CHAPTER 3

The Contributions of Ragnar Frisch to Economics and Econometrics

John S. Chipman

Ragnar Frisch opened his 1926 article "On a Problem in Pure Economics" with the following statement:

> Intermediate between mathematics, statistics, and economics, we find a new discipline which, for lack of a better name, may be called *econometrics*.
>
> Econometrics has as its aim to subject abstract laws of theoretical political economy or "pure" economics to experimental and numerical verification, and thus to turn pure economics, as far as possible, into a science in the strict sense of the word.

Thus we are here to celebrate the centennial of the birth of the founder of our subject, who gave it its name[1] and founded its journal.

Rather regrettably, but perhaps inevitably, the term "econometrics" has come to have a narrower meaning than Frisch originally intended: the study of statistical methods for the application of economic models. For that reason, the title of this chapter, instead of referring just to Frisch's contributions to econometrics in the narrower sense – which were many and profound – also refers to his contributions to economics, by which may be understood economic theory and policy, to which he made a large number of important contributions. I shall necessarily be quite selective, and rather than try to survey his huge output, which could be done only in a superficial way, I shall concentrate on what seem to me the most important and lasting of his contributions.

The history of economic thought can be, and often is, a dry subject, and if one limits oneself to the works of a single person, out of the context in which that person lived and worked, it can be dull. What is really much more interesting is how such a person interacted with others and with

This work was supported by a Humboldt Research Award for Senior U.S. Scientists. I wish to thank Dale Jorgenson for valuable discussions and Olav Bjerkholt for his valuable bibliographic help. Thanks are also due to the referee for helpful comments.

[1] This notwithstanding the fact that an earlier use of the term was subsequently made known to Frisch (1936c).

the economic environment. While I do not pretend to subscribe to, let alone understand, all of Hegelian philosophy, I believe that Hegel had an enormous insight into how knowledge progresses: by conflict. What makes the history of economic thought interesting, in my opinion, is the study of how truth comes out of controversy. In addition, conflict in itself is always interesting and makes the subject come alive. I shall therefore be especially interested in recounting the controversies in which Frisch was engaged, and in showing how they led to important advances in the subject.

I shall look at four fields in which Frisch made major contributions: (1) utility theory, index numbers, and welfare economics, (2) estimation of demand and supply functions, and statistical confluence analysis, (3) capital theory and dynamic economics, and (4) depression and circulation planning.

1 Utility Theory, Index Numbers, and Welfare Economics

1.1 *Measurable Utility, Price Indices, and Homothetic Preferences*

Frisch's first work on utility theory (1926a) was, in his words, an attempt to "realize the dream of Jevons." Jevons had stated (1911, pp. 146–7; 1871, p. 140) that

> the price of a commodity is the only test we have of the [marginal] utility of the commodity to the purchaser; and if we could tell exactly how much people reduce their consumption of each important article when the price rises, we could determine, at least approximately, the variation of the final degree of utility – the all-important element in Economics.

Frisch set himself the problem of objectively defining utility as a quantity. "The real advances in a science," he said, "begin on the day that it is realized that vague common sense notions must be replaced by notions capable of objective definition." He noted that while that had been the object of works by Edgeworth (1881), Fisher (1892), and Pareto (1906), definitive results had not been obtained. Indeed, those authors had provided the basic idea that Frisch was to use. Edgeworth postulated (1881, p. 99) that "just perceivable increments of pleasure are equatable." An alternative approach was followed by Fisher (1892, p. 17n). Pareto was still more explicit (1906, p. 252; 1909, p. 264):

> Moreover, man may know approximately whether in passing from combination I to combination II he experiences greater pleasure than in passing from combination II to another combination III. If this judgment could be made sufficiently precise we could, in the limit, determine whether passing from I to II provides equal pleasure to passing

> from II to III; in which case, passing from I to III provides double the pleasure that is obtained in passing from I to II. This would suffice to enable us to consider pleasure or ophelimity as a quantity.

Frisch proceeded to formulate his axioms of the first kind (comparisons of commodity bundles) and of the second kind (comparisons of pairs of commodity bundles) and sketched a proof of the measurability of utility. Some years later this was followed by Alt (1936).

Proceeding to the "marginal utility of money" he added some further (and more controversial) assumptions for tractability, notably that it would remain unchanged if prices varied in such a way as to leave "the mean value of its components" (i.e., the price level) unchanged. He therefore expressed it as a function of income (Y) and price level (P) (on the legitimacy of this step, more later). He assumed further that it would approach infinity for minimum-subsistence income, would approach zero for infinite income, and would be decreasing in income – all reasonable assumptions. However, he added the more questionable assumptions that the elasticity of the marginal utility of income is greater than unity (in absolute value) for sufficiently small incomes and approaches zero for indefinitely large incomes. This led him to a formula for the marginal utility of income that I shall write in the form[2]

$$\frac{\partial \bar{V}}{\partial Y} = \frac{c(P)}{\log Y - \log a(P)} \tag{1.1}$$

where I have replaced his parameters a (representing subsistence income) and c by functions of P. Since $\bar{V}_Y(Y, P)$ must be homogeneous of degree -1 in its two arguments, $a(P)$ and $c(P)$ must be homogeneous of degree 1 and -1, respectively. Assuming additively separable utility, that is,

$$U(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} u_i(x_i) \tag{1.2}$$

since the marginal utility of commodity i is proportional to its price, the factor of proportionality being the marginal utility of income, Frisch noted that

$$u_i'(x_i) = \bar{V}_Y(Y, P)p_i = \frac{p_i}{P}\bar{V}_Y\left(\frac{Y}{P}, 1\right) = \frac{p_i}{P}\frac{c(1)}{\log(Y/P) - \log a(1)}$$

expressing a relation among the three variables x_i (quantity of a commodity), p_i/P (its relative price), and Y/P (real income). Using French sugar data, he developed a method for estimating the coefficients $c(1)$ and $a(1)$.[3]

