduces the cost of equity because it acts as a loan from the firm to the government.

KEYWORDS: Cost of equity, taxation, weighted average cost of capital, uncertainty

JEL CLASSIFICATION NUMBERS: G31, H25

1 Introduction

One of the most frequently used formulae in corporate finance is the weighted average cost of capital, WACC.[1] In the standard formulation the cost of debt depends on taxes because of interest deductibility. But normally[2] no tax factor is introduced in connection with the cost of equity, the other of the costs of which the WACC is a weighted average. This paper argues that there should be such a factor, reflecting the corporate tax system, when the firm invests in risky assets subject to tax depreciation. The factor will be more important, the larger is the diversity of the tax rates and depreciation schedules under which the firm operates. This means it is important when different forms of capital, such as tangibles versus intangibles, have widely different depreciation schedules, and also when high tax rates are imposed, e.g., on natural resource extraction. The factor is crucial for discussions of corporate tax reforms.

There exist alternative methods of capital budgeting, not relying on the WACC. Two alternatives[3] are the adjusted present value (APV) of Myers (1974) and the flow to equity (FTE). The points to be made here have the same relevance for the FTE method as for the WACC. The APV approach, however, is usually applied in a way which is consistent with this paper, since it is realized that depreciation tax shields have a much lower systematic risk than pre-tax cash flows.[4] Even if firms adopt the APV method, the points to be made below are interesting if one wants to interpret and apply market data in capital budgeting.

Since depreciation tax shields have a lower systematic risk, and since they are proportional to investment, it is possible to write down their effect on the after-(corporate-)tax cost of equity for an unlevered firm. This is the contribution of the present paper. One result is that the higher the corporate tax rate, the lower is the after-tax cost of equity for depreciable assets.

The intuition behind the result is that different tax systems split the risks in the pre-tax cash flows differently between firms and the government. More specifically the result can

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[1]According to Brealey and Myers (2000), "In practical capital budgeting, a single discount rate is usually applied to all future cash flows" (p. 242). The survey of Graham and Harvey (2001) confirms this.

[2]Brealey and Myers (2000), p. 543, Ross, Westerfield, and Jaffe (1999), p. 305.

[3]This follows Ross, Westerfield, and Jaffe (1999), ch. 17.

[4]See, e.g., Lessard (1979), Summers (1987), and Brealey and Myers (2000, p. 566).

be explained with reference to two well-known facts. One is that leverage increases the cost of equity. The other is that the tax on non-financial cash flows (immediate expensing, no interest deductions) proposed by Brown (1948), with immediate, full loss offset and no depreciation schedule, does not affect the cost of equity. In fact, it does not affect investment decisions at all when there is value additivity, since the tax acts cash-flow-wise as just another shareholder.

With the Brown tax as a reference point, the introduction of depreciation schedules instead of immediate expensing is analogous in terms of risk to a loan from the firm to the tax authorities. There may be no interest paid, but the loan analogy shows the direction of the effect on the systematic risk of equity. Depreciation has the opposite effect of leverage, and thus reduces the systematic risk of equity.

The paper uses the Capital Asset Pricing Model (CAPM) to determine after-tax valuation when the tax position is certain, and an option pricing model when the position is uncertain. The valuation parameters (such as the interest rate, the market price of risk) are exogenous. This is realistic when the firm's shares are mainly held by investors in one large economy, and the valuation parameters are not influenced by foreign tax systems. In the model below, the firm's after-tax cost of equity is determined endogenously, while the systematic risk of the firm's pre-tax cash flows is exogenous.

The model extends to a CAPM setting the method for tax-determination of required rates of return which is well-known in tax research, starting with Hall and Jorgenson (1967).[5] For given market valuation parameters it is determined what pre-tax rate of return must be expected in order for a project to be exactly marginal. This paper goes a step further, and shows how the tax system affects the after-tax required expected rate of return.

Most of the previous literature has neglected the effects of a corporate income tax on the systematic risk of equity. While the effect of interest deductibility in reducing the cost of debt is well known, it is most often neglected that even in a fully equity financed firm,

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[5]That literature uses the term "cost of capital" for the required rate of return *before* any taxes. The terminology in the present paper is taken from corporate finance, so the weighted average cost of capital, and its components, the cost of equity and the cost of debt, are *after* corporate income tax, cf. Modigliani and Miller (1963).

the required after-tax expected rate of return is affected by taxes. If one considers a closed economy with the same corporate income tax applied (and effective) everywhere, the point is irrelevant, as the market already reflects the tax which applies.

Levy and Arditti (1973) observe that taxes with depreciation schedules affect the required expected rate of return after tax. Their model is an extension of Modigliani and Miller (M&M) (1963), introducing depreciable assets in the M&M model, but maintaining their assumption of perpetual projects. The relationship with the present paper will be discussed in section 5 below. Their result is quite different, and it is argued that the present paper's assumptions are more realistic in many situations.

Galai (1988) (very briefly, p. 81) and Derrig (1994) both discuss the effect of a corporate tax on the systematic risk of equity based on the CAPM.[6] They do not observe the necessity of solving for the expected rate of return of an after-tax marginal project. This is dicusssed in section 2.3 below.

Both Levy and Arditti (1973) and Derrig (1994) consider only one simple tax system, and assume that the firm is certain to be in tax position. This paper is an extension in both respects. Galai (1988) considers both risky debt and a risky tax position, but only one tax system.

Section 2 presents a two-period model in which the firm pays taxes with certainty. Section 3 introduces uncertainty about whether taxes are paid. Section 4 extends to a multi-period model, but without the uncertainty about taxes being paid. Section 5 compares with the alternative multi-period model of Levy and Arditti (1973). Section 6 illustrates some results with a numerical example. Section 7 gives some further discussion, and section 8 concludes.

2   The model

A firm invests in period 0 and produces in period 1, only. (A multi-period extension follows in section 4.) We consider a marginal investment project, i.e., one with an APV equal to zero. This is the standard method in an analysis of the effects of taxation on investment

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[6]The CAPM is presented in Assumption 1 and footnote 7 below.

when there are non-increasing returns to scale. Which project is marginal, is endogenous, determined by the tax system and other parameters in each case below. In this way we also characterize the minimum required expected return to equity in each case.

Assumption 1: *The firm maximizes its market value according to a tax-adjusted Capital Asset Pricing Model,*

$$E(r_i) = r_f\theta + \beta_i[E(r_m) - r_f\theta], \tag{1}$$

where $r_f > 0$ and $\theta \in (0,1]$.[7]

This allows for differences in the tax treatment on the hands of the firm's owners of income from equity and income from riskless bonds, reflected in the tax parameter θ.[8] In a discussion of taxation and the CAPM it seems reasonable to allow for this, but it has no consequences for the results which follow. The standard CAPM with $\theta = 1$ is all that is needed.

When various tax systems are considered below, these are assumed not to affect the capital market equilibrium. This will be a good approximation if they apply in small sectors of the economy (e.g., resource extraction), or abroad in economies which are small in relation to the domestic one. This is thus a partial equilibrium analysis.[9] For concreteness

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[7] Of course, r_i is the rate of return of shares in firm i, r_f is the riskless interest rate, r_m is the rate of return on the market portfolio, $\beta_i \equiv \text{cov}(r_i, r_m)/\text{var}(r_m)$, and E is the expectation operator. The original model is derived in Sharpe (1964), Lintner (1965), and Mossin (1966).

All variables are nominal. As long as the tax system is based on nominal values, the model is only consistent with a rate of inflation which is known with certainty, and fixed exchange rates. The underlying real CAPM would then be

$$\frac{1 + E(r_i)}{1 + \pi} = \frac{1 + r_f\theta}{1 + \pi} + \beta_i \left[\frac{1 + E(r_m)}{1 + \pi} - \frac{1 + r_f\theta}{1 + \pi} \right],$$

where π is the rate of inflation.

[8] A tax-adjusted CAPM appears, e.g., in Sick (1990) or Benninga and Sarig (1997, 1999). Rather strict assumptions are needed for Benninga and Sarig's Extended Miller Equilibrium, in which θ equals one minus the home country's corporate tax rate. While this is perhaps the most sophisticated tax-adjusted CAPM available, the more general formulation in Assumption 1 also covers other interesting situations. Under an imputation system one could have θ equal to one minus the corporate tax rate without any Miller equilibrium. This was the situation in Norway 1992–2000, cf. Sørensen (1994).

[9] The existing literature, Levy and Arditti (1973), Galai (1988), and Derrig (1994), is not explicit at this point, but neglects general equilibrium effects. As soon as one wants to examine the effects of changes in

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we use "home" and "domestic" about the economy where the firm's shares are traded. This economy may have a tax system, which is exogenously given in the analysis, and reflected in θ. This will be called a "personal" tax system, even though the owners may be firms or other institutions. For concreteness we use "abroad" and "foreign" about the sector whose corporate income tax we analyse.

A consequence of the CAPM is that the claim to any uncertain cash flow X, to be received in period 1, has a period-0 value of

$$\varphi(X) = \frac{1}{1 + r_f \theta}[E(X) - \lambda_\theta \operatorname{cov}(X, r_m)], \tag{2}$$

where $\lambda_\theta = [E(r_m) - r_f \theta]/\operatorname{var}(r_m)$. Equation (2) defines a valuation function φ to be applied below.

A product price, P, will most likely not have an expected rate of price increase which satisfies the CAPM.[10] A claim on one unit of the product will satisfy the CAPM, however, so that the beta value of P should be[11] defined in relation to the return $P/\varphi(P)$,

$$\beta_P = \frac{\operatorname{cov}(\frac{P}{\varphi(P)}, r_m)}{\operatorname{var}(r_m)}. \tag{3}$$

It is possible to express this more explicitly, without the detour via $\varphi(P)$, namely as

$$\beta_P = \frac{1 + r_f \theta}{E(P)\frac{\operatorname{var}(r_m)}{\operatorname{cov}(P, r_m)} - [E(r_m) - r_f \theta]}. \tag{4}$$

Assumption 2: *In period 0 the firm invests an amount $I > 0$ in a project. In period 1 the project produces a quantity $Q > 0$ to be sold at an uncertain price P. The joint probability distribution of (P, r_m) is exogenous to the firm, and $\operatorname{cov}(P, r_m) > 0$. There is no production flexibility; Q is fixed after the project has been initiated.*

The market value in period 0 of a claim to the revenue in period 1 is $Q\varphi(P)$ in case there are no taxes. There is no need to specify a "production function" relationship between

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tax rates, a general equilibrium analysis would be interesting. Alternatively one can explicitly concentrate on tax rates abroad or in a small sector, as in the present paper.

