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Existence of optimal nonanticipating controls in piecewise deterministic control prolems.

by Atle Seierstad, University of Oslo

Abstract Optimal nonanticipating controls are shown to exist in nonautonomous piecewise deterministic control problems with hard terminal restrictions. The assumptions needed are completely analogous to those needed to obtain optimal controls in deterministic control problems. The proof is based on well-known results on existence of deterministic optimal controls.

1 Introduction In this paper, optimal nonanticipating controls are shown to exist in nonautonomous piecewise deterministic control problems. The assumptions needed for obtaining existence are completely analogous to those needed in the simplest cases to obtain optimal controls in deterministic control problems, namely a common bound on admissible solutions, compactness of the control region and, essentially, convexity of the velocity set. The proof mainly involves standard arguments and include the use of well-known results on existence of deterministic optimal controls.

Existence theorems for nonrelaxed controls involving convexity condition are given in Dempster et. al. (1992), and for another type of condition in Forwick et al (2004), (for relaxed controls, see e.g. also Davis (1993)). In contrast to the works mentioned, the present paper treats nonautonomous problems and hard terminal restrictions, and obtains existence of optimal controls dependent on previous jump times, so-called nonanticipating controls.

First, systems where there are no jumps in the state variable are treated.

2. Sudden stochastic changes in the differential equation, continuous solutions.

Consider the following control problem

$\max_{u(.,.)} E[\int_0^T f_0(t, x^{u(.,.)}(t,\tau), u(t,\tau), \tau)dt + h^*(x(T,\tau))]$ (1)

subject to

$\dot{x} = f(t, x, u(t,\tau), \tau)$, $t \in J = [0,T], x(0) = x_0 \in \mathbb{R}^n, u(t,\tau) \in U \subset \mathbb{R}^r$ (2)

and, a.s.,

$x^i(T) = \bar{x}^i$, $i = 1, ..., n_1$, (3)

$x^i(T) \geq \bar{x}^i, i = n_1 + 1, ..., n_2 \leq n$ (4)

Here $f_0 : J \times \mathbb{R}^n \times U \times \Omega$, ($\Omega$ defined in a moment), $h^* : \mathbb{R}^n \to \mathbb{R}$, and $f : J \times \mathbb{R}^n \times U \times \Omega \to \mathbb{R}^n$, are fixed functions, moreover, the control region U, the initial point x_0, and the terminal time T are also fixed, whereas the control functions $u(t,\tau)$ are subject to choice. Certain stochastic time-points τ_i, $\tau_1 < \tau_2, ...$, influence both the the right hand side of the differential equation as well as the integrand in the criterion, as $\tau = (\tau_0, \tau_1, \tau_2, ...) \in \Omega = \{(\tau_0, \tau_1, \tau_2, ...) : \tau_i \in [0, \infty)\}, \tau_0 = 0$. Thus in this type of systems, the right hand side of the differential equation (as well as the integrand in the criterion) exhibits sudden changes at stochastic points in time τ_i. In concrete (economic) situations, these changes may be the result of earthquakes, inventions, sudden currency devaluations etc. Given $u(.,.)$ and τ, the differential equation is an ordinary deterministic equation with continuous solution $t \to x^{u(.,.)}(t,\tau)$. The solution depends of course on τ, (the stochastic variable), and what we obtain is pathwise solutions. A unique solution is assumed to exist on all J for all τ and all controls $u(.,.)$ of the type described below. The present type of systems might be termed continuous, piecewise deterministic. The points τ_i are random variables taking values in $[0,\infty)$, with probability properties as follows: Conditional probability densities $\mu(\tau_{j+1}|\tau_0,...,\tau_j)$ are given, (for $j = 0$, the density is simply $\mu(\tau_1)$, sometimes written $\mu(\tau_1|\tau_0)$, $\tau_0 = 0$). The conditional density $\mu(\tau_{j+1}|\tau_0,...,\tau_j)$ is assumed to be measurable with respect to $(\tau_1,...,\tau_{j+1})$, and integrable with respect to τ_{j+1}, with integral 1. We assume $\mu(\tau_{j+1}|\tau_0,...,\tau_j) = 0$ if $\tau_{j+1} < \max_{1\le i\le j}\tau_i$, for $j \ge 1$. This means that we need only consider the set Ω^* of nondecreasing sequences $\tau = (\tau_0, \tau_{1,}\tau_2,...)$, or even the set Ω' of strictly increasing sequences. Moreover, the existence of Lebesgue integrable functions $\mu^*_{j+1}(.)$ is assumed, such that, for all $(\tau_0,...,\tau_j), \mu(\tau_{j+1}|\tau_0,...,\tau_j) \le \mu^*_{j+1}(\tau_{j+1})$ a.e. For $\tau^j := (\tau_0,\tau_1,...,\tau_j)$, the conditional densities define simultaneous conditional densities $\mu(\tau_{j+1},...,\tau_m|\tau^j)$ $(\mu(\tau_1,...,\tau_m|\tau^0) = \mu(\tau_1,...,\tau_m))$, assumed to satisfy: For some $k_* \in (0,1)$, and some positive number Φ^*,

$$\Pr[t \in (\tau_m, \tau_{m+1}]|\tau^j] \le \Phi^*(k_*)^{m-j} \text{ for any given } t \in [0,\infty). \tag{5}$$

Property (5), used for $j = 0$, means that with probability 1, the sequences $(\tau_1,\tau_2,...)$ has the property that $\tau_i \to \infty$. The set of τ's in Ω' such that $\tau_i \to \infty$ is denoted Ω''. Below, it is assumed that any τ belongs to Ω''.

Let the term "nonanticipating function" mean a function $y(t,\tau) = y(t,\tau_0,\tau_1,...)$ that for each given $t \in [0,T]$, depends only on τ_i's $\leq t$. (Formally, we require $y(t,\tau_0',\tau_1',...) = y(t,\tau_0,\tau_1,...)$ if $\{i : \tau_i' \leq t\} = \{i : \tau_i \leq t\}$ and $\tau_i' = \tau_i$ for $i \in \{i : \tau_i \leq t\}$.) Here, $y(.,.)$ is assumed to take values in a Euclidean space (or even in a complete metric space $\bar{Y}$). Let $\mathcal{M}^{\text{nonant}}(J\times\,\Omega'',\bar{Y})$ be the set of functions being nonanticipating and simultaneous Lebesgue measurable on each set $J \times \Omega_i$, $\Omega_i := \{\tau \in \Omega'':\tau_i \leq T, \tau_{i+1} > T\}$, $i = 1,2,....$ [1]. Define $U' := \mathcal{M}^{\text{nonant}}(J\times\,\Omega'',U)$, ($U$ closed). From now on, all control functions $u(t,\tau)$ belong to U', they are called admissible if in addition the corresponding solutions satisfy (3) and (4).

As functions of (t,τ), f_0 and f are now assumed to be nonanticipating. Furthermore, $t \to f(t,x,u,\tau)$ and $t \to f_0(t,x,u,\tau)$ have one-sided limits at each point,

[1] These properties are essentially equivalent to progressive measurability with respect to the subfields Φ_t defined as follows: Let Φ_t, $t \in [0,T]$, be the σ-algebra generated by sets of the form $A = A_{B,i} := \{\tau := (\tau_1,\tau_2,...) \in \Omega'' : \tau_i \in B\}$, where B is either a Lebesgue measurable set in $[0,t]$, or $B = (t,\infty), i \in \{1,2,...\}$. A probability measure P, corresponding to the conditional densities $\dot{\mu}(\tau_{i+1}|\tau^i)$, is defined on $(\Omega'',\Phi), \Phi := \Phi_T$.

$f(t,x,u,\tau)$ and $f_0(t,x,u,\tau)$ are separately continuous in (x,u), and $f(t,x,u,\tau)$ and $f_0(t,x,u,\tau)$ are separately measurable in $\tau \in \Omega_i$ for each i. The continuity in (x,u) is independent of τ and t, and the onesided limits in t are uniform in τ. Finally, h^* is continuous. Let us call the above assumptions on f_0 and f for the General Assumptions. (These assumptions imply that e.g. f can essentially be written as $f(t,x,u,\tau) = \sum_{i\geq 0} f^i(t,x,u,\tau^i)1_{[\tau_i,\tau_{i+1}]}(t)$, $\tau = (\tau_0,\tau_1,...) \in \Omega''$ for certain functions $f^i(t,x,u,\tau^i)$, $i = 0,1,...$.)

The specific conditions needed in the first existence theorem are as follows:

There exists an admissible pair $x(.,.), u(.,.), (x(.,.) = x^{u(.,.)}(.,.))$,

thus $(x(.,.), u(.,.))$ satisfies (2),(3), and (4), with $u(.,.)$ in U', (6)

U is compact, (7)

and

$N(t,x,\tau) = \{(f_0(t,x,u,\tau) + \gamma, f(t,x,u,\tau)) : u \in U, \gamma \leq 0\}$ is convex for all (t,x,τ). (8)

Moreover, there exist positive numbers K_i and positive continuous functions $r_i^*(t)$, and a number $\bar{k} \in (0, 1/k_*)$, (for k_* , see (5)) with $\sup K_i/\bar{k}^i < \infty$, $\sup_{i,t\in[0,T]} r_i^*(t)/\bar{k}^i < \infty$, such that (9) and (10) below hold.

$|f(t,x,u,\tau)| \leq K_i$, $|f_0(t,x,u,\tau)| \leq K_i$, for all $(x,u,\tau) \in$

$\text{cl}B(x_0, r_i^*(t)) \times U \times \Omega''$, all $t \in (\tau_i, \tau_{i+1}) \cap J$. (9)

For any control $u(.,.) \in U'$ and any $\tau \in \Omega''$, a unique solution $x(t,\tau;\tau_i,\bar{x})$

of $\dot{x} = f(t,x,u(t,\tau),\tau)$ starting at $(\tau_i,\bar{x})$, $\bar{x} \in \mathrm{cl}B(x_0, r^*_{i-1}(\tau_i))$ exists for $t \in [\tau_i, T]$,

that satisfies $x(t,\tau;\tau_i,\bar{x}) \in \mathrm{cl}B(x_0, r^*_j(t))$ for all $t \in [\tau_j, \tau_{j+1}] \cap J$, $j \geq i \geq 1$.

Moreover, $x(t,\tau;\tau_0,x_0) \in \mathrm{cl}B(x_0, r^*_j(t))$ for all $t \in [\tau_j,\tau_{j+1}] \cap J$, $j \geq 0$. (10)

Theorem 1. If the General Assumptions are satisfied, and (6)-(10) hold, then an optimal solution exists.

Proof. Define $\tilde{U}$ to be the set all Lebesgue measurable functions from $J = [0,T]$ into U. A result from deterministic control theory is needed.

Proposition 1 Let $f(t,x,u) : J \times \mathbb{R}^n \times U \to \mathbb{R}^n$ have one-sided limits with respect to t and be, separately continuous in x, and in u. Let $h(x) : R^n \to R \cup \{-\infty\}$, be upper semicontinuous, (abbreviated usc), and let $g(t,x) : J \times \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ be usc in x, for a.e. t, and Borel measurable in (t,x) (i.e. $\{(t,x) : g(t,x) \leq r\}$ is a Borel set for each $r \in [-\infty,\infty)$). Consider the problem

$$\max_{x(.),u(.)}[\int_{t_0}^{T} g(t,x(t))dt + h(x(T))]$$

where the maximization is carried out over the set $A(t_0,x_0)$ of pairs $x(.), u(.)$ satisfying the following differential equation with side conditions:

For a.e. $t \in [t_0,T], \dot{x} = f(t,x,u(t))$, $x(t_0) = x_0$, $x(T) \in B :=$

$$\{x \in \mathbb{R}^n : x^i = \bar{x}^i, i = 1, ..., n_1, x \geq \bar{x}^i, i = n_1 + 1, ..., n_2\} \qquad (11)$$

Assume that U is compact, that $f(t, x, U)$ is convex for all t, x, that there exist an integrable function $\psi(t)$, a positive number K, and positive continuous function $r^*(t)$ and $r(t)$ such that $|f(t, x, u)| \leq K$ and $g(t, x) \leq \psi(t)$ for all $(t, u) \in J \times U, x \in \text{cl}B(0, r^*(t))$, and that all pairs $(x(t), u(t))$, $u(.) \in \tilde{U}$, satisfying the differential equation in(11) (with $x(t_0) = x_0$) on some interval $[t_0, T] = J_0$, $t_0 \in J$, $|x_0| \leq r(t_0)$, also satisfy $|x(t)| \leq r^*(t)$ for all $t \in J_0$.