Stimulated by Fisher's 1927 paper, Frisch undertook a more extensive investigation in his *New Methods of Measuring Marginal Utility* (1932c). Influenced by Birck's concept of a "general commodity" (1922, p. 53), he formulated the marginal utility of income more explicitly as a function of income and the "general price level" P. Defining "real income" by $R = Y/P$, because of homogeneity of degree 0 of the function $\bar{V}(Y, P)$ (Frisch dealt not with this function but only with its partial derivative with respect to income, Y) we have

$$\bar{V}(Y, P) = \bar{V}(Y/P, 1) = \bar{V}(R, 1) \equiv W(R) \tag{1.3}$$

Hence

$$W'(R) = \bar{V}_Y(Y/P, 1) = P\bar{V}_Y(Y, P)$$

leading to Frisch's basic formula [1932c, p. 16, formula (3.2)]

$$\frac{\partial \bar{V}(Y, P)}{\partial Y} = \frac{W'(R)}{P} = \frac{u_i'(x_i)}{p_i} \tag{1.4}$$

where x_i is the amount of the "commodity of comparison" (1932c, p. 8), and p_i is its price. He called $W'(R)$ the "real money utility," in contrast to the "nominal money utility" $\bar{V}_Y$ (1932c, p. 14). "Real income" R, according to this formula, may be interpreted as the quantity of the "general commodity," and P its price.

Frisch's formulation came under the forceful criticism of R. G. D. Allen (1933), whose paper, while ostensibly a review article of Frisch's work, was a very important contribution on its own. For the first time since (and independently of) Antonelli (1886), Allen formulated the concept of an "equilibrium utility function" (now known as "indirect utility function")

$$V(Y, p_1, p_2, \ldots, p_n)$$

and the accompanying partial differential equation[4]

$$\frac{\partial V}{\partial p_i} = -\frac{\partial V}{\partial Y}h_i \tag{1.5}$$

(1933, p. 190), where $h_i(Y, p_1, p_2, \ldots, p_n)$ is the Marshallian demand function for commodity i. Allen pointed out that whereas in equilibrium it is necessarily true that

[2] He specifically rejected Bernoulli's (1738) formula in which Y takes the place of $\log Y$ in (1.1), as well as a squared variant suggested by Jordan (1924) – see Frisch (1932c, p. 31).

[3] For good expositions of the procedure used by Frisch, see Marschak (1931, pp. 128–35), as well as the reviews of Frisch (1932c) by Bowley (1932) and Schultz (1933).

[4] Usually attributed to Roy (1942, p. 21). Roy (pp. 38–9) referred to Frisch but not to Allen. For the earliest derivation of (1.5), see Antonelli [1886, p. 17, equation (24)], or page 349 of the English translation.

$$\frac{\partial V}{\partial Y} = \frac{\sum_{i=1}^{n} u_i'(x_i)x_i}{\sum_{i=1}^{n} p_i x_i} \tag{1.6}$$

where the numerator on the right is the equilibrium marginal utility of the composite commodity and the denominator is its equilibrium amount, they cannot legitimately be interpreted as structural parameters; therefore the device of introducing a composite commodity entails a restrictive assumption on consumer behavior. In his words (p. 193):

> Professor Frisch's analogy with the marginal utility of a consumer's good and his introduction of a composite commodity only serve to hide the serious assumption that must be made before his statistical methods can be applied.

Unfortunately, however, Allen did not attempt to find out what that assumption was. It was left for Frisch himself to do so in his famous "Annual Survey" of the theory of index numbers (1936a), a paper subsequently characterized by Bergson (1936, p. 34n; 1966, p. 94n) as "chiefly a response to criticisms by Allen." Bergson himself set out to tackle this same problem.

The reasoning in these two papers is extremely difficult to follow, but I shall try to restate what is apparently claimed and sketch a supporting argument. The argument is entirely in terms of a single individual; the problem of aggregation over individuals is not taken up. The claim concerns two conditions:

1. Individual preferences can be represented by an additively separable, monotonic, and strongly concave utility function (1.2) (i.e., $u_i' > 0, u_i'' < 0$).
2. Denoting the demand functions generated by (1.2) by $h_i(Y, p)$, where p denotes the price vector, the indirect utility function obtained by composing (1.2) with these demand functions is separable as between income and the commodity prices; that is,

$$V(Y, p) \equiv \sum_{i=1}^{n} u_i[h_i(Y, p)] = \bar{V}[Y, P(p)] \tag{1.7}$$

The claim is that these two conditions together imply that individual preferences are homothetic. This is what Frisch described as "expenditure proportionality."[5]

In terms of concepts developed since Frisch's time, but undoubtedly influenced by and indeed implicit in Frisch's own work, we can define his concepts of price index and real income as follows. Let the *expenditure function* be defined as[6]

$$e(u, p) = \min\{Y\colon\ V(Y, p) \geqq u\} \tag{1.8}$$

Note from this definition that since $V(Y, p)$ is homogeneous of degree 0, $e(u, p)$ is homogeneous of degree 1 in p. Frisch (1936a, p. 11) then defines the general price or cost-of-living index at time t, following Bowley (1928, p. 223), as the proportionate "change in expenditure . . . necessary, after a change of prices, to obtain the same satisfaction as before," that is (pp. 15–16),

$$P(u; p^0, p^t) \equiv \frac{e(u, p^t)}{e(u, p^0)} \tag{1.9}$$

We verify that $P(u; p^0, \lambda p) = \lambda P(u; p^0, p)$, so (1.9) satisfies the property of the aggregator function $P(p)$ required for $\bar{V}(Y, P)$ to be homogeneous

---

[5] The superficial resemblance of this theorem to that of Houthakker (1960) and Samuelson (1965) – as amended by Hicks (1969) and Samuelson (1969) – requires some comment. The Houthakker-Hicks-Samuelson theorem states that (barring the exceptional cases brought to light by Hicks) condition 1 combined with the condition that the indirect utility function can be written in the form *(continued)*

$$V(Y, p) = \sum_{i=1}^{n} v_i(p_i/Y)$$

implies that preferences are homothetic. (The exceptional cases involve "parallel preferences.") In Frisch's theorem the second condition states that the indirect utility function can be expressed in the form

$$V(Y, p) = \bar{V}[Y, P(p)]$$

In one sense this is a weaker condition, since all that is involved is separability (not even additive) as between the income variable and the set of price variables. If $P(p)$ is homogeneous of degree 1, then

$$\bar{V}[\lambda Y, \lambda P(p)] = \bar{V}[\lambda Y, P(\lambda p)] = V(\lambda Y, \lambda p) = V(Y, p) = \bar{V}[Y, P(p)]$$

