THE ROCKING HORSE RELOADED

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Preface

To my father and Sir Isaac

This thesis is the result of work done by the author as a research assistant for Professor Olav Bjerkholt at the Department of Economics at the University of Oslo. Professor Bjerkholt's research in the history of Econometrics and more specifically in the re-organization of the Ragnar Frisch files produced the opportunity for a student to further examine Frisch's work. It was this author's privilege to seize said opportunity and work with Professor Bjerkholt's guidance to produce this document that represents nearly a year's worth of work in Frisch's studies in Macro-dynamics. My deepest gratitude must therefore go to Professor Olav for his counsel, guidance and for all the wisdom he shared with me during this process.

Special thanks must go to my friend Professor Drew T. Rodgers for his editorial assistance. The quality of the writing in this document is his doing and his support was key for my progress. I would also like to thank Alvaro Salas at the University of Caldas for his aid upon finding the solutions to that difficult characteristic equation, his help was vital to the development of this thesis. I would also like to thank Marius Lysebo and Professor Morten Hjorth-Jensen at The Physics Institute at the University of Oslo for the conversations entertained, they were of much more help than they may realize.

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Lastly, a thank-you to the master Ragnar Frisch and the Merlin Sir Isaac Newton. I thank you and salute you. You Sir Isaac, giant among men, sage, sorcerer and demigod to those of us who fight daily to shed light on truth and to eliminate superstition and darkness from our lives. To Ragnar Frisch, the most excellent of economists, for all your works and efforts. All my thanks are still less than your deserve.

Summary

This Master Thesis was written with the excellent supervision of Professor Olav Bjerkholt and is part of a project for the organization and the improvement of the catalogue for the Ragnar Frisch Archive. The subject of this thesis was chosen for its relevance to Economic history and for the history of methodology in Economic theory. The overall goal of the thesis is to revisit and analyse Ragnar Frisch's Macrodynamic model as published in 'Economic essays in honour of Gustav Cassel', London, George Allen and Unwin Ltd., 171-205 under the name: *Propagation problems and impulse problems in dynamic economics* (which will be referred to as PPIP throughout the paper). An added goal to the thesis was to inspect the economist Stefano Zambelli's criticism of PPIP in his two papers.*The Wooden Horse that Wouldn't Rock: Reconsidering Frisch*. UCLA Economics Working Papers 623, UCLA Department of Economics - 1991 and *A Rocking Horse That Never Rocked: Frisch's "Propagation Problems and Impulse Problems"* History of Political Economy, Spring 2007. The result of this paper is a challenge of Zambelli's conclusions regarding the oscillatory nature of the model, as well as a new development in adding new analysis in Frisch's *impulse mechanism*. The conclusions drawn from this thesis are that Zambelli's assertions that PPIP is overall a non-oscillatory model **across the board** are wrong, and that an analysis of the nature of the impulse mechanism allows for no doubt to be lifted as to the oscillatory qualities of the model. It is also a conclusion in this manuscript that Zambelli's criticism of the propagation mechanism are correct, but that more rigour ought to have been applied when producing the criticism accorded to Frisch. The Thesis is divided into 6 sections.

The first section contains a most brief introduction into the subject of Econonomic and Scientific thought and is to be received as an appetizer for the development ahead. The second section contains a short description of the historical context regarding the application of mathematics, or more importantly: the application of mathematical and statistical tools to the development of economics and specifically to the theory of business cycles. The relevance of the economists chosen was based firstly due to their impact on business cycle theory in the eyes of economic historians and secondly on the effect these had on Ragnar Frisch upon his development of PPIP.

As a third section one finds a thorough description of Frisch's model as written in PPIP. Full explanations as well as a complete development of the model are included here. Reproduction

of results for the sake of information and reference are created here as well as a few historical notes that will prove useful regarding the understanding of the model and how it came to be.

An entire section is assigned to Zambelli's reviews and opinions on PPIP as expressed in his two articles and this constitutes the main body of section four. In this section, a full reproduction of Zambelli's criticism regarding the trends for the general solutions for the variables utilized by Frisch are developed, as well as new judgement regarding Zambelli's attempts.

The fifth section is entirely the contribution of the author and attempts to enlighten two particular aspects of the problems proposed on PPIP. Firstly, the finding of the zeroes and its relevance to the final solution of the characteristic equation Frisch finds in his search for a solution to his 'Macro-Dynamic system giving rise to oscillations' are evaluated and expanded. Secondly, an in-depth analysis is done regarding the as-of-yet untouched perspective from PPIP regarding the origin of what Frisch referred to as the *Impulse Mechanism*. This is a mechanism Frisch barely touched in his article despite its paramount importance for the entirety of the model, and the inspection carried out in this thesis would go on to reveal that Frisch's discussion on the possible look of these impulse structures were not only relevant and applicable to his model, but ultimately adequate to generate the oscillations that PPIP claimed the model was capable of creating.

The last section contains closing arguments regarding Zambelli's articles and critics as well as the conclusions drawn from the discussions contained in the earlier sections. Lastly, one finds bibliographical reference.

The software used for this paper are the following privately owned programmes: Microsoft Office 2007, Matlab, Mathematica 7 and Texmaker for the generation of LaTex.

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The Rocking Horse reloaded

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1 Introduction: Ragnar Frisch - a man of method

'Since the ancients (as we are told by Pappus), made great account of the science of mechanics in the investigation of natural things; and the moderns, laying aside substantial forms and occult qualities, have endeavoured to subject the phenomena of nature to the laws of mathematics, I have in this treatise cultivated mathematics so far as it regards philosophy. The ancients considered mechanics in a twofold respect; as rational, which proceeds accurately by demonstration; and practical. To practical mechanics all the manual arts belong, from which mechanics took its name. But as artificers do not work with perfect accuracy, it comes to pass that mechanics is so distinguished from geometry, that what is perfectly accurate is called geometrical; what is less so, is called mechanical. But the errors are not in the art, but in the artificers'.

Extract from the Author's Preface to sir Isaac Newton's book: Principles of Philosophy, 1687; or as it is most widely known: *The Principia*

Newton's missive to the reader in the first lines of the book that has had the greatest influence on the lives of the people that inhabit the Earth, is a message of warning regarding the nature of mathematics and its application to our understanding of things. The message is the near impossibility of accomplishing mathematical perfection (let alone beauty) whilst using it for the purposes of broadening our understanding of the world. Sir Isaac Newton's success in employing mathematics to describe the Universe through the laws of the motions of the bodies is a gargantuan example of how powerful a tool mathematics is and despite a few minor corrections in terms of its scope, the treatise in the *Principia* remains ever relevant and successful in explaining the phenomena it embarks upon explaining. It is perhaps due to this success that so many have strived to harness its power for the understanding of other phenomena. Despite numerous attempts, Sir Isaac towers alone above all others, and the reason for this is simple: the Universe simply will not submit itself solely to mathematical analysis; the Universe is obstinate and other ingredients are required.

Economics is itself a science that studies the use and distribution of resources that are scarce through the use of mathematical assay, but its greatest difficulty as a science is the scope of interaction human behavior has on economic factors. This difficulty is a yoke borne by all economists, and a large amount of our economic theories have failed to successfully tackle economic problems due to this most exacting problem of the interaction between individuals and the economic machine. Despite this, in 1933, the economist Ragnar Frisch made an attempt (prompted by discussions with his fellow economists and by his own convictions) to construct what he called 'a macro-dynamic analysis' through the creation of a deterministic model of the economy, simplifying what he knew to be the perhaps insurmountable complexity of the true nature of an economy. The result ought to be seen more as an attempt to induce protocol and rigorous mathematical methodology to a science in its blooming stages, than an attempt to produce solid economic theory.

To account for Frisch in the history of economics one may commence with Professor Olav Bjerkholt at the University of Oslo who begins his memorandum from 2007 entitled *Ragnar Frisch's contribution to business cycle theory* stating that it is very difficult to classifiy Ragnar Frisch in the history of economics. I differ considerably in this view perhaps as a result of lack of immersion time in the subject of the history of economics. I believe it is very easy to place Frisch. He is quite simply if not the spiritual first, at least the standard-bearer of the implementation of rigorous method and scientific protocols to the subject of economics. He is the father of the experimental framework of the economist (econometrics) and a seeker of mathematical tools with which to express economic theory. If nothing more, his detractors must admit to him being the bane of those who would practice economics in innumeracy.

2 Frisch's PPIP in context

The earliest technical attempts by economists to study the cycles of commerce, and more specifically, the business cycles come from William Stanley Jevons and Henry Ludwell Moore. Jevons was (in the 1870s), the first to break away from the causal tradition of applied work which prevailed in the nineteenth century and combine theory with statistical data (Morgan, 1990). Jevons' inital hypothesis, later called *sunspot theory*, was that the sunspot cycles led to certain weather cycles which in turn would have an effect on the harvest. He was encouraged by Schuster who had already found a cycle in German wine vintages that matched the sunspot cycle. Through his work, Jevons found a coincidence in his setting of the sunspot cycle and economics cycles and this coincidence, through the use of *probability inference*, and for Jevons this became a *causal relationship* (Morgan, 1990). Jevons' conclusions were met with disdain from his contemporaries and Moore reminds us that much of this can be linked to Jevons' methodology being "beyond the pale" (Moran, 1990) of current thinking and its contrast to the prevailing structure of current economic thought which involved deduction from granted truths.

Henry Ludwell Moore would go further in his own theory to account for business cycles and surpass the apparent boldness of Jevons' claims. His theory was presented in two works, one in 1914 and the other in 1923. The former would find the weather accountable for the business cycles and the latter would find the direct causal participation of the planet Venus in the weather cycle (Morgan, 1990). The theory differed from Jevons' in that to Moore the causal relationship was of the essence and the main objective of the study.

Despite their pioneering qualities, Jevons' and Moore's theories, in particular Moore's Venus theory, had little influence on other economists. Morgan (1990) reminds us that (relating to their importance to the history of econometrics is that) Moore and Jevons had concentrated their efforts on statistical evidence of economic interactions and the analysis of these interactions (Morgan, 1990).

The next set of economists to be considered among those to practice "quantitative economics" were Clement Juglar, Wesley Mitchell and Warren M. Persons. Juglar was a contemporary of Jevons, whereas Mitchell and Persons were contemporaries of Moore (Morgan, 1990). Each of them pursued the use of data in an empirical form with different aims. Juglar strived to provide a convincing explanation of the cycle, Mitchell sought an empirical definition of the cyclical phenomenon, while Persons aimed at providing a representation of the cycle (Morgan, 1990). Juglar's work in 1862 would move on to be particularly influential, emanating through his study of a table of financial statistics for France.

In his book on crises from 1862, *Des crises commerciales et de leur retour periodique*, Juglar goes through the histories of all of the crises in the nineteenth century for three countries, France, England and the USA (Morgan, 1990). Of importance is the remark that events such as wars, famines and revolutions were causes "of the moment" that they were only "the last drop which caused the basin to overflow" (Juglar, 1862) as well as his overall conclusion that the common cause for all cycles were the changes in the conditions of credit. This he based on the regularities of the data he used, and more particularly to "the novel reason that a discussion of the monetary history of each crisis was repetitious and liable to cause ennui in the reader" (Morgan, 1990). Morgan notes in his 1990 book that the latter reason was proof on his conclusion through boredom. The exclamation mark at the end of Morgan's remark reminds us that indeed one could jokingly trace the origins of behavioral economics to certain peculiar observations, in this case Juglar's on the ennui created by repetition in monetary history.

Juglar introduced certain concepts that would influence many aspects of future studies in business cycles. The variation in length and amplitude of the cycles, as well as their regularity. Juglar describes this repetitiveness as "periodic" meaning that the sequences repeated themselves, not however that the cycles contained a period that would be determined. The impact of Juglar's work (and the influence Jevons' works might have had) are summarized well in Morgan's remark: "The novelty of both Juglar's and Jevon's work to contemporaries was that they used lots of data instead of a few odd numbers, that they were concerned with cycles and not crises, and that they used statistical data to recognize regularity in the behavior and causes of economic cycles" (Morgan, 1990).

Business cycles as studied from an analytic perspective resembling that of Jevons, Moore and Juglar would have to wait a few years. What is unique about that type of study, and what interests us here is the use of mathematics and statistical analysis resembling the methods Ragnar Frisch would eventually come to use when proposing his model from 'Propagation Problems and Impulse Problems'. Apart from a shy spell of analysis from Karl Marx' *Capital*, we find ourselves left with Knut Wickell's 1898 essay *Interest and prices* and his re-elaborated work in the second volume (*Money*) of his 1906 book *Lectures on Political Economy.*

Wicksell established a distinction between the money interest rate and the natural interest rate (Roncaglia, 2005). He distinguished the two by stating that the natural interest rate is determined by the 'real' variables which lead to an equilibrium for the economic system and it corresponded with 'the marginal productivity of capital as indicated by the marginalist theory of income distribution' (Roncaglia, 2005). According to Wicksell, the money rate was determined somewhat independently from the natural rate of interest and determined mainly on the money markets. This relationship was used by Wicksell to account for the 'cyclical oscillations of the economy and the inflationary or deflationary pressures on the general level of prices' (Roncaglia, 2005). Whenever the money rate of interest is thus lower than the natural one, entrepeneurs find it advantageous to take out loans and invest, thus giving rise to inflationary pressures; conversely, whenever the money rate of interest is higher than the natural rate, investments are discouraged and a deflationary pressure is generated (Roncaglia, 2005). In this model, Wicksell made the assumption that no changes took place in production techniques thus rendering the cycles as borne by the monetary variables. Wicksell's approach was used by a number of economists including Friedrich von Hayek and would later be taken up by the 'Swedish School' using sequential analysis.

The economist Friedrich von Hayek would carry on the analysis started by Wicksell regarding the generation of the cyclical oscillations in the economy through the difference between the money rate and the natural interest rate. Hayek would further the analysis and proceed to make a much deeper description of the two stages present in the cycles in the economy. These stages are firstly: the ascending stage, in which we depart from equilibrium (full use of all available resources) and according to Hayek (basing his analysis on Wicksell) entrepreneurs take up bank loans due to a money rate lower than the natural rate of interest. This leads to an increase in price levels due to the heightened demand. This increase in additional demand for investment goods generates in relative prices which leads to a lenghtening of the production process. The second stage of the trade cycle - the descending stage, is characterized by a real wage increase as a consequence of relative prices for investment goods being higher than consumption goods. This increase in the real wage subsequently leads to an increase in the demand for consumption goods which makes it advantageous to reduce the period of production leading to the descending aspect of the stage. Hayek's analysis built on Wicksell's and accounted (or included, depending on the perspective) for changes in technical changes, income distribution and relative prices. Despite his approach, the economist Piero Sraffa (allegedly prompted by Keynes according to Roncaglia (2005)) attacked the foundations of Hayek's approach and 'showed the non-existence of the natural rate of interest' (Roncaglia, 2005). Straffa argues that there are as many 'natural rates of interest' as there are commodities and accuses Hayek of not having understood the difference between a monetary economy and a barter economy (Roncaglia, 2005).

According to Roncaglia (2005), Hayek's response was feeble, and Sraffa's criticism would span a wider target than Hayek's structural explanation by rendering it impossible to reconcile influences of monetary factors on real variables within the acceptance of a marginalist theory of value (Roncaglia, 2005). By 1960, Sraffa had published his book which included all his essential criticism of Hayek and dealt the last blow to Hayek's proposals founded on their understanding of the average period of production. But alas, one more step must be taken in order to provide a clearer picture, a picture that deliberately ignores contributions made by other economists due to their apparent lack of influence upon the numerical approach taken by Frisch. Such economists ignored in this brief account are those who observed business cycles from an underconsumption perspective such as Keynes and Hobson, or those from the continental tradition such as Schumpeter and Gustav Cassel. However, one of the remaining schools is represented below in the form of Wesley C. Mitchell.

In 1913 the volume *Business Cycles* by Wesley C. Mitchell appeared and was the culmination of a series of efforts that started in 1908 (Schumpeter, 1950). Scumpeter (1950) tells us that Mitchell's efforts were not directed only to the evaluation of 'cyclical phenomena *per se*' (Schumpeter 1950), but towards a new view on economics that would be influenced by the development of the theory itself. At the time of Mitchell's *Business Cycles* there seemed to be not only a great many men approaching the problem of the cycle of trade and business by different paths, but also an almost equal amount of men reaching different 'explanations' (Schumpeter 1950). Mitchell himself, however, considered his direct predecessor in terms of approach to be Clement Juglar and followed Juglar in his understanding of the problem by upholding the notion that a crisis is the direct result of former prosperity (Schumpeter 1950), eventually placing the term 'cycle' as a replacement for 'crisis'. At the time, Schumpeter reminds us, the intellectual battle for the comprehension of these 'crises' of 'cycles' had two relatively discernable camps. The first camp represented those that would claim that the economic process was nonoscillatory and that the explanations of fluctuations must be sought in particular circumstances (Schumpeter 1950). The other camp would be the one to which Mitchell would pledge his allegiance; the camp where the 'economic process itself is essentially wavelike - that cycles are the form of capitalist evolution -' (Scumpeter 1950). Mitchell would state that the capitalist economy is a profit economy in which economic activity depends upon the factors that have an effect on the levels of profit. Mitchell declared that profits are the key to business fluctuations and this was a view that coincided with many economists in his day. Scumpeter (1950) stops the description of Mitchell's accomplishments at this stage and states that Mitchell would go no further or as written by Schumpeter: 'beyond this Mitchell did not commit himself' (Schumpeter 1950).

Mitchell would not connect profit with investment in this respect nor would he proceed any further in describing any other indicators as relevant. From Schumpeter's account from this work we gather that Mitchell's attempts were much more connected to a comprehension of the cycles themselves through strong statistical analysis. Mitchell's work would later go further however, in his book, co-authored with Burns entitled *Measuring Business Cycles* from 1946 where the attempts still seemed to be those of presenting the problem before us rather than attempting an explanation. This book however supercedes Frisch's model in youth (published at a later time) and will unfortunately be of little relevance for us now.