Define the set $C \subset J \times \text{cl}B(0, r(.)) := \{(t, x) : t \in J, x \in \text{cl}B(0, r(t))\}$ to be the set of points (t_0, x_0) in $J \times \text{cl}B(0, r(.))$ for which a pair $(x.), u(.))$, $u(.) \in \tilde{U}$ exists, satisfying (11). Let $V(t_0, x_0) := \sup_{(x(.),u(.)) \in A(t_0,x_0)} V^{x(.),u(.)}$, where $V^{x(.),u(.)} = \int_{t_0}^{T} g(t, x(t))dt + h(x(T))$, and where (t_0, x_0) belongs to C. For any $(t_0, x_0) \in C$, an optimal pair $(x(.), u(.)), u(.) \in \tilde{U}$ exists, satisfying (11), (perhaps the corresponding value of the criterion is $-\infty$). Moreover, C is closed, and $V(t_0{,}x_0)$ is usc on C. □

Proof of Proposition 1

For $k = 1, 2, ...$, when $k \to \infty$, let $t_0^k \to t_0, t_0, t_0^k \in J, I_k := [t_0^k, T], I = [t_0{,}T]$ and let $x_0^k \to x_0$, $x_0^k \in \text{cl}B(0, r(t_0^k))$, $x_0 \in \text{cl}B(0, r(t_0))$. Assume (A) that the sequence $(x^k(.), u^k(.))$ satisfies (11) for $(t_0, x_0) = (t_0^k, x_0^k)$ and (B) that $V^{x^k(.),u^k(.)} \to$

$\limsup_{(\tilde{t}_0,\tilde{x}_0)\to(t_0,x_0)} V(\tilde{t_0},\tilde{x}_0)$, $(\tilde{t},\tilde{x}_0) \in C$. By standard arguments, (see e.g. Cesari (1983), 8.3, 10.8), there exists a subsequence $x^{k_j}(.)$, a control function $u^*(.) \in \tilde{U}$, and a continuous function $x^*(.)$ such that $\sup_{t\in I_{k_j}\cap I} |x^{k_j}(t) - x^*(t)| \to 0$, and such that $(x^*(.), u^*(.))$ satisfies (11). Now, by slight misuse of notation, by upper boundedness of g and Fatou's lemma, $\limsup_j [\int_{t_0^{k_j}}^T g(t, x^{k_j}(t))dt + h(x^{k_j}(T))] \leq \int_J (\limsup_j g(t, x^{k_j}(t))1_{I_{k_j}})dt + \limsup_j h(x^{k_j}(T)) \leq \int_J g(t, x^*(t))1_I dt + h(x^*(T))$. Hence, $V(t_0,x_0)$ is usc. Dropping the assumption (B), we get that C is closed. If all $t_0^k = t_0$, and we assume that $V^{x^k(.),u^k(.)} \to V(t_0, x_0)$, then the above arguments give that $V(t_0, x_0) \leq \int_J g(t, x^*(t))1_I dt + h(x^*(T))$, hence $(x^*(.), u^*(.))$ is optimal. □

If V is defined to be equal to $-\infty$, for $(t_0, x_0) \in J\times$ cl$B(0, r(.)))\backslash C$, then V is usc on $(J\times$ cl$B(0, r(.))$.

Remark 1. In Proposition 1, assume that f, g and h contain an additional parameter $z \in \mathbb{R}^i$, $(f = f(t, x, u, z), g = (t, x, z), h = h(x, z))$. Asssume also that the conditions in Proposition 1 are satisfied for x replaced by (x, z) and for the differential equation in (11) augmented by the equation $\dot{z} = 0$, $z(t_0) = z_0$, $z(T)$ free. Proposition 1 implies that the value function $V(t_0, x_0, z_0)$ in this system is usc in $(t_0, (x_0, z_0)) \in J\times$ cl$B(0, r(.))$. Moreover, if we let t_0 be a continuous function of $z_0, t_0 = t_0(z_0) \in J$, and $r' := \min_{t\in J} r(t) > 0$, then $V(t_0(z_0), x_0, z_0)$ is usc for $(x_0, z_0) \in \{(x_0, z_0) : z_0 \in$ cl$B(0, r'), x_0 \in$ cl$B(0, r(t_0(z_0)))\}$.

An additional result is needed, in which the following entities occur. Let J^i be the product of J i times and let $Y \subset J^i$ be a closed set. Let a be a given vector in $\mathbb{R}^n$, and let $f(t,x,u,y) : J \times \mathbb{R}^n \times U \times Y$, be separately continuous in (x,y) and continuous in u, and, separately, let it have one-sided limits in t. Let $t_0(y)$ be a continuous function from Y into J. and let τ be a stochastic variable in $J^* := [t_0(y), \infty)$, with a given integrable density $\tau \to \phi_y(\tau)$, $y \in Y$ a parameter of the density, Assume that for any τ, $y \to \phi_y(\tau)$ is continuous, and, for some Lebesgue integrable function $\mu^*(.)$, that $|\phi_y(\tau)| \leq \mu^*(\tau)$, for all $y \in Y$. Let $W(\tau,x,y) : J \times \mathbb{R}^n \times Y \to \mathbb{R} \cup \{-\infty\}$ be Borel measurable in (τ,x,y). Let E^y be the expectation calculated by means of ϕ_y. Define $\hat{\tau} = \min\{\tau, T\}$, and let $\bar{U}(x_0,y) \subset \tilde{U}$ be the set of functions $u(.) \in \tilde{U}$, for which a solution $x^u(t)$ exists, satisfying

$$\dot{x} = f(t,x,u(t),y),\ t \in [t_0(y),T], x(t_0(y)) = x_0,\ x(T) \in B\ . \tag{12}$$

(uniqueness of solutions is assumed.) For $u = u(.) \in \bar{U}(x_0,y)$, define $V^u(x_0,y) := E^y[ax^u(\hat{\tau}) + W(\tau, x^u(\tau), y)1_{[t_0(y),T]}(\tau)]$, and $V(x_0,y) := \sup_{u \in \bar{U}(x_0,y)} V^u(x_0,y)$ ($= -\infty$ if $\bar{U}(x_0,y)$ is empty).

Proposition 2. Assume that U is compact. Assume also that there exist a positive number K, and positive continuous functions $r(.)$ and $r^*(t)$, such that $f(t,x,U,y)$ is convex for all (t,x,y) such that $(t,y) \in J \times Y, x \in \mathrm{cl}B(0,r^*(t)) \times U$, and such that $|f(t,x,u,y)| \leq K$ and $W(t,x,y) \leq K$ for all (t,x,u,y) such that

$(t, u, y) \in J \times U \times Y, x \in \mathrm{cl}B(0, r^*(t))$. Furthermore, assume that $W(\tau, x, y) : J \times \mathbb{R}^n \times Y \to \mathbb{R} \cup \{-\infty\}$ is usc in $(x, y) \in \mathrm{cl}B(0, r^*(t)) \times Y$, for each $\tau \in J$. For each $u(.) \in \tilde{U}$, for any $t_0 \in J$, any $x_0 \in \mathrm{cl}B(0, r(t_0))$, a unique solution $x^u(.)$ of the differential equation in (11), with $x^u(t_0) = x_0$ is assumed to exist, for which $\sup |x(t)| \le r^*(t)$. Let C be the set of points $(x_0, y) \in \mathrm{cl}B(0, r(t_0(.))) \times Y := \{(x_0, y) : y \in Y,\ x \in \mathrm{cl}B(0, r(t_0(y)))\}$ for which there exists a pair $x(.,.), u(.,.)$ satisfying (12), with $u(.)$ in $\tilde{U}$. Then C is closed and for any $(x_0, y) \in C$, there exists an optimal pair $x(.), u(.)$, $u(.) \in \tilde{U}$, satisfying (12), and having criterion value $V(x_0, y)$ (perhaps the criterion value is $-\infty$), and $V(x_0, y)$ is usc on $\mathrm{cl}B(0, r(t_0(.))) \times Y$.

Proof: This result follows from Proposition 1, and Remark 1, once it is observed that $V^u(x_0, y) =$

$$\int_{t_0(y)}^{T} \phi_y(\tau)[ax^u(\tau) + W(\tau, x^u(\tau), y)]d\tau + ax^u(T) \int_T^\infty \phi_y(\tau)dt \tag{13}$$

(by a suitable redefinition, it can be assumed that $Y \subset \mathrm{cl}B(0, r')$. □

Remark 3 Define $B = \mathbb{R}^n$ in case $\int_T^\infty \phi_y(\tau)dt = 0$, whereas B is as in (11) (indicated by $B \neq \mathbb{R}^n$) when $\int_T^\infty \phi_y(\tau)dt > 0$.Assume that this definition of B is used in (12) and hence in the definition of C. Then still C is closed in $\mathrm{cl}B(0, r(t_0(.))) \times Y$, an optimal pair exists for each $(x_0, y) \in C$, and $V(x_0, y)$ is usc on $\mathrm{cl}B(0, r(t_0(.))) \times Y$.

To see this, let $(x_j, y_j) \to (\hat{x}, \hat{y}), (x_j, y_j) \in C$. Now either $\int_T^\infty \phi_{y_j}(\tau)dt > 0$ for all j large enough, or $\int_T^\infty \phi_{y_{j_i}}(\tau)dt = 0$ for a subsequence y_{j_i}. In the former case, $(\hat{x}, \hat{y})$ belongs to C even for the stricter definition of C of Proposition 2 (i.e. $B \neq R^n$), and $\limsup_j V(x_j, y_j) \leq V(\hat{x}, \hat{y})$. In the latter case, as also $\int_T^\infty \phi_{\hat{y}}(\tau)dt = 0$, $(\hat{x}, \hat{y})$ belongs to the set C (for the present definition) and $\limsup_j V(x_j, y_j) \leq V(\hat{x}, \hat{y})$. Finally, defining $V(x_0, y) = -\infty$ if $(x_0, y) \in (\text{cl}B(0, r(t_0(.)) \times Y)\backslash C$, $V(x_0, y)$ is usc on $\text{cl}B(0, r(t_0(.))) \times Y$.

Continued proof of Theorem 1

It suffices to consider the special case where $ax^u(T)$ is maximized, a a fixed nonzero vector in $\mathbb{R}^n$. For simplicity, let $x_0 = 0$. Define $\hat{\tau}^k = \min\{T, \tau_k\}$ and $\Omega_k = \{\tau \in \Omega'' : \tau_{k+1} > T\}$. The central part of the proof of the theorem is the following: Let $V^{k,\infty}(x, \tau_k)$ be the supremum over controls of the conditional expectation of the criterion $ax^u(T, \tau)$ given that the process starts start at (τ_k, x), i.e. τ_k has just occured, and the state at which we start at that time is x. (A more precise definition is given below, we take the supremum only for controls that yield solutions satisfying the end conditions, but if no such controls exists, we let the supremum be equal to $-\infty$.) Then, as shows below, a relationship similar to the optimality equation in dynamic programming holds:

$V^{k,\infty}(x, \tau^k) =$

$$\sup_u E_{\tau_{k+1}}[a\int_{\hat{\tau}^k}^{\hat{\tau}^{k+1}} f(s,x^u(s),u(s),\tau)ds + V^{k+1,\infty}(x^u(\hat{\tau}^{k+1}),\tau^{k+1})\}|\tau^k] \quad (13^*).$$