[10] The product price has what McDonald and Siegel (1984) call an (expected-)rate-of-return shortfall.

[11] It is certainly possible to define $\beta_i = \operatorname{cov}(r_i, r_m)/\operatorname{var}(r_m)$ for any rate of return r_i. But the definition has little relevance for non-equilibrium rates of return, i.e., those for which (1) does not hold. (In the CAPM jargon, those which are not on the Security Market Line.) The formula for linear combination of betas is only true when the betas relate to equilibrium returns.

I and this market value. We shall only be concerned with the marginal project, which will be solved for in each case below. Of course, with no taxes the marginal project has $I = Q\varphi(P)$.

In the background there may be a "production function" specifying $Q\varphi(P)$ as a linear or concave function of I. It would be a problem, however, to use this method to analyze tax distortions in a general equilibrium framework. One would want to analyze distortions in production, while we shall only be able to analyze distortions in the ratio of $Q\varphi(P)$ to I. In a general equilibrium analysis, if taxes are changed in the home economy so that the market valuation function φ is affected (e.g., through changes in θ and/or the joint probability distribution of (P, r_m)), then $\varphi(P)$ will change, and this model is not able to detect the underlying production distortions.

In the present version of the model no explicit costs are specified in period 1. If $P > 0$ with full certainty, there can never be any reason for the firm to cancel production after P has become known.[12] A more complicated extension would include a cost and a real option, the option to shut down if the net after-tax cash flow becomes negative in period 1. The assumption of $\text{cov}(P, r_m) > 0$ can easily be relaxed. It is only a convenience in order to simplify the verbal discussions below.

Definition: The *relative distortion parameter* is defined as that ratio

$$\gamma_i \equiv \frac{Q\varphi(P)}{I} \tag{5}$$

which makes the net after-(corporate-)tax market value of the project equal to zero in case i below.[13]

The parameter is a useful summary measure of how the tax system in a particular case distorts the pre-tax productivity of the marginal project. It compares with a first-best (or no-tax) situation, and one should not use it to draw any conclusions about opimal taxation. Observe that γ_i is not one plus an expected rate of return. The numerator is already expressed as a market value in period 0. Instead γ_i measures the relative distortion from

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[12]Assume that PQ is replaced by $PQ - WL$, where W is a stochastic factor price, and L is the input quantity. The easy extension is based on the ratio $Q\varphi(P)/L\varphi(W)$ being a constant, say, $\kappa > 1$. Then β_P is replaced by $(\kappa/(\kappa - 1))\beta_P - (1/(\kappa - 1))\beta_W$.

[13]Equation (36) gives a multi-period extension of this definition.

the tax system in "one plus the required expected rate of return," with $\gamma_i = 1$ indicating the no-tax situation. One could call $\gamma_i - 1$ the tax wedge in "one plus the required expected rate of return."

Assumption 3: *A fraction* $(1-\eta) \in [0,1)$ *of the financing need in period 0 is borrowed. This fraction is independent of the investment decision and of the tax system. The loan B is repaid with interest with full certainty in period 1.*

The financing need is equal to I minus the immediate tax relief for investment, if any.[14]

Debt is introduced only because of the prominence of debt in the traditional literature on taxes and the cost of capital. In the present paper the results can be derived with zero debt. The assumptions of independence between financing and investment, and between (after-tax) financing and taxes, are those underlying the standard derivation of the WACC, and therefore the appropriate set of assumptions here.[15]

It should be kept in mind that Assumption 3 concerns the formal project-related borrowing by the firm. When applied to the subsidiary of a multinational, this may be tax motivated and differ from the net project-related borrowing undertaken by the multinational and its subsidiaries taken together. The lender to one subsidiary is often another subsidiary of the same multinational. Also, the possibility of transfer pricing is neglected here.

The assumption of default-free debt is also a common simplification, and should be acceptable for the purposes of this paper. Galai (1988) focuses on risky debt in a similar model.

2.1 Case 1: Tax position known with certainty

Assumption 4: *A tax at rate* $t \in [0,1)$ *will be paid with certainty in the production period. The tax base is operating revenue less* $(gr_f B + cI)$. *There is also a tax relief of* taI *in period 0. The constants g, c, and a are in the interval* $[0,1]$.

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[14]η is in this sense the ratio of equity to assets after tax, as will become clear below.

[15]Another defence for not introducing a more sophisticated theory of debt financing is that the survey by Graham and Harvey (2001) shows that most firms have a fixed target debt ratio.

This general formulation allows for accelerated depreciation with, e.g., $a > 0$ and $a + c = 1$, or a standard depreciation interpreted (since there is only one period with production) as $a = 0, c = 1$. There is usually full interest deduction, i.e., $g = 1$, but the Brown (1948) cash flow tax has $g = 0$, and some transfer pricing regulations might require $0 < g < 1$.

The cash flow to equity in period 1 is

$$X_{(1)} = PQ(1 - t) - (1 + r_f)B + r_f Bgt + tcI. \tag{6}$$

The market value of this is

$$\varphi(X_{(1)}) = Q\varphi(P)(1 - t) - \frac{1 + r_f(1 - tg)}{1 + r_f\theta}B + \frac{tcI}{1 + r_f\theta}. \tag{7}$$

For a marginal project the expression must be equal to the financing need after borrowing and taxes, $\eta I(1 - ta)$, and by definition $Q\varphi(P) = \gamma_1 I$, so that

$$\eta I(1 - ta) = \varphi(X_{(1)}) = \gamma_1 I(1 - t) - \frac{1 + r_f(1 - tg)}{1 + r_f\theta}B + \frac{tcI}{1 + r_f\theta}$$

$$= \gamma_1 I(1 - t) - \frac{1 + r_f(1 - tg)}{1 + r_f\theta}(1 - ta)(1 - \eta)I + \frac{tcI}{1 + r_f\theta}, \tag{8}$$

which implies

$$\gamma_1 = \frac{1}{1 - t}\left\{(1 - ta)\left[\eta + \frac{1 + r_f(1 - tg)}{1 + r_f\theta}(1 - \eta)\right] - \frac{tc}{1 + r_f\theta}\right\}. \tag{9}$$

The proof of the following proposition is in the appendix.

Proposition 1: *Under Assumptions 1–4: The relative distortion is given by (9). It is increasing in t for $a = 0$, and decreasing in t for $a = 1$. For each vector of parameters $(\eta, r_f, \theta, g, c)$, one has $\partial\gamma_1/\partial t = 0$ for one intermediate value $a \in (0, 1]$, which also depends on the tax rate t if $\eta < 1$ and $g > 0$.*

In the most clear-cut case depreciation is allowed without any immediate deduction, i.e., $a = 0$. As long as cI is less than I in present value, the tax system clearly entails a distortion compared with a no-tax situation, and more, the higher is t.

It is instructive first to consider the special cases of $t = 0$ and of a Brown (1948) cash flow tax. Setting $t = 0$ in (9) yields

$$\gamma_1 = \eta + \frac{1 + r_f}{1 + r_f\theta}(1 - \eta). \tag{10}$$

The corollary follows directly:

Corollary 1.1: *Under Assumptions 1–4: If $t = 0$, the relative distortion is given by (10), which strictly exceeds unity if $\eta < 1$ and $\theta < 1$. The same relative distortion occurs, independently of the tax rate, under a tax on non-financial cash flows, i.e., if $t > 0, a = 1, c = 0, g = 0$.*

As long as the capital market reflects a personal tax system which discriminates against interest income ($\theta < 1$), borrowing ($\eta < 1$) implies that less is invested ($\gamma_1 > 1$) than under no borrowing if there is no corporate tax. There is no additional distortion from a corporate cash flow tax.[16]

More generally, however, there is a distortion which depends on the tax rate, but which is independent of β_P.

For a standard corporate income tax without accelerated depreciation, we find

$$\gamma_1 = \frac{1}{1-t}\left[\eta + (1-\eta)\frac{1 + r_f(1-t)}{1 + r_f\theta}\right] - \frac{t}{(1-t)(1 + r_f\theta)}. \tag{11}$$

The corollary follows directly, cf. equation (A1) in the appendix.

Corollary 1.2: *Under Assumptions 1–4: If $a = 0$, $c = 1$, $g = 1$, the relative distortion is given by (11). It is strictly increasing in the tax rate and exceeds the value given in (10) as long as $t > 0$.*

Going back to the general formulation of case 1 (without specifying a, c, g, θ, η), the beta value of equity is a value-weighted average of the beta values of the elements of the cash flow. From (6) this is

$$\beta_{X1} = \frac{Q\varphi(P)(1-t)}{\varphi(X_{(1)})}\beta_P = \frac{\gamma_1(1-t)}{\eta(1-ta)}\beta_P. \tag{12}$$

The following is shown in the appendix:

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[16]When applied to firms facing $\theta < 1$ in their home capital markets, one could replace tg with a (perhaps partial) interest deductibility at a rate $1 - \theta$ to obtain $\gamma_1 = 1$. One can obtain an effect similar to the cash flow tax from a "Resource Rent Tax" (RRT), as proposed by Garnaut and Clunies Ross (1975). This has $a = 0$ and $c = 1 + r_f\theta$, so that the postponed deduction for investment costs is compensated by interest accumulation.