One last stop must be taken in the path towards Frisch's *Propagation Problems and Impulse Problems*, and that is his role played as opponent in the dissertation at Lund University in 1928 of Swedish economist Johan Åkerman. Bjerkholt (2007) recalls this exchange of ideas at the dissertation as one which aroused the appetite for Frisch to present his views regarding Business Cycles. It is also recalled by Bjerkholt (2007) that it was Åkerman's dissertation where Frisch obtained the reference of the rocking horse. Åkerman had spent one year at Harvard with Warren M. Persons and while there, he had studied the works of Wesley C. Mitchell at the National Bureau of Economic Research (Bjerkholt 2007). Apart from commending Åkerman's distinction of components for different wavelengths and for his attempt at a 'synthesis between the abstract economic theory and the concrete observation material found in economic material' Åkerman received little other appreciation from Frisch (Bjerkholt, 2007). Instead, Frisch used his propagation-impulse mechanism which he was at that point developing, to criticise Åkerman's inability to identify different components in the time series and to counterpoise Åkerman's general view of the cyclical movement as being a continuous process (Bjerkholt, 2007). Frisch insisted in the distinction between ' *impulse* phenomena and *rhythm* phenomena' (Bjerkholt, 2007) and was as well convinced that the bounded oscillations could not account for the observations but that instead free oscillations were the proper approach.

The stage is now ripe for PPIP's entry into the world, an environment in which PPIP would participate and take its role. In the period between 1931 and 1932 a discussion developed between Frisch and J.M.Clark where Frisch aimed to show that the conclusions drawn by Wesley Mitchell, Alvin Hansen and Clark himself were mistaken with regards to the causality in the accelerator model. The accelerator model was originated by Albert Aftalion in his paper from 1927 *The Theory of Economic Cycles Based on the Capitalistic Technique of Production* together with Clark 1917. These two last contributions built the most representative contributions to the accelerator model and in the Hansen-Mitchell-Clark idea that to account for a decrease in demand for the intermediate product it may only be needed to have the demand "Slacken its rate of growth" (Clark, 1917). According to them, this occurred because every producer of things to be sold to producers has two demands to meet. . . . Both these demands come ultimately from the consumer, but they follow different laws. The demand for maintenance and replacement of existing capital varies with the amount of the demand for finished products, while the demand for new construction or enlargement of stocks depends upon whether or not the sales of the finished product[s] are growing (Clark, 1917). Frisch (as we are told by Zambelli (1991)) pointed out the logical inconsistency within this statement by elucidating how this view would give rise to a single equation with two unknowns making the system indeterminate hence creating the need for revision and/or addition of another linear equation.

Frisch summarized the accelerator principle in an equation by which $x(t)$ is the yearly production of consumption of goods and $y(t)$ is the production of capital goods. This generates the following equation (in which m and μ are parameters):

$$y(t) = mx(t) + \mu\dot{x}$$

The above equation is of course one with two unknowns and therefore indeterminate unless further information is obtained. This extra information would have to be in the form of another linear equation or a functional expression for the demand $x(t)$. Clark would reply that this remark would be true only in the cases where the rate of growth of consumer demand were very small (Zambelli, 2007). Frisch would reply that not only any change in consumer-taking dominating the change in production replacement would bring about a total decline in production (Zambelli, 2007), but that a theory with such properties could not explain turning points on regular Business Cycles. As Zambelli (2007) points out, the inability to explain turning points is because 'there will be a little time interval of time after the point of fastest increase in consumer-taking where total capital production continues to increase, although the rate of increase in consumer-taking has slowed down. This little interval of time around the turning point in capital production is the critical interval of the business cycles. It is here that the enigma of the business cycle lies' (Zambelli, 2007). As told by Zambelli (2007), Clark would not reply with a retort but instead with a call to economists to solve the problem. It could be our understanding that Frisch took the call.

3 Frisch's Propagation problems and impulse problems in dynamic economics

Frisch's views on business cycles are usually understood to be those from his paper "Propagation problems and impulse problems in dynamics economics" from *Economic Essays in honour of Gustav Cassel* published in 1933. This version of his gathered ideas on the subject is simply the most known attempt Frisch provides at tackling a macro-dynamic model and observing the resulting dynamics. This model of course, departing from Frisch's convictions regarfing the oscillatory nature of the economy. We know from several sources including the archives he left behind, that there were a few presentations prepared before the Cassel article. We also know that this article was the only one ever published in English and therefore has become instrumental in the use it has been given to interpret Frisch's mind regarding Business Cycles.

The propagation and impulse model as known from the Cassel Festschrift is the last of several presentations (Bjerkholt, 2007). The first presentation was done at a Nordic meeting of economists that took place in Stockholm in June 1931. Frisch had been bestowed the responsibility of representing Norway with a contribution and he titled it *Business Cycles as a statistical and theoretical problem.* For this presentation he prepared a handout that reveals what would be the eventual focus of the Cassel paper. Frisch decides to stress 14 points which we shall list here. These points were contained within the Stockholm handout:

1. The connection between the economic-theoretical and the statistical parts of the problem. Necessary for business cycle theoretician to handle modern statistical tools.
2. Cycles of different kinds: short run cycles, long run cycles, etc.
3. Free and bound oscillations
4. Impulse problems and propagation problems with free oscillation. Perturbations and half-free oscillations
5. The business cycle must for the most part be dealt with as a free oscillation
6. The business cycle theory (understood as a theory for a free oscillation) must be *determined.* It must contain exactly the same number of conditions as variables.
7. None of the business cycle theories proposed until now, have been determined

8. In a determined business cycle theory at least one of the conditions must be *dynamic*. From this follows *inter alia* that a system of Walrasian equations can never lead to a business cycle theory. A dynamic condition is a condition which connects the values of a certain variable at two (or more) points in time

9. A complete business cycle theory comprises three problems:

   - The specification problem: The specification of the relevant variables
   - The determination problem: Analysis of the number and independence of the posed conditions and comparison with the number of variables.
   - The shape problem: Clarify that the posed conditions really lead to cyclical movements. This depends not only on upon which variables enter the condition(s), but also of the numerical relationship. Certain values of the numerical parameters characterizing the conditions will lead to cyclical movements. The numerical character of the conditions is thus essential.

10. An attempt at erecting a determined scheme for a business cycle theory [here followed as an addendum to point 10 a "draft of a dynamic equlibrium theory of business cycles", in fact a sketch of a macroeconomic framework with 38 endogenous variables and 37 equations]

Statistical part

11. The decomposition problem for statistical time series

12. The distinction between *prim*-relations and *confluent* relations (*phase* relations). Inflated and deflated phase-relations

13. Pitfalls to watch out for when trying to determine the numerical character of the economic-theoretical laws by means of statistical data. In principle the phase-relations can always be determined, but the prim-relations can only be determine in certain cases.

14. Conditions under which it is possible to determine the prim-relations statistically. Even if prim-relations cannot be determined statistically, there is a loophole: a systematic "interview" method which very likely will give useful results

Bjerkholt (2007) points out that the handout was more of a research program and that seemed to have been prepared on short notice. Bjerkholt also points out that this handout does not place any weight on obtaining the causes or the explanation of how business cycles come to pass. Instead, it seems to highlight the point that to understand business cycles one must spend time finding and understanding a structural framework that may generate cycles. Much better stated, Bjerkholt in 2007 writes:

> Frisch's presentation dealt only with the formal requirements of the theory (points 6-8), rather than with the economic explanations of real phenomena. Thus it is the overall explanatory paradigm, the logic and the components of the explanation, which is Frisch's message. In Frisch's view the real challenge in business cycle analysis was how theory must be formulated to explain cycles in a satisfactory way. Frisch was thus thinking, not in terms of theoretical explanations, but rather in terms of "theoretical structures" that with appropriate parameter values could generate cycles.

The handout and the lecture can be then understood as an ideological draft to what would be Frisch's approach when dealing with the problem and a description by Frisch of a protocol to follow whence dealing further with the problem. The extent of Frisch's intentions were quite big, and sadly only a fraction remain collected in the propagation problems and impulse problems paper of 1933.

The second presentation of the ideas Frisch held regarding Business Cycles are those at a lecture given at the University of Minnesota and had a significantly different approach. Frisch made a brief historical survey of Business Cycle theory to start with and then proceeded onto the discussion of six brief points that excluded out the "impulse" problem. Attention was given to the equilibrium properties of the model, as well as to the propagation structure (Bjerkholt, 2007). It is during this visit that Frisch wrote his criticism of J.M. Clark. Bjerkholt 2007 states:

> It was during Frisch's stay in Minnesota that he wrote his criticism of J.M.Clark, Frisch (1931d) in close contact with Hansen, whom it seems he had brought over to his side. The controversy with Clark has been interpreted as a theoretical controversy, but Frisch's aim was mainly to point out that Clark had not fulfilled the "determination" condition the Stockholm handout's point 10.

The third presentation of the ideas Frisch was working on (and ultimately the last before

PPIP) was largely unpublished until recently (Bjerkholt, Dupont 2009). These were a series of lectures given at the Poincare Institute at the University of Paris. The name of the lectures was *Problems and methods of econometrics* and these were given between the 24th of March and the 5th of April 1933. The eight lectures encompassed many issues related to the newly created discipline of econometrics, from the "Philosophical foundations of econometrics" to the "statistical construction of econometric functions". Frisch's fourth and fifth lecture entitled (respectively) *Oscillations in closed systems. The theory of crises* and *The creation of cycles by random shocks. Synthesis between a probabilistic point of view and the point of view of deterministic dynamic laws* contain most aspects regarding the overall goals Frisch would attempt when making an approach at business cycle theory. In particular, the fourth lecture contained the model proposed by Frisch in PPIP and the parameter values chosen by him. He had however not made the necessary calculations to present his results and instead issued a statement that his assistant at the University doing the calculations had not reached decisive results due to the fact that "a mistake has slipped into the calculations" (Bjerkholt, Dupont 2009).

Frisch's resulting thoughts were compiled in the Festschrift to celebrate the 70th birthday of Gustav Cassel and were the result of his presentations in Stockholm and Minnesota, and his synthesis found in the Poincare lectures. It was also the result of a compromise with the editor of the Festschrift who found that Frisch's first submitted draft was too long and had to be shortened. The cut was made by removing a section 2 titled *An example of micro-dynamic analysis* which was taken directly from the Poincare lectures. The Cassel Festschrift appeared in October 1933 (Bjerkholt, 2007).

3.1 The model

As we have read above, Frisch's PPIP is the resulting form of attempting to give an oscillatory structure to a *macrodynamic* phenomena. It is the result of a formulation shaped through a protocol Frisch supported greatly, namely attempting to produce a *theoretic* approach to a problem that seemed to present cyclical qualities, therefore rendering the approach as one that would explain oscillations. Frisch derives two systems, the latter of which, made up of a three difference and differential equation (or delay differential equations of second order) would, in Frisch's view, result in a system that would oscillate around an equilibrium making it subject to the simile of the *rocking horse*.

3.2 Free oscillations

According to Frisch in PPIP, the majority of economic oscillations seem to be explained most plausibly as free oscillations (Frisch, 1933). Free oscillations are understood, mainly as oscillations that allow for self-excitement. This self-excitement is due to the characteristics of the system and not by effects that may lie beyond the scope of the model. In this understading one can point at agricultural models as models which would not be self'exciting and therefore cannot be adequately understood as freely oscillatory. Frisch (1933) explains how within these systems of free oscillations there are essentially two mechanisms which determine tendency towards dampening and the amplitude with which it oscillates, namely the interior structure of the system and the intensity of the exterior impulse. It is important to note that Frisch makes it a point to mention that the system must be dealt with dynamically for "only that type of theory can explain how one situation can grow from the foregoing" (Frisch, 1933). Frisch wanted to have a system that worked in time as well as within an equilibrium, as for, him only such systems gave rise to oscillations and growth.

Frisch christens the problem of the intrinsic structure the "propagation problem" and the exterior impulse problem, the "impulse problem". These two mechanisms are best understood if seen from a classical mechanic perspective (a perspective he himself used in PPIP). The Propagation Mechanism, is seen as the physical qualities within, say, a pendulum that is removed from equilibrium and allowed to oscillate freely (considerations of friction added in a dampening term). It is the forces exerted by the gravitational pull and the tension of the cord from which the weight hangs that produce this restoration force that maintain the pendulum oscillating. Characteristics such as the weight of the pendulum, the length of the cord, the angle at which the pendulum is let go and so forth are the structure of the system. These physical conditions and their ability to constantly restore the weight back to its equilibrium position and then past it to once again start the cycle in the opposite direction are the propagation mechanism that produces the oscillations. An impulse mechanism however, is the setting-up physically of forces (applied as vectors or changes in the angular velocity) that create an impulse, this impulse is a change in velocity and therefore momentum of the weight. This mechanism, that is obviously foreign to the propagation mechanism is what is understood by the impulse mechanism.

It is after this assertion regarding the two mechanisms of his free oscillation approach that Frisch goes on to distinguish between two types of analysis the "macro-dynamic" and the "micro-dynamic". He distinguishes them by saying that a micro-dynamic type of analysis looks at the details of evolution of a simple market, the behavior of a given type of consumer and so on (Frisch, 1933). A macro-dynamic system, on the other hand, is distinguished by Frisch as one that tries to give an account of the fluctuations of the whole economic system taken in its entirety (Frisch, 1933). To Frisch, the dimension of such as macro-dynamic system could be boundless and that one could stick to a strictly *formal* analysis to give it a manageable amount of detail. It is for this purpose that he introduces *Le Tableau Economique.*

3.3 Le Tableau Economique

Frisch models a macro-dynamic system encompassing all the "important variables" (Frisch,1933) by introducing an illustration by which he attempts to model an entire economy and its functions. The model contains a series of stocks and flows that describe the functioniong of an economy. In this illustration there are three receptacles as he calls them, or Stocks: Forces of nature, the stock of capital goods and the stock of consumer goods. There are as well three machines: the Human machine, the machine producing capital goods and the machine producing consumer goods. We find the flow-diagram below:

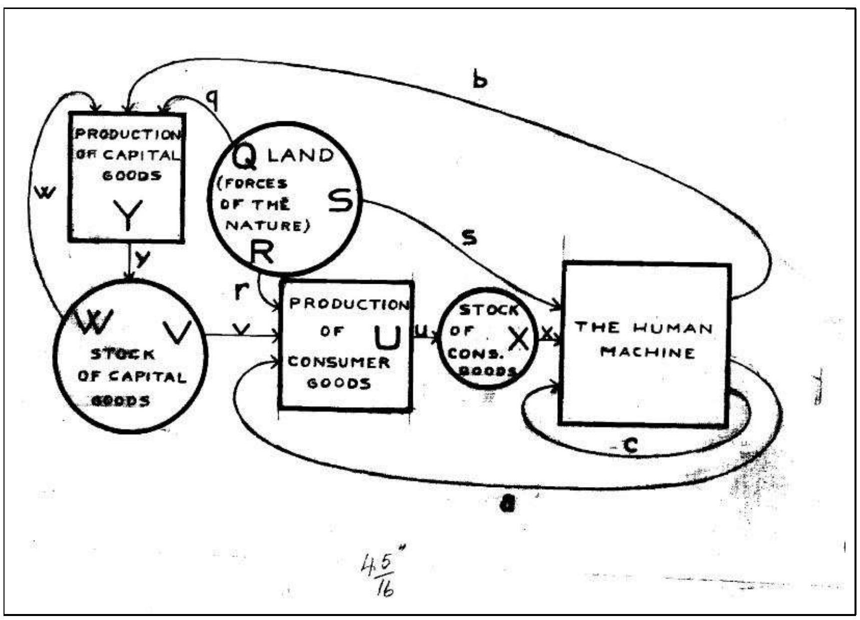

Figure 1: Le tableau Economique

All capital letters in his model are meant to represent stocks whereas all small letters represent flows. To explain the movements, the complete macro-dynamic problem consists in describing as realistically as possible the kind of relationships that exist between various magnitudes (Frisch, 1933). Once the illustration has made its point of modelling a macro-dynamic system of an economy, Frisch goes on to present a simplified system without oscillations.

3.4 A simplified system without oscillations

The assumptions made for this initial description are as follows: Firstly, he asserts that initially there will be no inventories, meaning that the yearly consumption is equal to yearly production of consumer goods (Frisch, 1933). Secondly, the depreciation of capital stock is defined as a two-part type of depreciation: a term for depreciation due to the use of capital goods in the production of consumers' goods and a second term for depreciation of capital goods used to produce other capital goods., For these he creates two constant depreciation coefficients h and k for the capital producing industry and the consumer industry respectively (Frisch, 1933). We then have three variables to define:

Z is Capital stock
x is Consumption
y is Investment decision

These are related through the first step which is the rate of increase of the capital stock given by:

$$\dot{Z} = y - (hx + ky) \tag{1}$$

Frisch proceeds with a zero capital accumulation assumption or a "stationary state" for which $\dot{Z} = 0$ leaving us with:

$$y = \frac{h}{1-k}x = mx \tag{2}$$

Where $m = \frac{h}{1-k}$ represents the total depreciation of capital stock.
Next is the definition of the production of capital goods. This is made up of two replacement components, a total depreciation as in equation (2) and an increase in capital needs which are determined by the increase of the production of consumer goods.

$$y = mx + \mu\dot{x} \tag{3}$$

Where μ is the size of capital stock that is needed directly and indirectly in order to produce one unit of consumption each year. Given this system, both Frisch (and Zambelli later) remind us that at this point there is need for another equation to make the system determinate. Some equation that defines x or y explicitly, 'we need to introduce an equation expressing the behavior of consumers' (Frisch, 1933). Enter the *encaisse desiree* or the cash on hand to model consumer behavior. It is made up of two parts: the need for cash required for the transaction of consumer goods and the cash needed for the transaction of production goods. Taking it we define the *encaisse desiree* as ω, we have:

$$\omega = rx + sy \tag{4}$$

Where r and s are 'properly defined constants' (Frisch, 1933). Frisch considers them as given by habits and the nature of the existing monetary institutions. Here comes an important step that points towards a possible non-linear relation: We understand from the system just described that, just as Frisch did, if we imagine a period of expansion in which the economic activity in both producer goods and consumer goods; the need for cash in hand will increase, in which case we must realize that the stock of money cannot increase *ad infinitum* (Frisch, 1933). At least it cannot in this system. Frisch assumes a *tension* that acts against further expansion. This tension is made explicit in equation (4). It seems plausible then that the *encaisse desiree* ω will enter as an *important factor* which after a certain point will tend to diminish the *rate of increase of consumption*. Frisch states that: 'Later, consumption may perhaps actually decline' (Frisch, 1933). Assuming as *first approximation* the relationship to be linear (Frisch, 1933), we have:

$$\dot{x} = c - \lambda\omega = c - \lambda(rx + sy) \tag{5}$$

Where c and λ are positive constants (Frisch, 1933). Both Velupillai (1987) and Zambelli (1991) point out that the initial approximation was never removed and the question of whether the *encaisse desiree* is indeed important, the linear formulation of it should have been kept (Zambelli, 1991). Using equations (3) and (5) we get an **autonomous two-variable** differential equation defined as:

$$\begin{aligned} y &= mx + \mu\dot{x} \\ \dot{x} &= c - \lambda(rx + sy) \end{aligned}$$

Which can be compounded unto:

$$\begin{aligned} \dot{x} &= c - \lambda(rx + s(mx + \mu\dot{x})) \\ &= c - \lambda rx - \lambda smx - \lambda s\mu\dot{x} \\ \dot{x}(1 + \lambda s\mu) &= c - \lambda x(r + sm) \end{aligned}$$

And now we can solve the autonomous differential equation.

$$\begin{aligned} \frac{dx}{dt}(1 + \lambda s\mu) &= c - \lambda x(r + sm) \\ \frac{(1 + \lambda s\mu)dx}{c - \lambda x(r + sm)} &= dt \\ x &= \frac{Ke^{\frac{-\lambda(r+sm)}{(1+\lambda s\mu)}t} + c}{\lambda(r + sm)} \end{aligned}$$

which gives:

$$x(t) = X_0 e^{\frac{-\lambda(r+sm)}{(1+\lambda s\mu)}t} + \frac{c}{\lambda(r + sm)} \tag{6}$$

where $X_0 = x(t_0) - \frac{c(1+\lambda sm)}{\lambda(r+sm)}$.