Hence $\bar{V}(Y, P)$ is homogeneous of degree 0. Conversely, if $\bar{V}(Y, P)$ is homogeneous of degree 0, then

$$\bar{V}[\lambda Y, \lambda P(p)] = \bar{V}[Y, P(p)] = V(Y, p) = V(\lambda Y, \lambda p) = \bar{V}[\lambda Y, P(\lambda p)]$$

which can hold for all λ if and only if $P(\lambda p) = \lambda P(p)$. Thus one need only require that the function $P(p)$ in (1.7) be positively homogeneous of degree 1. The conclusion of Frisch's theorem states that the preference ordering must be homothetic. But Bergson's extension of Frisch's theorem (shown later) proves as a consequence that the indirect utility function is *additively* separable as between income and the set of prices [see (1.19), where for $\beta \neq 0$ one can replace the indicated indirect utility function by its logarithm] and, in the Cobb-Douglas case ($\beta = 0$), additively separable in income and the individual prices.

[6] If p^t is the price vector in period t, then Frisch's notation for $e(u, p^t)$ is $\rho_t(I)$, where ρ denotes income and I ("indicator") denotes utility. Thus the concept is present in all but name.

of degree 0 (see footnote 5). The case of *expenditure proportionality* occurs when this function is independent of u. Frisch also defines *real income* in period t (p. 32) as money income in period t deflated by the above cost-of-living index; that is,

$$R(u;\, p^0, p^t) = \frac{Y^t}{P(u;\, p^0, p^t)} = e(u, p^0),$$
$$\text{since } Y^t = e(u, p^t) \tag{1.10}$$

Thus, real income, according to this definition, is independent of current prices and coincides with the expenditure function evaluated at base-year prices, $e(u, p^0)$.

Now the relation $R = e(u, p^0)$ from (1.10) can be inverted (because of monotonicity) to

$$u = W(R, p^0), \quad \text{where } R = e[W(R, p^0), p^0] \tag{1.11}$$

and the relation

$$Y^t = e(u, p^t) = e(u, p^0)P(u;\, p^0, p^t)$$

yields

$$\hat{Y}(R;\, p^0, p^t) = R\hat{P}(R;\, p^0, p^t) \tag{1.12}$$

where

$$\hat{Y}(R;\, p^0, p^t) = e[W(R, p^0), p^t] \quad \text{and} \quad \hat{P}(R;\, p^0, p^t) = P[W(R, p^0);\, p^0, p^t]$$

Differentiating (1.12) we obtain

$$\frac{\partial \hat{Y}}{\partial R} = \hat{P}\left(1 + \frac{R}{\hat{P}}\frac{\partial \hat{P}}{\partial R}\right)$$

Thus the marginal utility of income is

$$\frac{\partial V}{\partial Y} = \frac{\partial W/\partial R}{\partial \hat{Y}/\partial R} = \frac{\partial W/\partial R}{\hat{P}\left(1 + \dfrac{\partial \log \hat{P}}{\partial \log R}\right)} \tag{1.13}$$

This agrees with (1.4) if and only if

$$\frac{\partial \hat{P}}{\partial R} = \frac{\partial P}{\partial u}\frac{\partial W}{\partial R} = 0 \tag{1.14}$$

which occurs if and only if $\partial P(u;\, p^0, p^t)/\partial u = 0$ (expenditure proportionality). Thus, homotheticity of preferences is implied by (1.4), which is in turn implied by (1.7).[7] Conversely, if preferences are homothetic, the formula

$$\frac{u_i'(x_i)}{p_i} = \frac{\partial W(R, p^0)/\partial R}{\hat{P}(R;\, p^0, p^t)}$$

holds, where $\hat{P}$ is now independent of R.[8]

Now the assumptions of homotheticity and additive separability of the direct preference relation together have very stringent implications, as shown by Bergson (1936, p. 45; 1966, p. 111). Since homotheticity and separability imply that the marginal rates of substitution

$$R_{ij}(x_i, x_j) = \frac{\partial U(x)/\partial x_i}{\partial U(x)/\partial x_j} = \frac{u_i'(x_i)}{u_j'(x_j)}$$

are homogeneous of degree 0 in the quantities x_i, x_j, it follows by Euler's theorem that

$$0 = \frac{\partial R_{ij}}{\partial x_i}x_i + \frac{\partial R_{ij}}{\partial x_j}x_j = \frac{u_i''(x_i)}{u_j'(x_j)}x_i - \frac{u_i'(x_i)u_j''(x_j)}{u_j'(x_j)^2}x_j$$

whence, multiplying through by $u_j'(x_j)/u_i'(x_i)$, we obtain

$$\frac{x_i u_i''(x_i)}{u_i'(x_i)} = \frac{x_j u_j''(x_j)}{u_j'(x_j)} \quad \text{for all } i, j \tag{1.15}$$

Thus, each of the two expressions in (1.15) is a (negative) constant. This can be written as

$$-\frac{d \log u_i'(x_i)}{d \log x_i} = -\frac{x_i}{u_i'(x_i)}u_i''(x_i) = 1 + \beta \tag{1.16}$$

where $\beta > -1$. Integrating (1.16) for each i gives the marginal utility

$$u_i'(x_i) = A_i x_i^{-1-\beta} \qquad (A_i > 0)$$

and integrating this equation once again gives

[7] For the equivalence of homotheticity and expenditure proportionality, see Samuelson and Swamy [1974, p. 570, equation (2.5)] and Chipman and Moore (1980, p. 939, Proposition H6).