Proposition 2: *Under Assumptions 1–4: The beta value of equity is given by (12). It is decreasing in the tax rate, and strictly so when $c > 0$ or $(1 - \eta)g > 0$ (or both).*

The various effects are nicely separated: β_P/η is roughly the beta value corrected for the effect of leverage[17] (and exactly so when $\theta = 1 - tg$). Then the beta is multiplied by the distortion factor γ_1, and also by a factor $(1 - t)/(1 - ta)$ which reflects the deviation from a cash flow tax. The latter is related to the basic intuition mentioned in the introduction, that a depreciation deduction acts as a loan from the firm to the tax authorities, when compared with a cash flow tax. Any depreciation deduction would be sufficient to make β_{X1} decreasing in the tax rate. But some interest deductibility ($g > 0$) is also sufficient, given that there is some leverage ($\eta < 1$). The effect of a cash flow tax is summarized as follows:[18]

Corollary 2.1: *Under Assumptions 1–4: For $a = 1$, $c = 0$, $g = 0$, the beta of equity is independent of the tax rate t, and is given by $\beta_{X1} = \beta_P \gamma_1/\eta$, where γ_1 is given by (10).*

Since (12) contains tax parameters in γ_1, but also separately, there is clearly no monotonic relation between β_{X1} and γ_1 as tax systems change. As is seen from (9), a simultaneous change in t and c may leave γ_1 unchanged, but it will then change β_{X1}.

If a higher tax rate is introduced, e.g., to capture rent from natural resource extraction, this is often partly compensated by higher deductions ("uplift") in order to avoid a too high distortion.[19] The following corollary shows that the effects of this on the beta of equity depend on the method of compensation. The result follows directly from equation (12).

Corollary 2.2: *Under Assumptions 1–4: Consider a tax reform with $t > 0, a = 0$ and $\gamma_1 > 1$ at the outset. If an increase in t is compensated by a higher c so that γ_1 is kept constant, then also β_{X1} is reduced in proportion with $(1 - t)$.*

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[17] "Leverage" refers to borrowing as a fraction of $I(1 - ta)$, not as a fraction of I.

[18] While it is often stated that a RRT (see footnote 16) and a Brown cash flow tax have the same effects when tax positions are certain, this is not true for effects on the systematic risk of equity. For instance, when the project is fully equity financed ($\eta = 1$), the RRT has $\beta_{X1} = \beta_P(1 - t)$, while the cash flow tax has $\beta_{X1} = \beta_P$.

[19] Such deductions are well known, e.g., from petroleum taxation in the U.K., Denmark and Norway.

If instead an increase in t is compensated by a higher a, there will also be a reduction in β_{X1} if $g(1 - \eta) > 0$ or $c > 0$ (or both). (The proof is a bit tedious, and is left out.) These effects are illustrated in section 6 below.

Equation (12) (with (9) inserted) could be compared with the text book version. For instance, in Brealey and Myers (2000) p. 483 one finds $\beta_X = \beta_P/\eta$ when the debt is riskless and the "beta of assets" is interpreted as β_P. Using equation (12) instead, we find

$$\beta_{X1} < \frac{\beta_P}{\eta} \Leftrightarrow (1 - \eta)r_f(1 - tg - \theta) < \frac{tc}{1 - ta}. \tag{13}$$

Except for the case of a cash flow tax ($c = 0$), or other cases with very small c values, this inequality is likely to be satisfied, since the right-hand side is greater than or equal to tc, while the left-hand side is less than r_f. Thus β_{X1} here is lower than the text book version when β_P is interpreted as the text books' "beta of assets."

A more reasonable interpretation is that the "beta of assets" already includes a corporate income tax in some sector. In that case the correct beta of equity is lower (ceteris paribus) the higher is the tax rate for the project to be evaluated.

2.2   The weighted average cost of capital

Based on case 1, we can calculate the cost of equity and the WACC. The CAPM formula gives the cost of equity,

$$\frac{E(X_{(1)})}{\varphi(X_{(1)})} - 1 = r_f\theta + \beta_{X1}[E(r_m) - r_f\theta], \tag{14}$$

where β_{X1} is taken from (12), and $E(X_{(1)})/\varphi(X_{(1)})$ represents a marginal project.

As a control we may calculate the WACC directly. By standard definition the WACC is the expected rate of return for the marginal project after tax. More specifically, it is based on the after-tax cash flow in period 1 calculated as if there is no borrowing. The return should be calculated in relation to the total of debt and equity, and the tax advantage of borrowing shows up in the cost of debt. Thus,

$$1 + \text{WACC}_1 = \frac{E(X_{(1)}) + B(1 + r_f) - r_f Bgt}{\varphi(X_{(1)}) + B}. \tag{15}$$

The following corollary is shown in the appendix:

Corollary 2.3: *Under Assumptions 1–4: The weighted average cost of capital as defined by (15) can be expressed as in (16), with β_{X1} taken from (12).*

The WACC is given by:

$$1 + \text{WACC}_1 = 1 + \eta \left\{ r_f \theta + \beta_{X1}[E(r_m) - r_f \theta] \right\} + (1 - \eta) r_f (1 - tg). \tag{16}$$

On an abstract level this is consistent with the standard expression, one plus a value-weighted average of the costs of equity and debt, where the cost of debt is $r_f(1 - tg)$. Case 1 has shown, however, how the cost of equity is affected by the corporate tax applied in the sector. Equation (16) shows that this only happens when the cash flow has systematic risk. If $\beta_P = 0$, then also $\beta_{X1} = 0$, and the equity element in the WACC formula is not affected by the tax rate t.

2.3   A comparison with Derrig (1994)

Derrig's model can be compared with case 1 above by setting $\theta = 1$, $g = 1$, $a = 0$, and $\eta = 1$. The most obvious interpretation includes setting $c = 1$, but we shall see below that there exists another possibility.

Derrig does not consider an after-tax marginal project, and thus arrives at a somewhat different expression from that of case 1 above. The after-tax beta of Derrig's model can be reproduced with the present paper's notation as follows:

An amount $I = Q\varphi(P)$ is invested, and yields an after-tax income one period later of

$$QP - t[QP - I] = QP - t[QP - Q\varphi(P)], \tag{17}$$

i.e., only the net nominal income is taxed. This tax is paid with certainty, or received (if the expression in square brackets is negative). The value one period earlier of the after-tax income is

$$Q\varphi(P)(1 - t) + \frac{tQ\varphi(P)}{1 + r_f} = Q\varphi(P)\frac{1 + r_f - r_f t}{1 + r_f}. \tag{18}$$

The beta of this is

$$\frac{Q\varphi(P)(1 - t)}{Q\varphi(P)\frac{1 + r_f - r_f t}{1 + r_f}}\beta_P = \frac{(1 - t)(1 + r_f)}{1 + r_f(1 - t)}\beta_P, \tag{19}$$

decreasing in t, which is equation (17) in Derrig (1994).

But observe that the relation between the investment in period 0 and the market value in period 0 of the after-tax income in period 1 was not determined endogenously in this derivation. In fact, the only role of the statement "An amount $I = Q\varphi(P)$ is invested," is to determine the deduction granted in the tax base in period 1. This deduction has a different relation to the tax base than in case 1 above. Following the same exposition, case 1 goes like this:

An amount $I = Q\varphi(P)/\gamma_1$ is invested, and yields an after-tax income one period later of

$$QP - t[QP - I] = QP - t[QP - Q\varphi(P)/\gamma_1], \tag{20}$$

i.e., only the net nominal income is taxed. This tax is paid with certainty, or received (if the expression in square brackets is negative). The value one period earlier of the after-tax income is

$$Q\varphi(P)(1 - t) + \frac{tQ\varphi(P)}{\gamma_1(1 + r_f)} = Q\varphi(P)\frac{(1 - t)\gamma_1(1 + r_f) + t}{\gamma_1(1 + r_f)}. \tag{21}$$

The beta of this is

$$\frac{Q\varphi(P)(1 - t)}{Q\varphi(P)\frac{\gamma_1(1-t)(1+r_f)+t}{\gamma_1(1+r_f)}}\beta_P = \frac{\gamma_1(1 - t)(1 + r_f)}{\gamma_1(1 - t)(1 + r_f) + t}\beta_P. \tag{22}$$

The solution for γ_1 which ensures that the project is marginal is shown in (9), which yields

$$\gamma_1 = \frac{1 + r_f - t}{(1 - t)(1 + r_f)}. \tag{23}$$

Inserting this gives

$$\beta_{X1} = \frac{1 + r_f - t}{1 + r_f}\beta_P. \tag{24}$$

For $t > 0, r_f > 0$, this is strictly greater than the value in (19) found by Derrig (1994).

It is, however, possible to reproduce Derrig's result as a special case of case 1 by considering a contrived tax system with $c = \gamma_1$. This reintroduces a relationship for period 1 between the pre-tax income and the deduction in the tax base which is the same as in Derrig's model. Preserving the other parameter values of this section yields (from (9))

$$\gamma_1 = \frac{1}{1 - t}\left(1 - \frac{tc}{1 + r_f}\right) = \frac{1 + r_f - tc}{(1 - t)(1 + r_f)}. \tag{25}$$

Introduce $c = \gamma_1$, and then solve for γ_1:

$$\gamma_1 = \frac{1 + r_f}{1 + r_f - r_f t}. \tag{26}$$

13

From (12) we now find

$$\beta_{X1} = \frac{(1-t)(1+r_f)}{1+r_f - r_f t},\tag{27}$$

which is Derrig's result. This shows that the effect of Derrig's not solving for a marginal project shows up through the ratio of the before-tax income to the deduction.

3   Extending the model: Uncertain tax position

The results for case 1 above are based on the assumption that the firm is certain to be in tax position in period 1. While the tax element tPQ is perfectly correlated with the operating revenue, the values of the depreciation deduction and interest deduction were assumed to be certain, relying on the firm being in a certain tax position.

Most corporate income taxes have imperfect loss offset. If the tax base is negative one year, there is no immediate refund. The loss may under some systems be carried back or forward, but there are usually limitations to this, and the present value is not maintained.[20] In a two-period model the loss carry-forward cannot be modeled in detail. [21] An extreme assumption which yields an analytical solution, is that in these cases, there is no loss offset at all.[22] The cash flow to equity in period 1 is then

$$PQ - B(1+r_f) - t\chi(PQ - gBr_f - cI),\tag{28}$$

where χ is an indicator variable, $\chi = 1$ when the firm is in tax position in period 1, $\chi = 0$ if not. We shall consider an even more extreme version, Assumption 5, which replaces Assumption 4.