This solution is a first order linear equation that produces only *monotonic* evolutions. Copying Zambelli's tactic of using Frisch's numerical values we can see the evolution in the graph below (the values are $\lambda = 0.05$, $r = 1$, $s = 1$, $m = 0.5$, $\mu = 10$ and $c = 0.165$).

3.5 A Macro-Dynamic System Giving Rise to Oscillations

To create such a system Frisch introduces the concept of the carry-on activity which is derived from Aftalion (1927) in which a distinction is made between capital goods whose production has already begun and the resources required to carry them to completion in other periods.This concept is founded on the idea that investments made take time to mature, indeed investments that are time-consuming. Under Frisch's treatment this period between invesment and maturity is namely ε. This carry-on activity is defined in PPIP as:

$$z_t = \int_0^\infty D_\tau y_{t-\tau} d\tau \tag{7}$$

where D_τ is the 'advancement function' which is explained by Frisch as the 'amount of production activity needed at the point of time $t+\tau$ in order to carry on the production of a unit of capital goods started at the point of time t. (Frisch 1933). Frisch defines D_τ as a 'box function'(Zambelli 1991):

$$D_\tau = \begin{cases} \frac{1}{\varepsilon} & 0 < \tau < \varepsilon \\ 0 & \tau > \varepsilon \end{cases}$$

This new term renders our *encaisse desiree* different for 'it is now z that will occur instead of y', giving us the consumption function:

$$\dot{x} = c - \lambda(rx + sz) \tag{8}$$

Now, upon differentiation of equations (7) and (8) we obtain the two following equations:

$$\dot{z}_t = \frac{1}{\varepsilon}(y_t - y_{t-\varepsilon})$$
$$\ddot{x}_t = -\lambda(r\dot{x}_t + s\dot{z}_t)$$

This renders a new determinate system of equations. In Zambelli (1991) the system is referred to as SYSTEM 1 a notation we will here uphold for the purposes of reference and comparison. SYSTEM 1 is therefore:

$$\ddot{x}_t = -\lambda(r\dot{x}_t + s\dot{z}_t) \tag{9}$$

$$y = mx_t + \mu\dot{x}_t \tag{10}$$

$$\dot{z}_t = \frac{1}{\varepsilon}(y_t - y_{t-\varepsilon}) \tag{11}$$

Solving the system yields a second order differential first order difference equation in x:

$$\varepsilon\ddot{x}_t + \lambda(r\varepsilon + s\mu)\dot{x}_t + \lambda smx_t - \lambda smx_{t-\varepsilon} - \lambda s\mu\dot{x}_{t-\varepsilon} = 0 \tag{12}$$

Using thid equqation, we obtain a trascendental function. This trascendental function is:

$$\varepsilon(-\beta + i\alpha)^2 + \lambda(\varepsilon r + s\mu)(-\beta + i\alpha) + \lambda sm - (\lambda s\mu(-\beta + i\alpha) + \lambda sm)e^{-\varepsilon(-\beta + i\alpha)} = 0$$

which can be simplified to:

$$\varepsilon\rho^2 + \lambda(\varepsilon r + s\mu)\rho + \lambda sm - (\lambda s\mu\rho + \lambda sm)e^{-\varepsilon\rho} = 0 \tag{13}$$

where $\rho = -\beta + i\alpha$.

This characteristic equation has a countable infinity of solutions (zeros) and a solution that takes all the components into account is:

$$x_t = a_0 + \sum_{k=1}^{\infty} a_k e^{-\rho_k t} \tag{14}$$

where the a_ks are determined by the initial conditions. If we assume this type of progression/solution for each variable (x, y and z) and describe them as time series we obtain an expression of the above solution as:

$$\begin{cases} x_t = a_* + \sum_{k=0}^{\infty} a_k e^{\rho_k t} \\ y_t = b_* + \sum_{k=0}^{\infty} b_k e^{\rho_k t} \\ z_t = c_* + \sum_{k=0}^{\infty} c_k e^{\rho_k t} \end{cases} \tag{15}$$

where ρ_k are complex numbers and where a, b and c are also constants. The characteristics of ρ_k can be found to be determined by the structural constants ε, λ, μ, s, m and r; whereas the coefficients a, b and c will depend on the initial conditions. Frisch uses a set of relations between the coefficients to determine how the ρ_k will depend on the structural components. Differentiating all equations in (15) and inserting the results into each of the equations (8), (10) and (11) one obtains the following relations between a, b and c:

$$\begin{cases} \frac{c_k}{a_k} &= -\frac{\lambda r + \rho_k}{\lambda s} \\ \frac{b_k}{a_k} &= m + \mu\rho_k \\ \frac{c_k}{b_k} &= \frac{1-e^{\varepsilon\rho_k}}{\varepsilon\rho_k} \end{cases} \tag{16}$$

Through this solution Frisch then finds that all the ρ_k must be 'roots of the following characteristic equation' (Frisch,1933). It is owrth noticing as well that this equation is now a result of Frisch's treatments from equations (16), but that this equation can also be trivially obtained by manipulating equation (13):

$$\frac{\varepsilon\rho}{1-e^{\varepsilon\rho}} = -\lambda s \frac{m + \mu\rho}{r\lambda + \rho} \tag{17}$$

At this point Frisch reminds us that this equation may have real or imaginary roots. He goes on to re-arrange the expression by inserting $\rho = -\beta + i\alpha$ and the obvious $i = (-1)^{(\frac{1}{2})}$ into equation (17) to simplify the numerical computation. This leads him to two expressions from which we can obtain the roots. The two expressions are:

$$1 + \lambda s \mu e^{\varepsilon\beta} \frac{sin\varepsilon\alpha}{\varepsilon\alpha} = \frac{m\frac{\varepsilon^2}{\mu^2}(m - \lambda r\mu)}{(\varepsilon\beta - m\frac{\varepsilon}{\mu})^2 + (\varepsilon\alpha)^2} \tag{18}$$

$$-\frac{\varepsilon\beta - \lambda r\varepsilon + m\frac{\varepsilon}{\mu}}{\varepsilon\beta - m\frac{\varepsilon}{\mu}} + \lambda s\mu\frac{1 - e^{\varepsilon\beta}cos\varepsilon\alpha}{\varepsilon\beta - m\frac{\varepsilon}{\mu}} = \frac{m\frac{\varepsilon^2}{\mu^2}(m - \lambda r\mu)}{(\varepsilon\beta - m\frac{\varepsilon}{\mu})^2 + (\varepsilon\alpha)^2} \tag{19}$$

Upon making a tabulation of the zeros or solutions to equations (18) and (19), Frisch goes on to explain how each of these zeros are compounded into trends. He considers the components, for whom he gives the index $j = 0, 1, 2, 3...$ where $j = 0$ is the first trend for each of the variables x, y and z. What Frisch is after when he speaks of trends are basically those components from the set of equations (15) ordered from longest to shortest wavelength. These wavelengths are determined by the real part of the zeroes (namely the αs). He expresses these first trends thus:

$$\begin{cases} x_0(t) &= a_* + a_0 e^{\rho_0 t} \\ y_0(t) &= b_* + b_0 e^{\rho_0 t} \\ x_0(t) &= c_* + c_0 e^{\rho_0 t} \end{cases} \tag{20}$$

And we must remember that, in the case of consumption x these trends are constructed by:

$$x_t = a_* + \sum_{k=0}^{\infty} a_k e^{-\rho_k t} = (a_* + a_0 e^{-\rho_0 t}) + \sum_{k=1}^{\infty} a_k e^{-\rho_k t} = x_0(t) + \sum_{k=1}^{\infty} a_k e^{-\rho_k t} \tag{21}$$

These are, as Frisch explains, just the first terms in the composite expressions (Frisch, 1933). Here we see that the dampening exponent ρ_0 is the first root from the characteristic equation (13). Frisch claims that the additive constants a_*, b_* and c_* are also determined by the structural coefficients λ, ε, etc. Frisch also states that if $t \longrightarrow \infty$ the functions (20) above will approach the stationary state levels a_*, b_* and c_*. According to Frisch (1933), since the derivatives will vanish we can obtain from our initial equations (15) and (20)

$$b_* = ma_* \qquad c_* = b_* \qquad \lambda r a_* + \lambda s c_* = c \tag{22}$$

and as an example he gives c the value $c = 0.165$ which determines the three constants $a_* = 1.32$, $b_* = 0.66$ and $c_* = 0.66$. Frisch also states that the coefficients a_0, b_0 and c_0 are not uniquely determined by the structural coefficients (Frisch, 1933). These are instead determined by initial conditions. Once initial conditions can be placed on, say a_0, both b_0 and c_0 will follow suit.

Frisch then finds that upon gathering the results on his zeros, the periods on each cycle can be found revealing the length of each cycle. The first cycle is 8.57 years, the second cycle is 3.50 years and a tertiary cycle with a period of 2.20 years.

Frisch makes an important consideration when it comes to the features of the cycles. He expresses the cycles anew by using a trigonometric extension of the Euler's formula. These new components and expressions for the variables look thus in Frisch 1933:

$$\begin{cases} x_j(t) & = A_j e^{-\beta_j t} sin(\phi_j + \alpha_j t) \\ y_j(t) & = B_j e^{-\beta_j t} sin(\psi_j + \alpha_j t) \\ x_j(t) & = C_j e^{-\beta_j t} sin(\theta_j + \alpha_j t) \\ (j & = 1, 2, 3, ...) \end{cases} \tag{23}$$

We are now reminded that what have now become frequencies (α) and what has become the dampening coefficients (β) are determined by the different solutions or zeros to equation (13), whereas the phases ϕ, ψ and θ and the amplitudes A, B and C are influenced by the initial conditions (Frisch, 1933). He commences for the primary cycle ($j = 1$) with two conditions, namely: $x_1(0) = 0$ and $\dot{x}_1(0) = \frac{1}{2}$. This leads to $\phi_1 = 0$ and $A_1 = \frac{1}{2\alpha_1}$. He imposes the same conditions on the secondary cycle. These conditions are put out to be: $x_2(0) = 0$ and $\dot{x}_2(0) = \frac{1}{2}$. For all these conditions, by 'virtue of' (9), (10) and (11) the phases and amplitudes in y and z are found. When these conditions from the set of equations (20) are taken into account we get a set of relations for the amplitudes and phases that looks thus:

$$\begin{cases} B sin(\psi - \phi) & = A\mu\alpha \\ B cos(\psi - \phi) & = A(m - \mu\beta) \end{cases} \tag{24}$$

$$\begin{cases} C sin(\theta - \phi) & = -\frac{A\alpha}{\lambda s} \\ C cos(\theta - \phi) & = \frac{A(\beta - \lambda r)}{\lambda s} \end{cases} \tag{25}$$

These equations, as Frisch points out, not do they become valid for all $j = 1, 2, 3, ...$, but they show that whatever lag exists between the variables (x, y and z) are independent of the initial conditions and depend only on the structural coefficients of the system (Frisch, 1933). Frisch also points out that from (23) and (24) we can also obtain:

$$\begin{cases} tg(\psi - \phi) & = \frac{\mu\alpha}{m - \mu\beta} \\ tg(\theta - \phi) & = -\frac{\alpha}{\beta - \lambda r} \end{cases} \tag{26}$$

Frisch also gives us the relations between the amplitudes (where the square roots are taken as positive (Frisch, 1933)):

$$\begin{cases} |B| & = \sqrt{(\mu\alpha)^2 + (m - \mu\beta)^2} |A| \\ |C| & = \frac{\sqrt{\alpha^2 + (\beta - \lambda r)^2}}{\lambda s} |A| \end{cases} \tag{27}$$

Frisch's multiple results for all the zeroes, amplitudes, coefficients and phases are given below in table (1):

-	Trend	Primary Cycle	Secondary Cycle	Tertiary Cycle
	$(j = 0)$	$(j = 1)$	$(j = 2)$	$(j = 3)$
Frequency ... α		0.73355	1.79775	2.8533
Period ... $p = \frac{2\pi}{\alpha}$		8.5654	3.4950	2.2021
Damping Exponent ... β	$\rho_0 = -0.08045$	0.371335	0.5157	0.59105
Damping factor per period $e^{-\frac{2\pi\beta}{\alpha}}$		0.0416	0.1649	0.2721
Amplitude (x) ... A	-0.32	0.6816	0.27813	0.17524
Phase (x) ... ϕ		0	0	0
Amplitude (y)...B	0.09744	5.4585	5.1648	5.0893
Phase (y)... ψ		1.9837	1.8243	1.7582
Amplitude (z)... C	0.12512	-10.662	-10.264	-10.147
Phase (z)... θ		1.9251	1.7980	1.7412

With the set of equations (22) looked as solutions for the system obtained we can start to replicate Frisch's results. It is now also important to mention the nature and explanations of the values given to the constants μ, s, m, r, λ and ε. During Frisch's development of the model it was clear that the values of these constants were not obvious to Frisch at first and that these values were the result of endless calculations and trial-and-error attempts as evidenced by his notes taken while preparing the manuscript, and it is thanks to these notes that now lie in the Ragnar Frisch Archive at the University of Oslo that insight into these as well as other matters that further light has been shed on Frisch's development of PPIP. Frisch accounts for the value $\mu = 10$ as a condition where the total capital stock is ten times as large as the annual production (Frisch, 1933). Also he puts $m = 0.5$ and adds that it means that the direct and indirect yearly depreciation of the capital stock caused by its use in the production of the national income is one-half of that income, i.e. 20 percent of the capital stock (Frisch, 1933). The effects of the *encaisee desiree* are summarized as $\lambda = 0.05$, $r = 1$ and $s = 1$ (Frisch, 1933).

Frisch makes an interesting claim right after setting these values; he claims that 'There is, however, reason to believe that these latter constants will not affect very strongly the lenght of the cycles obtained' (Frisch, 1933). This statement is in fact true, but as evidenced by his several attempts with the model using several values of these 'latter constants' one can see that those values do affect the oscillatory characteristic of the propagation mechanism (but not the lenght of the cycles as such, which is indeed strange). Evidently the effect they have is not sufficient to render the independent harmonic solutions non-oscillatory, but they are enough to drive Frisch to attempt at least 6 different values for each constant which are the attempts recorded in his notes for PPIP.

When it comes to Frisch's setting of the value for the constants, Thalberg (1998) has important remarks. Using the numerical values described above, Frisch finds a primary cycle with a length of 8.5 years, a secondary cycle with a length of 3.5 years and a third cycle with a length of 2.2 years. All cycles are heavily damped, and according to Frisch they corresponded well to the observed long and short business cycles. Thalberg (1998) states ' the results certainly seem impressive, but also puzzling, in view of the model's drastic simplifications and the fact that the inserted numerical values were very rough guesses'. Thalberg (1998) is particularly skeptical about the value of $\varepsilon = 6$ which is the length of completion of capital goods and to Thalberg this length is 'much too long' (Thalberg, 1998). Thalberg (1998) also points out that 'Despite his strong desire, he (Frisch) was unable to work out the total picture (i.e., the sum of all components)' (Thalberg, 1998). This is of coruse due to the fact that there are simply an infinite amount of trends where each trend corresponds to a zero for our characteristic equation.

We now commence to replicate Frisch's results, yet firstly, we need to impose on these trends the same initial conditions as constructed by Frisch in 1933. Frisch imposed the first initial condition that x_0 should be unity at origin, and that on each cycle in x the condition that it shall be zero at origin and with velocity $= \frac{1}{2}$. These initial conditions provide us the following functions:

$$\begin{cases} x_0 &= 1.32 - 0.32e^{-0.8045t} \\ y_0 &= 0.66 + 0.09744e^{-0.08045t} \\ z_0 &= 0.66 + 0.12512e^{-0.08045t} \end{cases} \tag{28}$$

$$\begin{cases} x_1 &= 0.6816e^{-\beta_1 t} sin(\alpha_1 t) \\ y_1 &= 5.4585e^{-\beta_1 t} sin(1.9837 + \alpha_1 t) \\ z_1 &= -10.662e^{-\beta_1 t} sin(1.9251 + \alpha_1 t) \end{cases} \tag{29}$$

$$\begin{cases} x_2 &= 0.27813e^{-\beta_2 t}sin(\alpha_2 t) \\ y_2 &= 5.1648e^{-\beta_2 t}sin(1.8243 + \alpha_2 t) \\ z_2 &= -10.264e^{-\beta_2 t}sin(1.7980 + \alpha_2 t) \end{cases} \quad (30)$$

$$\begin{cases} x_3 &= 0.17524e^{-\beta_3 t}sin(\alpha_3 t) \\ y_3 &= 5.0893e^{-\beta_3 t}sin(1.7582 + \alpha_3 t) \\ z_3 &= -10.147e^{-\beta_3 t}sin(1.7412 + \alpha_3 t) \end{cases} \quad (31)$$

In the plots below, we must see that the thick lines correspond to the evolution of x, the thin lines correspond to y and the dashed lines correspond to the evolution of z. Let us now go onto the plots.