[8] This provided Frisch's answer to Allen's objection that equation (1.6) was only an equilibrium condition and not a structural relationship. In his words (1936a, p. 34n): "(1.13) – here derived as a theoretical consequence – should completely meet Allen's objection. . . . (1.13) shows that my original formula does hold under expenditure proportionality, which was assumed in the statistical work in *New Methods*. . . ."

$$u_i(x_i) = \begin{cases} \alpha_i x_i^{-\beta} + \gamma_i & \text{for } \beta < 0 \text{ and } \alpha_i = -A_i/\beta \\ -\alpha_i x_i^{-\beta} + \gamma_i & \text{for } \beta > 0 \text{ and } \alpha_i = A_i/\beta \\ \alpha_i \log x_i + \gamma_i & \text{for } \beta = 0 \text{ and } \alpha_i = A_i \end{cases}$$

Summing these over all n commodities and dropping the spurious constant terms γ_i, we obtain for the utility function (1.2)

$$U(x) = \begin{cases} -(\operatorname{sgn}\beta)\sum_{i=1}^{n} \alpha_i x_i^{-\beta} & \text{for } \beta \neq 0 \\ \sum_{i=1}^{n} \alpha_i \log x_i & \text{for } \beta = 0 \end{cases} \tag{1.17}$$

The first of these, of course, will be recognized, after taking its absolute value and raising it to the power $-1/\beta$, as the Arrow–Solow constant-elasticity-of-substitution (CES) function introduced by Arrow et al. (1961, p. 226n) for the two-commodity case and generalized by Uzawa (1962) and McFadden (1963) to the n-commodity case, where the elasticity of substitution is $\sigma = 1/(1+\beta)$.[9] The exponential of the second is the "Cobb-Douglas" function to which the CES reduces as $\beta \to 1$. We verify that

$$\frac{\partial U}{\partial x_i} = |\beta| \frac{\alpha_i}{x_i^{1+\beta}} > 0 \quad \text{and} \quad \frac{\partial^2 U}{\partial x_i^2} = -|\beta|(1+\beta)\frac{\alpha_i}{x_i^{2+\beta}} < 0 \quad \text{for } \beta \neq 0$$

as well as

$$\frac{\partial U}{\partial x_i} = \frac{\alpha_i}{x_i} > 0 \quad \text{and} \quad \frac{\partial^2 U}{\partial x_i^2} = -\frac{\alpha_i}{x_i^2} < 0 \quad \text{for } \beta = 0$$

as desired. Equating the ratios of these marginal utilities to the corresponding price ratios and substituting in the budget equation $\Sigma_{i=1}^{n} p_i x_i = Y$, we obtain, upon adopting the normalization $\Sigma_{i=1}^{n} \alpha_i = 1$, the demand functions

$$x_i = h_i(Y, p) = \frac{Y}{\alpha_i^{-\frac{1}{1+\beta}} p_i^{\frac{1}{1+\beta}} \sum_{j=1}^{n} \alpha_j^{\frac{1}{1+\beta}} p_j^{\frac{\beta}{1+\beta}}} \quad \text{for } i = 1, 2, \ldots, n, \ -1 < \beta < \infty \tag{1.18}$$

Substituting (1.18) into (1.17), we obtain the indirect utility function

$$V(Y, p) = \begin{cases} -(\operatorname{sgn}\beta) Y^{-\beta} \left(\sum_{i=1}^{n} \alpha_i^{\frac{1}{1+\beta}} p_i^{\frac{\beta}{1+\beta}} \right)^{1+\beta} & \text{for } \beta \neq 0 \\ \log Y + \sum_{i=1}^{n} \alpha_i \log \alpha_i - \sum_{i=1}^{n} \alpha_i \log p_i & \text{for } \beta = 0 \end{cases} \tag{1.19}$$

[9] It is also the same as the "generalized weighted mean" of Hardy, Littlewood, and Pólya [1934, p. 13, formula (2.2.5)].

The marginal utility of income then becomes

$$\frac{\partial V}{\partial Y} = \begin{cases} \dfrac{|\beta| \left(\sum_{i=1}^{n} \alpha_i^{\frac{1}{1+\beta}} p_i^{\frac{\beta}{1+\beta}} \right)^{1+\beta}}{Y^{1+\beta}} & \text{for } \beta \neq 0 \\ \dfrac{1}{Y} & \text{for } \beta = 0 \end{cases} \tag{1.20}$$

It follows from (1.20) that under the conditions implied by Frisch's assumptions, his functional form (1.1) for the marginal utility of income is inadmissible. Since this form no longer appears in the "Annual Survey" (Frisch, 1936a), it can be assumed that he became aware of this inconsistency.

The hypothesis of additively separable utility adopted by Fisher and Frisch came under strong criticism by Samuelson (1947), who noted that the hypothesis had strong empirical implications, namely (p. 177),

> if we are given as empirical observational data the two expenditure paths corresponding to the changes in quantities with income in each of two respective price situations, then from these observations, and these alone, the whole field of indifference curves can be determined by suitable extrapolation.

However, Arrow (1960) later came to Frisch's defense: "The sharpness of the implications of a hypothesis are a virtue, not a vice, provided of course the implications are not refuted by evidence" (p. 177). Samuelson returned to this subject in his obituary article (1974, pp. 11–15): "Arrow's 1960 appreciation of Frisch suggests that my own earlier criticisms have been too strong. He may well be right."

1.2 *Laspeyres and Paasche Bounds to the Cost-of-Living Index*

One of the most important contributions of Frisch's paper on index numbers was his analysis of inequalities bounding the cost-of-living index by the Laspeyres and Paasche price indices. Not having access to the original article in Russian by Konüs (1924), first brought to light by Bortkiewicz (1928), hence basing himself on the subsequent detailed exposition by Bortkiewicz (1932, pp. 18–20), he set forth and proved the following propositions attributed to Konüs by Bortkiewicz:

1. If $x^1 = h(Y^1, p^1)$ and $p^1 \cdot x^0 = p^1 \cdot x^1 = Y^1$, then denoting $u^1 = U(x^1) = V(Y^1, p^1)$ we have

$$P(u^1; p^0, p^1) \leqq \frac{p^1 \cdot x^0}{p^0 \cdot x^0} \tag{1.21}$$

That is, the "true" cost-of-living index is bounded above by the Laspeyres price index.

2. If $x^0 = h(Y^0, p^0)$ and $p^0 \cdot x^1 = p^0 \cdot x^0 = Y^0$, then denoting $u^0 = U(x^0) = V(Y^0, p^0)$ we have

$$P(u^0; p^0, p^1) \geqq \frac{p^1 \cdot x^1}{p^0 \cdot x^1} \tag{1.22}$$

That is, the "true" cost-of-living index is bounded below by the Paasche price index.

Frisch noted that if both inequalities are satisfied simultaneously, then under the stated conditions the Laspeyres and Paasche indices are necessarily the same, and moreover x^1 and x^0 must lie on the same indifference surface (and indeed, if the demand function is single-valued, $x^1 = x^0$); consequently (p. 25), "the simultaneous fulfillment of both Konüs conditions is, therefore, a trivial case, when the points compared lie in the same indifference map."