($E_{\tau_{k+1}}$ means expectation with respect to τ_{k+1}, i.e., with τ_{k+1} as integration variable.) Here the supremum is taken over all deterministic functions $u(.)$ for which the corresponding deterministic solutions $x^u(t)$ satisfy the terminal conditions and starts at $(\hat{\tau}_k, x)$. Generally, $V^{k,\infty}(x,\tau^k) = 0$ if $\tau_k \geq T \Leftrightarrow \hat{\tau}_k = T$. Let us then construct the optimal controls by induction. (Below, this construction is repeated, with more detailed arguments.) By Remark 3, there exists a control $u_0(t) = u_{0,\tau^0}(t)$ with corresponding solution $x_{0,\tau^0}(t)$, ($x_{0,\tau^0}(0) = x_0$), yielding the supremum in (13*) for $k = 0$, and such that $x_{0,\tau^0}(T) \in B$. By induction, for each τ^{k-1} such that $\tau_{k-1} \in (\tau_{k-2}, T)$, assume $u_{k-1,\tau^{k-1}}(t)$ defined, with corresponding solution $x_{k-1,\tau^{k-1}}(t)$ satisfying $x_{k-1,\tau^{k-1}}(T) \in B$ if $\Pr[\tau_k > T|\tau^{k-1}] > 0$. By Remark 3, for each τ^k such that $\tau_k \in (\tau_{k-1}, T)$, there exists a control function $u_{k,\tau^k}(t)$ with corresponding solution $x_{k,\tau^k}(t)$, starting at $(\tau_k, x_{k-1,\tau^{k-1}}(\tau_k))$ and satisfying $x_{k,\tau_k}(T) \in B$ if $\Pr[\tau_{k+1} > T|\tau^k] > 0$, that yields the supremum in (13*). So $u_{k,\tau^k}(t)$ exists for all k.Using (13*) for $k = 0, 1, 2, ...$, for any given k, $V^{0,\infty}(0,0) = E[a\sum_{j=0}^{k}\int_{\hat{\tau}^j}^{\hat{\tau}^{j+1}} f(s, x_{j,\tau^j}(s), u_{j,\tau^j}(s), \tau)ds + V^{k+1,\infty}(x_{k,\tau_k}(\hat{\tau}^{k+1}), \tau^{k+1})]$. When $k \to \infty$, as $E[V^{k+1,\infty}\ (x_{k,\tau_k}(\hat{\tau}^{k+1}), \tau^{k+1})] \to 0$, we get $V^{0,\infty}(0,0) =$
$E[a\sum_{j=0}^{\infty}\int_{\hat{\tau}^j}^{\hat{\tau}^{j+1}} f(s, x_{j,\tau^j}(s), u_{j,\tau^j}(s), \tau)ds]$. Hence, the control $u^*(t,\tau)$ defined by $u^*(t,\tau) = u_{k,\tau^k}(t)$ for $t \in (\tau_k, \tau_{k+1})$ is optimal. (It is admissible because if $T \in (\tau_k, \tau_{k+1})$, then $x^*(T,\tau) = x_{k,\tau^k}(T) \in B$ if $\Pr[\tau_{k+1} > T|\tau^k] > 0$, hence $x^*(T,\tau) \in B$ a.s. if $T \in (\tau_k, \tau_{k+1})$.)

Three properties remains to be proved. (A): For any k, $V^{k,\infty}(x,\tau^k)$ is Borel measurable in (x,τ^k) on a Borel set of full measure in $\mathrm{cl}B(0,r^*_{k-1}(.))\times\Omega_k := \{(x,\tau^k) : \tau^k\in\Omega_k,\, x\in \mathrm{cl}B(0,r^*_{k-1}(\tau_k))\}$, and usc in $x\in clB(0,r^*_{k-1}(\tau_k))$ for a.e τ^k, (B): (13*), and (C): The controls $u_{k,\tau^k}(t)$ can be chosen to be simultaneously Borel measurable in (τ^k,t).

Proof of (A) and (B). The function $f(t,x,u,\tau)$ is simultaneous measurable in $(t,x,u,\tau)\in J\times\mathrm{cl}B(0,r^*_k(t))\times U\times\Omega_k$ for each k, so $\tau\to f(.,.,.,\tau) : \Omega_k\to L_1([J\times\ \mathrm{cl}B(0,r^*_k(.))]\times U,\mathbb{R}^n)$ is measurable ($J\times\mathrm{cl}B(0,r^*_k(.)) = \{(t,x) : t\in J, x\in\ \mathrm{cl}B(0,r^*_k(t))\}$), hence by Lusin's theorem for Banach space valued functions, an increasing sequence of closed sets D^k_j, $j = 1,2,...$, in Ω_k exists such that $\tau\to f(.,.,.,\tau)$ is continuous on D^k_j, with meas$(\Omega_k\backslash D^k_j) < 1/j$. The sets D^k_j are also chosen so that $\mu(.|\tau^k)\to L_1(J,\mathbb{R})$ is continuously dependent on $\tau\in D^k_j$. For a sequence $\tau^{(i)}\in\Omega''$, let $\tau^{(i)}\to\tau\in\Omega''$ mean that $\tau^{(i)}_j\to\tau_j$ for each j. When $\tau^{(i)}\to\tau\in D^k_j$, $\tau^{(i)}\in\Omega_k$, then $f(s,x,u,\tau^{(i)})\to f(s,x,u,\tau)$ for all (s,x,u), such that $s\in(0,T)$ is a continuity point of $t\to f(t,x,u,\tau)$ and $(x,u)\in \mathrm{cl}B(0,r^*_k(s))\times U$. To see this, using the particular continuity properties of f, if for some such point (s,x,u), this convergence fails, then for some ε, for some subsequence i_j $|f(s,x,u,\tau^{(i_j)}) - f(s,x,u,\tau)| > 5\varepsilon$, and for some δ, $|f(s',x,u,\tilde\tau) - f(s,x,u,\tilde\tau)| < \varepsilon$ for $|s'-s| < \delta$ for all $\tilde\tau$, and for some δ', $|f(s',x',u',\tilde\tau) - f(s',x,u,\tilde\tau)| < \varepsilon$ when $(x',u')\in \mathrm{cl}B((x,u),\delta')$ for all $(s',\tilde\tau)$. Then, $|f(s',x',u',\tau^{(i_j)}) - f(s',x',u',\tau)| > \varepsilon$ for $|s'-s| < \delta$, $(x',u')\in \mathrm{cl}B((x,u),\delta')$, but this means that L_1 - convergence of $f(.,.,.,\tau^{(i_j)})\to f(.,.,.,\tau)$ fails. By the same properties, for a.e. s, $(x,\tau)\to f(s,x,u,\tau)$ is continuous on

$\mathrm{cl}B(0, r_k^*(.)) \times D_j^k$.

Let C_k be the set of points (x, τ), $\tau^k \in \Omega_k$, $x \in \mathrm{cl}B(0, r_{k-1}^*(\tau_k))$ for which a deterministic control $u(.) \in \tilde{U}$ exists, such that the solution $x(t) = x^u(t; \tau_k, x)$ on $[\tau_k, T]$ of $\dot{x} = f(t, x, u(t), \tau)$, $x(\tau_k) = x$, satisfies $x(T) \in B$ if $\Pr[\tau_{k+1} > T|\tau^k] > 0$, with no condition on $x(T)$ if this inequality fails, and let U^{k,x,τ^k} be all controls of this type. In case $\Pr[\tau_{k+1} > T|\tau^k] =$, $C_k = \mathrm{cl}B(0, r_{k-1}^*(.)) \times \Omega_k := \{(x, \tau),$ $\tau \in \Omega_k$, $x \in \mathrm{cl}B(0, r_{k-1}^*(\tau_k))\}$ and $U^{k,x,\tau^k} = \tilde{U}$. By Remark 3, $C_j^k := \{(x, \tau) \in C_k, \tau \in D_j^k\}$ is closed.

Below, we will need the following definitions: For $u(.) \in U^{N,x,\tau^N}$, let $V_u^{N,N}(x, \tau^N) :=$

$$E_{\tau_{N+1}}[(a \int_{\hat{\tau}^N}^T 1_{[T,\infty)}(\tau_{N+1}) f(\check{s}, x^u(\check{s}, \tau; \tau_N, x), u(\check{s}), \tau^N) d\check{s} | \tau^N] \tag{14}$$

$$V^{N,N}(x, \tau^N) = \sup_{u \in U^{N,x,\tau^N}} V_u^{N,N}(x, \tau^N) \tag{15}$$

For $k \leq N$, by backwards induction, for $u(.) \in U^{k-1,x,\tau^{k-1}}$, define

$$V_u^{k-1,N}(x; \tau^{k-1}) := E_{\tau_k}[a \int_{\hat{\tau}^{k-1}}^{\hat{\tau}^k} f(\check{s}, x^u(\check{s}, \tau; \tau_{k-1}, x), u(\check{s}), \tau) d\check{s} +$$
$$V^{k,N}(x^u(\hat{\tau}^k, \tau; \tau_{k-1}, x), \tau^k) | \tau^{k-1}] \tag{16}$$

$$V^{k-1,N}(x; \tau^{k-1}) := \sup{}_{u \in U^{k-1,x,\tau^{k-1}}} V^{k-1,N}(x; \tau^{k-1}) \tag{17}$$

All the time, the convension is used that when taking supremum over an empty set, we get $-\infty$.

Define $B_j^k := \mathrm{cl}B(0, r_{k-1}^*(.)) \times D_j^k$. By Remark 3, $V^{N,N}(x, \tau^N)$ is usc in $(x, \tau) \in B_j^N$, for any j, so $V^{N,N}(x, \tau^k)$ is Borel measurable on $B_N := \cup_j B_j^N$, and usc in $x \in \mathrm{cl}B(0, r_{N-1}^*(\tau_N))$ for a.e. τ^N. By induction, if $(x, \tau) \to V^{k,N}(x, \tau^k)$ is Borel measurable on $B_k := \cup_j B_j^k$, and usc in $x \in \mathrm{cl}B(0, r_{k-1}^*(\tau_k))$ for a.e. τ^k, then by Remark 3, $(x, \tau) \to V^{k-1,N}(x, \tau^{k-1})$ is usc on B_j^{k-1}. So $V^{k-1,N}(x, \tau^{k-1})$ is Borel measurable on $B_{k-1} := \cup_j B_j^{k-1}$, and it is usc in $x \in \mathrm{cl}B(0, r_{k-2}^*(\tau_{k-1}))$ for a.e $\tau \in \Omega_{k-1}$.

For any given admissible control $u(t, \tau) \in U'$, let us prove the following inequality by backwards induction. $E[1_{[T,\infty)}(\tau_{N+1}) a x^u(T, \tau)] \leq$
$E[\sum_{0 \leq j \leq k-1} \int_{\hat{\tau}^j}^{\hat{\tau}^{j+1}} a f(s, x^u(s, \tau), u(s, \tau), \tau) ds] +$
$E[V^{k,N}(x^u(\hat{\tau}^k, \tau), \tau^k)], k < N$ (18)

Let $u = u(t, \tau)$ be any admissible control with solution $x^u(t, \tau)$ of (2) satisfying (3) and (4). Let us show that, for all $\tau \in \Omega_k$, a.s., $(x^u(\tau_k, \tau), \tau) \in C_k$. Let the deterministic $\hat{u}(.)$ equal $u(t, \tau)$, for $t \in [\tau_k, T]$, $\tau_{k+1} \geq T$. Since $x^u(T, \tau) \in B$ a.s., then for all τ^k, a.s., if $\Pr[\tau_{k+1} > T | \tau^k] > 0$, we have that for all $\tau_{k+1} \geq T$, $x^{\hat{u}}(T, \tau_k, x^u(\tau_k, \tau)) = x^u(T, \tau) \in B$. Then, for all τ^k, a.s., $\hat{u}(.) \in U^{k, x^u(\tau_k, \tau), \tau^k}$ and, evidently, the assertion follows.

Now, a.s. in $\tau \in \Omega_N, V^{N,N}(x^u(\hat{\tau}^N, \tau), \tau^N) \geq$
$E[\int_{\hat{\tau}^N}^{\hat{\tau}^{N+1}} a 1_{[T,\infty)}(\tau_{N+1}) f(s, x^u(s, \tau), u(s, \tau), \tau) ds | \tau^N]$,

since, a.s., $\hat{u}(.) \in U^{N,x^u(\tau_N,\tau),\tau^N}$, where $\hat{u}(.)$ is the deterministic control that equals $u(t,\tau)$, for $t \in [\tau_N, T]$, when $\tau_{N+1} \geq T$. Furthermore,

$E[1_{[T,\infty)}(\tau_{N+1})ax^u(T,\tau)|\tau^N] = E[\sum_{0\leq j\leq N}\int_{\hat{\tau}^j}^{\hat{\tau}^{j+1}} af(s,x^u(s,\tau),u(s,\tau),\tau)ds|\tau^N]$.

Replacing the last term by the greater term $V^{N,N}(x^u(\hat{\tau}^N,\tau),\tau^N)$, we get, for $\tau \in \Omega_N$, that

$E[1_{[T,\infty)}(\tau_{N+1})ax^u(T,\tau)|\tau^N] \leq$

$E[\sum_{0\leq j\leq N-1}\int_{\hat{\tau}^j}^{\hat{\tau}^{j+1}} af(s,x^u(s,\tau),u(s,\tau),\tau)ds|\tau^N] + V^{N,N}(x^u(\hat{\tau}^N,\tau),\tau^N)$.