Assumption 5: *The tax base in period 1 is operating revenue less $(gr_f B + cI)$. When this is positive, there is a tax paid at a rate t. When it is negative, the tax system gives no*

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[20]There are exceptions, such as the system proposed by Garnaut and Clunies Ross (1975).

[21]For a numerical, multi-period approach, see Lund (1991).

[22]Observe that for a sufficiently small project, there is no such thing as a partial loss offset within one period. Since we are only interested in identifying the borderline case, the after-tax marginal project, it would be quite arbitrary to introduce a large loss project of which one part exhausts a tax liability from other activity, while the remaining part has no loss offset. While this is certainly possible in practice, we concentrate on the extremes.

loss offset at all. There is also a tax relief of taI in period 0. The constants g, c, and a are in the interval [0, 1].

The assumption is extreme, exaggerating the probability of a negative tax base. The marginal project is supposed to make up the whole tax base, which will not be the case if there are decreasing returns to scale. The average rate of return from the project is likely to be substantially higher than the marginal rate of return, in particular in natural resource extraction. The marginal project is more realistically seen as a sub-project within a larger project, "the last amount to be invested."

More generally, the firm (i.e., the subsidiary) is likely to consist of several projects. Under Assumption 2, these would just be more projects with infra-marginal rates of return. Outside the model there are also sunk costs from periods before period 0, and there is flexibility, so there may be a variety of projects. Some of these could contribute negatively to the tax base, but in those cases, there may be flexibility which allows the firm to cancel that contribution. Flexibility would increase the probability of being in tax position.

Nevertheless we will consider Assumption 5 because it leads to an analytical solution, and because it gives an upper bound on the distortion. The case may be called the constant-returns-to-scale (CRS), stand-alone case. The solution is found using option valuation methods, first applied to tax analysis by Ball and Bowers (1983).[23]

Assumption 6: *A claim to a period-1 cash flow* $\max(P - K, 0)$, *where* K *is any positive constant, has a period-0 market value according to the model in McDonald and Siegel (1984).* The value can be written as

$$\varphi(P)N(x_1) - \frac{K}{1 + r_f\theta}N(x_2), \tag{29}$$

where

$$x_1 = \frac{\ln(\varphi(P)) - \ln(K/(1 + r_f\theta))}{\sigma} + \sigma/2, \quad x_2 = x_1 - \sigma, \tag{30}$$

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[23]The application of option valuation does not imply that there is a real option, i.e., some flexibility in the project. It is simply the tax cash flow which resembles that from a European call option at the option's expiration date. Other applications of the method are Majd and Myers (1987) and Lund (1991). MacKie-Mason (1990) combines real options and tax options.

N is the standard normal distribution function, and σ is the instantaneous standard deviation of the price. The formula is modified here with $r_f\theta$ as the risk free interest rate. If $\theta < 1$, this must rely on an assumption that anyone who trades in securities is more heavily taxed on their interest income than on their equity income.

As pointed out by McDonald and Siegel, there are two alternative derivations of their valuation formula, one based on absence of arbitrage, the other on an intertemporal version of the CAPM.[24] $N(x_2)$ can be interpreted as the probability that $P > K$, but with an adjustment. The drift of the price process must be adjusted to what it would have been if all investors had been risk neutral.[25] When $\beta_P > 0$, this means that $N(x_2) < \Pr(P > K)$. $N(x_1)$ exceeds $N(x_2)$, reflecting that $E(P|P > K) > E(P)$, so that the whole formula is the present value of $E[\max(0, P - K)]$ under the adjusted price process.[26]

Equations (30) originate from the geometric Brownian motion price process which underlies the original Black and Scholes (1973) option theory. Equation (29) is much more general, however, and holds for any price process which does not allow arbitrage, with its adjusted probabilities replacing $N(x_1)$ and $N(x_2)$.[27] This more general interpretation is sufficient here.

3.1 Case 2: CRS, stand-alone, uncertain tax position

Based on Assumptions 1, 2, 3, 5, and 6, the beta and distortion can be derived as follows. The cash flow to equity in period 1 is a special case of (28),

$$X_{(2)} = PQ - B(1+r_f) - t\max(0, PQ - gBr_f - cI) = PQ - B(1+r_f) - tQ\max(0, P - \frac{gBr_f + cI}{Q}),$$
(31)

with $(gBr_f + cI)/Q$ replacing K. The valuation in period 0 of a claim to this is

$$\varphi(X_{(2)}) = Q\varphi(P) - \frac{B(1+r_f)}{1+r_f\theta} - tQ\left[\varphi(P)N(x_1) - \frac{gBr_f + cI}{Q(1+r_f\theta)}N(x_2)\right]$$

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[24] The first requires spanning, i.e., the existence of forward contracts for the output, or the ability to create these from other securities. The second places more restrictions on preferences and the joint distribution of asset returns.

[25] See, e.g., Cox and Ross (1976), Constantinides (1978), or any text book in option theory.

[26] It is possible that $N(x_1) > \Pr(P > K)$, which happens when β_P is small and σ is large.

[27] This was realized by Cox and Ross (1976), and is elaborated upon in advanced text books in finance theory, e.g., in Björk (1998).

$$= Q\varphi(P)(1 - tN(x_1)) - \frac{B}{1 + r_f\theta}(1 + r_f(1 - tgN(x_2))) + \frac{tcIN(x_2)}{1 + r_f\theta}. \tag{32}$$

For a marginal project the market value must be equal to the financing need after borrowing and taxes, and by definition $Q\varphi(P) = \gamma_2 I$, so that

$$\eta I(1 - ta) = \varphi(X_{(2)}) = \gamma_2 I(1 - tN(x_1)) - \frac{1 + r_f(1 - tgN(x_2))}{1 + r_f\theta}(1 - ta)(1 - \eta)I + \frac{tcIN(x_2)}{1 + r_f\theta}, \tag{33}$$

which implies

$$\gamma_2 = \frac{1}{1 - tN(x_1)}\left\{(1 - ta)\left[\eta + \frac{1 + r_f(1 - tgN(x_2))}{1 + r_f\theta}(1 - \eta)\right] - \frac{tcN(x_2)}{1 + r_f\theta}\right\}. \tag{34}$$

The effect of the uncertain tax position shows up only through the multiplication of the (period-1) tax rate with $N(x_1)$ or $N(x_2)$. One could say that this is similar to a reduced tax rate in period 1, but the difference between the two probabilities is crucial. If uncertainty goes to zero, so that both probabilities go to unity, γ_2 approaches γ_1.

The beta value of equity is equal to the beta value of a portfolio with value $\varphi(X_{(2)})$, with $Q(1 - tN(x_1))$ claims on P and the rest in risk free assets,

$$\beta_{X2} = \frac{Q\varphi(P)(1 - tN(x_1))}{\varphi(X_{(2)})}\beta_P = \frac{\gamma_2(1 - tN(x_1))}{\eta(1 - ta)}\beta_P. \tag{35}$$

The results can be summarized as follows:

Proposition 3: *Under Assumptions 1–3, 5, 6: The relative distortion is given by (34). The beta of equity is given by (35), and is decreasing in $N(x_2)$.*

The product $\gamma_2(1 - tN(x_1))$, which appears in (35), is equal to the expression in curly brackets in (34), which does not contain $N(x_1)$. It is easily seen from this that as $N(x_2)$ is reduced (from unity, which is its implicit value in (12)), β_{X2} is increased. The possibility of being out of tax position increases the systematic risk of the equity position, whether there is borrowing ($\eta < 1$) or not.

The effect on γ_2 is less transparent, since $(1 - tN(x_1))$ appears in the denominator, and this is increased as uncertainty increases. This counteracts the increase in the expression in curly brackets in (34). For those two terms which contain $N(x_2)/(1 - tN(x_1))$, however, it is possible to show that this fraction decreases with uncertainty.[28] The two terms have

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[28] From the fact that $t < 1 < (1 - N(x_2))/(N(x_1) - N(x_2))$, one can conclude that the fraction $N(x_2)/(1 - tN(x_1))$ is less than $1/(1 - t)$, which is its value when the tax position is certain.

negative sign, so this contributes to increasing γ_2. But if η is large and/or θ is small, the effect of uncertainty is difficult to determine without considering numerical examples.

4 Extending the model: Many periods, certain tax position

A multi-period extension without production flexibility and with the firm always in tax position can be derived from the following assumptions:

Assumption 7: *In period 0 the firm invests an amount $I > 0$ in a project. In period T the project produces a quantity $Q_T \geq 0$ to be sold at an uncertain price P_T, for all periods $T \geq 1$. The joint probability distribution of all prices and all rates of return on the market portfolio is exogenous to the firm. There is no production flexibility; $\{Q_T\}_{T=1}^{\infty}$ is fixed after the project has been initiated.*

It will be apparent below that there is no need to specify any particular production profile. At this point it is not specified whether the profile ends in finite time or continues indefinitely.

Assumption 8: *The tax system and the risk free interest rate are constant over time. As seen from period 0, claims to one unit of output to be delivered in different future periods all have the same beta, β_P, irrespective of the period of delivery. Such claims are valued in period 0 by some well defined, linear valuation function, $\varphi(P_T)$. The valuation sums with infinitely many terms have finite values.*

This formulation is sufficient to solve the model, and avoids any specification of, e.g., the output price process or the term structure of convenience yields. Only valuation as of period 0 is of interest. The assumption of finite values implies, i.a., a positive after-tax riskless interest rate, and it restricts the possible growth in the Q_T sequence.

Assumption 9: *A fraction $(1-\eta) \in [0,1)$ of the financing need in period 0 is borrowed. This fraction is independent of the investment decision and of the tax system. In each*

subsequent period, a fraction $\mu \in (-r_f\theta, 1)$ of the remaining loan is repaid, and interest is paid.

Since the product price is likely to have autocorrelation, the market value of the firm's remaining production will be stochastic and change over time. Thus it would be complicated to assume that the leverage is a constant fraction of this value. Instead an exponentially decreasing loan is assumed for simplicity. If $\theta = 1 - tg$, the assumption will have no importance, since the net after-tax value of the loan will be zero. There is no assumption that $\mu \geq 0$, so the loan may be increasing in nominal terms, as will be apparent below. The after-tax present value should be finite, however, requiring $\mu > -r_f\theta$.