Frisch contrasted this with the limits obtained by Haberler (1927, pp. 89–92), which he interpreted (rightly, in my opinion) as follows:

1. If $x^0 = h(Y^0, p^0)$, then, denoting $u^0 = U(x^0) = V(Y^0, p^0)$,

$$P(u^0; p^0, p^1) \leqq \frac{p^1 \cdot x^0}{p^0 \cdot x^0} \tag{1.23}$$

That is, the change in the cost of living from situation (Y^0, p^0) to situation (Y^1, p^1), defined as the ratio $\bar{Y}^1/Y^0$, where $\bar{Y}^1$ is the hypothetical expenditure in period 1 that would make $(\bar{Y}^1, p^1)$ indirectly indifferent to (Y^0, p^0), is bounded above by the Laspeyres price index.

2. If $x^1 = h(Y^1, p^1)$, then, denoting $u^1 = U(x^1) = V(Y^1, p^1)$,

$$P(u^1; p^0, p^1) \geqq \frac{p^1 \cdot x^1}{p^0 \cdot x^1} \tag{1.24}$$

That is, the change in the cost of living from situation (Y^0, p^0) to situation (Y^1, p^1), defined as the ratio $Y^1/\bar{Y}^0$, where $\bar{Y}^0$ is the hypothetical expenditure in period 0 that would make $(\bar{Y}^0, p^0)$ indirectly indifferent to (Y^1, p^1), is bounded below by the Paasche price index.

He noted that under expenditure proportionality – an assumption that originally was only implicit (Haberler, 1927), but which was subsequently made explicit (Haberler, 1929, p. 8) in response to Bortkiewicz's charge that the result was simply fallacious (1928, pp. 428–9) – because $P(u; p^0, p^1)$ is then independent of u, the two limits reduce to the double limit

$$\frac{p^1 \cdot x^1}{p^0 \cdot x^1} \leqq P(u; p^0, p^1) \leqq \frac{p^1 \cdot x^0}{p^0 \cdot x^0} \tag{1.25}$$

That (1.25) holds when x^0 and x^1 lie on the same indifference surface, and hence $u = U(x^0) = U(x^1)$, as is immediately obvious from (1.23) and (1.24), was shown by Keynes (1930, I, p. 110) [who referred to Haberler (1927) and Pigou (1929)], Bortkiewicz (1932, p. 21), and Allen (1933, p. 204), but as Frisch (1936a, p. 26) pointed out, "none of these three authors noted the perfectly trivial character of" (1.25) in this case. Because $e(u; p^0) = p^0 \cdot x^0$ and $e(u; p^1) = p^1 \cdot x^1$ we then have simply $P(u; p^0, p^1) = p^1x^1/p^0x^0$, and the bounds (1.25) are superfluous. Even Staehle (1935, pp. 169, 172), who had provided a detailed exposition of Haberler's 1927 and 1929 contributions (Staehle, 1934, pp. 76–9), thought that the condition $U(x^0) = U(x^1)$ was *necessary* as well as sufficient for (1.25) to hold. In this he was influenced by Bortkiewicz (1928). Writing decades later, Allen (1949, 1975, pp. 65–72) showed no evidence of having assimilated the results discovered by Haberler (1929) and proved by Frisch (1936a), namely, that homotheticity of preferences is necessary and sufficient for (1.25) to be true for any two arbitrary equilibrium situations (p^0, x^0) and (p^1, x^1).

A signal service to the profession was provided by Schultz (1939a) in arranging for the publication of an English translation of Konüs's 1924 article and pointing out that its contents had been greatly distorted by Bortkiewicz (1928, 1932). It turned out that Konüs had in fact obtained the Haberler conditions (1.23) and (1.24) three years before Haberler. He had also sought conditions under which the standards of living would be equivalent in two different situations (the part of his treatment summarized by Bortkiewicz), but had himself pointed out, as Frisch later showed, that the true cost-of-living index in such a situation would be simply the ratio of expenditures. He went further, however, in seeking conditions under which (1.25) would hold for some standard of living u' intermediate between $u^0 = U(x^0)$ and $u^1 = U(x^1)$. He did not, however, obtain the general Haberler-Frisch homotheticity result. The closest he came to this was a characterization, in collaboration with Buscheguennce (1925) (Konüs and Buscheguennce, 1926), of preferences under which Fisher's "ideal index" (the square root of the products of the Laspeyres and Paasche indices) would be an exact cost-of-living index – preferences generated by a homogeneous quadratic utility function of the form $U(x) = (x'Ax)^{1/2}$ – as well as conditions (Cobb-Douglas preferences) under which a geometric price index would be an exact cost-of-living index (cf. Diewert, 1976; Afriat, 1977). In these cases the Laspeyres-Paasche bounds would of course be unnecessary.

Schultz's 1939 exposition (1939a) was astonishing in one respect. It praised Konüs's work as a forerunner of the results of Allen and Staehle – results that (at least in the case of Allen) Frisch had characterized as "perfectly trivial" (1936a, p. 26). On the other hand, it made no mention of Frisch (1936a) or of Haberler (1927, 1929) (except as the subject of Bortkiewicz's 1928 review). From then on, it seems that the most important contributions to index-number theory – the Haberler-Frisch propositions concerning the implicit assumption of homotheticity underlying the *economic* theory of index numbers – were buried alive, so to speak, and had to be rediscovered.