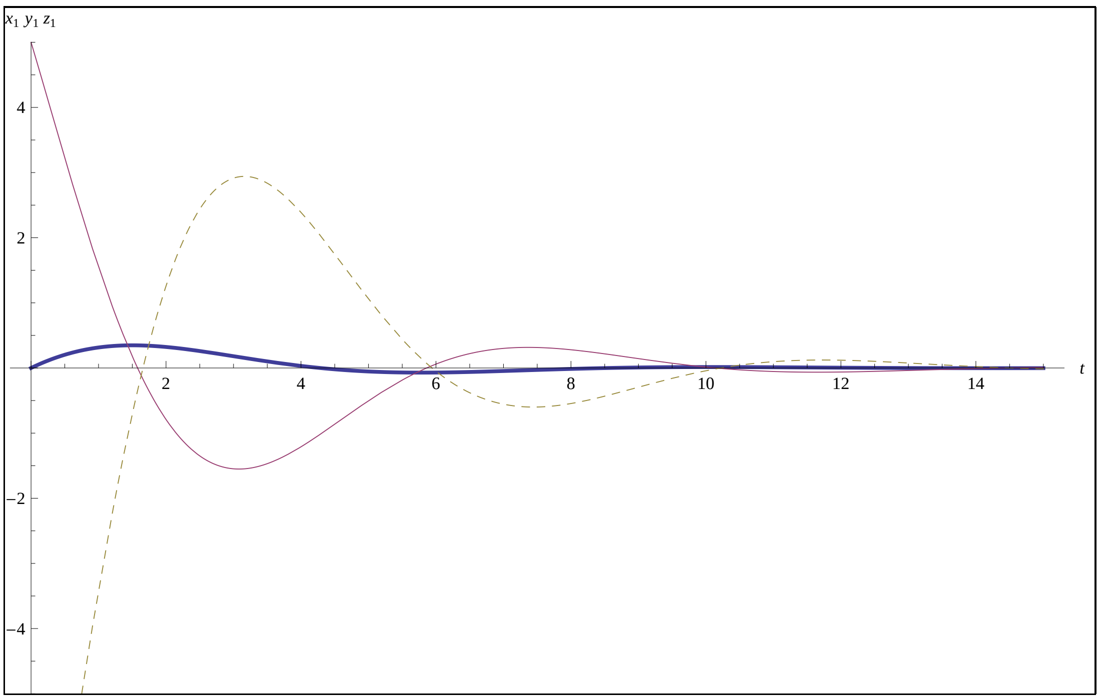

Figure 2: plots forx_1, y_1 and z_1

These plots represent the exact same results as presented by Frisch in PPIP 1933. Frisch's analysis of figures 2, 3, and 4 is that the shorter cycles are not as heavily damped as the long cycle (Frisch, 1933). He further adds that the lags between the variables x, y and z are, roughly speaking, the same in all three cycles (Frisch, 1933). Frisch further analyzes the results by comparing firstly, consumption and production starting. He notes that apart from the heavy dampening in figure 4, the relation between x and y is very much that which had been thought of beforehand, namely that a peak in consumption comes after the peak in production.

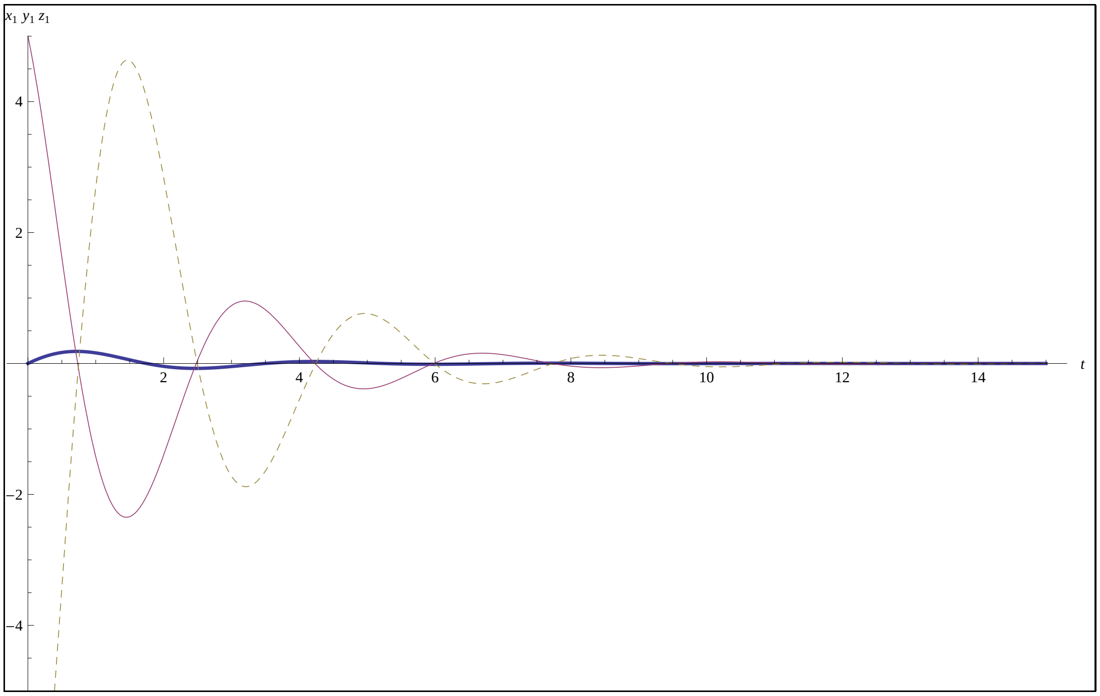

Figure 3: plots forx_2, y_2 and z_2

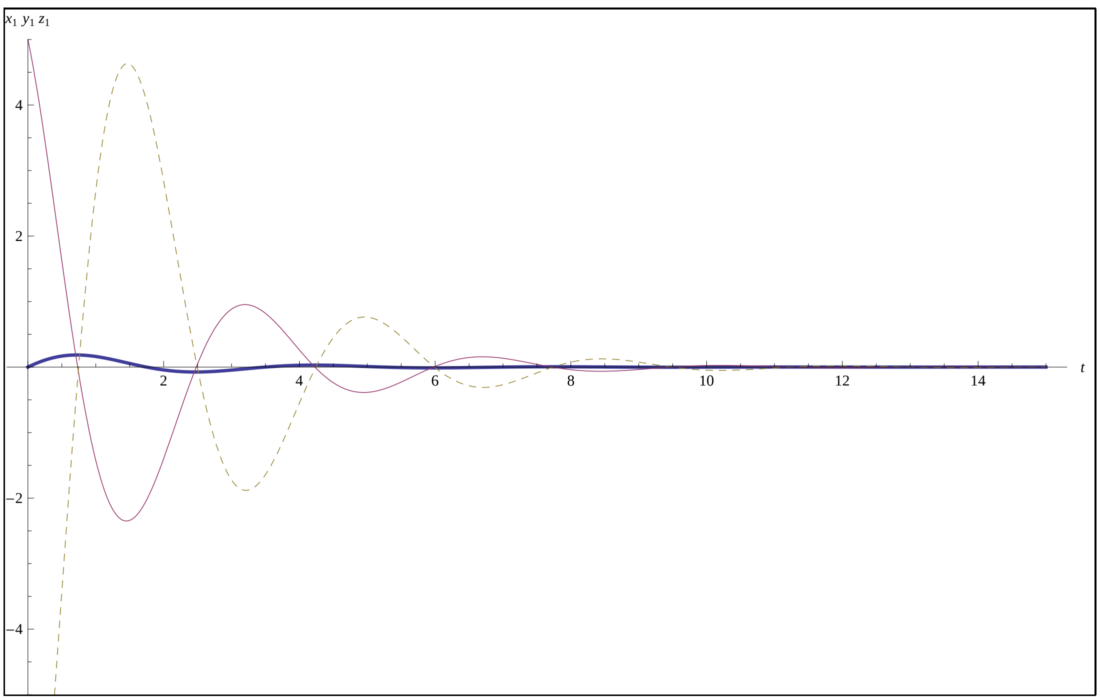

Figure 4: plots forx_3, y_3 and z_3

Frisch also had stated earlier in the development of the model that a comparison between production and *the rate of increase of consumption* a synchronism can be found in that the maximum rate of consumption occurs at the same moment as the the peak in the actual size of capital production (Frisch, 1933). This was a fact that had been observed, as Frisch points out, by Mitchell and is explained by saying that it is consumption which exerts an influence on production (Frisch, 1933). Frisch explains this observation in his own results in PPIP clarifying that we are probably looking at a highly capitalistic society where the annual depreciation is relatively small (Frisch, 1933); a fact understood by the relatively large size of μ in comparison to m.

Frisch proceeds with his analysis of the graphical results by comparing consumption and the carry-on activity. Frisch notes that consumption, in his plots, is 'leading by a considerable span of time' (Frisch, 1933) which suggests that in the depression portion of the plots, the carry-on activity starts to increase only when the upswing in consumption is well underway, and the carry-on activity continues to increase even after consumption has started to decline (Frisch, 1933).

Frisch finishes his analysis of the model he described by making a comparison between a step-by-step computation of the primary cycle vs. the primary cycle being directly computed by formula. The results are, as one would expect, exceedingly close suggesting a sound construction in the model.

The rest of Frisch's PPIP goes on to consider the *other* aspect of the model, but this aspect is only treated as an intellectual discussion with two *hors d'oeuvre* of mathematical analysis. The author of this paper will look at these two aspects later but it is worth mentioning that they both include considerations regarding the movement of pendula and their dynamics. The first consideration is the effect of numerous random shocks upon a free pendulum, and the second one is an attempt at adding Schumpeter's innovations as an explanation for continued shocks in the economy. It can be argued that the second train of thoughts, where a valve is attached to a pendulum and left to the whims of the current flowing from above (the current supposed to represent the innovations) is almost analogous in description and much simpler to manage as the problem of the double-pendulum. These considerations, which are of great importance for they do indeed represent the other half of the model, were not analyzed in depth by Frisch and left to be further analyzed either by himself or others. We will also find that Zambelli (1991) bases his argument on the non-cyclical nature of Frisch's model by criticizing only the propagation portion of it, and, as we will find, a rather incomplete statement for we have no way of knowing as of yet whether a *complete* dynamic system by Frisch including both the propagation structure *and* the impulse structure would indeed give rise to oscillations.

4 Zambelli's criticism

4.1 Zambelli's examination

In his 1991 article Zambelli proceeds in much the same manner we have proceeded here. A full description of the model is made (however brief) and Frisch's results are also reproduced using the exact same values for the constants (namely: $\lambda = 0.05$, $r = 1$, $s = 1$, $m = 0.5$, $\mu = 10$, and $\varepsilon = 6$). One of the first impressions one gets of Zambelli's analysis is the assumption that the reader is all too familiar with the procedures that are to be used and that the mathematical analysis undergone in many aspects can be taken for granted. Take for example his claim that, upon describing his 'MAIN CRITICAL POINTS AND REMARKS' in Zambelli 1991, Zambelli says that '..with the aid of a computer I have stimulated Frisch's examples, recomputed the roots and reproduced the same values and graphs....'. It is perhaps understandable that Zambelli would expect his audience to be well-versed in mathematics but upon finding the roots of a characteristic equation such as equation (13) the algorithm used (being surely different from that Frisch used) should have been explained and the results clarified. This paper contains a description of one such algorithm, and despite the willing omission from Zambelli, the zeroes found for equation (13) stand as the same ones both Frisch and Zambelli used. It then becomes unimportant how these zeroes were found for, as far as we know, three different attempts have been made to find them (Frisch's, Zambelli's and our present one here) and they all coincide. Despite this, Zambelli should have made an attempt at describing the resulting plot of the inclusion of a great many more zeroes (and therefore other cycles), if only to prove that it is the first three zeroes that have any affect in the resulting dynamic. This is a result that will be presented later in this paper.

First true criticism comes from Zambelli (1991) shortly after Frisch's results have been reproduced (as they have been here) and the question is asked whether these represent 'plausible histories' (Zambelli, 1991). Zambelli reminds us that the evolutions of the cycles in the variables were generated imposing specific initial conditions and that given the importance of the carry-on activity z as well as the fact that present conditions are the result of investment decisions that took place during the last $-\varepsilon$ years, the past history is essential for the determination of the evolution of the cycles (Zambelli, 1991). Zambelli continues describing these evolutions where he claims that the past histories are consistent with Frisch's actual simulations. But essentially, what are these past histories? The past histories are simply an evaluation of the evolution of the found solutions in "past times", namely, commencing the plots not at a time $t_0 = 0$, but a time $t = -6$. The results from the examination of the "past histories" are shown below (identical results to those of Zambelli 1991):

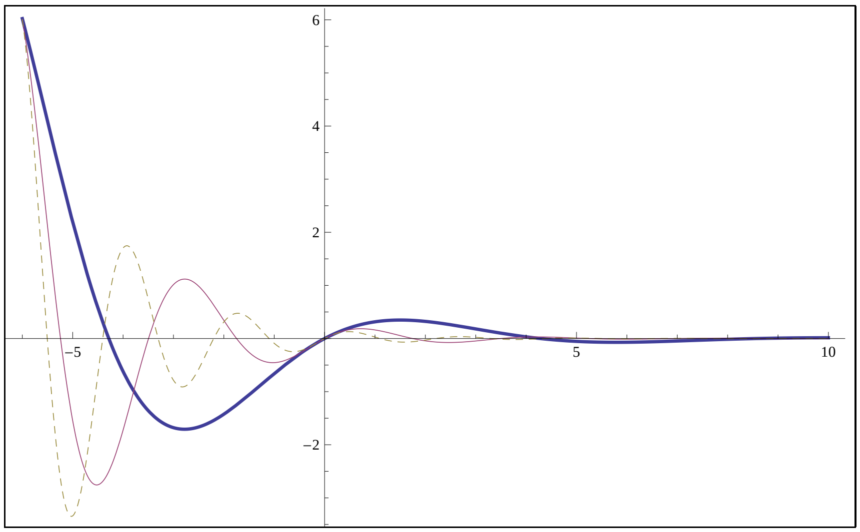

Figure 5: plot for the single cycle inclusion of past histories of x

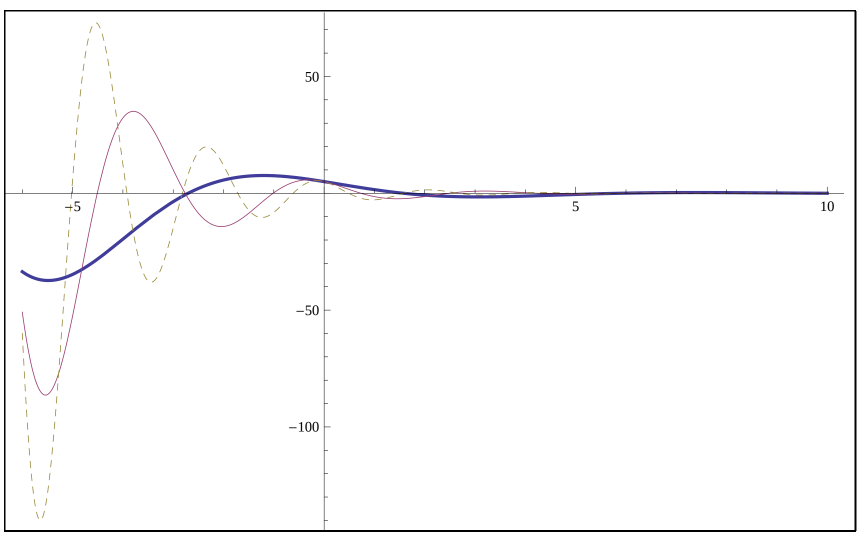

Figure 6: plot for the single cycle inclusion of past histories of y

Zambelli's next criticism is however much deeper and with true consequences for the comprehension of Frisch's 'system giving rise to oscillations'. The sum of the trend, primary, secondary and tertiary components of Frisch's model are not shown in Frisch's PPIP. In fact, the abscence

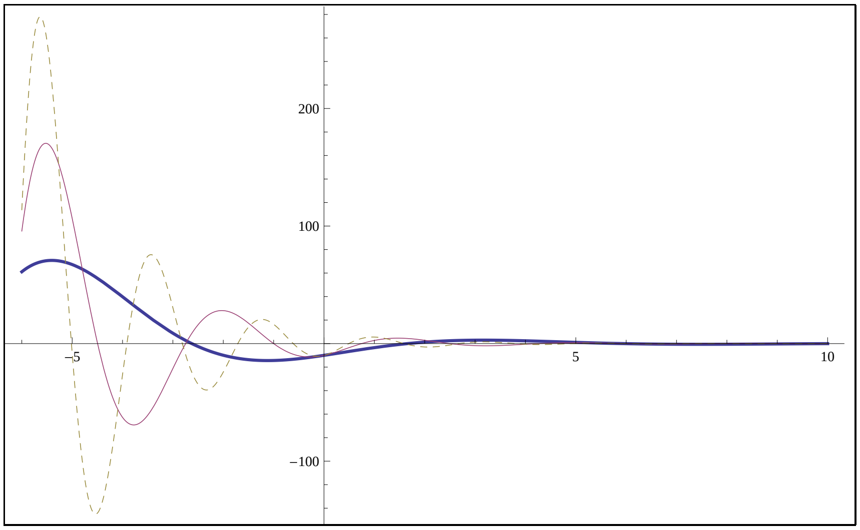

Figure 7: plot for the single cycle inclusion of past histories of z

of such analysis can be seen as a great flaw in Frisch's description of the model. Despite this, Zambelli does present us with these evolutions and they are reproduced here. Zambelli does as well offer us with the closing argument regarding Frisch's overall accomplishments: the evolutions for the aggregated cycles as well as their past histories are made and plotted, putting together thus an entire picture of the model. In figures 8, 9 and 10 we find the aggregated evolutions for the three variables x, y and z, whereas in figures 11, 12 and 13 we find the evolutions of the aggregated variables x, y and z as well as their respective past histories.