Assumption 10: *A tax at rate $t \in [0, 1)$ will be paid with certainty in all production periods. The tax base in period T is operating revenue less $(gr_f B_{T-1} + c_T I)$, where B_{T-1} is the loan remaining from the previous period. There is also a tax relief of taI in period 0. The constants g, a, and all c_T (for $T \geq 1$) are in the interval $[0, 1]$.*

The schedule of tax depreciation allowances is the rather general $\{c_1 I, c_2 I, \ldots\}$, and it turns out that only the present value of this will be important.

The models of M&M (1963) and Levy and Arditti (1973) assume continuous reinvestment and a stochastic revenue which has the same probability distribution every period. The present model can be extended to accomodate such assumptions. This will be case 4. The realism and relevance of the models will be considered thereafter.

4.1 Case 3: Many periods, certain tax position

Based on Assumptions 1, 7, 8, 9, and 10, the beta and distortion can be derived as follows. Define the relative distortion as

$$\gamma_3 \equiv \frac{\sum_{T=1}^{\infty} Q_T \varphi(P_T)}{I}. \tag{36}$$

when the project is exactly marginal after tax.

The cash flow to equity in any period T (for $T \geq 1$) is

$$X_T = P_T Q_T (1-t) - r_f(1-tg)B_0(1-\mu)^{T-1} - B_0(1-\mu)^{T-1}\mu + tc_T I. \tag{37}$$

The market value in period 0 of this sequence is

$$\sum_{T=1}^{\infty} \varphi(X_T) = (1-t) \sum_{T=1}^{\infty} Q_T \varphi(P_T) - B_0 \sum_{T=1}^{\infty} \frac{[r_f(1-tg) + \mu](1-\mu)^{T-1}}{(1+r_f\theta)^T} + tI \sum_{T=1}^{\infty} \frac{c_T}{(1+r_f\theta)^T}.$$
(38)

Writing the last of these sums, the present value of (the tax value of) the depreciation schedule, as tIA, the whole expression can be rewritten as

$$\sum_{T=1}^{\infty} \varphi(X_T) = (1-t) \sum_{T=1}^{\infty} Q_T \varphi(P_T) - B_0 \frac{\mu + r_f(1-tg)}{\mu + r_f\theta} + tIA.$$
(39)

For a marginal project, this must be equal to a fraction η of the after-tax financing need $I(1-ta)$, while $B_0 = (1-\eta)I(1-ta)$, which yields

$$\eta I(1-ta) = \gamma_3 I(1-t) - (1-\eta)I(1-ta)\frac{\mu + r_f(1-tg)}{\mu + r_f\theta} + tIA.$$
(40)

This gives

$$\gamma_3 = \frac{1}{1-t} \left\{ (1-ta) \left[\eta + \frac{\mu + r_f(1-tg)}{\mu + r_f\theta}(1-\eta) \right] - tA \right\}.$$
(41)

The structure of the solution (9) is easily recognized. Obviously, for many existing depreciation schedules, the distortion due to $A < 1$ is substantial, and higher than in the two-period model. The effect of any difference between $1 - tg$ and θ is strengthened by $\mu < 1$, as compared with the two-period model.

The following proposition is proved in the appendix:

Proposition 4: *Under Assumptions 1, 7–10: Let A be the defined as in (39) above. The relative distortion is given by (41). If $A(1 + r_f\theta/\mu) \leq 1$, then this distortion is increasing in the tax rate for $a = 0$, decreasing for $a = 1$. The beta of equity is given by (42). It is decreasing in the tax rate, and strictly so when $A > 0$ or $(1-\eta)g > 0$ (or both).*

The condition on μ and A which is made in order to sign $\partial\gamma_1/\partial t$, means that the results from case 1 carry over if there is a sufficiently fast repayment of the loan and a sufficiently low present value of depreciation deductions. Case 1 would correspond to $\mu = 1$ and $A(1 + r_f\theta) = c$.

Due to Assumption 8, the beta is found as

$$\beta_{X3} = \frac{(1-t)\sum_{T=1}^{\infty} Q_T \varphi(P_T)}{\sum_{T=1}^{\infty} \varphi(X_T)} \beta_P = \frac{\gamma_3(1-t)}{\eta(1-ta)} \beta_P.$$
(42)

The structure of the solution is the same as in case 1, but the two multi-period effects on the distortion (which were mentioned above) show up through γ_3. A stronger distortion due to the depreciation schedule implies an increased β_{X3}.

4.2 Case 4: Perpetual reinvestment, certain tax position

The next assumption is introduced in addition to those definining case 3. It modifies Assumption 7 by introducing reinvestment as a necessary means to maintain the given production sequence, so case 4 is not a special case of case 3.

Assumption 11: *The inflation rate is $\pi \in [0, r_f\theta)$. The real value of the loan is maintained, so that $\mu = -\pi$. Output is a constant, $Q_T = Q > 0$, for all periods $T \geq 1$. In order to maintain this output profile, the firm must commit to reinvesting $\xi(1+\pi)^{T-1}I$ in each period $T \geq 1$. The tax depreciation schedule is $c_T = \nu(1+\pi)^{T-1}$, i.e., a fraction ν of the nominal value of the invested capital, which is maintained in real value due to reinvestment equal to the physical depreciation. The constants ξ and ν are in the interval $(0,1)$.*

The important extension is that the firm commits to reinvestment. This is interpreted as maintaining the real value of the capital equipment. Without reinvestment this would depreciate at a rate ξ, which may or may not coincide with the tax depreciation rate ν. The assumption about maintaining the loan in real terms is not essential, as will become clear. There is also an assumption of a positive after-tax real interest rate, $r_f\theta > \pi$.

The reinvestment is a non-stochastic sequence, fixed after the project has been initiated. This is in line with Levy and Arditti (1973).

Based on Assumptions 1, 7, 8, 9, 10, and 11, the beta and distortion can be derived. The definition of a relative distortion is not obvious anymore, unfortunately. (Should the present value of the pre-tax reinvestment sequence be subtracted in the numerator or added in the denominator?) For this reason and for notational simplicity, ignore that present value for the moment, and define

$$\gamma_{4p} \equiv \frac{Q\sum_{T=1}^{\infty}\varphi(P_T)}{I}. \tag{43}$$

when the project is exactly marginal after tax. The subscript "p" means pseudo, since γ_{4p} does not measure the relative distortion.

The cash flow to equity in any period T (for $T \geq 1$) is

$$X_T = P_T Q(1-t) - r_f(1-tg)B_0(1+\pi)^{T-1} + B_0(1+\pi)^{T-1}\pi + t\nu(1+\pi)^{T-1}I - \xi(1+\pi)^{T-1}I(1-ta).$$

(44)

The immediate tax refund fraction a is assumed to be applied in all periods.

The market value in period 0 of this sequence is

$$\sum_{T=1}^{\infty} \varphi(X_T) = (1-t)Q\sum_{T=1}^{\infty} \varphi(P_T) - B_0\frac{r_f(1-tg)-\pi}{r_f\theta - \pi} + I\frac{t\nu - \xi(1-ta)}{r_f\theta - \pi}.$$

(45)

For a marginal project, this must be equal to a fraction η of the after-tax financing need $I(1-ta)$, while $B_0 = (1-\eta)I(1-ta)$, which yields

$$\eta I(1-ta) = \gamma_{4p}I(1-t) - (1-\eta)I(1-ta)\frac{r_f(1-tg)-\pi}{r_f\theta - \pi} + I\frac{t\nu - \xi(1-ta)}{r_f\theta - \pi}.$$

(46)

This gives

$$\gamma_{4p} = \frac{1}{1-t}\left\{(1-ta)\left[\eta + \frac{r_f(1-tg)-\pi}{r_f\theta - \pi}(1-\eta)\right] - \frac{t\nu - \xi(1-ta)}{r_f\theta - \pi}\right\}.$$

(47)

The beta of the cash flow to equity is

$$\beta_{X4} = \frac{\gamma_{4p}(1-t)}{\eta(1-ta)}\beta_P.$$

(48)

The final term within the curly brackets in (47) is the combined effect of reinvestment and tax depreciation deductions. Consider the case without immediate tax relief, i.e., with $a = 0$. Except if the tax rate is very high and depreciation deductions generous, one will then have $t\nu < \xi(1-ta) = \xi$, so that the final term represents an addition to γ_{4p} and β_{X4}, as opposed to cases 1 and 3. This is the consequence of the commitment to a riskless reinvestment sequence, which increases the risk of the cash flow.

The numerator in the final term, $t\nu - \xi(1-ta)$, shows that when $\nu = \xi$, the tax with depreciation deductions acts as a cash flow tax in periods $T \geq 1$. $\nu = 0$ and $a = 1$ will have the same effect as $\nu = \xi$ and $a = 0$. The important difference is of course in period 0, when a cash flow tax gives the refund $taI = tI$. Assuming perpetual reinvestment is

thus equivalent to assuming a cash flow tax without the refund in the initial period, which illustrates the extreme character of the assumption.

The effect of the tax rate on beta is given by

$$\frac{\partial \beta_{X4}}{\partial t} = \frac{\beta_P}{\eta(r_f \theta - \pi)} \left[-(1 - \eta) r_f g - \frac{\nu}{(1 - ta)^2} \right] \leq 0. \tag{49}$$

While the reinvestment sequence contributes to increasing beta as compared with case 3, it is still the case that the tax system contributes to decreasing beta.

The results can be summarized as follows:

Proposition 5: *Under Assumptions 1, 7–11: The beta of equity is given by (48). It is decreasing in the tax rate, and strictly so when $\nu > 0$ or $(1 - \eta)g > 0$ (or both).*

5 A comparison with Levy and Arditti (1973)

The model of Levy and Arditti (1973) is obtained as a subcase of case 4 above by letting $a = 0$, $\pi = 0$, $g = 1$, $\theta = 1$, $\nu = \xi$, and ρ a risk-adjusted discount rate for the output price. From (47) we can now calculate

$$\rho \gamma_{4p} - \xi = \frac{1 - t(1 - \eta)}{1 - t} \rho + \xi \left(\frac{\rho - r_f}{r_f} \right), \tag{50}$$

where the right-hand side is easily recognized as the right-hand side of equation (10) in Levy and Arditti (1973). The left-hand side, $\rho \gamma_{4p} - \xi$, is the difference between the required rate of expected return from the revenue stream, $\rho \gamma_{4p}$, and the depreciation rate, so it is a required net-of-depreciation expected rate of return, just as in Levy and Arditti.