Rediscoveries there were, because truth always waits to be discovered. One was that of Malmquist (1953, p. 215) – though only a very acute reader would be able to discern homotheticity in the purely technical assumption he provided. The first systematic rediscovery appears to have been that of Pollak (1971), which was not published until 1983 (in a very obscure volume), and again in 1990. Shortly after came that of Afriat (1972). Resurrection of the idea came with Samuelson and Swamy (1974) and Samuelson (1974), although Frisch and Haberler still were not given their full due. The Laspeyres and Paasche bounds (1.23) and (1.24) were attributed by Diewert (correctly, as original discoverer) to Konüs (Diewert, 1981, p. 168; 1990, p. 85); the condition for the double inequality (1.25) to hold was correctly attributed to Frisch (Diewert, 1981, p. 168), though no mention was made of Haberler. The theorem that the cost-of-living function $P(u; p^0, p^1)$ is independent of u if and only if the preference ordering is homothetic was attributed to Malmquist (1953), Pollak (1971), and Samuelson and Swamy (1974) by Diewert (1981, p. 166), though in a footnote (p. 200) he remarked as follows: "It seems clear that earlier researchers such as Frisch (1936, p. 25) also knew this result, but they had some difficulty in stating it precisely, since the concept of homotheticity was not invented until 1953 (Shephard, 1953; Malmquist, 1953)." It is perhaps true that the *word* "homotheticity" did not enter the vocabulary until 1953, but the concept was surely well understood by Haberler (1929), Frisch (1936a), Bergson (1936), and Samuelson (1942). But the bulk of the profession apparently was not ready to accept the need to postulate severe restrictions on preferences in order to justify the use of index numbers in economic analysis. Thus it was the fate of this true genius, Frisch, that much of his work was misunderstood and buried by his contemporaries because it was too advanced for that time and had to await rediscovery.

1.3 *The Double-Expenditure Method*

One of the most novel ideas presented in Frisch (1936a) was the double-expenditure method (pp. 27–30). The problem posed was this: Suppose we are given data on a quantity vector $x^0 = (x_1^0, x_2^0, \ldots, x_n^0)$ and a price vector $p^0 = (p_1^0, p_2^0, \ldots, p_n^0)$ observed at time 0, where x^0 is consumed at prices p^0. This is called the base-period quantity and price. Suppose we are also given a price vector p^1. The problem is to find a commodity

bundle x^1 consumed at prices p^1 that is indifferent to the bundle x^0. Stimulated by a formulation of Bowley (1928), Frisch took a second-order Taylor approximation of a utility function around the point x^0:

$$U(x^1) - U(x^0) = \sum_{i=1}^{n} U_i(x^0)(x_i^1 - x_i^0) + \frac{1}{2}\sum_{i=1}^{n}\left[\sum_{j=1}^{n} U_{ij}(x^0)(x_j^1 - x_j^0)\right](x_i^1 - x_i^0) \tag{1.26}$$

where $U_i(x) = \partial U(x)/\partial x_i$ and $U_{ij}(x) = \partial^2 U(x)/\partial x_i \partial x_j$. Noting that the quadratic term in the Taylor expansion of $U_i(x^0)$ is

$$U_i(x^1) - U_i(x^0) = \sum_{j=1}^{n} U_{ij}(x^0)(x_j^1 - x_j^0) \tag{1.27}$$

and substituting the left member of (1.27) into the bracketed term of (1.26), we obtain

$$\begin{aligned} U(x^1) - U(x^0) &= \tfrac{1}{2}\textstyle\sum_{i=1}^{n}[U_i(x^0) + U_i(x^1)](x_i^1 - x_i^0) \\ &= \tfrac{1}{2}\textstyle\sum_{i=1}^{n}(\omega^0 p_i^0 + \omega^1 p_i^1)(x_i^1 - x_i^0) \\ &= \omega^1 Y^1 + \omega^0 Y^0 + \omega^0 \textstyle\sum_{i=1}^{n} p_i^0 x_i^1 - \omega^1 \textstyle\sum_{i=1}^{n} p_i^1 x_i^0 \end{aligned} \tag{1.28}$$

where

$$\omega^t = V_Y(Y^t, p^t) \quad \text{where} \quad V_Y(Y, p) = \frac{\partial V(Y, p)}{\partial Y} \tag{1.29}$$

is the marginal utility of income in period t, and Y^t is period-t income, the quantities consumed satisfying the budget constraint $\Sigma_{i=1}^n p_i^t x_i^t = Y^t$. To obtain x^1, the expression (1.29) must be set equal to zero.

Now Frisch introduces another approximation, namely, $\omega^1 Y^1 = \omega^0 Y^0$, and notes that it is satisfied exactly if preferences are homothetic ("expenditure proportionality"), appealing to formula (1.13). Adopting this assumption, (1.28), when set equal to zero, becomes, using the budget constraint,

$$\sum_{i=1}^{n} p_i^1 x_i^1 \cdot \sum_{i=1}^{n} p_i^0 x_i^1 = \sum_{i=1}^{n} p_i^0 x_i^0 \cdot \sum_{i=1}^{n} p_i^1 x_i^0 \tag{1.30}$$

The term on the left Frisch denotes D_{01} and calls the *double expenditure* along Engel curve 1 ($\{x\colon (\exists Y) x = h(Y, p^1)\}$), with Engel curve 0 ($\{x\colon (\exists Y) x = h(Y, p^0)\}$) as a base. Likewise the term on the right, D_{10}, is the double expenditure along Engel curve 0, with 1 as a base. Intuitively, at any point along Engel curve 1 we may ask the following: (1) What is the cost of purchasing this bundle at prices p^1? (2) What would be the cost of purchasing this bundle at prices p^0? The product of these costs is the double expenditure. A point x^1 on path 1 is indifferent to a point x^0 on path 0 if and only if (under this approximation) its double expenditure D_{01} relative to path 0 is equal to the double expenditure D_{10} of point x^0 relative to path 1. The concept can be applied to different markets (e.g., different countries) in place of different time periods with suitable interpretations (cf. Menderhausen, 1938; Frisch, 1937). Frisch subsequently (1938), following some further comments by Bowley (1938), carried out some simulations with some two-commodity utility functions to see how good an approximation his method gave; the results were certainly very favorable.

The following year, Wald (1939) raised an objection to Frisch's method. He pointed out that taking a Taylor approximation of the utility function to two terms amounted to assuming that the utility function was quadratic; hence "it is superfluous to make additional assumptions, because the polynomial assumption already suffices for the unique determination of the index" (p. 329). He proceeded to carry out this determination and showed that the utility function could be approximated by a quadratic function in the neighborhood of comparison points. Frisch, in a footnote to Wald's paper (p. 329n), made the point that "my additional 'superfluous' assumption may indeed in many cases correct for part of the error committed by assuming the indicator as a polynomial" and reported that he had experimented with Wald's method and found that the goodness of approximation was about the same for the two methods, but that "Dr. Wald's method proved to be much more laborious." Except for some interesting comments by Samuelson and Swamy (1974), that appears to be where the subject has rested![10]

1.4 *Fisher's Tests for Internal Consistency of Index Numbers*

An important early contribution of Frisch was his demonstration (1930) that if prices and quantities are chosen arbitrarily, there exists no relative index number (comparing situations at two points of time) that satisfies simultaneously several of Fisher's (1922) tests. That paper drew the attention of Subramanian (1934), who found technical problems with Frisch's proofs, but Frisch's reply (1934d) seemed to put the matter at rest. The issue was revived by Swamy (1965), who provided a rigorous proof of the incompatibility of four of Fisher's tests. Eichhorn (1976) subsequently furnished proofs that dispensed with continuity and differentiability assumptions, and the topic has been treated at length by Eichhorn and Voeller (1976). A recent survey has been provided by Balk (1995).