Using that $V^{N,N}(x^u(\hat{\tau}^N,\tau),\tau^N)$ vanishes when $\tau_N \geq T$, (in which case the inequality is an equality), by taking expectations on both sides, (18) follows for $k = N$. Now, for $k < N$, since $(x^u(\tau_k,\tau),\tau) \in C_k$ a.s. in $\tau \in \Omega_k$, , then, a.s. in $\tau \in \Omega_k$, $\hat{u}(.) \in U^{k,x^{\hat{u}}(\tau_k),\tau^k}$, where $\hat{u}(.)$ equals $u(t,\tau)$, for $t \in [\tau_k, T]$ when $\tau_{k+1} > T$. Then evidently, for all $\tau \in \Omega_k$, a.s., $V^{k,N}(x^u(\hat{\tau}^k,\tau),\tau^k) \geq$

$$E[a\int_{\hat{\tau}^k}^{\hat{\tau}^{k+1}} af(s,x^u(s,\tau),u(s,\tau),\tau)ds + V^{k+1,N}(x^u(\hat{\tau}^{k+1},\tau),\tau^{k+1})|\tau^k] \qquad (19).$$

In fact, (19) holds for all $\tau \in \Omega''$, since both sides of (19) are zero if $\tau_k > T$. Assume now that (18) holds for k replaced by $k+1$, $k+1 \leq N$, and let us prove (18) as written. The induction hypthesis implies the first inequality below, and (19) implies the second one: $E[1_{[T,\infty)}(\tau_{N+1})ax^u(T)] \leq$

$E[\sum_{0\leq j\leq k-1}\int_{\hat{\tau}^j}^{\hat{\tau}^{j+1}} af(s,x^u(s,\tau),u(s,\tau),\tau)ds]+$

$E[E[\int_{\hat{\tau}^k}^{\hat{\tau}^{k+1}} af(s,x^u(s,\tau),u(s,\tau),\tau)ds|\tau^k]]$

$+E[EV^{k+1,N}(x^u(\hat{\tau}^{k+1},\tau),\tau^{k+1})|\tau^k]] \leq$

$E[\sum_{0\leq j\leq k-1}\int_{\hat{\tau}^j}^{\hat{\tau}^{j+1}} af(s,x^u(s,\tau),u(s,\tau),\tau)ds]+$

$V^{k,N}(x^u(\hat{\tau}^k,\tau),\tau^k)]$.

So (18) has been proved by induction.

For $u \in \tilde{U}$, define

$\hat{V}_u^{N,N}(x,\tau^N) := E[\int_{\hat{\tau}^N}^{\hat{\tau}^{N+1}} af(s,x^u(s,\tau^N;\tau_N,x),u(s),\tau)ds|\tau^N]$.

Define $\hat{K}_i = |a|K_i$. For any $(x,\tau) \in \mathrm{cl}B(0,r^*_{N-1}(\tau))\times\Omega_N$, note that $|V_u^{N,N}(x,\tau^k) -\hat{V}_u^{N,N}(x,\tau^k)| \le E[\hat{K}_N 1_{[0,T]}(\tau_{N+1})|\tau^N]$. Similarly, for any $(x,\tau) \in \mathrm{cl}B(0,r^*_N(\tau))\times \Omega_{N+1}$, $|V_u^{N+1,N+1}(x,\tau^{N+1}) -\hat{V}_u^{N+1,N+1}(x,\tau^{N+1})| \le E[\hat{K}_{N+1}1_{[0,T]}(\tau_{N+2})|\tau^{N+1}]$. Also, $|V_u^{N+1,N+1}(x,\tau^{N+1})| \le \hat{K}_{N+1}[1_{[0,T]}(\tau_{N+1})$, ($V_u^{N+1,N+1}$ vanishes if $\tau_{N+1} \ge T$), so $V^{N+1,N+1}(x,\tau^{N+1}) \le \hat{K}_{N+1}[1_{[0,T]}(\tau_{N+1})]$, and we also have $V^{N+1,N+1}(x,\tau^{N+1}) \ge -\hat{K}_{N+1}[1_{[0,T]}(\tau_{N+1})]$, if $V^{N+1,N+1}(x,\tau^{N+1})$ is finite ($\Leftrightarrow U^{N+1,x,\tau^{N+1}} \neq \emptyset$). Hence, if $V^{N+1,N+1}(x,\tau^{N+1})$ is finite,

$$|V^{N+1,N+1}(x,\tau^{N+1})| \le \hat{K}_{N+1}[1_{[0,T]}(\tau_{N+1})], \tag{19*}$$

By (16), $V_u^{N,N+1}(x,\tau^N) = \hat{V}_u^{N,N}(x,\tau^N)+E[V^{N+1,N+1}(x^u(\hat{\tau}^{N+1},\tau;x,\tau^N),\tau^{N+1})|\tau^N]$,

so for $\beta(x,\tau^N) = E[V^{N+1,N+1}(x^u(\hat{\tau}^{N+1},\tau;x,\tau^N),\tau^{N+1})|\tau^N]$, if $\beta(x,\tau^N)$ is finite, then $|V_u^{N,N+1}(x,\tau^N) - V_u^{N,N}(x,\tau^N)| = |V_u^{N,N+1}(x,\tau^N) - \hat{V}_u^{N,N}(x,\tau^N) + \hat{V}_u^{N,N}(x,\tau^N) - V_u^{N,N}(x,\tau^N)| = |\beta(x,\tau^N) + \hat{V}_u^{N,N}(x,\tau^N) - V_u^{N,N}(x,\tau^N)| \le$

$|\beta(x,\tau^N)| + E[\hat{K}_N 1_{[0,T]}(\tau_{N+1})|\tau^N] \le E[(\hat{K}_N + \hat{K}_{N+1})1_{[0,T]}(\tau_{N+1})|\tau^N] =: \alpha(\tau^N)$.

Hence, $V_u^{N,N+1}(x,\tau^N) \le V_u^{N,N}(x,\tau^N)+\alpha(\tau^N)$, (which also holds if $V_u^{N,N+1}(x,\tau^N)$ is nonfinite), and $V_u^{N,N}(x,\tau^N) \le V_u^{N,N+1}(x,\tau^N)+\alpha(\tau^N)$ if $V_u^{N,N+1}(x,\tau^N)$ is finite, (then $\beta(x,\tau^N)$ is finite). Thus $\sup_{u\in U^{N,x,\tau^N}} V_u^{N,N+1}(x,\tau^N) = V^{N,N+1}(x,\tau^N) \le$

$\sup_{u\in U^{N,x,\tau^N}} V_u^{N,N}(x,\tau^N)+\alpha(\tau^N)=V^{N,N}(x,\tau^N)+\alpha(\tau^N)$,

and, symmetrically, $V^{N,N}(x,\tau^N)\le V^{N,N+1}(x,\tau^N)+\alpha(\tau^N)$ if $V^{N,N+1}(x,\tau^N)$ is finite.

The next to last inequality also holds if U^{N,x,τ^N} is empty. Define $\alpha(\tau^{N-1}):=E[\alpha(\tau^N)|\tau^{N-1}]$. The two last inequalities imply the two inequalities in what follows:

$V_u^{N-1,N}(x,\tau^{N-1})-\alpha(\tau^{N-1})=E[\int_{\hat\tau^{N-1}}^{\hat\tau^N} af(s,x^u(s,\tau),u(s,\tau),\tau)ds+$

$V^{N,N}(x^u(\hat\tau^N,\tau;\tau_{N-1},x),\tau^N)-\alpha(\tau^N)|\tau^{N-1}]\le V_u^{N-1,N+1}(x,\tau^{N-1})=$

$E[\int_{\hat\tau^{N-1}}^{\hat\tau^N} af(s,x^u(s,\tau;x,\tau_{N-1}),u(s,\tau),\tau)ds+V^{N,N+1}(x^u(\hat\tau^N,\tau;\tau_{N-1},x),\tau^N)|\tau^{N-1}]\le$

$E[\int_{\hat\tau^{N-1}}^{\hat\tau^N} af(s,x^u(s,\tau),u(s,\tau),\tau)ds+V^{N,N}(x^u(\hat\tau^N,\tau;\tau_{N-1},x),\tau^N)+\alpha(\tau^N)|\tau^{N-1}]=$

$V_u^{N-1,N}(x,\tau^{N-1})+\alpha(\tau^{N-1})$,

so

$V^{N-1,N}(x,\tau^{N-1})-\alpha(\tau^{N-1})\le V^{N-1,N+1}(x,\tau^{N-1})\le V^{N-1,N}(x,\tau^{N-1})+\alpha(\tau^{N-1})$, where $\alpha^N(\tau^{N-1})=E[\alpha^N(\tau^N)|\tau^{N-1}]$,

(the second inequality holds also if $U^{N-1,x,\tau^{N-1}}$is empty, the first one holds if $V^{N-1,N+1}(x,\tau^{N-1})$ is finite; then $V^{N,N+1}(x,\tau^{N+1})$ is finite as.in $\Pr[.|\tau^{N-1}]$).

This evidently continues backwards, (i.e., for $N-1$ replaced by $N-2$, $N-3$, and so on), so for $\alpha(\tau^k)=E[\alpha(\tau^{k+1})|\tau^k]=E[E[\alpha(\tau^{k+2})|\tau^{k+1}]|\tau^k]=E[\alpha(\tau^{k+2})|\tau^k]=$ $....=E[\alpha(\tau^N)|\tau^k]$,

$V^{k,N}(x,\tau^k)-\alpha(\tau^k)\le V^{k,N+1}(x,\tau^k)\le V^{k,N}(x,\tau^k)+\alpha(\tau^k)$,

(the first inequality holds if $V^{k,N+1}(x,\tau^k)$ is finite).

Completely analogously, we get

$$V_u^{k,N}(x,\tau^k) - \alpha(\tau^k) \le V_u^{k,N+1}(x,\tau^k) \le V_u^{k,N}(x,\tau^k) + \alpha(\tau^k),$$

(the first inequality holds if $V_u^{k,N+1}(x,\tau^k)$ is finite).

Let $A := \sup_i K_i/\bar{k}^i < \infty$. By (5), $E1_{[0,T]}(\tau_{N+1})|\tau^N]|\tau^k] \le \sum_{m=N+1}^{\infty} \Pr[T \in [t_m, t_{m+1})|\tau^k] \le \Phi^* k_*^{N+1-k}/(1-k_*)$. Hence, $E[\alpha(\tau^N)|\tau^k] = E[E[(\hat{K}_N + \hat{K}_{N+1})1_{[0,T]}(\tau_{N+1})|\tau^N]|\tau^k] \le (\hat{K}_N + \hat{K}_{N+1})\Phi^* k_*^{N+1}/k_*^k(1-k_*) \le A(\bar{k}^N + \bar{k}^{N+1})\Phi^* k_*^{N+1}/k_*^k(1-k_*) = L_k(\bar{k}k_*)^N$, where $L_k := A(1/\bar{k}+1)\Phi^*/k_*^k(1-k_*)$. By repeated use of the last "double inequality" , for $\alpha_N^k = \sum_{M=N+1}^{\infty} L_k(\bar{k}k_*)^M = L_k(\bar{k}k_*)^{N+1}/(1-\bar{k}k_*)$ and for $N' > N$, we get the "iterated double inequality"

$$V^{k,N}(x,\tau^k) - \alpha_N^k \le V^{k,N'}(x,\tau^k) \le V^{k,N}(x,\tau^k) + \alpha_N^k, \qquad (19^{**})$$

(the first inequality holding if $V^{k,N'}(x,\tau^k)$ is finite, note that then $V^{k,M}(x,\tau^k)$ is finite for M such that $N \le M \le N'$, by the next inequality). In fact these two inequalities even hold for α_N replaced by $\sum_{M=N}^{N'} L_k(\bar{k}k_*)^M$. Completely analogously, we get another "iterated double inequality"

$$V_u^{k,N}(x,\tau^k) - \alpha_N^k. \le V_u^{k,N'}(x,\tau^k) \le V_u^{k,N}(x,\tau^k) + \alpha_N^k, \qquad (19^{***}),$$

(the first inequality if $V_u^{k,N'}(x,\tau^k)$ is finite). Note for later use that when $N \to \infty$, $E[\alpha(\tau^N)|\tau^k] \to 0$, $E[\alpha_N^N 1_{[0,T]}(\tau^N)] = \sum_{m=N}^{\infty} \alpha_N^N \Pr[T \in [t_m, t_{m+1})|\tau^0] = [L_N(\bar{k}k_*)^{N+1}/(1-\bar{k}k_*)]\Phi^* k_*^N/(1-k_*). \to 0$, and $E[|x^u(T,\tau)|1_{[0,T]}(\tau_{N+1})] \to 0$. For the last result, define $A' := \sup_{i,t\in[0,T]} r_i^*(t)/\bar{k}^i < \infty$, and note that

$|x^u(T,\tau)| \le r^*_m(T)$, when $T \in (\tau_m, \tau_{m+1})$. Thus, $E[|x^u(T,\tau)|1_{[0,T]}(\tau_{N+1})] \le$
$\sum_{m=N+1}^{\infty} r^*_m(T)\Pr[T \in [t_m, t_{m+1}] \le$
$\sum_{m=N+1}^{\infty} \Phi^* A'(\bar{k}k)^m_* \le \Phi^* A'(\bar{k}k_*)^{N+1}/(1-\bar{k}k_*)$.