A comparison between cases 1 and 3 on one hand, and case 4 and this special subcase on the other, shows that the assumption of reinvestment is absolutely crucial for the results. Reinvestment at a rate ξ more than counterbalances tax deductions at a rate ξt (and most likely also in a more general case, at a rate νt), so that the effect of assets being depreciable *and* reinvested is to increase the risk of the cash flow to equity, not to decrease it.

While both of these alternative assumptions may have practical relevance, there are reasons to believe that case 3 is at least as relevant as the perpetual reinvestment case,

case 4. Case 4 relies on a commitment to reinvestment which is almost unheard of in practice when it comes to real investment projects.

It may be argued that actual projects will often be of an intermediate kind, with some non-perpetual commitment to reinvestment. This may be true, but there is often the possibility in some periods of deciding against reinvestment. At the point in time of each such reinvestment decision, that period's reinvestment is a new project. That is the relevant point in time for choosing the right cost of equity, or WACC, since these concepts are applied to decisions. If the project initiated by that particular reinvestment is irreverisble and inflexible, it may be analyzed along the lines of case 3,[29] and case 4 becomes irrelevant.[30]

Even if some reinvestment is highly likely, because it may be very productive as compared with letting the project deteriorate, this is no argument against viewing reinvestment as another case-3 type project.

Another observation which modifies the extreme case 4 result, is that even if there is perpetual commitment, many investment factor prices will actually have a positive beta, and few are likely to have negative betas. This is outside the model, as it violates the assumptions, both of case 4 and of Levy and Arditti (1973). The tax depreciation deductions may still be riskless, while the negative reinvestment stream covaries with revenues, which reduces the risk of the net cash flows.[31]

5.1 A firm with overlapping marginal projects

The discussion so far has clarified what determines the required expected rate of return for an investment decision. Next one might ask what will be the realized rate of return when a firm has a sequence of overlapping projects. Will this reflect the higher beta values

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[29] As opposed to the analysis of real options, neglected both here and in Levy and Arditti (1973).

[30] In addition to real options, a further discussion of this could also include capital risk, which is particularly relevant when there is the possibility of selling the ownership to the project after it has been initiated.

[31] This points to the arbitrariness of the distinction between reinvestment and the use of input factors. As mentioned in footnote 12, one may rewrite the model with (gross revenue minus operating costs) replacing gross revenue, $P_T Q_T$.

of the Levy and Arditti (1973) assumptions or the lower beta values of case 3? We shall not discuss the possibility that the firm invests in projects which are not marginal. To disentangle the marginal from the average rate of return is a more general problem, not specific to the topics raised in this paper.

Consider a firm which invests in one new marginal project each period. As seen from its initiation period, each project is of the type denoted case 3 above.

Consider first a firm which invests in the first of such a sequence of projects now, and has the opportunity to invest in new marginal projects in each of the coming years. The expected rate of return to equity between this period and the next will reflect β_{X3} as derived in (42). The reason is as follows: One plus the expected rate of return is equal to the ratio of the expected value of equity as evaluated in the next period, to the market value of a claim to the same as evaluated in this period. In this connection, next period's value of equity includes the cash flow from the first project to equity in that period. The value of equity in the next period does not reflect any value of the opportunity to invest in new marginal projects in that period or later. This is because these projects are all marginal, so their values are zero.

Consider then a firm which is some years into such a sequence, so that it has a portfolio of projects, one initiated last year, one the year before, and so on. The expected rate of return to equity for such a firm will be a value-weighted average of the expected rates of return to equity from each project. So far there has been no assumption to determine the relative sizes of these projects, and any such assumption will be rather arbitrary. Thus it is difficult to give any precise indication of the expected rate of return to equity for the whole firm. However, it is possible to say something about the expected rate of return over the next period for the remaining part of each of the projects, based on some simplifying, but not quite arbitrary, assumptions.

Assumption 12: *Relative changes in the output price are stochastically independent over time and have a constant expected value of $1 + \alpha \geq 1$. Their joint probability distribution with the rate of return on the market portfolio is also the same every period, so that $E_{T-1}(P_T)/\varphi_{T-1}(P_T)$ is a constant, $1 + \rho > 1 + r_f\theta$, where φ_{T-1} is the valuation as of period $T-1$, and E_{T-1} is the expectation conditional on P_{T-1}, for all T. Output from the*

marginal project initiated in period $-\tau$ is $Q(1-\omega)^{T-1}$ in each subsequent period $-\tau+T$, for $T \geq 1$. Tax depreciation deductions each period are a fraction μ of the nominal remaining tax-depreciated value of the invested capital. The constants ω and μ are in the interval $(0, 1)$.

We shall be concerned with valuation as of periods 0 and 1. The initiation period for a project is denoted $-\tau$, with $\tau \geq 0$. Since the scale of a marginal project is not determined in this model, we arbitrarily fix the project's output sequence, $Q, Q(1-\omega), Q(1-\omega)^2, \ldots$, while the investment in period $-\tau$, $I_{-\tau}$, is determined so that the project is marginal.

In period 1, the market value of remaining production in that period and the years to come from a project initiated in period $-\tau$ is

$$\sum_{T=1}^{\infty} Q(1-\omega)^{\tau+T-1} P_1 \left(\frac{1+\alpha}{1+\rho}\right)^{T-1} = \frac{QP_1(1+\rho)}{\rho-\alpha+\omega(1+\alpha)}(1-\omega)^{\tau}, \qquad (51)$$

while one period earlier, in period 0, a claim to this has the market value

$$\frac{QP_0(1+\alpha)}{\rho-\alpha+\omega(1+\alpha)}(1-\omega)^{\tau}. \qquad (52)$$

The tax-depreciated value of capital will decrease at the same nominal rate, μ, as the loan. This is a simplification in order to have only two different time paths for the elements of the project's cash flow. As seen from period 1, the present value of remaining depreciation deductions, including the present period's, is

$$I_{-\tau} \sum_{T=1}^{\infty} \frac{\mu(1-\mu)^{\tau+T-1}}{(1+r_f\theta)^{T-1}} = I_{-\tau}\mu(1-\mu)^{\tau}\frac{1+r_f\theta}{r_f\theta+\mu}. \qquad (53)$$

The constant A, defined in connection with (39), takes the value

$$A = \frac{\mu}{\mu+r_f\theta} \qquad (54)$$

for this depreciation schedule.

Based on assumptions 1, 7, 8, 9, 10, and 12, the expected rate of return to equity can be derived. The investment decision which was made in period $-\tau$, determined

$$I_{-\tau} = \frac{QP_{-\tau}(1+\alpha)}{\gamma_3[\rho-\alpha+\omega(1+\alpha)]} \qquad (55)$$

for the marginal project, cf. (36) and (52).

The derivation in the appendix shows that the expected rate of return to equity between periods 0 and 1 for this project depends on the relative values of the remaining output on one hand and the remaining loan and depreciation deductions on the other. The ratio of these values will depend partly on the difference between the decline rates of output, ω, and the loan and depreciation deductions, μ, and partly on the realized rate of output price increase between period $-\tau$ and period 0. If

$$\frac{P_0}{P_{-\tau}}(1-\omega)^{\tau} = (1-\mu)^{\tau}, \tag{56}$$

there has been a balanced development of the value elements, so that the expected rate of return is the same as it was for the first period after initiation.

Proposition 6: *Under Assumptions 1, 7–10, 12: Assume the value elements of a project have had a balanced development as given by (56). The beta of equity for the remainder of a project is then given by $\beta_{J3} = \beta_{X3}$, where the latter is given by (42). If there is an imbalance,*

$$\frac{P_0}{P_{-\tau}}(1-\omega)^{\tau} > (1-\mu)^{\tau}, \tag{57}$$

and if

$$Z \equiv \frac{[\mu + r_f(1-tg)](1-\eta)(1-ta) - t\mu}{\mu + r_f\theta} > 0, \tag{58}$$

then $\beta_{J3} < \beta_{X3}$. If one of the two inequalities (57) and (58) is reversed, then $\beta_{J3} > \beta_{X3}$, but if both are reversed, then $\beta_{J3} < \beta_{X3}$.

The proof is in the appendix. The intuition behind the result can be explained, e.g., by considering the case with $Z > 0$ and $P_0(1-\omega)^{\tau}/P_{-\tau} > (1-\mu)^{\tau}$. A positive Z means that the loan repayment dominates the depreciation deductions. When the project was started in period $-\tau$, this implied an expected rate of return to equity exceeding ρ due to leverage. But an imbalanced development, where the output has retained a larger fraction of its absolute value than has the (net negative) non-stochastic cash flow element(s), means that the expected rate of return for the remainder of the project is reduced and gets closer to ρ.

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para-	$t = 0.25$	$t = 0.5$	$t = 0.75$
meters	$\beta_P(1-t) = 0.6$	$\beta_P(1-t) = 0.4$	$\beta_P(1-t) = 0.2$
$a = 0$	$\gamma_1 = 1.014$	$\gamma_1 = 1.043$	$\gamma_1 = 1.129$
$c = 1$	$\beta_{X1} = 0.6086$	$\beta_{X1} = 0.4172$	$\beta_{X1} = 0.2258$
$a = 0$	$\gamma_1 = 1.014$	$\gamma_1 = 1.014$	$\gamma_1 = 1.014$
	$c = 1$	$c = 1.197$	$c = 1.262$
	$\beta_{X1} = 0.6086$	$\beta_{X1} = 0.4057$	$\beta_{X1} = 0.2029$
$c = 1$	$\gamma_1 = 1.014$	$\gamma_1 = 1.014$	$\gamma_1 = 1.014$
	$a = 0$	$a = 0.1882$	$a = 0.2509$
	$\beta_{X1} = 0.6086$	$\beta_{X1} = 0.4479$	$\beta_{X1} = 0.2499$

Table 1: Distortion and equity beta with no borrowing, various tax parameters

6   A numerical example

This section gives a numerical example which illustrates Corollary 2.2. The example illustrates how depreciation deductions or the immediate tax relief may be used to counteract the distortionary effect of high tax rates in, e.g., resource extraction, and what effect this could have on the equity beta.