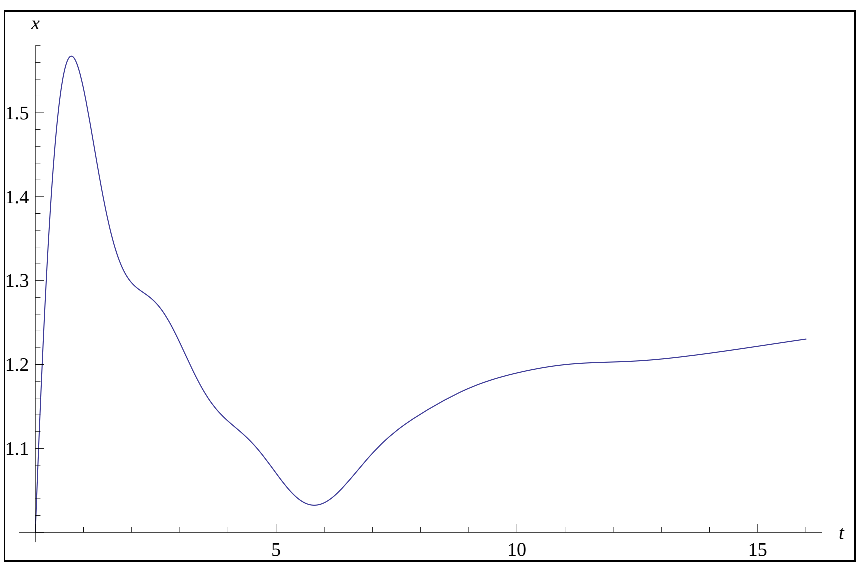

Figure 8: plot for aggregated first trends of x: $x_0 + x_1 + x_2 + x_3$

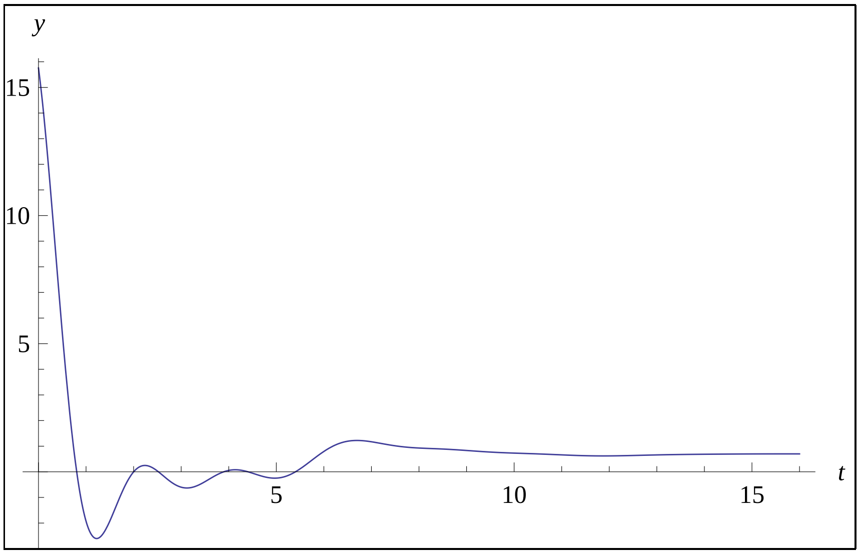

Figure 9: plot for aggregated first trends of y: $y_0 + y_1 + y_2 + y_3$

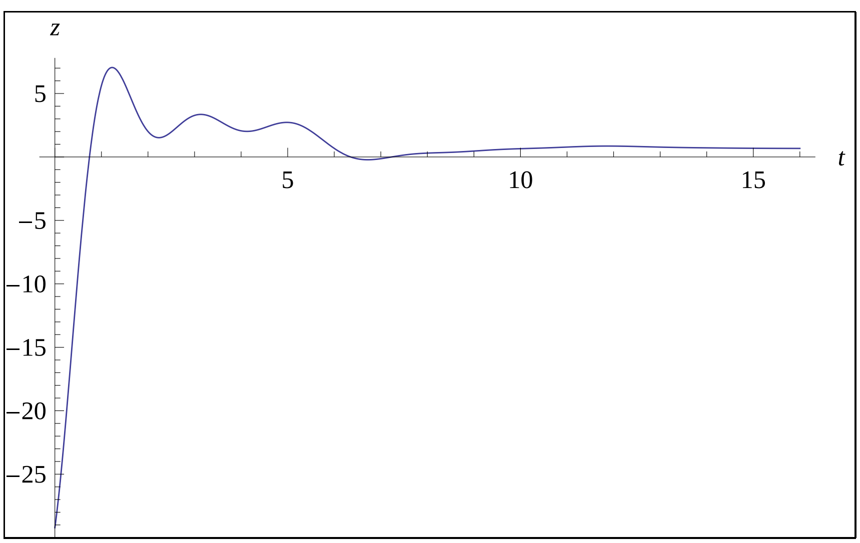

Figure 10: plot for aggregated first trends of z_3: $z_0 + z_1 + z_2 + z_3$

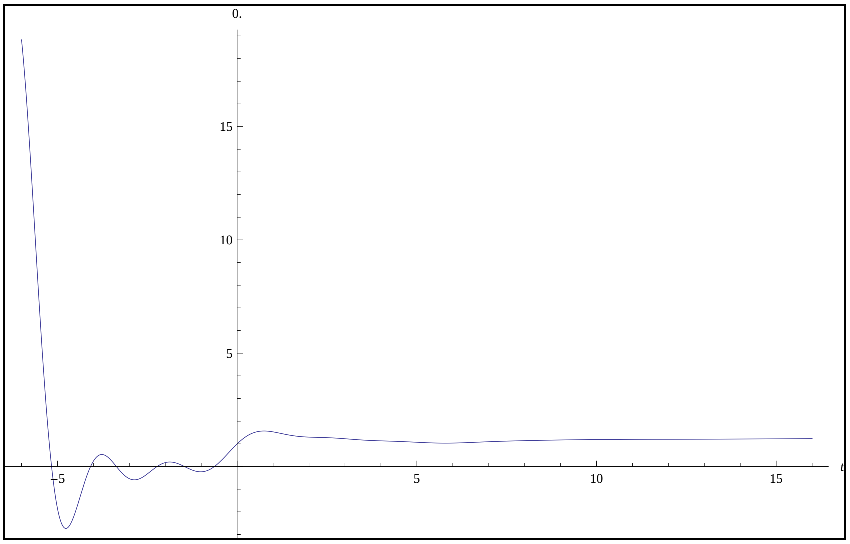

Figure 11: plot for the inclusion of histories aggregated x: $x_0 + x_1 + x_2 + x_3$

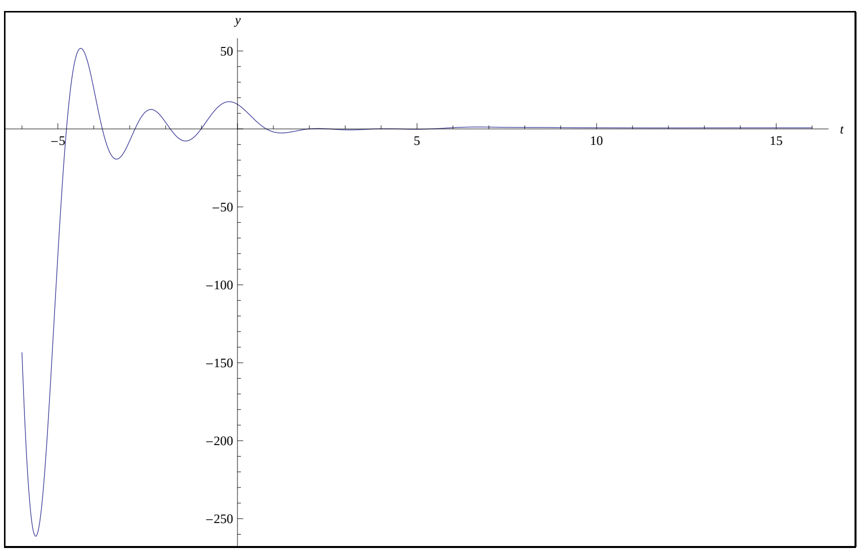

Figure 12: plot for the inclusion of histories to aggregated y: $y_0 + y_1 + y_2 + y_3$

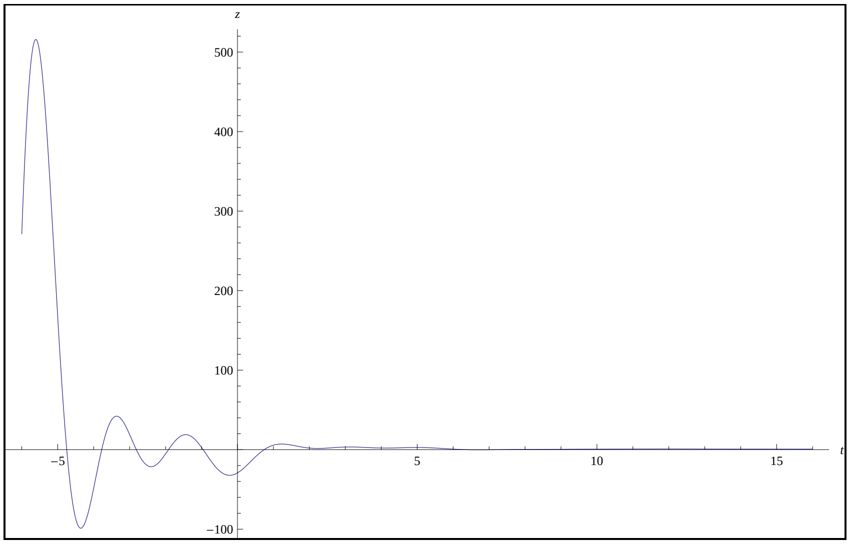

Figure 13: plot for the inclusion of histories to aggregated z_3: $z_0 + z_1 + z_2 + z_3$

Zambelli argues that in particular figures 11, 12 and 13 one can see that in that example some cyclical resemblance in the interval $[t_0, t_0 + 16]$ is maintained but that it implies a rather unlikely oscillating revolution around the interval $[t-\varepsilon, t_0]$ (Zambelli, 1991). In general, Zambelli rightfully establishes that Frisch had missed the objective of providing a complete answer to the solution of the system proposed and that instead, partial results more in line with examples is what Frisch presented us. Zambelli also points out that the use of just three cycles could be detrimental to procuring a complete solution; he states: '.. there can easily be other components with higher frequencies and also higher amplitudes and therefore the approximating error could be high' (Zambelli, 1991). A clear example of this is given when we observe figure 5 where in the interval $[-6, -4]$ the tertiary cycle exhibits wider amplitudes than the primary cycle (Zambelli, 1991). Perhaps the most important argument in favour of Zambelli's criticism comes from an analysis that can be made upon a qualitative comparison of the two systems proposed by Frisch. The first system (more importantly, equation (6)) which does not give rise to oscillations should have been compared qualitatively with SYSTEM 1 by adding all the terms on equation (7) thus producing a complete solution. Zambelli points out that it is a fact that the sum of trigonometric or independently harmonic functions could very well give rise to monotonic behaviour, and this is in fact the case in the reproductions of SYSTEM 1.

Zambelli suggests that a full examination of all the terms in the set of equations (15) must be made and that a decomposition in harmonics of the evolution of the aggregated magnitudes must be made in the interval $[t-\varepsilon, t_0]$. Only thus, Zambelli suggests, can we truly procure an explicit solution to the system. Noting the difficulty involved in not only decomposing the harmonics but in attempting to include all the zeroes from equation (15) to provide a full solution, Zambelli provides an alternative: numerical integration using a trapezoidal algorithm as well as using the Euler method for obtaining a different set of expressions. Zambelli departs from our known SYSTEM 1:

$$y_t = mx_t + \mu\dot{x}_t \tag{32}$$

$$\dot{x}_t = c - \lambda(rx_t + sy_t) \tag{33}$$

$$\dot{z}_t = \frac{1}{\varepsilon}(y_t - y_{t-\varepsilon}) \tag{34}$$

The system described above is approximated through numerical integration. Equation (32) is computed using the Euler method, whereas equation (33) is computed with the Newton-Cotes formulas (known, as Zambelli points out, as the trapezoidal algorithm). The resulting system is thus:

$$y_t = mx_t + \mu\dot{x}_t \tag{35}$$

$$x_{t+h} = (c - \lambda(rx_t + sy_t))h + x_t \tag{36}$$

$$z_{t+h} = \frac{h}{2\varepsilon}(y_{t-\tau} + 2y_{t-\tau+h} + ... + 2y(t-h) + y_t) \tag{37}$$

Zambelli then sets this new system in equilibrium, as described in Zambelli (1991) and then a shock or impulse removes the system from equilibrium. The shocks examined by Zambelli are reduced to only a 10 percent increase in consumer-taking. Secondly, the return to equlibrium is classified, namely, the evolutions are observed. Zambelli, in numerous attempts observes only monotonic returns to equilibrium, and therefore concludes that the system does not give rise to oscillations and that the horse does not *rock*.

But are these returns truly monotonic? During the writing of this thesis it was found that even without Zambelli's numerical integration methods, this same return can be shown by establishing new initial conditions to the system. It is easy to see from the sets of equations (28) through (31) that our variables x, y and z have equlibrium conditions at the points: $x = 1.32$, $y = 0.66$ and $z = 0.66$. Using these equlibrium points we change Frisch's initial conditions for finding the coefficients a_0, b_0 and c_0. These initial conditions are now: $x_0(0) = 1.32$, $x_j(0) = 0$ and $\dot{x}_j(0) = \frac{1}{2}$. Similar conditions using the equilibrium variables as initial conditions are imposed on y for this examination (both x and y) are the variables explored here. We notice in figures 14 and 15 that these particular changes in what seem to be Frisch's initial conditions (which seem rather arbitrary) does not affect the behavior of the evolutions. In fact, for x the change is a displacement upwards, where as for y the displacement is minimal as can be asserted from the difficult distinguishing the two on the plot! These results do not strengthen Zambelli's argument about a monotonic return to equlibrium.

Zambelli's criticism does become evident however when these shocks become applied in a manner that more coincides with an economic perspective. It would be an important result to apply these new initial conditions on the evolution fo the variables as they were approaching its equilibrium points. From Frisch's assertions in PPIP, shocks applied to this stage must feed the oscillations once more and we should see an oscillating return to equilibrium. These new initial conditions were thus applied to a point in time when the approach towards equilibrium seems certain, namely at $t = 45$ and thus we split the evolution in the variables as 'before' and 'after ' the shock; where the interval before the shock is plotted as we have done before with the sets of equqations (28) through (31) plus two more trends, and the second stage starting at$t = 45$ where the conditions on this new evolution are given as: $x_0(45) = 1.32(1 + 0.25)$, all other trends remaining the same. Figures 16 and 17 (where the interval for y is starte at $t = 2$ to ease the viewing of the plot) confirm Zambelli's findings. The returns to equlibrium after these 25 percent shocks are clearly monotonic revealing the propagation structure as inherently non-oscillatory.

