Correspondence between Ragnar Frisch and Alexander Craig Aitken

Aitken, Alexander C. to Frisch, Ragnar, 06.09.1930

2 Sycamore Terrace
Corstorphine
Midlothian
Scotland
Sept. 6, 1930

Professor Ragnar Frisch.

My dear Professor Frisch,

Through the kindness of my friends Professor E. J. Whittaker and Dr. G. J. Lidstone I have lately had the great pleasure of reading papers by yourself, one on the use of the calculus of matrices in statistical problems of correlation and dispersion, the other "on approximation to a certain type of integrals", $\int f(x)g(x)dx$. Being greatly interested in questions of algebra and statistics, I should feel it a privilege to have any offprints you may be able to spare. I enclose some small notes of my own, which I hope you will accept.

Yours very truly,
A. C. Aitken (sign.)

Frisch, Ragnar to Aitken, Alexander C., 26.09.1930

September 26, 1930

Dr. A. C. Aitken,
2 Sycamore Terrace,
Corstorphine,
Midlothian, Scotland

My dear Doctor Aitken:

Thank you very much for the reprints of your papers which you recently sent to me. Your use of matrices interests me greatly. Some of the results which you have obtained seem to me to be susceptible of application to certain problems I have encountered in the analysis of statistical time series, on the basis of the method I have developed in my paper, "Changing Harmonics and Other General Types of Components in Empirical Series", I refer particularly to the solution of the principal equation (page 228).

If there could be developed a practical method of tracing the roots of this equation as time series, very great progress could be made toward the further application of my time series method to the more complicated cases where more than two essential components are present. I hope that you will give this problem your closest consideration. With the apparatus and technic[!] which you have at your disposal in your knowledge of your theory of matrices and the theory of approximation to the roots of algebraic equations, you should be in a position to obtain very significant results along this line. I hope that this problem will interest you and that you will let me know your reaction to it.

In accordance with your request, I am sending you my paper "On Approximation to a Certain Type of Integrals" as well as my paper on "Correlation and Scatter in Statistical Variables". Since you are interested in mathematical statistics I would particularly call you attention to my criticism of partial correlation and my attempt at developing the new method of linear regression which is studied in the latter paper. You might also be interested in the problem of invariance of linear regressions, which is discussed in this paper.

Very sincerely yours,

Ragnar Frisch
Visiting Professor,

Yale University.

Home address:
100 Howe Street,
New Haven, Conn.

Aitken, Alexander C. to Frisch, Ragnar, 01.11.1930

2 Sycamore Terrace
Corstorphine
Midlothian
SCOTLAND
November 1, 1930

My dear Professor Frisch,

I was very glad to receive those reprints of your papers, and am engaged in reading them with the utmost interest. I find that your ideas and projects, though doubtless arrived at along lines of different interest and viewpoint from my own, are extraordinarily related, and congenial to me; and I wish to congratulate you on the success with which you have carried them through to valuable conclusions.

It occurred to me in reading your work that there is often an underlying determinantal theorem which I myself have used a great deal, a theorem originally due to F. Schweins, 1825, and referred to in Sir Thomas Muir's "History of Determinants", Vol. I, Part I, p. 172. I myself prefer to adopt a rather different enunciation and proof from Muir, and would put the matter thus:

A very large number of problems involve finding the coefficients of a polynomial by means of simultaneous linear equations, and so involve calculating the quotients of determinants of the same order differing from each other in one column only. A very desirable expansion would be that in which the annexing of an additional last row and column to the determinants, the same in each, would cause merely an additional term in the expansion. Such an expansion is a Scheins expansion. We shall prove for example that

$$Q_4 \equiv \frac{|\theta_1 b_2 c_3 d_4|}{|a_1 b_2 c_3 d_4|} = \frac{\theta_1}{a_1} + \frac{b_1 \cdot |\theta_1 a_2|}{a_1 \cdot |a_1 b_2|} + \frac{|b_1 c_2||\theta_1 a_2 b_3|}{|a_1 b_2||a_1 b_2 c_3|} + \frac{|b_1 c_2 d_3||\theta_1 a_2 b_3 c_4|}{|a_1 b_2 c_3||a_1 b_2 c_3 d_4|}$$

Now

$$Q_4 - Q_3 = \begin{vmatrix} |\theta_1 b_2 c_3 d_4| & |\theta_1 b_2 c_3| \\ |a_1 b_2 c_3 d_4| & |a_1 b_2 c_3| \end{vmatrix} \div \{|a_1 b_2 c_3||a_1 b_2 c_3 d_4|\}.$$

But the compound determinant on the right is the "extensional", through the "extension" $(b_2 c_3)$, of $\begin{vmatrix} |\theta_1 d_4| & \theta_1 \\ |a_1 d_4| & a_1 \end{vmatrix}$, or $|\theta_1 a_4| \cdot d_1$. Hence, by the theorem of extensionals, we have its value as

$$|\theta_1 b_2 c_3 a_4||d_1 b_2 c_3|,$$

or, by compensating interchanges of columns, $|\theta_1 a_2 b_3 c_4||b_1 c_2 d_3|$. Hence

$$Q_4 - Q_3 = \frac{|\theta_1 a_2 b_3 c_4||b_1 c_2 d_3|.}{|a_1 b_2 c_3||a_1 b_2 c_3 d_4|},$$

Which with the analogous results for $Q_3 - Q_2$, $Q_2 - Q_1$, Q_1, gives by addition the required result of Schweins.