The example has $r_f = 0.06$, $E(r_m) = 0.09$, and $\beta_P = 0.8$. The equilibrium in the home country has $\theta = 0.75$. The first set of calculations has $\eta = 1$.

Three different values of the tax rate t will be considered, $t = 0.25$, $t = 0.5$, and $t = 0.75$. With $c = 1$ the lower tax rate gives $\gamma_1 = 1.014$, and $\beta_{X1} = 0.6086$. Increasing the tax rate will increase the distortion and reduce the equity beta. The numbers are given in the first lines of table 1.

The first lines of the table give values for the case $a = 0$, $c = 1$. Even when the high tax rate of $t = 0.75$ gives a distortion of 13 percent in the required expected one plus rate of return, the equity beta is quite close to $\beta_P(1-t)$. The subsequent lines consider two different cases for which the distortionary effect of high tax rates are counteracted by adjusting either a or c, while the other of these two parameters is kept at its initial level.

para-	$t = 0.25$	$t = 0.5$	$t = 0.75$
meters	$\beta_P \frac{(1-t)}{\eta} = 1.2$	$\beta_P \frac{(1-t)}{\eta} = 0.8$	$\beta_P \frac{(1-t)}{\eta} = 0.4$
$a = 0$	$\gamma_1 = 1.014$	$\gamma_1 = 1.029$	$\gamma_1 = 1.072$
$c = 1$	$\beta_{X1} = 1.217$	$\beta_{X1} = 0.8230$	$\beta_{X1} = 0.4287$
$a = 0$	$\gamma_1 = 1.014$	$\gamma_1 = 1.014$	$\gamma_1 = 1.014$
	$c = 1$	$c = 1.182$	$c = 1.242$
	$\beta_{X1} = 1.217$	$\beta_{X1} = 0.8115$	$\beta_{X1} = 0.4056$
$c = 1$	$\gamma_1 = 1.014$	$\gamma_1 = 1.014$	$\gamma_1 = 1.014$
	$a = 0$	$a = 0.1751$	$a = 0.2352$
	$\beta_{X1} = 1.217$	$\beta_{X1} = 0.8893$	$\beta_{X1} = 0.4926$

Table 2: Distortion and equity beta with 50% borrowing, various tax parameters

To avoid distortionary effects, rent tax systems typically admit extra depreciation-type allowances ("uplift") or accelerated depreciation, or both. The two alternative ways of doing this in the table are just the extreme possibilities, relying on only one of the two. The parameters are adjusted so that the distortion is as in the upper left corner, $\gamma_1 = 1.014$.

The table shows that β_{X1} is close to $\beta_P(1-t)$, and that none of the two alternative parameter adjustments alters this very much. For $t = 0.75$ an immediate tax relief, $a = 0,25$, goes in the direction of a cash flow tax with a higher β_{X1}, while $c > 1$ goes in the opposite direction. The distortion is the same by construction.

Table 2 shows the same experiments for a firm which borrows 50 percent of its financing need. The same qualitative results hold: The beta of equity is roughly proportional to $(1-t)/\eta$.

7   Discussion

The concept of a cost of equity, or a weighted-average cost of capital, is used for capital budgeting, and may also be used for valuation of the equity in a firm. In both cases expected after-tax cash flows are used. Some of the well-known weaknesses of this approach have not

been considered here. The approach is known to be misleading when taxes are non-linear functions of pre-tax cash flows,[32] or when there are real options.[33]

Instead this paper demonstrates another weakness, that the application of the same cost of equity across different tax systems is misleading. Since the cost of equity is one component in the WACC, the application of the same WACC is also misleading. This is of interest for a firm considering projects under various tax systems. It is also of interest for tax authorities considering tax reforms. In forecasting the firms' behavior under different hypothetical tax systems, one should not apply the same cost of equity under all of them.

Much of the literature considers only the case of a firm which will be in tax position with certainty in all future periods. While it is commonplace in this literature to include the effect of interest deductibility in the cost of debt, as in equation (16), it is not common to include the effect of depreciation deductions in the cost of equity. This is surprising since depreciation deductions are proportional to investment, while it is much less obvious that borrowing is proportional to investment. There are other arguments for not using a constant cost of equity or WACC. But if such constant discount rates are used, there seem to be good reasons to include the effect of depreciation deductions, better than those for interest deductions.

A constant cost of equity or WACC may be consistent with the more general APT method, given that all elements of a cash flow stay in the same relation to each other through time. This was illustrated by the overlapping-projects case above. In general there are good reasons to try to use the more general method. But there are reasons to consider the cost of equity or WACC in some situations. In particular, it is important to know how to use market data as input in valuation procedures. In addition to the standard method for "unlevering" beta values,[34] this paper has shown the need to "untax" them.

A well-known textbook in finance, Brealey and Myers (2000), gives a detailed discussion of the effects of interest deductibility on the WACC. There is also a discussion of the valuation of depreciation deductions, consistent with the present paper, but there is no recommendation to tie this value to the WACC:

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[32]See, e.g., Bradley (1998).

[33]See, e.g., Laughton (1998).

[34]See, e.g., Brealey and Myers (2000), p. 231, or Ross, Westerfield, and Jaffe (1999), sect. 17.7.

Capital projects are normally valued by discounting the total after-tax cash flows they are expected to generate. Depreciation tax shields contribute to project cash flow, but they are not valued separately; they are just folded into project cash flows along with dozens, or hundreds, of other specific inflows and outflows. The project's opportunity cost of capital reflects the average risk of the resulting aggregate. (P. 566.)

This may be a true description of practice in many firms. Since the depreciation deductions are proportional to investment, there are good reasons to revise this practice. Summers (1987) discusses this, argues that depreciation deductions are practically risk free, and discusses optimal taxation given that firms' behavior is inconsistent with finance theory.

The practical relevance of this paper depends on the diversity of the tax rates and/or deduction parameters a firm is facing. If the firm is subject to the same effective tax rates everywhere, and the same deduction parameters, these will be reflected in the observed cost of equity in the market for shares in the firm. This may be applied without considering the tax effects. If there is diversity, the observed cost of equity will be some average, and is less useful. Even when the same tax effective rate applies to all activities in a firm, one can easily make mistakes, e.g., in disentangling the leverage effect from the cost of equity. This may lead to the wrong conclusion that $\beta_P = \eta \beta_{X1}$, with the right-hand side being observable.

The model has given analytical results for some simple cases. A more realistic model will be more complicated, and it may be impossible to solve analytically. Then one may use Monte Carlo simulation to estimate the values of the various elements of the cash flow. It is important to realize that future tax payments have unique risk characteristics, which may be related to the risks of the pre-tax cash flows in complicated ways.

Tax analysis has mainly focused on the wedge between pre-tax and after-tax required expected rates of return. Under full certainty the required after-tax rate of return is often taken as given, and the tax system results in another required pre-tax rate of return. Under uncertainty this is more complex, since the after-tax required expected rate of return depends on the risk characteristics of the return. These characteristics depend on

the taxes. Thus the tax system affects the after-tax required expected rate of return. This happens even if general-equilibrium effects are ignored, i.e., even if $r_f, E(r_m), \operatorname{cov}(r_m, P)$ are assumed to be unaffected by the taxes.

The papers by Levy and Arditti (1973) and Derrig (1994) do not explicitly assume partial equilibrium. But in their discussions of the effects of taxes, they clearly make the same assumptions at this point as does the present paper. They do not have the reference to a multinational firm, as does the title of the present paper. There are two distinct reasons for this reference. One is the partial character of the model. No conclusions can be drawn here about tax systems' effects on the equilibrium in the capital market. The other is the existence of many different tax systems, in different countries (or sectors), which makes the analysis relevant as a good approximation for multinationals.

The model assumes that only the foreign tax system applies at the margin. Whether this is true, will depend on double taxation treaties, tax rates, and the firm's activities at home and abroad. The foreign system is more likely to apply at the margin when it has higher rates than the home system.

Required expected rates of return after tax in various tax regimes are typically observed as (averages of) realized rates of return after tax, or realized beta values. In addition to the problems we have shown so far in the use of these under other tax regimes, there is the problem that they may include realized rent. Unless all after-tax rents have already been capitalized in the market value of shares, this is typically the case in resource extraction in countries where the firms do not pay any up-front fee to reflect the resource value. In that case the realized rates of return are average rates of return, while the required rate of return is a marginal rate.

8   Conclusion

From a CAPM-type model of investment under uncertainty, the paper derives the cost of equity, i.e., the required after-tax expected rate of return to equity for a firm operating under various foreign tax systems. When the firm's shares are traded in a capital market which is unaffected by the foreign tax systems, analytical expressions for the cost of equity

are found. When the firm is in a certain tax position, a tax-adjusted CAPM is used. When the tax position is uncertain, an option valuation method is used.

It is clearly demonstrated that the cost of equity depends on the tax system, even for fully equity financed projects. This is neglected in much of the literature. It is argued here that the most relevant model for project investment decisions does not involve perpetual reinvestment. In such a model it is shown that for a standard corporate income tax the main factor which reduces the cost of equity is the depreciation deduction system. Compared with a neutral cash flow tax, this reduces the systematic risk of equity because it acts as a loan from the firm to the tax authorities. Thus the effect is the opposite of leverage. The possibility of being out of tax position counteracts this effect.

Tax analysis has mainly focused on the tax system's influence on the required expected pre-tax return from a project, taking the after-tax required expected return as given. This paper shows that also the after-tax required expected return is affected by the tax system. This is crucial for the analysis of effects of changes in tax systems.