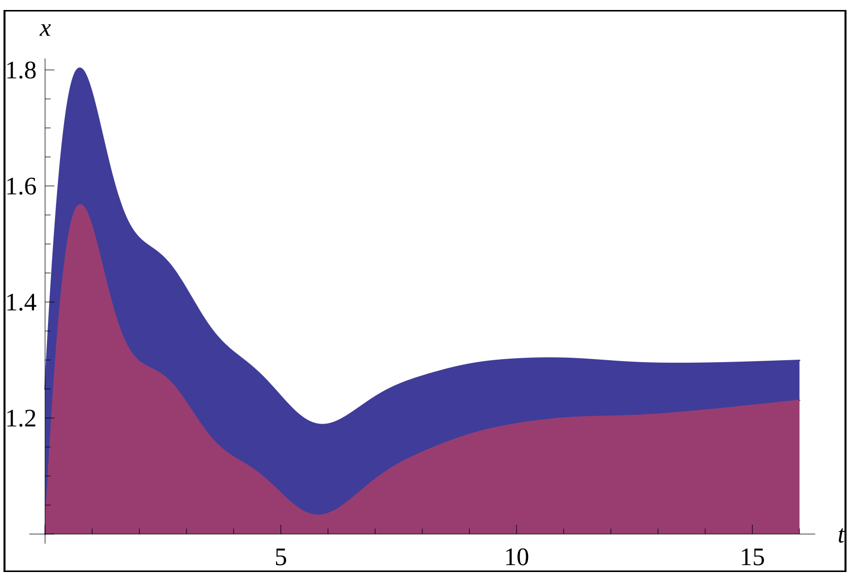

Figure 14: new initial conditions at eq. $x_0(0) = 1.32$

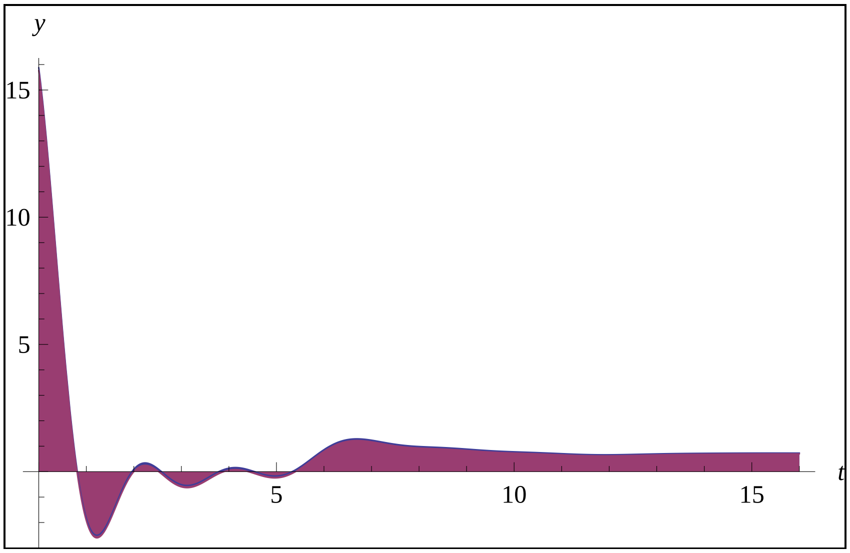

Figure 15: new initial conditions at eq. $y_0(0) = 0.66$

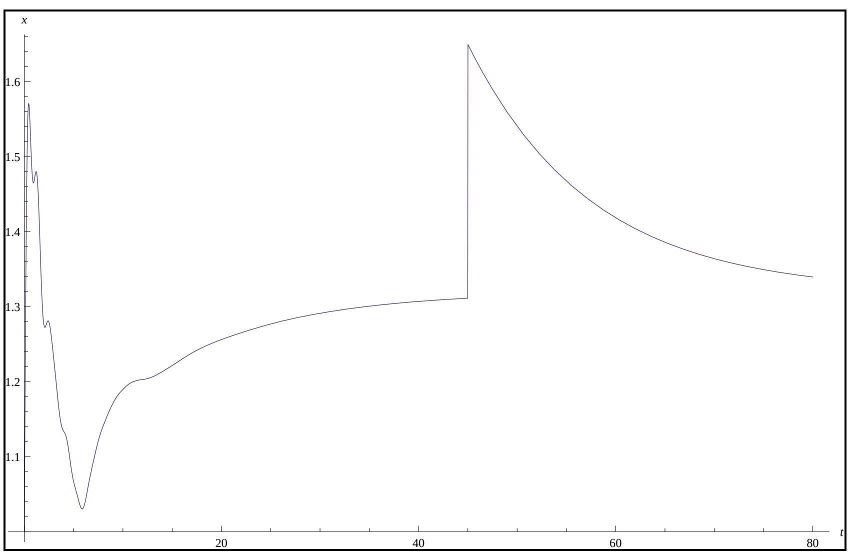

Figure 16: new initial conditions at eq. $x_0(45) = 1.32(1 + 0.25)$

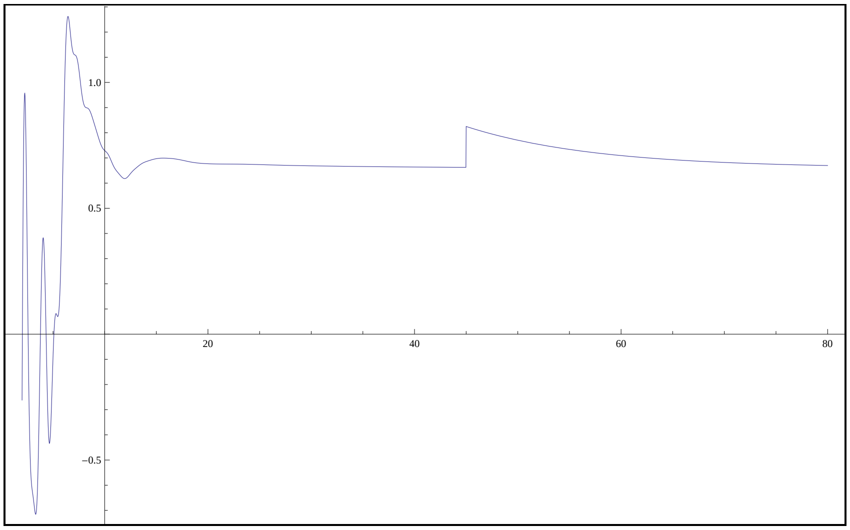

Figure 17: new initial conditions at eq. $y_0(45) = 0.66(1 + 0.25)$

4.2 Zambelli's questions

In his article from the Journal of History and political Economy, towards the end of the article, Zambelli asks: "Would Frisch have played the same role in defining research directions in economic theory? Would he have gotten the same support from the Cowles and Rockefeller foundations? Would he have been able to support Tinbergen's League of Nation's project in the same way as he did? Would he have had the same impact on his contemporaries as he did? Would research on nonlinear business cycle theory have been different?" (Zambelli, 2007)

Strong questions such as these are rarely found in journals of reputation and seriousness, and it seems rather that Zambelli was particularly pleased with having found a small flaw in a model put together by a man who had contributed so much to Economics. Given Frisch's vast and plentitudinous array of high-quality contributions and his resolve to publish and research, to question whether Frisch would have had the same effect based on a flaw within one of his models (and also making the assumption that no other Economist ever allows for flaws in their own models) is a rather strong chain of wonderement, yet understandable given the scope of the problem found.

Zambelli then, through meticulous calculation, rightly pointed out an important flaw in Frisch's model regarding its oscillatory nature. Through the appilcation of one-time shocks to the variables x and y he demonstrated (as we have also shown here) that the model has a monotonous return to equilibrium thus revealing the propagation structure as inherently non-oscillatory. This however does not render the entirety of the model itself non-oscillatory. A whole half of the model remained unexplored by Zambelli (1991) and (2007), namely: the impulse mechanism. This mechanism should be explored before making a generalized statement about the true value of Frisch's PPIP, and it is the task of this paper to examinate it here in section 5.2.

5 PPIP Reloaded

5.1 The Zeroes

We take it upon ourselves now to examine more closely the solution to the SYSTEM 1. The first aspect of both Zambelli and Frisch's propositions worthy of examination is the solution to the characteristic equation (13). According to Frisch's proposed soluton and Zambelli's statement this equation has a "countable infinity of solutions" (Zambelli, 1991). This was at first taken as mistaken upon first consideration when this paper approached the problem. This equation seemed to have a maximum of two solutions. The first steps towards verification were on this path. In fact, equation (13) can be written as:

$$\varepsilon\rho^2 + \lambda(\varepsilon r + s\mu)\rho + \lambda sm = (\lambda s\mu\rho + \lambda sm)e^{-\varepsilon\rho} \tag{38}$$

This is an equation that can be solved geometrically by finding the meeting points between the curves:

$$f(\rho) = \varepsilon\rho^2 + \lambda(\varepsilon r + s\mu)\rho + \lambda sm \tag{39}$$

$$g(\rho) = (\lambda s\mu\rho + \lambda sm)e^{-\varepsilon\rho} \tag{40}$$

Here $f(\rho)$ is a curve that opens upwards, whereas $g(\rho)$ is a curve that becomes annuled when $t \longrightarrow +\infty$. On the other hand we also see that,

$$\lim_{\rho\to\infty} g(\rho) = 0$$

We can also observe that

$$g'(\rho) = (\lambda s\mu - \varepsilon(\lambda s\mu\rho + \lambda sm))e^{-\varepsilon\rho} \tag{41}$$

Therefore,

$$g'(\rho) = 0 \Longleftrightarrow \lambda s\mu - \varepsilon(\lambda s\mu\rho + \lambda sm) = 0 \Longleftrightarrow \rho = \frac{1}{\mu\varepsilon}(\mu - m\varepsilon) \tag{42}$$

It is clear that $g(\rho)$ is increasing for $\rho < \frac{1}{\mu\varepsilon}(\mu - m\varepsilon)$ and decreasing for $\rho > \frac{1}{\mu\varepsilon}(\mu - m\varepsilon)$. This hints at that $g(\rho)$ has an absolute maximum at $\rho = \frac{1}{\mu\varepsilon}(\mu - m\varepsilon)$. These findings allow us to conclude that the curves $g(\rho)$ and $f(\rho)$ meet in no more than two different points. For there to be points in common between these two curves it is inecessary that g's maximum be "higher" than f's minimum. g's maximum is:

$$g_{max} = g(\frac{1}{\mu\varepsilon}(\mu - m\varepsilon)) = \frac{s\lambda\mu}{\varepsilon}e^{\frac{m\varepsilon}{\mu}-1}$$

whereas f's minimum is:

$$f_{min} = f(-\frac{\lambda(\varepsilon r + s\mu)}{2\varepsilon}) = ms\lambda - \frac{\lambda^2(r\varepsilon + s\mu)^2}{4\varepsilon}$$

Thus, we find that a necessary condition for the curves to meet is:

$$\frac{s\lambda\mu}{\varepsilon}e^{\frac{m\varepsilon}{\mu}-1} > ms\lambda - \frac{\lambda^2(r\varepsilon + s\mu)^2}{4\varepsilon} \tag{43}$$

This condition will be fulfilled if ε is *sufficiently small* given that:

$$\begin{aligned} \lim_{\varepsilon\to 0^+} \frac{s\lambda\mu}{\varepsilon}e^{\frac{m\varepsilon}{\mu}-1} &= +\infty \\ \lim_{\varepsilon\to 0^+} \left(ms\lambda - \frac{\lambda^2(r\varepsilon + s\mu)^2}{4\varepsilon} \right) &= -\infty \end{aligned}$$

Let us consider the values that both Frisch and Zambelli had ordained for their computations:

$$m = 0.5, \quad r = 1, \quad s = 1, \quad \varepsilon = 6, \quad \lambda = 0.05, \quad \textit{and} \quad \mu = 10. \tag{44}$$

For these values we obtain:

$$\frac{s\lambda\mu}{\varepsilon}e^{\frac{m\varepsilon}{\mu}-1} = 0.0413821$$

whereas

$$ms\lambda - \frac{\lambda^2(r\varepsilon + s\mu)^2}{4\varepsilon} = -0.00166667$$

Upon graphical inspection of the functions $g(\rho)$ and $f(\rho)$ we find:

Figure 18 is telling us that there are two zeros in the case for our equation (37). These are $\rho = 0$ and $\rho = -\frac{1}{20}$. We need however more powerful arguments to disclose whether or not these are the only zeros to be found for this characteristic equation (37). We remember that we need to find the zeros for the equation:

$$\varepsilon\rho^2 + \lambda(\varepsilon r + s\mu)\rho + \lambda sm - (\lambda s\mu\rho + \lambda sm)e^{-\varepsilon\rho}$$

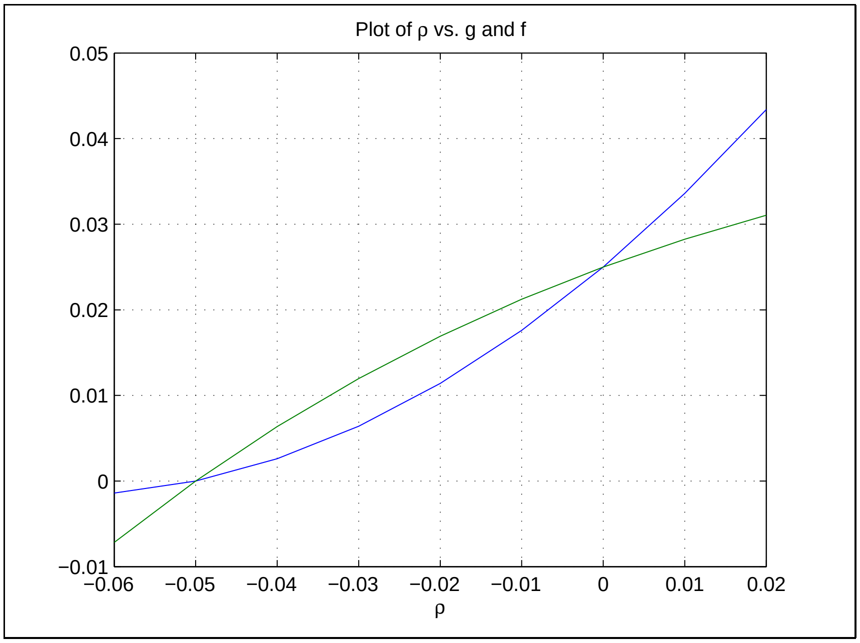

Figure 18: ρ

which after introducing the values of the constants $\lambda = 0.05$, $r = 1$, $s = 1$, $m = 0.5$, $\mu = 10$ and $\varepsilon = 6$ given to us by Frisch and doing a little factorization thanks to *Mathematica 7* we obtain:

$$\frac{1}{40}e^{-6\rho}(1 + 20\rho)(-1 + e^{6\rho} + 12e^{6\rho}\rho) = 0$$

from which it is clear that $\rho = -\frac{1}{20}$ is a solution to the characteristic equation. This result suggests that the other solutions are obtained by solving the equation:

$$-1 + e^{6\rho} + 12e^{6\rho}\rho = 0$$

It is obvious here as well that a solution to this equation is $\rho = 0$. The hypothesis pursued here is therefore that equation (37) does not admit any further solutions. Once more, we can commence for the case in which ρ is a real number. Let us plot in *Mathematica 7* the absolute value of the function:

$$h(\rho) = -1 + e^{6\rho} + 12e^{6\rho}\rho = 0$$

where $\rho = x + \sqrt{-1}y = x + iy$. This plot looks like this: We can look closer at that zero or "sucking black hole" better from another angle by evaluating Figure 20 . Figure 19 and 20 illustrate our point greatly. We can see that the "sucking black hole" corresponds to $\rho = 0$. For all other values of ρ the surface is above the xy plane which tells us that $h(\rho)$ is different from

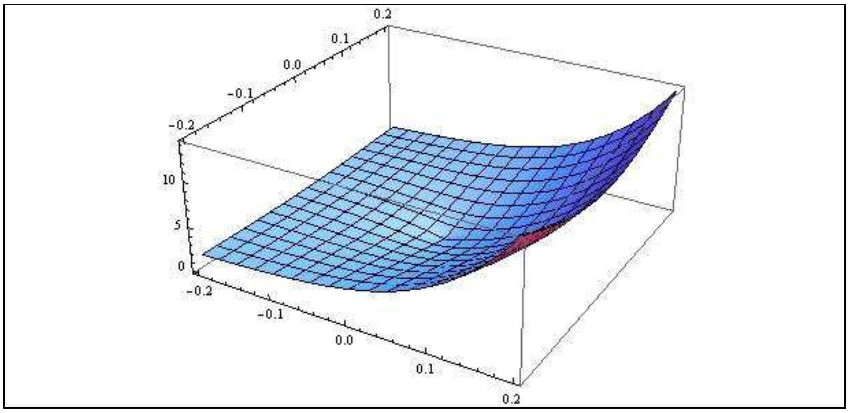

Figure 19: $h(\rho)$ (1)

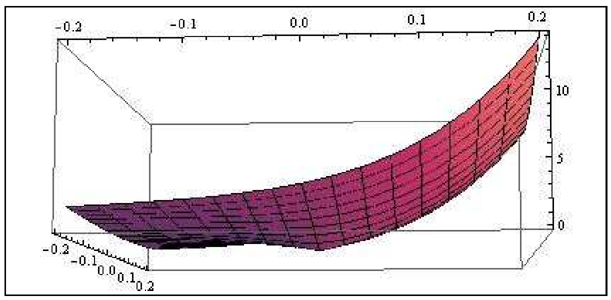

Figure 20: $h(\rho)$ (2)

zero. If we were to extend the range we would continue to see this trend. Now let us evaluate the entirety of the equation we are interested in, the equation that generates the complex zeroes we are looking for, namely:

$$f_*(\rho) = (-1 + e^{6\rho} + 12e^{6\rho}\rho) = 0 \tag{45}$$

If we decide to plot this entire equation, as we do in Figure (17) we confirm our suspicions even further. In Figure 21 we see the two zeros we have been speaking of, namely $\rho = 0$ and $\rho = -\frac{1}{20}$. But does this eliminate the idea that there could be further zeros? In both Frisch's evaluation as well as Zambelli's we are told that we are to be looking at zeros of the form: $\rho = -\beta + \alpha i$. In Frisch 1933 there is a slightly obscure reference as to how these complex zeros are calculated and Frisch rightly finds that these zeros follow a distinct 'trend'. He proceeds at tabulating three of them against their respective ρs and then he carries on with the paper. In Zambelli 1991 absolutely no method for finding these zeros is given and instead Zambelli criticises Frisch for offering no theoretical or empirical reasons for setting the ordered trends the way he does. Furthermore, Zambelli claims that he has 'stimulated' Frisch's examples and

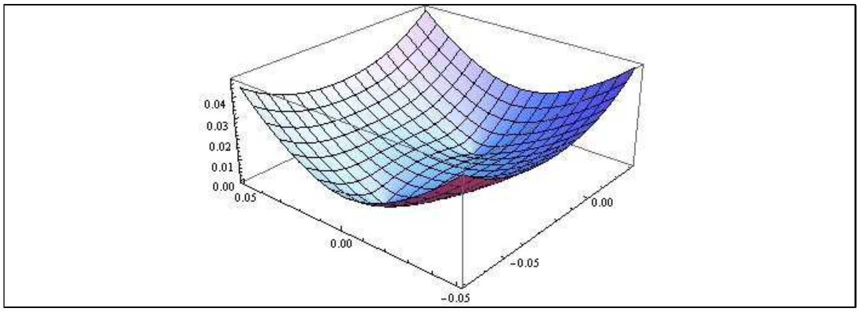

Figure 21: $f_*(\rho)$

recomputed the roots. One must make the criticism that due to the difficult nature of the algorithm to find these zeros, an explanation as to how these were found should have been given. It is worth noticing that during the course of the writing of this thesis a huge amount of zeros were indeed found by evaluating further parts of the complex plane, and not only are there indeed 'countable' and 'infinite' zeroes, but we can also see a pattern in how these zeros arise.

To find these zeros, we must remember we are seeking zeros for the following characteristic equation:

$$f(\rho) = \varepsilon\rho^2 + \lambda(\varepsilon r + s\mu)\rho + \lambda sm - (\lambda s\mu\rho + \lambda sm)e^{-\varepsilon\rho}$$

It was found, after an evaluation using *Mathematica 7*'s FindRoot function, that the complex numbers that are susceptible or suspect of being zeros are those whose real part is zero or negative given that it is easy to show that no complex number with a positive real part could be a zero of the above characteristic equation. It was thus that we used the algorithm FindRoot in *Mathematica 7* for complex numbers ρ of the type $n\frac{\pi}{2}$, where $n = 0, 1, 2, 3, ...$ It was then found that one of Frisch's observations should have been given much attention, namely that if a certain $\rho = -\beta + \alpha i$ is a solution to the characteristic equation (37), then $\rho = -\beta - \alpha i$ also is a solution. Looking at figures 22 and 23 we see results of some of our evaluations.