Note that, by (19**), $W^{k,N+1} := V^{k,N+1} - \sum_{j=0}^{N} L_k(\bar{k}k_*)^j \le V^{k,N} - \sum_{j=0}^{N-1} L_k(\bar{k}k_*)^j =: W^{k,N}$, so the sequence $\{W^{k,N}\}_N$ is decreasing, hence $\lim_N W^{k,N}$ exists, and then also $\lim_N V^{k,N}$ exists. In fact, by the iterated double inequality (19**), $V^{k,k}(x,\tau^k) - \alpha^k_k \le V^{k,\infty}(x,\tau^k) \le V^{k,k}(x,\tau^k) + \alpha^k_k$ (the first inequality if $V^{k,\infty}(x,\tau^k)$ is finite). Similarly, $\lim_N V^{k,N}_u$ exists. We need to show $\lim_N \sup_u V^{k,N}_u = \sup_u \lim_N V^{k,N}_u$. Now, by the iterated double inequality (19***), $\alpha^k_N + V^{k,\infty}_u \ge V^{k,N}_u$, so $\alpha^k_N + \sup_u V^{k,\infty} \ge V^{k,N}$ and hence $\sup_u V^{k,\infty}_u \ge \lim_N V^{k,N}$, (supremum over U^{k,x,τ^k}). On the other hand, $V^{k,N}_u \le V^{k,N}$, so $V^{k,\infty}_u \le V^{k,\infty}$ and $\sup_u V^{k,\infty}_u \le V^{k,\infty}$. Hence, the equality claimed follows.

Note that, by (16), and Fatous' lemma, $V^{k,\infty}_u(x,\tau^k) =$

$$E_{\tau_{k+1}}[a\int_{\hat{\tau}^k}^{\hat{\tau}^{k+1}} f(s,x^u(s;\tau_k,x),u(s),\tau)ds + V^{k+1,\infty}(x^u(\hat{\tau}^{k+1};\tau_k,x),\tau^{k+1})|\tau^k] \quad (20).$$

so $V^{k,\infty}(x,\tau^k) =$

$$\sup_{u\in U^{k,x,\tau^k}} E_{\tau_{k+1}}[a\int_{\hat{\tau}^k}^{\hat{\tau}^{k+1}} f(s,x^u(s;\tau_k,x),u(s),\tau)ds +$$
$$V^{k+1,\infty}(x^u(\hat{\tau}^{k+1},\tau;\tau_k,x),\tau^{k+1})|\tau^k] \quad (21).$$

Even $V^{k,\infty}$ is usc on B^k_j. To see this, let $(\bar{x},\bar{\tau}) \in B^k_j$, and let $(x_j,\tau_{(j)}) \to (\bar{x},\bar{\tau})$, $(x_j,\tau_{(j)}) \in B^k_j$, the sequence so chosen that $V^{k,\infty}(x_j,\tau_{(j)}) \to$
$\limsup_{(\check{x},\tau^k)\in B^k_j,(\check{x},\tau^k)\to(\bar{x},\bar{\tau})} V^{k,N}(\check{x},\tau^k)$. If the last entity equals $-\infty$, there is

nothing to prove. If not, $V^{k,\infty}(x_j,\tau_{(j)}) > -\infty$ for $j \geq$ some j^*. Then $V^{k,N'}(x_j,\tau_{(j)}) > -\infty$ for $N' = N'(j)$, $j \geq j^*$, N' large enough, in fact for all $N \geq k$, by the "iterated double inequality" (19**) above. Then, for N^* such that $\alpha_N^k \leq \varepsilon/4$ for $N \geq N^*$, by this inequality, for any j, $V^{k,N}(x_j,\tau_j) - \varepsilon/4 \leq V^{k,\infty}(x_j,\tau_j) \leq V^{k,N}(x_j,\tau_j) + \varepsilon/4$. For some j_N, $V^{k,N}(x_j,\tau_j) \leq V^{k,N}(\bar{x},\bar{\tau}^k) + \varepsilon/2$ when $j \geq j_N$, ($V^{k,N}(\check{x},\tau^k)$ is usc in $(\check{x},\tau^k) \in B_j^k$). This means that for all N, $V^{k,N}(\bar{x},\bar{\tau}^k) > -\infty$ and that $V^{k,N^*}(\bar{x},\bar{\tau}^k) \leq V^{k,\infty}(\bar{x},\bar{\tau}^k) + \varepsilon/4$, so letting $N' \to \infty$ in the iterated double inequality (19**), we get, for $j \geq j_{N^*}$, that $V^{k,\infty}(x_j,\tau_j) \leq V^{k,N^*}(x_j,\tau_j) + \varepsilon/4 \leq V^{k,N^*}(\bar{x},\bar{\tau}^k) + \varepsilon/4 + \varepsilon/2 \leq V^{k,\infty}(\bar{x},\bar{\tau}^k) + \varepsilon/4 + \varepsilon/2 + \varepsilon/4 \leq V^{k,\infty}(\bar{x},\bar{\tau}^k) + \varepsilon$. Thus, $(\check{x},\tau) \to V^{k,\infty}(\check{x},\tau^k)$ is usc on B_j^k and hence Borel measurable in $(\check{x},\tau) \in B_k = \cup_j B_j^k$. Furthermore, $V^{k,\infty}(x,\tau^k)$ is evidently usc. in $x \in \mathrm{cl}B(0,r_{k-1}^*(\tau_k))$ for a.e. τ^k.

Note that $E[ax^u(T,\tau)|1_{[T,\infty)}(\tau_{N+1})] \to E[ax^u(T,\tau]$ when $N \to \infty$, (we proved above that $E[|x^u(T,\tau)|1_{[0,T)}(\tau_{N+1})] \to 0$). By (18) and the monotone convergence theorem (cf. the $W^{k,N}$'s introduced above), $E[ax^u(T,\tau)] \leq$

$$E[\textstyle\sum_{0\leq j\leq k-1}\int_{\hat{\tau}^j}^{\hat{\tau}^{j+1}} af(s,x^u(s,\tau),u(s,\tau),\tau)ds] + E[V^{k,\infty}(x^u(\hat{\tau}^k,\tau),\tau^k)]. \quad (22)$$

Let us use (21) to define, by induction, Lebesgue measurable controls $u_k(t,\tau^k)$ that will turn out to give the optimal control: Due to (22) and the existence of an admissible solution, $V^{0,\infty}(0,0)$ is finite. Define $(u_0(t,\tau^0),x_0(t,\tau^0))$ to be a control in $U^{0,0,\tau^0}$ with corresponding solution $x_0(t,\tau^0) := x_0(t,\tau^0;0,0)$ yielding supremum for $k = 0$ in (21), (such a control exists in $U^{0,0,\tau^0}$, by Remark 3). Trivially, $(t, \tau) \to u_0(t, \tau^0)$ is Lebesgue measurable. By induction, assume, for $j \leq k-1$ and for some Lebesgue measurable set $M_j \subset \Omega_j$ of full measure in Ω_j ($\Pr[\Omega_j \backslash M_j] = 0$), that for each $\tau \in M_j$ a pair $(u_j(t, \tau^j), x_j(t, \tau^j))$ exists such that $V^{j,\infty}(x_j(\tau_j, \tau^j), \tau^j)$ is finite, and such that the pair yields supremum in (21) for k replaced by j, with $(x, \tau_j) = (x_j(\tau_j, \tau^j), \tau_j)$, $(x_j(\tau_j, \tau^j) = x_{j-1}(\tau_j, \tau^j)$, $x^u(., \tau; \tau_j, x) = x_j(., \tau^j))$, and with $u_j(., \tau^j) \in U^{j, x_{j-1}(\tau_j, \tau^{j-1}), \tau_j}$, $(t, \tau^j) \to u_j(t, \tau^j)$ Lebesgue measurable. By the induction hypothesis, $V^{k-1,\infty}(x_{k-1}(\hat{\tau}_{k-1}, \tau^{k-1}), \tau^{k-1})$ is finite on M_{k-1}. Since $U^{k-1, x_{k-1}(\tau_{k-1}, \tau^{k-1}), \tau_{k-1}}$ is nonempty for $\tau^{k-1} \in M_{k-1}$ (it contains $u_{k-1}(., \tau^{k-1})$), then, by (19*), $V^{k-1,k-1}(x_{k-1}(\tau_{k-1}, \tau^{k-1}), \tau^{k-1})$ is bounded on M_{k-1}, and $V^{k-1,N'}(x_{k-1}(\hat{\tau}_{k-1}, \tau), \tau^{k-1})$ is finite for large N', then, by the iterated double inequality (for $N = k$), $V^{k-1,\infty}(x_{k-1}(\tau_{k-1}, \tau^{k-1}), \tau^{k-1})$ is a bounded function on M_{k-1}. Then

$$
\begin{aligned}
&1_{M_{k-1}} V^{k-1,\infty}(x_{k-1}(\hat{\tau}_{k-1}, \tau^{k-1}), \tau^{k-1}) = \\
&1_{M_{k-1}} E[a \textstyle\int_{\hat{\tau}^{k-1}}^{\hat{\tau}^k} f(s, x_{k-1}(s, \tau^{k-1}), u_{k-1}(s, \tau^{k-1}), \tau) ds + \\
&V^{k,\infty}(x_{k-1}(\hat{\tau}^k, \tau^{k-1}), \tau^k)\} | \tau^{k-1}]
\end{aligned}
\tag{23}
$$

Taking expectation ($E[.|\tau^0]$) on both sides yields a finite expression also on the right hand side. This means that $1_{M_{k-1}} V^{k,\infty}(x_{k-1}(\hat{\tau}^k, \tau^{k-1}), \tau^k)$ is a.s. finite, (otherwise $E[E[1_{M_{k-1}} V^{k,\infty}(x_{k-1}(\hat{\tau}^k, \tau^{k-1}), \tau^k)\} | \tau^{k-1}]]$ would not be finite). I.e. a measurable subset M_k of full measure in Ω_k exists such that $V^{k,\infty}(x_{k-1}(\tau^k, \tau^{k-1}), \tau^k)$ is finite for $\tau^k \in M_k$. Thus, for $\tau \in M_k$, by Remark

3 and (21) holding for k, a control $u_{k,\tau^k}(.) \in U^{k,\tau^k,x_{k-1}(\tau_k,\tau^{k-1})}$ with corresponding solution $x_{k,\tau^k}(.)$ satisfying $x_{k,\tau^k}(\tau_k) = x_{k-1}(\tau_k, \tau^{k-1})$ exists, yielding supremum in (21). Then the following equality is satisfied for $\tau \in M_k$: $V^{k,\infty}(x_{k-1}(\tau^k, \tau^{k-1}), \tau^k) =$

$$E_{\tau_{k+1}}[a \int_{\hat{\tau}^k}^{\hat{\tau}^{k+1}} f(s, x_{k,\tau}(s), u_{k,\tau^k}(s), \tau)ds + V^{k+1,\infty}(x_{k,\tau^k}(\hat{\tau}^{k+1}), \tau^{k+1})|\tau^k] \quad (24)$$

((24) reduces to $0 = 0$ when $\tau_k \geq \hat{\tau}^k = T$.)