Appendix

Proof of Proposition 1

From (9) we find

$$\frac{\partial \gamma_1}{\partial t} = \frac{[1 + \eta r_f \theta + (1-\eta) r_f (1-g) - c] - a[1 + \eta r_f \theta + (1-\eta) r_f (1 - g + g(1-t)^2)]}{(1-t)^2 (1 + r_f \theta)}.$$

(A1)

The denominator is positive. When $a = 0$, the remaining terms in the numerator are positive by assumption (since $c \le 1$ and $\eta > 0$). When $a = 1$, the numerator simplifies to $-c - (1-\eta) r_f g (1-t)^2$, which is non-positive. The value of a which makes $\partial \gamma_1 / \partial t = 0$ is the ratio of the first term in square brackets to the second term in square brackets. This is clearly in $[0, 1]$. It contains the tax rate t only in one place, and will depend on t if $(1 - \eta) \ne 0$ and $g \ne 0$ ($r_f > 0$ is already assumed), Q.E.D.

Proof of Proposition 2

From insertion of (9) into (12) we find

$$\frac{\partial \beta_{X1}}{\partial t} = \frac{\beta_P}{\eta(1 + r_f\theta)}\left[-(1-\eta)r_f g - \frac{c}{(1-ta)^2}\right] < 0. \tag{A2}$$

The first fraction is strictly positive (and finite) by assumption. The terms in square brackets (minus signs included) are non-positive, and the result follows, Q.E.D.

Proof of Corollary 2.3

This shows how to derive (16) from (15), using (6) and (9). The numerator in (15) is

$$E(X_{(1)}) + B(1 + r_f(1 - tg)) = Q\varphi(P)\frac{E(P)}{\varphi(P)}(1-t) - B(1 + r_f(1-tg)) + tcI + B(1 + r_f(1-tg))$$

$$= \gamma_1 I(1-t)\{1 + r_f\theta + \beta_P[E(r_m) - r_f\theta]\} + tcI, \tag{A3}$$

where $\gamma_1 = Q\varphi(P)/I$ has been introduced. The denominator is $\varphi(X_{(1)}) + B = I(1 - ta)$, so that

$$1 + \text{WACC}_1 = \frac{\gamma_1(1-t)}{1-ta}\{1 + r_f\theta + \beta_P[E(r_m) - r_f\theta]\} + \frac{tc}{1-ta}$$

$$= \frac{\gamma_1(1-t)}{1-ta}(1 + r_f\theta) + \eta\frac{\gamma_1(1-t)}{\eta(1-ta)}\beta_P[E(r_m) - r_f\theta] + \frac{tc}{1-ta}. \tag{A4}$$

The second of these three terms goes directly into (16). The first and third can be rewritten as

$$\frac{\gamma_1(1-t)}{1-ta}(1 + r_f\theta) + \frac{tc}{1-ta} = (1 + r_f\theta)\left[\eta + \frac{1 + r_f(1-tg)}{1 + r_f\theta}(1-\eta) - \frac{tc}{(1 + r_f\theta)(1-ta)}\right] + \frac{tc}{1-ta}$$

$$= \eta(1 + r_f\theta) + (1-\eta)(1 + r_f(1-tg)) = 1 + \eta r_f\theta + (1-\eta)r_f(1-tg), \tag{A5}$$

which comprises the remaining terms of (16), Q.E.D.

Proof of Proposition 4

From (41) we find

$$\frac{\partial \gamma_1}{\partial t} = \frac{[\mu + \eta r_f\theta + (1-\eta)r_f(1-g) - A(\mu + r_f\theta)] - a[\mu + \eta r_f\theta + (1-\eta)r_f(1-g+g(1-t)^2)]}{(1-t)^2(\mu + r_f\theta)}. \tag{A6}$$

The denominator is positive. When $a = 0$, we must show that the remaining terms in the numerator are non-negative. The factors $(1-g)(1-\eta)$ may be positive or zero, and η may be close to zero. It is thus sufficient that $\mu - A(\mu + r_f\theta) \geq 0$, which is the condition given in the Proposition.

When $a = 1$, the numerator simplifies to $-A(\mu + r_f\theta) - (1-\eta)r_f g(1-t)^2$, which is non-positive, and the first part of the Proposition is proved.

From (42) we find

$$\frac{\partial \beta_{X3}}{\partial t} = \frac{\beta_P}{\eta(\mu + r_f\theta)} \left[-(1-\eta)r_f g - \frac{A(\mu + r_f\theta)}{(1-ta)^2} \right] \leq 0, \tag{A7}$$

Q.E.D.

Proof of Proposition 6

Under Assumption 12, define $J_{-\tau,T}$ as the market valuation as of period T of cash flows to equity in period T and all subsequent periods from a project initiated in period $-\tau < 0 < T$. We need the ratio of $E_0(J_{-\tau,1})$ to $\varphi_0(J_{-\tau,1})$, i.e., one plus the expected rate of return to equity from the remainder of the project. From this we also derive the beta of equity.

Solve (55) for Q and insert into (51) to find

$$J_{-\tau,1} = \frac{I_{-\tau}\gamma_3 P_1(1+\rho)(1-\omega)^\tau(1-t)}{P_{-\tau}(1+\alpha)}$$

$$- \frac{(1-\eta)I_{-\tau}(1-\mu)^\tau(1-ta)[\mu + r_f(1-tg)](1+r_f\theta)}{\mu + r_f\theta} + \frac{tI_{-\tau}(1-\mu)^\tau\mu(1+r_f\theta)}{\mu + r_f\theta}, \tag{A8}$$

where the third term is taken from (53). The second term, the value of remaining loan repayments, is as in (39), with $I_{-\tau}(1-\mu)^\tau$ replacing B_0 as the loan remaining after period 0, and with an additional factor $(1+r_f\theta)$, since (A8) gives the valuation in period 1.

Introduce γ_3 from (41), and find

$$J_{-\tau,1} = I_{-\tau} \left\{ \left[(1-ta)\left(\eta + \frac{\mu + r_f(1-tg)}{\mu + r_f\theta}(1-\eta)\right) - \frac{t\mu}{\mu + r_f\theta} \right] \frac{P_1(1+\rho)(1-\omega)^\tau}{P_{-\tau}(1+\alpha)} \right.$$

$$\left. - \frac{(1-\mu)^\tau(1+r_f\theta)}{\mu + r_f\theta} [(1-\eta)(1-ta)[\mu + r_f(1-tg)] - t\mu] \right\} \tag{A9}$$

In order to simplify the expression, define

$$Z \equiv \frac{[\mu + r_f(1-tg)](1-\eta)(1-ta) - t\mu}{\mu + r_f\theta}, \tag{A10}$$

and observe that for this depreciation schedule, (41) and (54) give

$$\gamma_3 = \frac{1}{1-t}\left[(1-ta)\eta + Z\right]. \tag{A11}$$

The constant Z is positive if the repayment of the project's loan exceeds the tax value of the depreciation deductions in present value terms, but negative otherwise.

We can rewrite $J_{-\tau,1}$ as

$$J_{-\tau,1} = I_{-\tau}(1-\omega)^\tau \left\{ \frac{P_1(1+\rho)}{P_{-\tau}(1+\alpha)}(1-ta)\eta + \left[\frac{P_1(1+\rho)}{P_{-\tau}(1+\alpha)} - \left(\frac{1-\mu}{1-\omega}\right)^\tau (1+r_f\theta)\right] Z \right\}. \tag{A12}$$

This gives

$$E_0(J_{-\tau,1}) = I_{-\tau}(1-\omega)^\tau \left\{ \frac{P_0(1+\rho)}{P_{-\tau}}(1-ta)\eta + \left[\frac{P_0(1+\rho)}{P_{-\tau}} - \left(\frac{1-\mu}{1-\omega}\right)^\tau (1+r_f\theta)\right] Z \right\}, \tag{A13}$$

and

$$\varphi_0(J_{-\tau,1}) = I_{-\tau}(1-\omega)^\tau \left\{ \frac{P_0}{P_{-\tau}}(1-ta)\eta + \left[\frac{P_0}{P_{-\tau}} - \left(\frac{1-\mu}{1-\omega}\right)^\tau\right] Z \right\}. \tag{A14}$$

If the two elements of the cash flow have had a balanced development between periods $-\tau$ and 0, so that $P_0(1-\omega)^\tau/P_{-\tau} = (1-\mu)^\tau$, then it is easy to solve for the expected rate of return and for beta. One plus the expected rate of return becomes

$$\frac{E_0(J_{-\tau,1})}{\varphi_0(J_{-\tau,1})} = 1 + \rho + \frac{\rho - r_f\theta}{(1-ta)\eta}Z. \tag{A15}$$

Observe that

$$\beta_P = \frac{\rho - r_f\theta}{E(r_m) - r_f\theta}, \tag{A16}$$

and use this to find the beta of $J_{-\tau,1}$ between periods 0 and 1,

$$\beta_{J3} = \frac{\frac{E_0(J_{-\tau,1})}{\varphi_0(J_{-\tau,1})} - 1 - r_f\theta}{E(r_m) - r_f\theta} = \frac{\rho - r_f\theta}{E(r_m) - r_f\theta}\left[1 + \frac{1}{(1-ta)\eta}Z\right] = \frac{(\rho - r_f\theta)(1-t)\gamma_3}{[E(r_m) - r_f\theta](1-ta)\eta} = \beta_{X3}, \tag{A17}$$

thus the equation for the balanced case is proved.

With the balanced case as a point of departure, we consider the unbalanced case with $P_0(1-\omega)^\tau/P_{-\tau} \neq (1-\mu)^\tau$. It can be shown that

$$\frac{\partial\left[\frac{E_0(J_{-\tau,1})}{\varphi_0(J_{-\tau,1})}\right]}{\partial\left[\left(\frac{1-\mu}{1-\omega}\right)^\tau \frac{P_{-\tau}}{P_0}\right]} > 0 \tag{A18}$$

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if and only if $Z > 0$. This implies that if

$$Z[P_0(1-\omega)^\tau/P_{-\tau} - (1-\mu)^\tau] > 0, \tag{A19}$$

then the equations (A15) and (A17) are replaced by

$$\frac{E_0(J_{-\tau,1})}{\varphi_0(J_{-\tau,1})} < 1 + \rho + \frac{\rho - r_f\theta}{(1-ta)\eta}Z \tag{A20}$$

and

$$\beta_{J3} < \beta_{X3}. \tag{A21}$$

If the inequality in (A19) is reversed, then the inequalities in (A20) and (A21) are also reversed, Q.E.D.

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