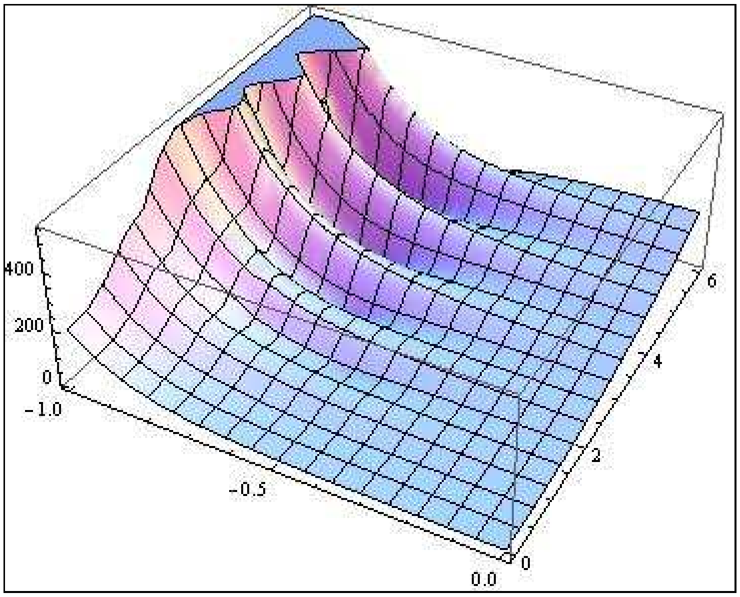

Figure 22: $f(\rho)$ for $x, -1, 0, y, 0, 6$

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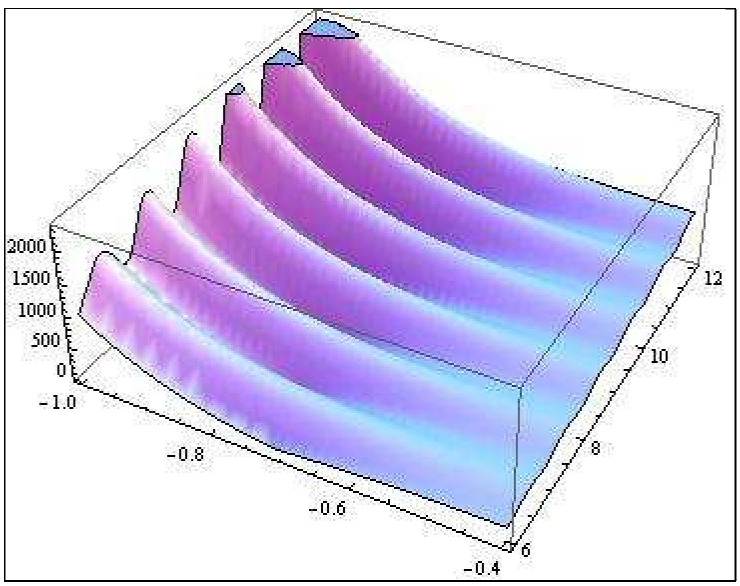

Figure 23: $f(\rho)$ for $x, -1, -0.4, y, 6, 12$

We must remember that the solution we are looking for is namely the solution of the equation:

$$(-1 + e^{\varepsilon\rho} + 2\varepsilon e^{\varepsilon\rho}\rho) = 0$$

This equation can be expanded and by replacing $\rho = -\beta + i\alpha$, we obtain the following:

$$Cos(\varepsilon\beta) - 2\varepsilon\beta Cos(\varepsilon\alpha) + 2\varepsilon\alpha i Cos(\varepsilon\alpha) + iSin(\varepsilon\alpha) - 2\varepsilon\beta i Sin(\varepsilon\beta) - 2\varepsilon\alpha Sin(\varepsilon\alpha) = e^{\varepsilon\beta}$$

This equation can be decomposed into two parts after a simple transformation and some algebra: The real part and the imaginary part. The imaginary part of the equation must equal zero, thus giving us the equation:

$$\varepsilon\beta = \varepsilon\alpha\frac{Cos(\varepsilon\alpha)}{Sin(\varepsilon\alpha)} + 1/2$$

The second equation, after replacing $\varepsilon\beta$ from the above equation is:

$$Cos(\varepsilon\alpha) - 2\varepsilon(\varepsilon\alpha\frac{Cos(\varepsilon\alpha)}{Sin(\varepsilon\alpha)} + 1/2) - 2\varepsilon\alpha Sin(\varepsilon\alpha) = e^{\varepsilon(\varepsilon\alpha\frac{Cos(\varepsilon\alpha)}{Sin(\varepsilon\alpha)}+1/2)}$$

For the solution of the above equations we need a numerical algorithm that can slowly look at the different solutions. A certain pattern emerges, and Frisch pointed at this pattern which can be observed in the plots.The 3D plots in figures (22) and (23) show us the pattern Frisch spoke of. What is important here is to remind the reader that Frisch in PPIP said that: A good guidance in the search for roots is the fact that the solutions in α are approximately the minimum points of the function" (Frisch, 1933):

$$\frac{sin(\varepsilon\alpha)}{\varepsilon\alpha}$$

In fact, while rummaging through Frisch's own notes on PPIP [2] it was found that Frisch had a very accurate idea of the pattern that the zeores would have in the complex plane. In fact, by observing Frisch's own notes in figure 24 we can see he was very much on target at understanding this pattern. It becomes clear then that Frisch was aware that a great many zeroes were to be

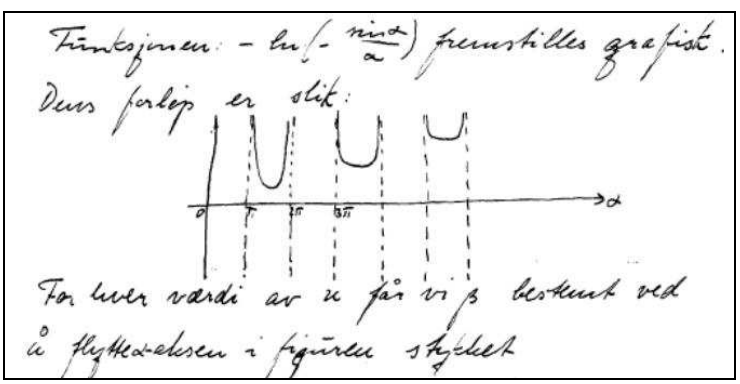
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Figure 24: Frisch's description of the pattern for the zeroes

[2]This thesis as mentioned before is part of a research project to re-organize and file The Frisch Archive at the University of Oslo. During this process many helpful manuscripts were found that would contribute to the writing of this thesis

found and it is certain from the notes found in the Frisch Archive that he indeed intended to add much more to the solution. The difficulty involved in finding these zeroes and calculating them compounded the problem by making it perhaps inefficient to produce all possible zeroes or solutions. He was pressed for time to turn in the article for the Cassel Festschrift. It is however strange that Frisch did not make either an attempt at producing plots for the aggregate solutions nor for the provision of the inclusion of the past histories. It can be speculated that in hopes of making a point regarding the methodological approach to economic problems (an approach he very strongly advocated) he chose to not present the most complicated and ellaborate parts of the model. Whether this represents an attempt to hide results that would stand to be questioned or simply that he considered the more complete and telling aspects of the individual harmonics to be indeed the evolutions he was looking for will remain clouded in mystery.

Upon inspection of Zambelli's argument that all 'zeroes' ought to be included and that a more complete solution should be presented, we present here the addition of two further zeroes as added and aggregated to the original plots. The zeroes used are of course those found using *Mathematica 7*. These trends are thus:

$$x_4 = 0.100891e^{-0.681943t} sin(4.95582t) \tag{46}$$

$$y_4 = 5.04046e^{-0.681943t} sin(1.44397 + 4.95582t) \tag{47}$$

$$z_4 = -10.0809e^{-0.681943t} sin(1.44437 + 4.95582t) \tag{48}$$

$$x_5 = 0.083278e^{-0.713801t} sin(6.00395t) \tag{49}$$

$$y_5 = 5.03044e^{-0.713801t} sin(1.46068 + 6.00395t) \tag{50}$$

$$z_5 = -10.0609e^{-0.713801t} sin(1.46066 + 6.00395t) \tag{51}$$

The plots below provide an overlook of the resulting aggregate evolutions for x y and z when adding first the fourth zero and then the fifth.

As we can see the oscillatory nature of the evolutions of the variables x, y and z cannot be exactly discarded with complete conviction, for adding trends changes their appearence and it would be a matter of subjective judgement just how many trends are sufficient. Frisch specifically picks three trends but fails to compose them in an aggregate plot thus revealing the weakness of the model. This may have been deliberate but this weakness alone cannot

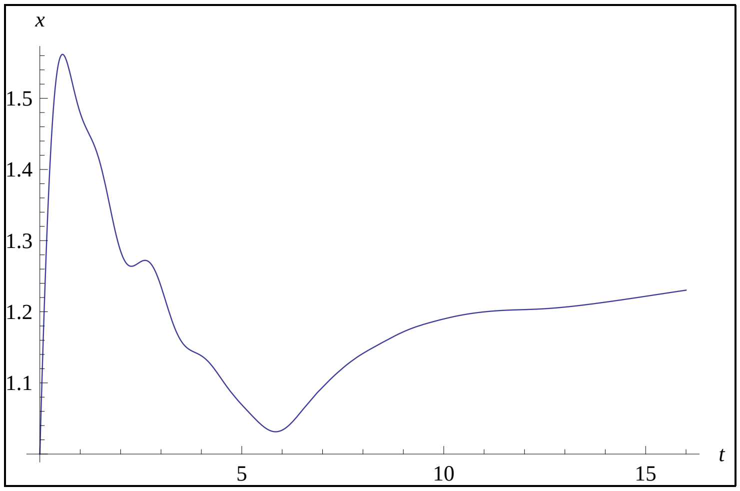

Figure 25: aggregate x with 4 trends

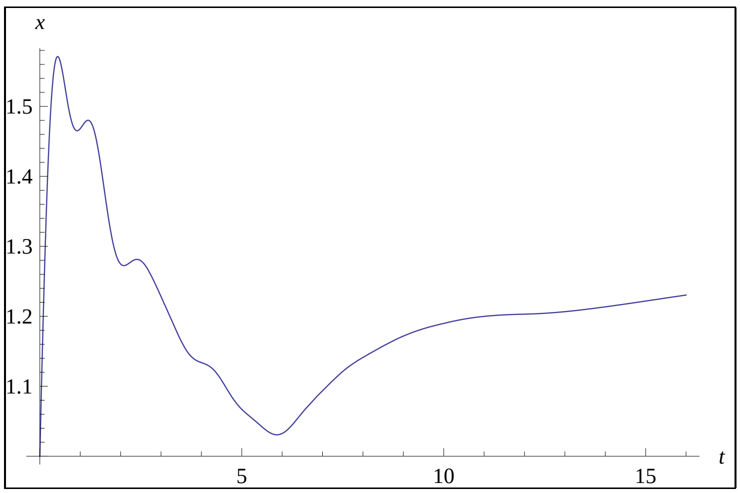

Figure 26: aggregate x with 5 trends

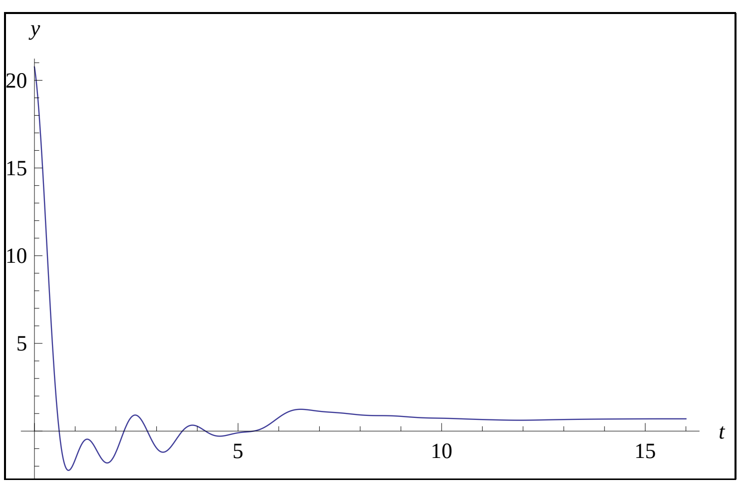

Figure 27: aggregate y with 4 trends

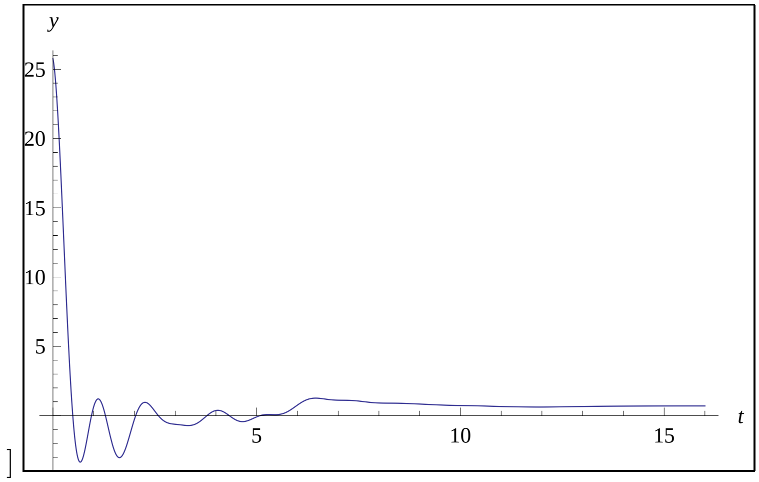

Figure 28: aggregate y with 5 trends

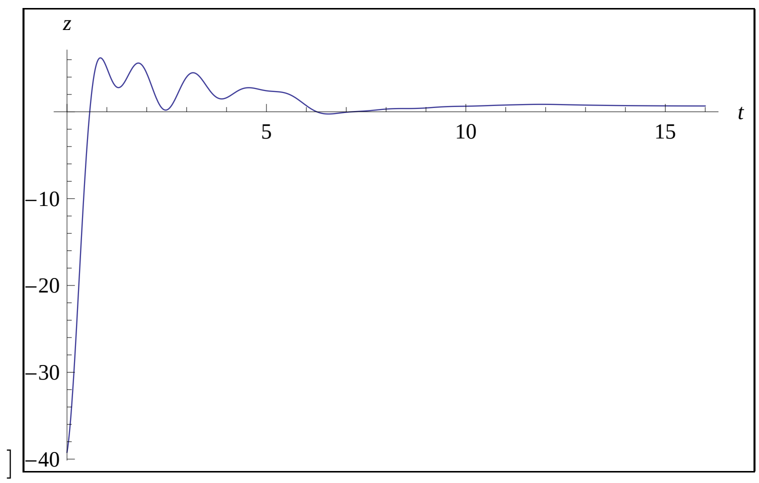

Figure 29: aggregate z with 4 trends

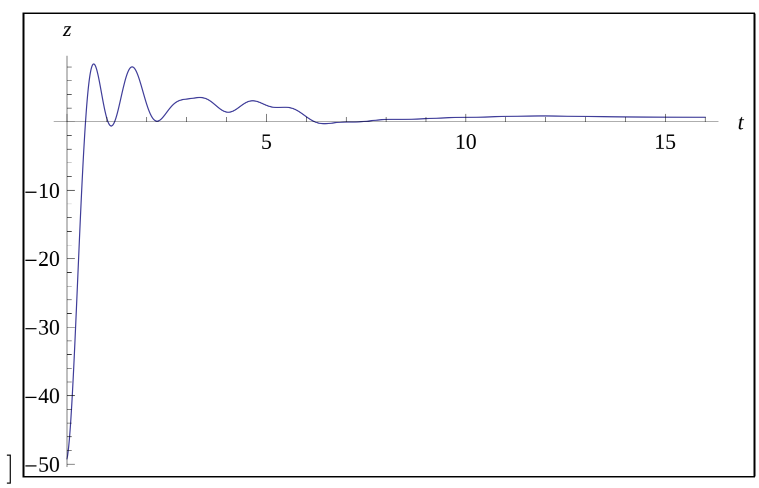

Figure 30: aggregate z with 5 trends

be sufficient to discard the model as non-oscillatory. Not only is there further and better analysis required for the judgement of the *propagation mechanism* (even further than Zambelli's algorithms) but the entire other half of the model regarding the *impulse mechanism* ought to be evaluated before judgement is passed.

5.2 Impulses

One of the most important aspects in Frisch's PPIP is one that was simply not included in the *Cassel Festschrift*. Not only was it not included, it hasn't been evaluated at all by anyone who has attempted an analysis of Frisch's model for business cycles. Klein (1998) provides a very short explanation for the *source* of the idea as to how to generate cycles through the use of the summation of random shocks. Klein (1998) states that this idea of these cumulating shocks were inspired by the works of Eugen Slutsky (1937) and G.U. Yule (1927). Klein (1998) explains the influence Slutsky and Yule exerted on Frisch by pointing at Frisch's use of a combination of these 'cumulative shock' ideas and Wicksell's propagation and impulse effects, both of which make their way into Frisch's model. Klein (1998) proceeds with a description of these Slutskian and Yulean shocks but does not attempt an application of them to Frisch's SYSTEM 1 and therefore leaves us hungry for more.

Frisch spent a conisderable amount of energy and time developing a model that would consist of two essential structures: a propagation mechanism and an impulse mechanism; hence the name *Propagation problems and Impulse problems*. What many have failed to see is that Frisch's argument regarding his model is incomplete! Frisch never presented a formal explanation for the sources of the impulse mechanisms and kept to a vague description of where impulses may come from in other phenomena. The phenomena he explores are classical problems in classical mechanics, namely: simple harmonic motion in a pendulum. In the fifth section of PPIP we find his description for *Erratic Shocks as a Source of Energy in maintaining Oscillations*. It is here that Frisch procures a description of the sources for impulses for an oscillating pendulum.

It is important to point out that this aspect of the impulse mechanism was raised in a remark done by Bjorn Thalberg at the end of Zambelli (1992). Thalberg proposed that in order to 'analyse this question generally, one may start with Frisch's suggestion regarding the cumulation effects of the shocks'. this cumulation is of course an effect proposed by Frisch and one we will study here, partly as a response to Thalberg in Zambelli (1992) and a general interest in attempting to come closer to a judgement on the nature of the model.