Samuelson and Swamy (1974), by removing the assumption that prices

[10] An interesting discussion of Wald's approach and its relation to that of Buscheguennce (1925) has been provided by Afriat (1977, pp. 133–40).

and quantities can be chosen arbitrarily, and taking account of the fact that quantities are chosen optimally at given prices, were able to find index-number formulas that satisfied "the spirit" of Fisher's tests "in the only case in which a single index number of cost of living makes economic sense – namely the 'homothetic' case" (p. 567). A valuable discussion has been provided by Samuelson (1974, pp. 15–21).

Going back to Allen (1933), it is clear that from a welfare point of view what is really sought is an indirect utility function, and it is only in the case of homothetic preferences that such a function has the form $V(Y, p) = Y/C(p)$, where $C(p)$ can be interpreted as a cost-of-living function. There are other cases where it makes more sense to *subtract* a cost-of-living index from income.[11] It is ironic that Frisch's 1936 approach to index-number theory, combined with his criterion of "expenditure proportionality," should in the end have furnished the required solution to his 1930 impossibility theorem!

1.5 *Taxation and Welfare*

Fisher (1927) constructed an ingenious example of three households, with households 1 and 3 living in the same district and thus facing the same prices but having different incomes, and household 2 facing different prices. He assumed that all three had identical preferences, representable by an additively separable utility function, and moreover that for each of two commodity groups the shares of expenditure were constant irrespective of prices and income [so that we are in the case $\beta = 0$ of (1.17)]. By assuming that households in a sample could be found such that households 1 and 2 consumed the same amount of food and households 2 and 3 consumed the same amount of housing, he was able to show that from this information one could deduce the marginal utility of income and the income of each household, and thus the elasticity of the marginal utility of income with respect to income. The object of this exercise was to determine the just degree of progression of an income tax. By the "principle of equal sacrifice" he meant that the subjective sacrifices of different households should be equated, these being defined as the product of the marginal utility of income and the amount of the tax payment. This kind of welfare economics has, of course, been pretty much discredited since the time of Lionel Robbins, but even if it is accepted, Fisher's assumption that the taxes are sufficiently small so as not to appreciably affect income and thus the marginal utility of income, even if realistic in 1927, would certainly not be so today. In any case, Fisher used this principle to show how one could calculate the optimal

[11] If preferences are of the "parallel" form with respect to commodity 1, then a representation of indirect preferences is given by $V(Y, p) = [Y - C(p)]/p_1$. Cf. Chipman and Moore (1980, p. 941).

rates at different incomes given information on prices and consumption of different households.

In "*Sur un problème*" (Frisch, 1926a) there is no indication that Frisch's research had similar goals; rather, it was a study in positive economics, with much of it devoted to the problem of statistical estimation of the marginal utility of income as a function of income and the price level. He had, in that paper, referred to Jordan (1924), who had associated mathematical expectation with equal taxes, Bernoullian moral expectation with proportional taxes, and his proposed "harmonic expectation" with progressive taxes. However, Frisch was interested only in the empirical realism of the functional forms. In *New Methods* (1932c), however, he devoted an entire chapter (ch. 11) to "Money Utility and the Income Tax." But he was far more cautious than Fisher, considering in turn the principles of equal sacrifice and proportional sacrifice and several others, ending with a Rawlsian "minimum-sacrifice" principle. Finally he insisted on specifying a particular "justice-definition" and insisted that "our statistically determined money utility curve *in itself* neither proves nor disproves the 'justice' of a progressive income tax, it will do so only when a particular form of [justice-definition] is used" (p. 133).

An important controversy in which Frisch was engaged was his 1939 debate with Hotelling about the welfare effects of excise taxes. Following Marshall (1890), but using a more sophisticated argument, Hotelling (1938) claimed that any system of ad-valorem excise taxes would be worse than a proportional income tax. Frisch objected to this conclusion and found a slip in Hotelling's argument. The argument, like all the previous ones discussed in this section, is stated in terms of a single consumer:

Suppose our single individual consumes n commodities in amounts x_i with prices p_i. Prior to the imposition of excise taxes, the individual consumes a bundle x^0 at prices p^0 and income Y^0 so as to maximize a utility function $U(x)$ subject to the budget constraint $p^0 \cdot x^0 = Y^0$. After the introduction of taxes, market (tax-inclusive) prices and after-tax income are p^1 and Y^1, respectively, and a bundle x^1 is chosen that maximizes $U(x)$ subject to $p^1 \cdot x^1 = Y^1$. The government collects $R = (p^1 - p^0)\cdot x^1 - (Y^1 - Y^0)$ in net revenues. Because the government is assumed by Hotelling to collect $(p_i^1 - p_i^0)x_i^1$ in taxes on commodity i, p_i^0 must be identified with the production cost after the tax as well as with the market price (= production cost) before the tax, that is, the tax does not affect pre-tax unit production costs, so that supplies are infinitely elastic; perhaps there is a single factor of production in the economy.[12]

[12] This results, as is well known, when there are constant returns to scale and no joint production; cf. Samuelson (1951), and for an elementary exposition, Chipman (1953).