We want to choose $u_{k,\tau^k}(.)$ to be simultaneously Lebesgue measurable in (t, τ^k), in which case we write $u_k(t, \tau^k)$ instead of $u_{k,\tau^k}(.)$ (and $x_k(t, \tau^k)$ for the corresponding solution). For $\tau \in M_k$, let $U^k_{\tau^k}$ be the set of controls in $U^{k,\tau^k,x_{k-1}(\tau_k,\tau^{k-1})}$ for which (24) is satisfied. Define $u^{k-1}(t, \tau^{k-1}) = u_j(t, \tau^j)$ for $t \in (\tau_j, \tau_{j+1}], j \leq k-1$, and write $x^{k-1}(t, \tau^{k-1})$ for the corresponding solution, starting at $(0, 0)$. Let $H^k_j, j = 1, 2, , ...$be measurable sets in Ω_k such that meas$(\Omega_k \backslash H^k_j) < 1/j$ and such that, by Lusin's theorem, $\tau \to u^{k-1}(., \tau^{k-1})$:$H^k_j \to L_1(J, R^r)$ is continuous. Let $\tau_{(n)} \to \tau$, $\tau_{(n)} \in F^k_j := M_k \cap D^k_j \cap H^k_j$. Moreover, let $((\tau_{(n)})_k$ be the k-th component of $\tau_{(n)}$, and $\tau_{(n)})^k$ the components no. $0, ..., k$, and assume that $u_{k,\tau_{(n)^k}}(.) \to u(.)$ in measure, $u_{k,\tau_{(n)^k}}(.) \in U^k_{\tau_k}$. Then it is easily seen that $x^{k-1}(t, (\tau_{(n)})^{k-1}) \to x^{k-1}(t, \tau^{k-1})$ uniformly in $t \in (\tau_{k+1}, T]$ and that the solution $x_{k,(\tau'_{(n)})^k}(t)$ corresponding to $u_{k,(\tau_{(n)})^k}(.)$, (which satisfies $x_{k,(\tau_{(n)})^k}((\tau_{(n)})_k) = x^{k-1}((\tau_{(n)})_k, (\tau_{(n)})^{k-1})$, converges to $x_{k,\tau^k}(t) := x^u(t, \tau; x_{k-1}(\tau_k, \tau^{k-1}))$ for all $t > \tau_k$. Thus, a.s. in τ, $x_{k,\tau^k}(t) \in B$ if $\tau_{k+1} > T$, provided $\mu(\tau^k) \Pr[\tau_{k+1} > T|\tau^k] > 0$, since this inequality must hold for large

n, by continuity in D_j^k. Hence, u belongs to $U^{k,\tau^k,x_{k-1}(\tau_k,\tau^{k-1})}$. Thus, when $\tilde{U}$ is furnished with the metric of convergence in measure, (in which it is separable and complete), the multifunction $\tau \to U^k_{\tau^k}$ is outer semi-continuous, (has the closed graph property), and hence is Lebesgue measurable on each F_j^k, and therefore Lebesque measurable on the set $M^k := \cup_j F_j^k$ of full measure. By Kuratowski's measurable selection theorem, for each $\tau^k \in M^k$, a function $u_k(.,\tau^k) \in U^k_{\tau^k}$ exists such that $\tau \to u_k(.,\tau^k)$ is measurable on M^k. Then $(t,\tau) \to u_k(t,\tau^k)$ is Lebesgue measurable. Let $x_k(t,\tau)$ correspond to $u_k(t,\tau)$. Obviously, $(u_k(.,\tau^k), x_k(.,\tau^k))$ is defined a.s. and satisfies (21) for $(x,\tau^k) = (x_{k-1}(\tau_k,\tau^{k-1}),\tau^k)$, $\tau \in M_k$. As $x_{k-1}(\tau_k,\tau^{k-1}) = x_k(\tau_k,\tau^k)$ for $\tau_k \le T$, $V^{k,\infty}(x(\tau_k,\tau^k),\tau^k)$ is finite on M_k.

Define $u^*(t,\tau) = u_j(t,\tau)$ and $x^*(t,\tau) = x_j(t,\tau)$ if $t \in (\tau_j,\tau_{j+1}]$. Evidently, using (23) for $j = 0,1,...,k$, we get

$$V^{0,\infty}(0,0) = \sum_{j=0}^k E[E[a\int_{\hat\tau^j}^{\hat\tau^{j+1}} f(s,x^*(s,\tau),u^*(s,\tau))ds|\tau^j]|\tau^0]+$$
$$E[V^{k+1,\infty}(x_k(\hat\tau^{k+1},\tau^k),\tau^{k+1})\}|\tau^k]|\tau^0]$$

By (19**), holding also for $N' = \infty$, the results $\lim E_{k\to\infty}[\alpha_k^k 1_{[0,T]}(\tau^k)] = 0$ and $0 \le E[E[\hat{K}_{k+1}1_{[0,T]}(\tau_{k+1})|\tau^k]|\tau^0] \le E[\alpha(\tau^k)|\tau^0] \to 0$ when $k \to \infty$ (see comments subsequent to (19***)), and (19*) (for $N = k$), the last term (i.e. $E[V^{k+1,\infty}(x_k(\hat\tau^{k+1},\tau^k),\tau^{k+1})\}|\tau^k]|\tau^0]$) goes to zero when $k \to \infty$, so letting $k \to \infty$, we get

$$V^{0,\infty}(0,0) = \sum_{j=0}^\infty E[a\int_{\hat\tau^j}^{\hat\tau^{j+1}} f(s,x^*(s,\tau),u^*(s,\tau))ds|\tau^0].$$

Hence, $u^*(.,.)$ is optimal. (Note that $x^*(t,\tau)$ does satisfy (3) and (4), recall that '$x_k(T,\tau^k) \in B$ when $\Pr[\tau_{k+1} > T|\tau^k] > 0 > 0$, and notice that $\Pr[x^*(T,\tau) \in B] = \sum_k \Pr[x^*(T,\tau) \in B, T \in [\tau_k,\tau_{k+1})] = \sum_k \Pr[x_k(T,\tau^k) \in B, T \in [\tau_k,\tau_{k+1})] = \sum_k \Pr[x_k(T,\tau^k) \in B, T < \tau_{k+1}, \tau_k \le T] = \sum_k \Pr[x_k(T,\tau^k) \in B|T < \tau_{k+1}, \tau_k \le T] \Pr[T < \tau_{k+1}|\tau_k \le T] \Pr[\tau_k \le T] = \sum_k \Pr[T < \tau_{k+1}|\tau_k \le T] \Pr[\tau_k \le T] = \sum_k \Pr[T \in [\tau_k,\tau_{k+1})] = 1$.

Remark 4 Below we need the following modifications of (9) and (10). For each τ, each $j \ge i$, for some $\tau'_j \in [\tau_j,\tau_{j+1})$, the solution $x(t,\tau;\tau_i,\bar{x})$ belongs to $\mathrm{cl}B(x_0, \max\{nr^*_{j-1}(t), nr^*_j(t)\}) \subset \mathbb{R}^n$ for $t \in (\tau_j,\tau'_j]$ and to $\mathrm{cl}B(x_0, r^*_j(t))$ for $t \in (\tau'_j,\tau_{j+1}]$ (instead of to $\mathrm{cl}B(x_0, r^*_j(t))$ for all $t \in (\tau_j,\tau_{j+1}]$). Moreover, (9) must be changed as follows: $|f_0(t,x,u,\tau)|, |f(t,x,u,\tau)| \le K_j$ for $(x,u,\tau) \in \mathrm{cl}B(x_0, \max\{nr^*_{i-1}(t), nr^*_i(t)\}) \times U \times \Omega$ when $t \in [\tau_j,\tau']$, and

$|f_0(t,x,u,\tau)|$,

$|f(t,x,u,\tau)| \le K_j$ for $(x,u,\tau) \in \mathrm{cl}B(x_0, r^*_i(t)) \times U \times \Omega$ when $t \in (\tau'_j,\tau_{j+1})$.

(Then still the start points $x_{k-1,\tau^{k-1}}(\tau_k)$ belong to $\mathrm{cl}B(x_0, r^*_{k-1}(\tau_k))$ and f_0 and f are bounded by K_k along the solutions $x_{k,\tau^k}(t)$ as before, both properties being used in the proof.)

The following observations are also needed. For a given closed set A in $\mathbb{R}^n$ containing x_0, assume in (10) (as modified) that if $\bar{x} \in A \cap \mathrm{cl}B(x_0, r^*_{i-1}(\tau_i))$, then $x(\tau_{i+1},\tau;\tau_i,\bar{x}) \in A \cap \mathrm{cl}B(x_0, r^*_i(\tau_j))$ (this need not hold for τ_{i+1} in any

$\Pr[.|\tau^i]$ - nullset Z, i.e. $\Pr[\tau_{i+1} \in Z|\tau^i] = 0]$). Moreover (10) (as modified above) holds only for such $\bar{x}$.

The proof is a trivial modification of the one above, (all $V^{k,N}(x, \tau^k)$, and $V^{k,\infty}(x, \tau^k)$ will only be defined for $x \in A$).

Finally, it is not necessary to assume uniqueness of solutions of (2) (or in (10)). It was done just to save a few words in the proof; uniqueness is not assumed in the crucial Proposition 1.

3 Piecewise continuous systems. Let us now consider piecewise continuous systems, where the state jumps at the times τ_i introduced in Section 2 above. Hence, to the setup in Section 2, add the feature that

$$x(\tau_i+, \tau) = \hat{g}(\tau_i, x(\tau_i-, \tau), i). \tag{25}$$

So now, $t \to x(t, \tau)$ is only absolutely continuous (and governed by the differential equation in (2)) between the points τ_i, with left and right limits at each $\tau_i, i = 1, 2, ...$ satisfying (25). We take $t \to x(t, \tau)$ to be left continuous. The functions f_0 and f satisfy the General Assumptions as before, and $\hat{g}$ is continuous. It is assumed that, for some constants α, κ, α_g, and κ_g, for all $(t, x, u, \tau) \in J \times \mathbb{R}^n \times U \times \Omega''$, $|f(t, x, u, \tau)| \le \alpha + \kappa|x|$, $|f_0(t, x, u, \tau)| \le \alpha + \kappa|x|$, and $|\hat{g}(t, x, i)| \le \alpha_g + \kappa_g|x|$ (for all i).

Theorem 2 Assume that the components g^m of $g := \hat{g} - x$ satisfy $g^m \equiv 0$ for $m = 1, ..., n_1$, and $g^m \geq 0$ for $m = n_1 + 1, ..., m_2$. Assume also that k_* in (5) satisfies $k_* < 1/\kappa_g$. Assume, finally, that an admissible solution $(x(t,\tau), u(t,\tau))$ of (2) exists, that U is compact, and that $N(t, x, \tau)$ is convex (see (8)). Then there exists an optimal control pair $(x^*(t,\tau), u^*(t,\tau))$.

Note. If the assumptions on the components g^m, $m = 1, ..., n_2$ fail, then we run the risk that no admissible solution exists.

It is not difficult to carry out essentially the same proof as above even in the present jump situation, it would add some few more details. However, being more than long enough, we did not want to the proof to become even longer by adding in these extra details. So, instead we shall use Theorem 1 in an suitably rewritten system to obtain Theorem 2, even if that necessitates some tedious, mainly "book-keeping" arguments.

Proof: Theorem 1 holds for any norm $|x|$ on $\mathbb{R}^n$ equivalent to the Euclidean norm, and given this norm, we shall use the max-norm $|(z, y)| = \max\{|z|, |y|\}$ on $\mathbb{R}^n \times \mathbb{R}^n$. Define $\bar{x} = (\bar{x}^1, ..., \bar{x}^{n_2}, 0, ..., 0) \in \mathbb{R}^n$, and let us introduce translated trajectories $\check{x}(t,\tau) := x(t,\tau) - \bar{x}$, governed by the system $d\check{x}/dt = \check{f}(t, \check{x}, u, \tau) := f(t, \check{x} + \bar{x}, u, \tau)$, $\check{x}(0) = \check{x}_0 := x_0 - \bar{x}$, $\check{x}(\tau_i+, \tau) = \check{g}(\tau_i, \check{x}(\tau_i-, \tau), i) := \hat{g}(\tau_i, \check{x}(\tau_i-, \tau) + \bar{x}, i) - \bar{x}$, with criterion integrand $\check{f}_0(t, \check{x}, u, \tau) := f_0(t, \check{x} + \bar{x}, u, \tau)$; note that

$|\check{f}(t,\check{x},u,\tau)| \le \check{\alpha}+\kappa|\check{x}|$, $|\check{f}_0(t,\check{x},u,\tau)| \le \check{\alpha}+\kappa|\check{x}|$, $|\check{g}(\tau_i,\check{x}(\tau_i-,\tau),i)| \le \check{\alpha}_g + \kappa_g|\check{x}|$, $\check{\alpha}=\alpha+\kappa|\bar{x}|$, $\check{\alpha}_g = |\bar{x}|+\alpha_g+\kappa_g|\bar{x}|$. The end condition on $\check{x}(T,\tau)$ is $\check{x}^m(T,\tau)=0$ a.s. for $m=1,...,n_1$, $\check{x}^m(T,\tau)\ge 0$ a.s. for $m=n_1+1,...,n_2$. Below, we write $x, x_0, f, f_0, g, \alpha$, and α_g instead of $\check{x}, \check{x}_0, \check{f}, \check{f}_0, \check{g}, \check{\alpha}$, and $\check{\alpha}_g$.