Frisch commences his analysis of this cumulative behavior of the shocks by considering an oscillating pendulum whose movement is hampered by friction. For y indicating the deviation of the pendulum from its vertical position, he explains the equation describing the movement of the pendulum:

$$\ddot{y} + 2\beta\dot{y} + (\alpha^2 + \beta^2)y = 0 \tag{52}$$

Here α and β are constants where β is a constant expressing the strength of the friction. Frisch states rightly that it is easily shown that the solution to this equation (51) is a function of the form:

$$He^{-\beta t}sin(\phi + \alpha t) \tag{53}$$

Here, the amplitude H and the phase ϕ are determined by the initial conditions. Frisch continues with the idea that writing the solution somewhat differently would help the reader see how the initial solutions determine the curve. He re-constructs the solution in the following form:

$$y(t) = P(t - t_0)y_0 + Q(t - t_0)\dot{y}_0 \tag{54}$$

Here, given that y_0 and $\dot{y}_0$ are the values of y and $\dot{y}$ respectively at the point in time when $t = t_0$ we find that $P(\tau)$ and $Q(\tau)$ are two functions independent of the initial conditions. These functions are given as:

$$P(\tau) = \frac{\sqrt{\alpha^2 + \beta^2}}{\alpha}e^{-\beta\tau}sin(v + \alpha\tau) \tag{55}$$

$$Q(\tau) = \frac{1}{\alpha}e^{-\beta\tau}sin(\alpha\tau) \tag{56}$$

where

$$sin(v) = \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}} \qquad cos(v) = \frac{\beta}{\sqrt{\alpha^2 + \beta^2}} \tag{57}$$

With this model thus described, Frisch provides a short analysis of what whould happen should the system be hit by certain shocks.

> Suppose that the pendulum starts... at the point in time t_0 and is hit at the points in time $t_1, t_2, ..., t_n$ by shocks which may be directed in either the positive or the negative sense and that may have arbitrary strengths. Let y_k and $\dot{y}_k$ be the ordinate and the velocity of the ordinate immediately before it is hit by the shock

> numer k. The ordinate y_k is not changed by the shock, but the velocity is suddenly changed from y_k to $\dot{y}_k + e_k$, where e_k is the strength of the shock; mechanically expressed it is the quantity of motion divided by the mass of the pendulum.

Frisch considers separately the effects produced by the two terms $\dot{y}_k$ and e_k. He offers the understanding that we can consider $\dot{y}_k$ and e_k as two independent contributions to the latter ordinates of the variable. He describes this resulting movement as:

> .. the fact of the shock may simply be represented by letting the original pendulummove on undisturbed by letting a *new* pendulum start at the point of time t_k with an ordinate equal to zero and a velocity equal to e_k. This argument may be applied to all the points of time. We simply have to start in each of the points of time $t_1, t_2, ..., t_n$ a new pendulum with an ordinate equal to zero and a velocity equal to the strength of the strength of the shock occurring at that moment, and then let all these pendula continue their undisturbed motion into the future.

In other words, the ordinate will simply be:

$$y(t) = P(t - t_0)y_0 + Q(t - t_0)\dot{y}_0 + \sum_{k=1}^{n} Q(t - t_k)e_k \tag{58}$$

At certain points very far away from the initial point t_0 and given that there is actual dampening (positive β) then the influence of the inital conditions will be negligible and the ordinate will be:

$$y(t) = \sum_{k=1}^{n} Q(t - t_k)e_k \tag{59}$$

This suggests that the ordinate of the pendulum $y(t)$ at a given moment is simply the *cumulation* of the effects of the impulses, this cumulation being given according to a system of weights. These weights are simply the shape of the function $Q(\tau)$. That is to say:

> *the system of weights in the operator will simply be given by the shape of the time curve that would have been the solution of the determinate dynamic system in case the movement had been allowed to go undisturbed.*

It is here that Frisch ends his argumentation as to the possible structure of the impulses or shocks. He adds further notes regarding a problem for a pendulum to which a constant current of schumpeterian innovations is added (modelled by water falling from a tank and applying torque to the pendulum). The problem presented is analogous to the double pendula system which is known that for certain initial phases can produce chaotic behavior which is quite similar in appearance to the fluctuations of the stock market and to business cycles in general. But that is essentially the end of the discussion. Frisch does not humour us with how the impulse structure can be included in his model leaving it thereafter essentially incomplete on this account. However, given that his description can be worked into the SYSTEM 1 proposed by Frisch we proceed now with the inclusion of a description of the shocks.

Firstly, we remember the set of equations (23) which is a solution for consumption for a given trend j:

$$x_j(t) = A_j e^{-\beta_j t} sin(\phi_j + \alpha_j t) \tag{60}$$

Which is an equation of the same form as equation (52):

$$He^{-\beta_j t} sin(\phi + \alpha t) \tag{61}$$

If we remember from earlier discussion, our equation (52) can be decomposed into two independent equations $P(\tau)$ and $Q(\tau)$ where $Q(\tau)$ has been defined above. Taking into account points in time far away from $t = t_0$ and acknowledging for all trends (now noted as i) we can express our solution for consumption as:

$$x(t) = \sum_{i=1}^{m} \sum_{k=1}^{n} R_i e^{-\beta_j (t-t_k)} sin(\omega_i + \alpha_i (t - t_k)) e_k \tag{62}$$

Where i indicates the trend and k the shock (impulse) number. Variables y and z have identical solutions. But it is perhaps made clearer by placing the sum of the shocks first, namely:

$$x(t) = \sum_{k=1}^{n} \sum_{i=1}^{m} R_i e^{-\beta_j (t-t_k)} sin(\omega_i + \alpha_i (t - t_k)) e_k \tag{63}$$

At this point it is worth mentioning that during the course of the writing of this paper this solution was proposed to be the final solution that would generate the oscillations that Frisch allegedly never came to create because of the absence of a formal analysis of the *Impulse Mechanism* or rather, an analysis of the shocks. At this point in time, before plots had been evaluated for this final solution, the author of this thesis, as part of his work in organizing and archiving the Frisch Archive at the University of Oslo stumbled upon a series of lecture notes prepared by Frisch in the fall of 1933 and the Spring of 1934 (Therefore after the publication of PPIP) and within these lecture notes, the exact same solution proposed here is found as Frisch's solution to the shocks problem. However flattering it may be for the author, one must agree that it is rather surprising to see a contribution to Frisch's work being trumped by Frisch himself from beyond the grave.

Given that the solutions are identical, and for the ease of the reader, Frisch's newly created notation will not be used and instead the solution will be presented with the same notation as we have used thus far. It is also worth adding that in his solutiom Frisch included the first two harmonics in the sum, this is a step will follow as well.

Including the first two trends in equation (62) we get:

$$x(t) = \sum_{k=1}^{n}(\sum_{\tau=1}^{n}(Q_{1x}(t-\tau)e_1^{\tau} + Q_{2x}(t-\tau)e_2^{\tau})) \tag{64}$$

Now, upon plotting these results, much care must be made when it comes to the distribution of the shock term e_k for it was found that depending on how these shocks come about the model may 'oscillate' in a given manner. In particular, attention was given to Frisch's proposed method of shocks in his lecture notes (which are unfortunately not available save at the Archive). Frisch writes that the shocks were 'two erratic series' accumulated through the ending ciphers of the Norwegian lottery. In this attempt these shocks were modeled by using random integers being drawn and then simply accumulated with their given weights. These weights, we must remember, are given by the shape of the time curve as it is given assuming the movement was allowed to go undisturbed. Three different shock distributions are illustrated for the purpose of the analysis.

Close analysis of these three figures reveals an important result. However possible it is for thse systems of weights to generate oscillations for the evolution of consumer demand x, the model requires further specification as to the nature of the shocks which Frisch mentioned 'does not interest us at the moment' (Frisch, 1933). Of course he dismissed the importance they had in PPIP due to the fact that he was not after solving the problem in this particular fashion just yet, instead, Frisch was just making a proposal as to how they could be solved. However, in his lecture notes one finds an incredible amount of random shocks observed, and especially a very strange time-span is plotted (the only plot one finds in his notes). Frisch picks the time-span of 35 to 60 years, possibly because in his model this is the section that most looked like the

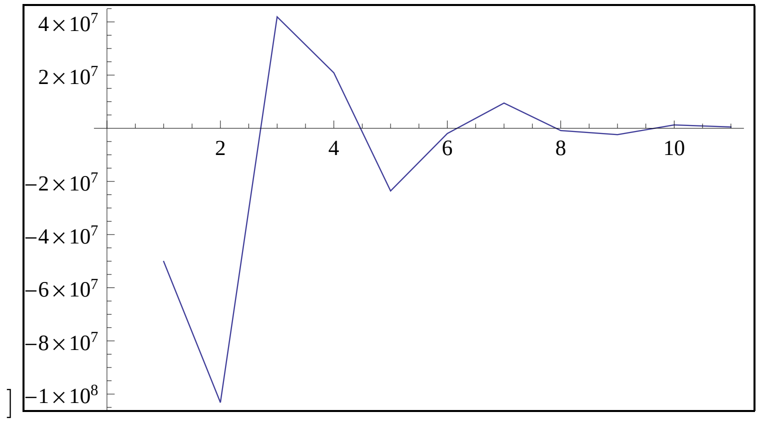

Figure 31: random shocks on aggregate x for time span of 10 to 20 years

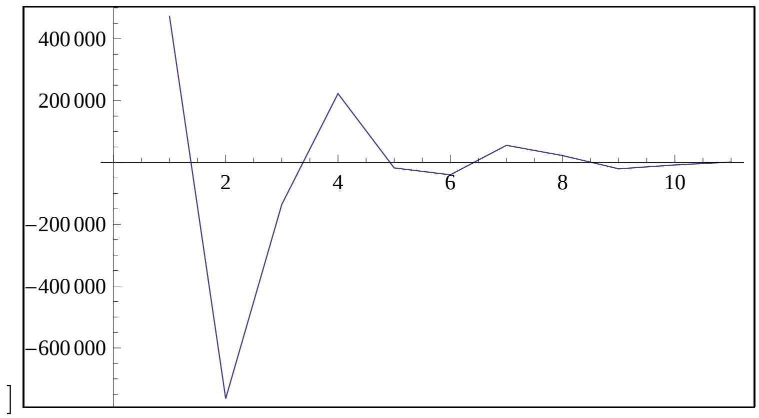

Figure 32: random shocks on aggregate x for time span of 20 to 30 years

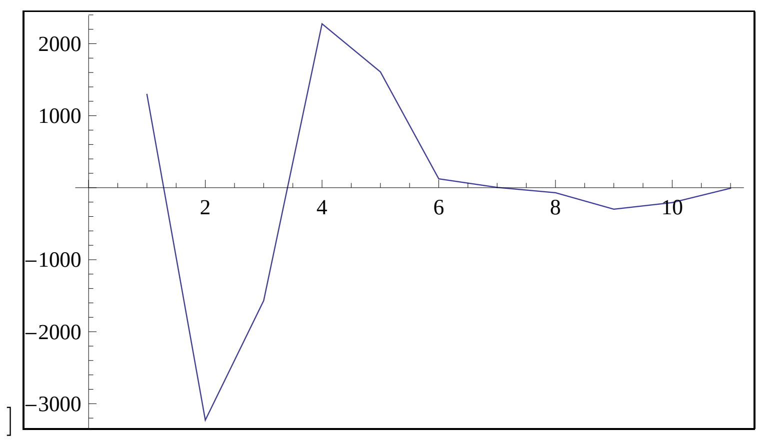

Figure 33: random shocks on aggregate x for time span of 30 to 40 years

behavior of an economy. Despite this, we must remember that upon the use of our model here we have lokked at a more standard time-span, where we have reduced the evolution of consumer demand $x(t)$ to the following equation (where only the first two trends are added):

$$x(t) = \sum_{\tau=1}^{n} (Q_{1x}(t-\tau)e_1^{\tau} + Q_{2x}(t-\tau)e_2^{\tau}) \tag{65}$$

Here, Q_{1x} represents the function $Q(\tau)$ for the first trend in x and Q_{2x} represents the second trend in x. Using this exact expression above and replacing for the $Q(\tau)$ produces de evolutions shown above but under the conditions of a certain species of shocks. The shocks used in this case were four different shocks over the course of 40 years. All Shocks are equidistant and they reveal a powerful picture of the nature of the model. Every figure shows a different time-span, and although the oscillations do seem to dissipate towards zero, closer examination of the next time span reveals that the oscillations do not disappear, quite the contrary, they remain as strong as ever! Despite this, the average *amplitude* of the evolution is **constantly decreasing**. That Frisch obtained such a powerful picture in the figure below must have been the consequence of using sufficient (and differently-distributed) shocks and looking at the evolution at a very particular section of it. Our three figures shown here point towards the bigger picture. The evolutions maintain an oscillatory nature, but their amplitude decreases constantly. The entire model containing the shocks *does oscillate* but at a constantly decreasing amplitude. In Zambelli, this aspect was never taken into account and instead a very quick judgement of SYSTEM 1 was given without a full disclosure of all the trends. We have now here observed that upon Frisch's own suggestion (and indeed, as would later prove: his own action!) the cumulation of the effects of the shocks according to a system of weights determined by the shape of the time curve in the system would create a constantly swinging system.

The strong conclusion one can draw here is that to obtain a fuller picture of the model, deeper investigation as to other possible characteristics for the shocks could be in order. However, it may well be hard to escape the strength of the shape of the time curve and the symptom of constantly decreasing amplitudes may be endemic. This evaluation however goes beyond the scope, possibility and time resources of this paper.

6 Conclusions

In writing *Propagation Problems and Impulse Problems*, Ragnar Frisch attempted to make a point regarding the appropiate approach an economist should make regarding methodology

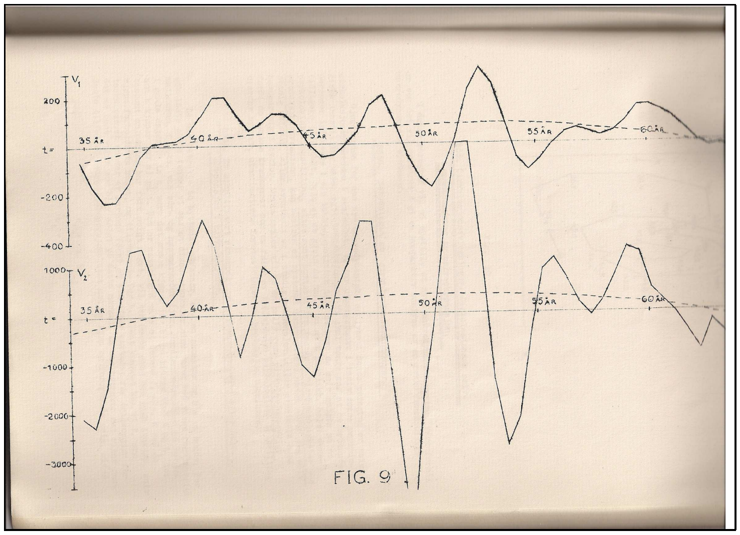

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Figure 34: Frisch's evolution for x in span of 35-60 years and over 120 shocks

when approaching an economic problem. Frisch understood the importance of the use of mathematics and how lightly one must tread when deriving analysis purely on mathematical grounds or deducing economic theory from conjecture only. PPIP was an attempt at solidifying a macrodynamical approach based on a system of deterministic equations put together after a draft of a simplified economy. The resulting system seemed in PPIP to be solid in terms of generating oscillations and the important distinction between a *propagation mechanism* and an *impulse mechanism* survive unscathed. Despite his attempts, Frisch overlooked the inclusion of all the trends he had obtained and therefore could not see how the resulting aggregates of his variables x, y and z did not exactly produce the oscillatory model he was after. Even further, Frisch missed the importance of the initial conditions he established for the finding of the trends in equations (28) through (31); and given another set of initial conditions, particularly applied to the system in its equilibrium stages, we find the non-oscillatory nature of his *propagation mechanism* to be quite evident.

Stefano Zambelli rightly points towards Frisch's flaw and extends on it through the use of his numerical integration algorithms where his conclusions attempt to portray the model as a non-oscillatory model. In his conclusions, Zambelli correctly points out that the *propagation mechanism* alone can account for only oscillatory behavior in y (given certain conditions for λ and ε and the trends $x_0(0)$ and $y_0(0)$) and that it does not alone account for oscillatory behavior in x and z. Despite this, upon evaluation of Zambelli's methods it becomes clear that his numerical algorithms, in particular the Euler approximating procedure rely too heavily on an adequate time-step size h which Zambelli fails to explain. This step size could change the outlook of the evolutions and given Zambelli's zeal to prove his point, one could only wish he had gone through more attempts at evaluating the consequences of different step sizes h. It is still not clear whether one can judge the aggregated evolutions of the three variables with further trends included such as the ones in figures 25 to 30 as non-oscillatory and it may just be a matter of subjectiveness at this point, given that we may not produce plots for an infinite number of trends. They are however, contrary to Zambelli (1991) not likely to become completely flat, since having added 2 more trends or zeroes only seemed to accentuate fluctuating behavior, not dampen it further. A key factor is however, that applying shocks to the variables x and y at their equilibrium points in the evolutions does generate monotonous returns to equilibrium once more, confirming Zambelli's remarks. It must be remembered however, that these shocks are unique, for they represent unlikely shocks in an economy and were, as Frisch expresses in PPIP, not the likely source of shocks. Frisch states the sources of the shocks to come from the derivatives of the variables, or their changes rather, namely: $\dot{x}$ and $\dot{y}$ and that these shocks have an overall cumulative effect.

The most important aspect of the evaluation made in this paper is above all the realization that both Frisch's analysis as well as Zambelli's criticism were incomplete. To fully understand and value the model proposed by Frisch one needs to inspect the *impulse mechanism* as well before one can pass judgement on its nature. In the numeric research done here we have shown that the simplest set of four shocks, added through a system of weights resulting from Frisch's own understanding of the propagation of shocks in simple harmonic motion in classical mechanics can very well generate oscillatory behavior, albeit one in which the amplitudes are constantly decreasing. This inspection, as well as viewing Frisch's example in his own lecture notes, can make one suspect that a more thorough description of the shocks could easily generate pronounced oscillations that would be maintained producing that mechanical behavior that has the appearence of chaotic evolutions one so often sees in the data obtained from economic surveys of the economy. This would of course be an improvement on a model that, as shown here, can already on its own generate oscillatory behavior despite its reducing amplitudes. Indeed we may now say 'E pur si muove', or in other and better words: the rocking horse does rock.

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