When I was first working under Professor Whittaker some years ago, he pointed out to us in his bi-weekly lectures that the expansion of a polynomial, fitted by least squares, as a series of Tchebychef's orthogonal polynomials was really a Schweins' expansion, and that the orthogonal properties were intuitively evident from the determinantal expression, e.g. for equidistant intervals, Tchebychef's polynomial was of the form

$$T_r(x) = c_r \begin{vmatrix} 1 & x & x^2 \cdots\cdots\cdots x^r \\ S_0 & S_1 & S_2 \cdots\cdots\cdots S_r \\ S_1 & S_2 & S_3 \cdots\cdots S_{r+1} \\ & & \cdots \\ & & \cdots \\ S_{r-1} & S_r & S_{r+1} \cdots\cdots S_{2r-1} \end{vmatrix}, \quad \text{where } S_r = \sum_{m=1}^{n} m^r,$$

and for non-equidistant intervals and weighted observations there were orthogonal polynomials of similar form, S_r being then replaced by the weighted moments M_r. It was commented on that Newton's interpolation formula of divided differences was a particular simple type of Schwein's expansion, the polynomials being "alternant" determinants,

$$N_r(x) = c_r \begin{vmatrix} 1 & x & x^2 \cdots x^r \\ 1 & a & a^2 \cdots a^r \\ 1 & b & b^2 \cdots b^r \\ & & \cdots \\ & & \cdots \end{vmatrix},$$

whereas the ordinary expansion of the numerator determinant of the quotient in question according to elements of the first column simply yielded Lagrange's formula. Recurrence-relations, properties like Rodrigues' in regard to the Legendre polynomials and so on flowed easily from determinantal algebra.

It had been my wish to generalize all this to the case of expansions by orthogonal polynomials under very general operations which would include summation (as for the Tchebychef polynomials) and integration (as for the Legendre polynomials) as very special cases, and even to show that, now that we have highly efficient multiplication machines at our disposal, the calculation of determinants is far from being the laborious affair it has hitherto been considered; - indeed recently, as the result of a challenge, I evaluated a determinant of the fifth order, with elements to 4 digits, on an "Archimedes" machine in 5 minutes, by a special reduction which obtained en route every minor of the determinant formed from adjacent rows and columns; the value of this for calculations of correlations you can see at once. Unfortunately my ambitions were disposed of by a severe nervous breakdown and long protracted insomnia in 1927, and only after three years is the former energy beginning to assert itself again, and I am not yet able to undertake any work which calls for "la longue haleine".

Your paper on "Changing Harmonics, etc" I find in some ways most interesting of all, and very specially §§ 2 to 4 of it. It suggests to me one or two problems in regard to the solving of algebraic equations which are in determinant shape, and I think I see the proper line of approach.

I have recently found, but am as yet unable to demonstrate, the rules for finding the canonical form (Latent roots and invariant-factor formulation) of compound matrices and all these other kinds which arise in the theory of the invariants on n-ary forms and families of multilinear forms, supposing the canonical form of the original matrices given. It is curious that one application of this theory, in compound matrices, is in the practical method of root-squaring for algebraic equations, for the difficult case of complex roots equal in modulus.

I am,

Yours sincerely,

A. C. Aitken (sign.)

P.S. Sylvester's Theorem, which you have occasionally used, is in essence an "extensional theorem", and from this point of view intuitive.

Frisch, Ragnar to Aitken, Alexander C., 20.11.1930

November 20, 1930

Dr. A. C. Aitken,
2 Sycamore Terrace,
Corstorphine,
Midlothian, Scotland

My dear Dr. Aitken:

Thank you very much for your letter of November 1. I find that the points you have raised are very interesting and have immediate connection with my own work on time series. I am sorry that so far I have not yet had an opportunity to study very carefully the various points you raise, but I expect to be able to do so shortly. I will then take the matter up again and you will hear from me.

You may be interested to know that I have recently been carrying on quite extensive numerical work along the lines of my time series method. A number of friends and students in my class this semester at Yale have volunteered to apply this method, namely the following:

Dr. Brouwer of Yale Observatory: "Uranus Longitude Residuals", yearly data from 1836 to date,

Professor C. W. Cobb of Amherst College: "Freight car Loadings in the United States", monthly data from October 1917 to date,

Professor Thompson of the University of Allahabad, India; at present staying at Yale: "Wheat Prices in Europe", yearly data from 1536 to date,

Mr, H. M. Cleland: "Pig Iron Production in the United States", monthly data from 1885 to data,

Mr. H. Edmiston: "United States Bureau of Labor Statistics Index to Wholesale Prices", monthly data from 1900 to data,

Mr. J. R. Wolf: "Rainfall at Boston, Massachusetts", yearly data from 1818 to date.

Furthermore, Professor Joseph Schumpeter (at present Visiting Professor at Harvard) is going to use the method in his study of the "Longer Fluctuations in the General Price Level in the United States".

May I again urge your closest attention to the problem of developing an approximate and shorthand method for solving the key equation? A practical solution of this problem would be of great importance for the analysis of time series.

With all best wishes, I am

Sincerely yours,

Ragnar Frisch
Visiting Professor of Economics
Yale University

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