A. Assume first that there exist three sequences of positive numbers M_i, K_i, and positive continuous nondecreasing functions $r_i^*(.), i=0,1,...$, with $r_{i-1}^*(.) \ge r_i^*(.)$ for all i, $\sup_{i,t\in[0,T]} r_i^*(t)/\bar{k}^i < \infty$, $\sup_i K_i/\bar{k}^i < \infty$ for some $\bar{k}\in(0,1/k_*)$, and $\sum\sqrt{M_i}<\infty$, such that $|f(t,x,u,\tau)|\le K_i$ and $|f_0(t,x,u,\tau)|\le K_i$ for all $(x,u,\tau)\in J\times \text{cl}B(x_0,r_i^*(t))\times U\times\Omega''$, for all $t\in(\tau_i,\tau_{i+1}]$, and such that, for any control $u(.,.)\in U'$ and any $\tau\in\Omega''$, with $\tau_k\in(0,T)$, and any $\tilde{x}\in \text{cl}B(x_0,r_{k-1}^*(\tau_k))$, a solution $x(t,\tau;\tau_k,\tilde{x})$, $t\in[\tau_k,T]$, of $\dot{x}=f(t,x,u(t),\tau)$ exists, starting at $(\tau_k,\tilde{x})$ (i.e. $x(\tau_k-,\tau;\tau_k,\tilde{x})=\tilde{x}$) and satisfying the jump condition (25) for $i\ge k$. By assumption, this solution satisfies $x(t,\tau;\tau_k,\tilde{x})\in \text{cl}B(x_0,r_j^*(t))$ for $t\in(\tau_j,\tau_{j+1}]$. Assume moreover that $|\hat{g}(t,x,i)-x|\le M_i$ when $|x|\le nr_{i-1}^*(t)$. (These conditions are called the Auxiliary Conditions.) This jumping system can be rewritten as a nonjumping system as follow:

Let $M=\sum_{i=1}^{\infty}(M_i+\sqrt{M_i})$, $M_0=0$, $a_j := \sum_{i=0}^{j}(M_i+\sqrt{M_i})$, and $b_j := M_j+\sqrt{M_j}$. For $i=1,2,...$, let $\sigma_i := \sigma_i(\tau_i) := \tau_i + a_{i-1}$ if $\tau_i<T$, and $\sigma_i := \sigma_i(\tau_i) := T+M+\tau_i$ if $\tau_i\ge T$, moreover, let $\sigma_0=0$. There is an one-one correspondence between the σ_i's and the τ_i's. Note that $\sigma_i<T+M \Leftrightarrow \tau_i<T$. In an obvious way, the densities $\mu(\tau_k|\tau^{k-1})$ will give rise to densities $\mu^*(\sigma_k|\sigma^{k-1})$, $k=0,1,2..$, that, by the way, are equal to zero on $[\sigma_{k-1},\sigma_{k-1}+b_k]$.

Let $\sigma = (\sigma_0, \sigma_1, ...)$ and let $v(t, \sigma_0, \sigma_1, ...)$ take values in U, be nonanticipating and simultaneously measurable on each set $[0, T+M] \times \Omega_i', \Omega_i' := \{(\sigma_0, \sigma_1, ...) : \sigma_{i+1} > T+M\}$. (The set of such controls is denoted U''.) For $t \in [0, T+M]$, define $h_0(t, z(.), v, \sigma_0, \sigma_1, ...) = \sum_{i=0}^{\infty} f_0(t - a_i, z(t), v, \tau_0, \tau_1, ...) 1_{(\sigma_i + b_i, \sigma_{i+1}]}(t)$ and, for $g := \hat{g} - x$,

$h(t, z(.), v, \sigma_0, \sigma_1, ...) = \sum_{i=0}^{\infty} f(t - a_i, z(t), v, \tau_0, \tau_1, ...) 1_{(\sigma_i + b_i, \sigma_{i+1}]}(t) +$
$\sum_{i=1}^{\infty} g(\tau_i, z(\sigma_i), i) 1_{(\sigma_i, \sigma_i + M_i]}(t)/M_i.$

Then, for any given $v(t, \sigma)$, let $z^v(t, \sigma) := z(t, \sigma)$, for $t \in [0, T+M]$, be the solution - continuous in t - of the retarded equation

$$\dot{z}(t, \sigma) = h(t, z(.), v(t, \sigma), \sigma),\ z(0, \sigma) = x_0, \tag{26}$$

Define, for $s \in [0, T]$,

$$x(s, \tau) = \sum_{i=0}^{\infty} z(s + a_i, \sigma) 1_{(\sigma_i + b_i, \sigma_{i+1}]}(s + a_i), \tag{27}$$

and

$$u(s, \tau) = \sum_{i=0}^{\infty} v(s + a_i, \sigma) 1_{(\sigma_i + b_i, \sigma_{i+1}]}(s + a_i). \tag{28}$$

Now, $z(t, \sigma)$ satisfies $\dot{z}(t, \sigma) = f(t - a_i, z(t, \sigma), v(t, \sigma), \tau_0, \tau_1, ...)$ for $t \in (\sigma_i + b_i, \sigma_{i+1}], t \le T$. Assume $\tau_i < T$. Then, for $t' \in [\tau_i, \tau_{i+1}], t' \le T, x(t', \tau) - x(\tau_i+, \tau) =$

$z(t' + \sigma_i + b_i, \sigma) - z(\sigma_i + b_i, \sigma) = \int_{\sigma_i + b_i}^{t' + \sigma_i + b_i} f(t - a_i, z(t, \sigma), v(t, \sigma), \tau) dt =$

$\int_{\tau_i}^{t'} f(s, z(s + a_i, \sigma), v(s + a_i, \sigma), \tau) ds = \int_{\tau_i}^{t'} f(s, x(s, \tau), u(s, \tau), \tau) ds.$

Note that $z(t, \sigma)$ is constant on $(\sigma_i + M_i, \sigma_i + b_i)$. Moreover, for $\tau_i < T$,

$x(\tau_i +, \tau) - x(\tau_i -, \tau) = z(\sigma_i + M_i, \sigma) - z(\sigma_i, \sigma) =$

$\int_{\sigma_i}^{\sigma_i + M_i} (1/M_i) g(\tau_i, z(\sigma_i, \sigma), i + 1) =$

$g(\tau_i, z(\sigma_i, \sigma), i) = g(\tau_i, x(\tau_i -, \tau), i).$

Hence, $(x(., \tau), u(., \tau))$ satisfies (2) and (25). Symmetrically, if $(x(.), u(.))$ satisfies (2) and (25), there is a pair $(z(., .), v(., .))$ satisfying (26), $(u(s, \tau)$ and $v(., \sigma)$ again related as in (28)).

Now, (26) is a retarded differential equation. There would be no problem if Theorem 1 was proved for nonjumping states governed by retarded equations, (and the proof would be almost the same). But let us stick to ordinary equations: We shall work with two states, z, developing as before, and y, being equal to z, except on each $(\sigma_i, \sigma_i + M_i]$, where it is constant and equals $z(\sigma_i, \sigma)$, and on each $(\sigma_i + M_i, \sigma_i + b_i]$ where it develops in such a manner that it reaches the constant value z has on $(\sigma_i + M_i, \sigma_i + b_i]$ before then end of the interval, (in particular, $y(\sigma_i + b_i, \sigma) = z(\sigma_i + b_i, \sigma)$).

Define $h_1(t, z, y, v, \sigma_0, \sigma_2, ...) =$

$\sum_{i=0}^{\infty} f(t-a_i, z, v, \tau_0, \tau_1, ...)1_{(\sigma_i+b_i,\sigma_{i+1}]}(t) + \sum_{i=1}^{\infty} g(\tau_i, y, i)1_{(\sigma_i,\sigma_i+M_i]}(t)/M_i$,

and $h_2(t, z, y, v, \sigma_0, \sigma_1, ...) =$

$\sum_{i=0}^{\infty} f(t-a_i, z, v, \tau_0, \tau_1, ...)1_{(\sigma_i+b_i,\sigma_{i+1}]}(t) + \sum_{i=1}^{\infty} H(z, y)1_{(\sigma_i+M_i,\sigma_i+b_i]}(t)$,

where $H(z, y)$ has the components $H^m := H^m(z^m, y^m) := -2(y^m - z^m)^{1/2}$ if $y^m \geq z^m, H^m := 2(z^m - y^m)^{1/2}$ if $y^m < z^m$, $m = 1, ..., n$. Evidently, H is continuous. The equations governing z and y are $\dot{z} = h_1(t, z, y, v, \sigma)$ and $\dot{y} = h_2(t, z, y, v, \sigma)$, $z(0) = y(0) = x_0$. Define $\gamma_i = z^m(\sigma_i) - z^m(\sigma_i + M_i)$ and note that $|\gamma_i| = |z^m(\sigma_i) - z^m(\sigma_i + M_i)| \leq |\int_{\sigma_i}^{\sigma_i+M_i} (1/M_i)g^m(\tau_i, z(\sigma_i, \sigma), i)| \leq \int_{\sigma_i}^{\sigma_i+M_i} 1dt = M_i$, when $\sigma_i < T+M$. Now, the equation $\dot{y}^m = H^m(z^m, y^m), y^m(\sigma_i+M_i) = z^m(\sigma_i)$ given, has the unique solution $y^m(t) = (-t + \sigma_i + M_i + \sqrt{\gamma_i})^2$ $+z^m(\sigma_i + M_i)$ on $(\sigma_i + M_i, \sigma_i + M_i + \sqrt{\gamma_i}] \subset (\sigma_i + M_i, \sigma_i + M_i + \sqrt{M_i}]$ if $\gamma_i \geq 0$, and if $\gamma_i < 0$, then $y^m(t) = -(-t + \sigma_i + M_i + \sqrt{-\gamma_i}\,)^2$ $+z^m(\sigma_i + M_i)$ on $(\sigma_i + M_i, \sigma_i + M_i + \sqrt{-\gamma_i}]$, whereas $y^m(t) = z^m(\sigma_i + M_i)$ on $(\sigma_i + M_i + \sqrt{|\gamma_i|}, \sigma_i + b_i]$, recall that $z(t)$ is constant on $(\sigma_i + M_i, \sigma_i + b_i]$. Define the continuous function $r_i^{**}(t)$ by $r_i^{**}(t) = r_i^*(t - a_i)$ for $t \in [a_{i-1}, T + a_{i-1}]$, with $r_i^{**}(t)$ constant on $[0, a_{i-1}]$ and on $[T + a_{i-1}, T + M]$. When $t \in [\sigma_j + b_j, \sigma_{j+1})$ and $(\bar{z}, \bar{y}) = (\bar{x}, \bar{x}) \in \text{cl}B((z_0, y_0), r_{i-1}^{**}(\sigma_i))$ (so $\bar{x} \in \text{cl}B(x_0, r_{i-1}^*(\tau_i))$), then $(z(t; \sigma_i, (\bar{z}, \bar{y})), y(t; \sigma_i, (\bar{z}, \bar{y})))$ and $(z(t; 0, (z_0, y_0)), y(t; 0, (z_0, y_0)))$ belong to $\text{cl}B((z_0, y_0), r_j^{**}(t))$, where $z(t; \sigma_i, (\bar{z}, \bar{y})) := y(t; \sigma_i, (\bar{z}, \bar{y})) := x(t - a_j; \tau_i, \bar{x}) \in$

$\mathrm{cl}B(x_0, r_j^*(t - a_i)) = \mathrm{cl}B(x_0, r_j^{**}(t))$. Moreover, when $t \in [\sigma_j, \sigma_j + b_j)$, as $t - a_{j-1} \geq \tau_j$ and $r_{j-1}^*(.) \geq r_j^*(.)$, then $z^m(t;\sigma_i,(\bar z,\bar y)), y^m(t;\sigma_i,(\bar z,\bar y)) \in [x^m(\tau_j-;\tau_i,\bar x), x^m(\tau_j+;\tau_i,\bar x)] \subset \mathrm{cl}B(x_0^m, r_{j-1}^*(\tau_j)) = \mathrm{cl}B(x_0^m, r_{j-1}^{**}(\tau_j + a_{j-1})) = \mathrm{cl}B(x_0^m, r_{j-1}^{**}(\sigma_j)) \subset \mathrm{cl}B(x_0^m, r_{j-1}^{**}(t))$, so $(z(t;\sigma_i,(\bar z,\bar y)), y(t;\sigma_i,(\bar z,\bar y)))$ $\in \mathrm{cl}B((z_0,y_0), nr_{j-1}^{**}(t))$. Similarly, for $z_0 = y_0 = x_0$, when $t \in [\sigma_j, \sigma_{j+1} + b_j)$, $(z(t;0,(z_0,y_0)), y(t;0,(z_0,y_0))) \in \mathrm{cl}B(x_0, nr_{j-1}^{**}(t))$. Define $r_i'' := \max_t r_{i-1}^{**}(t)$. Due to the Auxiliary Conditions, this system (i.e. (h_0,h_1,h_2)) satisfies all conditions placed upon a nonjumping system in Theorem 1, combined with Remark 4 above: The property $|h_0(t,z,u,\sigma)|, |h_1(t,z,y,u,\sigma)|, |h_2(t,z,y,u,\sigma)| \leq 4\max\{1, n^2 r_i'', K_i\}$ holds for $t \in (\sigma_i, \sigma_i + b_i)$, $z, y \in \mathrm{cl}B(x_0, \max\{nr_{i-1}^{**}(t), nr_i^{**}(t)\})$, and for $t \in (\sigma_i + b_i, \sigma_{i+1})$, $z, y \in \mathrm{cl}B(x_0, r_i^{**}(t))$, ($|H^m| \leq 4nr_i''$ when $|z|, |y| \leq nr_i''$, $t \in (\sigma_i + M_i, \sigma_i + b_i]$). Finally, in this nonjumping system, the criterion is $E\int_0^{T+M} h_0(t,z,v,\sigma)dt$. Theorem 1, with Remark 4 ($A = \{(x,x) : x \in \mathbb{R}^n\}$), implies the existence of an optimal control $u_*(t,\sigma)$ in this system, which implies the existence of an optimal control $u^*(t,\tau)$ in the original jumping system.

B. Consider next the case where $|g| \leq M_i$, $\sum \sqrt{M_i} < \infty$ is not satisfied. For any i, there exist positive nondecreasing continuous functions $r_i(.)$ and positive numbers K_i such that $|f_0(t,x,u,\tau)|, |f(t,x,u,\tau)| \leq K_i$ when $x \in \mathrm{cl}B(x_0, r_i(t))$, $(u,\tau) \in U \times \Omega''$, $t \in (\tau_i, \tau_{i+1})$, with $\sup_i K_i/\bar k^i < \infty$, $\sup_{i,t\in[0,T]} r_i(t)/\bar k^i < \infty$. Moreover, the following property holds: For any $u(.,.) \in U'$, for any $\tau \in \Omega''$, for any $\bar x \in \mathrm{clB}(x_0, r_{i-1}(\tau_i))$, there exists a solution $x^u(t,\tau;\tau_i,\bar x)$ of (2), (25) on $[\tau_i, T]$ with $x^u(\tau_i-,\tau;\tau_i,\bar x) = \bar x$, such that $|x^u(t,\tau;\tau_i,\bar x) - x_0| \leq r_j(t)$ when

$t \in (\tau_j, \tau_{j+1})$. Also $|x^u(t,\tau;0,x_0) - x_0| \le r_j(t)$ when $t \in (\tau_j, \tau_{j+1})$.

To see this, for a monent consider translated solutions of the form $\tilde{x}(t,\tau) = x(t,\tau) - x_0$, governed by $d\tilde{x}/dt = \tilde{f}(t,\tilde{x},u) := f(t,\tilde{x}+x_0,u,t)$, $\tilde{x}(0) = 0$, with criterion integrand $\tilde{f}_0(t,\tilde{x},u) = f(t,\tilde{x}+x_0,u)$, and with jump function $\tilde{g}(t,\tilde{x},i) = \hat{g}(t,\tilde{x}+x_0,i) - x_0$. Note that $|\tilde{f}(t,\tilde{x},u)|$, $|\tilde{f}_0(t,\tilde{x},u)| \le \alpha + \kappa|x_0| + \kappa|\tilde{x}| = \alpha' + \kappa|\tilde{x}|$, where $\alpha' := \alpha + \kappa|x_0|$. Similarly, $|\tilde{g}(t,\tilde{x},i)| \le \alpha'_g + \kappa_g|\tilde{x}|$, where $\alpha'_g = |x_0| + \alpha_g + \kappa_g|x_0|$. Choose numbers $\kappa' > \kappa$ and $\kappa'_g > \kappa_g$, $\kappa'_g \ge 1$, such that $k_* < 1/\kappa'_g$, and let $\beta := \max\{\alpha'_g/(\kappa'_g - \kappa_g), \alpha'/(\kappa' - \kappa)\}$. When $|\tilde{x}| \ge \beta$, then $\kappa'_g|\tilde{x}| \ge \alpha'_g + \kappa_g|\tilde{x}|$ and $\kappa'|\tilde{x}| \ge \alpha' + \kappa|\tilde{x}|$. For any $u(.,.)$, when $|\bar{x}| \le \beta(\kappa'_g)^{k-1}\ e^{\kappa'\tau_k}$, then $|\tilde{x}^u(\tau_k+,\tau;\tau_k,\bar{x})| \le \beta(\kappa'_g)^k\ e^{\kappa'\tau_k}$, and for $t \in (\tau_k, \tau_{k+1}]$, by Gronwall's inequality, $|\tilde{x}^u(t,\tau;\tau_k,\bar{x})| \le \beta(\kappa'_g)^k e^{\kappa'\tau_k} e^{\kappa'(t-\tau_k)} = \beta(\kappa'_g)^k e^{\kappa' t}$, where $\tilde{x}^u(t,\tau;\tau_k,\bar{x})$ is any solution of the equation $d\tilde{x}/dt = \tilde{f}(t,\tilde{x},u)/dt$, $\tilde{x}(\tau_k-) = \bar{x}$ combined with the jump equation given by $\tilde{g}$. (The existence of piecewise continuous solutions $t \to \tilde{x}^u(t,\tau;\tau_k,\bar{x})$ of these two equations follows from standard global existence theorems for ordinary diffential equations.) Moreover, $|\tilde{x}^u(\tau_{k+1}+,\tau;\tau_k,\bar{x})| \le |\tilde{g}(\tau_{k+1},\tilde{x}^u(\tau_{k+1}-,\tau;\tau_k,\bar{x}),k+1)| \le \alpha'_g + \kappa_g\beta(\kappa'_g)^k\ e^{\kappa'\tau_{k+1}} \le \kappa'_g\beta(\kappa'_g)^k e^{\kappa'\tau_{k+1}} = \beta(\kappa'_g)^{k+1} e^{\kappa'\tau_{k+1}}$, so for $t \in (\tau_{k+1}, \tau_{k+2})$,
$|\tilde{x}^u(t,\tau,;\tau_k,\tilde{x})| \le [\beta(\kappa'_g)^{k+1}\ e^{\kappa'\tau_{k+1}}]e^{\kappa'(t-\tau_{k+1})} = \beta(\kappa'_g)^{k+1}e^{\kappa' t}$.

Continuing in this manner, it is easily seen that for $i > k$, when $t \in (\tau_i, \tau_{i+1})$ and $|\bar{x}| \le \beta(\kappa'_g)^k e^{\kappa'\tau_k}$, then $|\tilde{x}^u(t,\tau;\tau_k,\bar{x})| \le \beta(\kappa'_g)^k e^{\kappa'\tau_k}(\kappa'_g)^{i-k}e^{\kappa'(t-\tau_k)}) \le \beta e^{\kappa' t}(\kappa'_g)^i =: r_i(t)$. Finally, put $K_i = \alpha' + \kappa \sup_{t\in[0,T]} r_i(t)$. The existence of functions $r_i(t)$ and numbers K_i with the above properties has then been shown, (for any $\bar{k} \in (k_*, 1/\kappa_g)$. Note also that $\beta \ge \alpha_g/(\kappa'_g - \kappa_g)$, hence $|\hat{g}(t,x,i)| \le$

$\alpha_g + \kappa_g|x| \leq \kappa'_g\beta'$ when $|x| \leq \beta'$, $\beta' \geq \beta$, so

$$|\hat{g}(t,x,i)| \leq \kappa'_g n\beta(\kappa'_g)^{i-1}e^{k't} = \kappa'_g n r_{i-1}(t), \text{ when } |x| \leq n r_{i-1}(t) \qquad (29).$$

Choose a decreasing sequence $d_i \in (0,1]$, with $d_0 = 1$, such that $M_i := d_{i-1}(\kappa'_g + 1)n\sup_{t\in[0,T]} r_{i-1}(t)$ satisfies $\sum_{i=1}^{\infty}\sqrt{M_i} < \infty$ and such that $d_{i-1}(\kappa'_g)^{i-1} \geq d_i(\kappa'_g)^i$, i.e. $d_{i-1}r_{i-1}(.) \geq d_i r_i(.)$. Consider now the system

$$\dot{y} = \hat{f}(t,y,u,\tau) := \textstyle\sum_{i=0}^{\infty} d_i f(t, y/d_i, u, \tau)1_{[\tau^i,\tau^{i+1})}(t),\ y(0) = x_0,$$

$\hat{f}_0(t,y,u,\tau) := \sum_{i=0}^{\infty} f_0(t,y/d_i,u,\tau)1_{[\tau^i,\tau^{i+1})}(t)$, with jumps governed by $y(\tau_i+) = \breve{g}(\tau_i, y(\tau_i-), i) := d_i\hat{g}(\tau_i, y(\tau_i-)/d_{i-1}, i)$ and with end conditions $y^m(T) = 0$, $m = 1,...,n_1$, $y^m(T) \geq 0$, $m = n_1+1,...,n_2$. Evidently, $|\hat{f}_0 1_{[\tau^i,\tau^{i+1})}(t)| = |f_0(t,y/d_i,u,\tau)|1_{[\tau^i,\tau^{i+1})}(t) \leq K_i$ when $|y| \leq d_i r_i(t)$, (then $|y/d_i| \leq r_i(t)$) Moreover, $|\hat{f}1_{[\tau^i,\tau^{i+1})}(t)| = |d_i f(t,y/d_i,u,\tau)|1_{[\tau^i,\tau^{i+1})}(t) \leq d_i K_i \leq K_i$, when $|y| \leq d_i r_i(t)$. Because $y^u(t,\tau;\tau_k,\tilde{y}) = d_j x^u(t,\tau;\tau_k,\tilde{x})$ for $t \in (\tau_j, \tau_{j+1}]$, $j \geq k$, when $\tilde{y} = d_k\tilde{x}$ then $|y^u(t,\tau;\tau_k,\tilde{y})| \leq d_j r_j(t)$ for $t \in (\tau_j, \tau_{j+1}]$ when $|\tilde{y}| \leq d_k r_k(t)$. Finally, by (29), when $|y|/d_{i-1} \leq n r_{i-1}(t)$, then $|\breve{g}(t,y,i) - y| = |d_i\hat{g}(t,y/d_{i-1},i) - y| \leq d_i\kappa'_g n r_{i-1}(t) + d_{i-1}n r_{i-1}(t) \leq d_{i-1}n r_{i-1}(t)(\kappa'_g+1) \leq M_i$. For $r_i^*(t) = d_i r_i(t)$, the system $(\hat{f}_0, \hat{f}, \breve{g})$ satisfies the Auxiliary Conditions in A, so an optimal pair $(y^*(.,.), u^*(.,.))$ exists. Defining $x^*(t) = y^*(t)/d_i$ for $t \in (\tau_i, \tau_{i+1}]$, then $(x^*(.,.), u^*(.,.))$ is optimal in the original jumping system, ((3) and (4) are satisfied because $y^*(t,\tau)$ satisfies $y^{*i}(t,\tau) = 0, i = 1,...,n_1$ and

$y^{*i}(t,\tau) \geq 0, i = n_1+1,...,n_2$ a.s.. Thus $\sum_i x^*(T,\tau)d_i 1_{[\tau_i,\tau_{i+1})}(T)$, and so also $x^*(T,\tau)d_i 1_{[\tau_i,\tau_{i+1})}(T)$ and hence $x^*(T,\tau)$, satisfy the same relationships a.s.

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