Let the *ad valorem* excise-tax rate on commodity i and a proportional income-tax rate be denoted

$$t_i = p_i^1/p_i^0 - 1 \quad \text{and} \quad t_0 = 1 - Y^1/Y^0 \tag{1.31}$$

respectively (negative taxes are interpreted as subsidies). The government's net revenues are

$$R = \sum_{i=1}^{n} t_i p_i^0 x_i^1 + t_0 Y^0 = 0 \tag{1.32}$$

assumed zero because the government distributes the entire proceeds of these excise taxes back to the consumer (or taxes the consumer if these are negative). The consumer's budget constraint after the imposition of the taxes is

$$\sum_{i=1}^{n}(1 + t_i)p_i^0 x_i^1 = p^1 \cdot x^1 = Y^1 = (1 - t_0)Y^0 \tag{1.33}$$

Equations (1.33) and (1.32) together imply

$$Y^0 - \sum_{i=1}^{n} p_i^0 x_i^1 = \sum_{i=1}^{n} t_i p_i^0 x_i^1 + t_0 Y^0 = R = 0$$

that is, that x^1 satisfies the budget constraint

$$p^0 \cdot x^1 = Y^0 \tag{1.34}$$

and hence x^1 was in the consumer's original budget set.

According to Hotelling (1938, p. 252), setting aside the "infinitely improbable . . . contingency" that x^0 and x^1 lie on the same indifference surface, it follows that "if a person must pay a certain sum of money in taxes, his satisfaction will be greater if the levy is made directly on him as a fixed amount than if it is made through a system of excise taxes which he can to some extent avoid by rearranging his production and consumption."

It was pointed out by Frisch (1939a,b) that Hotelling implicitly assumed that $x^1 \neq x^0$, whereas if the system of excise taxes were *uniform* (i.e., $t_i = t$ for $i = 1, \ldots, n$), then it would follow that $x^1 = x^0$, and Hotelling's conclusion would not follow. The reason for this is that under a system of *uniform (ad-valorem)* excise taxes, the consumer's post-tax budget constraint (1.33) becomes

$$(1 + t)p^0 \cdot x^1 = (1 - t_0)Y^0, \quad \text{hence } p^0 \cdot x^1 = \frac{1 - t_0}{1 + t} Y^0 \tag{1.35}$$

On the other hand, the government's budget constraint (1.32) becomes

$$tp^0 \cdot x^1 = -t_0 Y^0, \quad \text{hence } p^0 \cdot x^1 = -\frac{t_0}{t} Y^0 \tag{1.36}$$

Putting (1.35) and (1.36) together, we conclude that

$$\frac{1-t_0}{1+t} = -\frac{t_0}{t}, \quad \text{implying } t_0 = -t \tag{1.37}$$

Substituting (1.37) back into the consumer's post-tax budget constraint (1.35), we obtain

$$(1+t)p^0 \cdot x^1 = (1+t)Y^0$$

which is a multiple of, *hence identical with*, his pre-tax budget constraint (1.34). Therefore, so long as demand is single-valued (as Hotelling assumed), it must follow that $x^1 = x^0$. Therefore, a system of *uniform (ad-valorem)* excise taxes is equivalent to a proportional income tax. Hotelling (1939) conceded the point.

1.6 *The "Complete Scheme"*

In a return to utility theory, Frisch (1959) developed his "complete scheme" for computing own- and cross-elasticities of demand in a complete system of demand functions. He observed that it was generally more difficult to estimate cross-elasticities than own-elasticities and that the imposition of restrictions on the forms of preference relations would facilitate the drawing of conclusions concerning these price elasticities from information about budget shares and income elasticities. The two principal restrictions were (1) the assumption that market demand is derivable from aggregable rational preferences and (2) the assumption of "want-independence" as between certain groups of commodities.

To formulate the concept of want-independence, Frisch proceeded as follows. Let the system

$$v_i = \frac{\partial}{\partial x_i} U(x_1, x_2, \ldots, x_n) \qquad (i = 1, 2, \ldots, n) \tag{1.38}$$

be regarded as a mapping from the n commodity quantities x_i to the n marginal utilities v_i. The elasticities

$$v_{ij} = \frac{\partial v_i(x)}{\partial x_j} \cdot \frac{x_j}{v_i(x)} = \frac{\partial^2 U(x)}{\partial x_i \partial x_j} \cdot \frac{x_j}{\partial U(x)/\partial x_i} \tag{1.39}$$

are called the "utility accelerations." Frisch then considered the mapping inverse to (1.38). This involves the implicit assumption that such an inverse exists. For example, if (to fulfill his first criterion) individual preferences are assumed to be identical and homothetic, then a homogeneous-of-degree-1 utility indicator $\bar{U}(x)$ representing these preferences must be ruled out, since its Hessian determinant – which is the Jacobian determinant of the mapping (1.38) – would then vanish. Consequently, Frisch implicitly had to assume that these preferences are represented by a strongly increasing and concave function $U(x) = f[\bar{U}(x)]$, where $f(u)$ is such that $f'(u) > 0$ and $f''(u) < 0$. In the case, for example, of the homogeneous-of-degree-1 CES utility function $\bar{U}(x) = (\Sigma_{i=1}^n \alpha_i x_i^{-\beta})^{-1/\beta}$ (where $\beta > -1$ and $\Sigma_{i=1}^n \alpha_i = 1$), one could choose $f(u) = -(\text{sgn}\,\beta)u^{-\beta}$ for $\beta \neq 0$ and $f(u) = \log u$ for $\beta \to 0$ to obtain the Bergson family $U(x)$ given by (1.17). With this assumption, the conditions of Gale and Nikaido (1965) are satisfied, and one can define the inverse mapping

$$x_i = \xi_i(v_1, v_2, \ldots, v_n) \qquad (i = 1, 2, \ldots, n) \tag{1.40}$$

and the corresponding elasticities

$$\xi_{ij} = \frac{\partial \xi_i(v)}{\partial v_j} \cdot \frac{v_j}{\xi_i(v)} \tag{1.41}$$

which Frisch described as the "want elasticities."

Frisch stated (1959, p. 182) that "although not invariant under a general transformation of U, the magnitude ξ_{ij} expresses a very realistic fact: It answers the question: *is the want for good j elastic or not with respect to the quantity i*?" Frisch also defended his adherence to a particular cardinal utility indicator in the following terms (p. 178):

> To proceed from assumptions about an abstract theoretical set-up and from them to draw conclusions about the observable world and to test – by rough or more refined means – whether the conformity with observations is "good" enough, is indeed the time honoured procedure that all empirical sciences, including the natural sciences, have used. I shall therefore not plead guilty of heresy even if I do work with choice-theory concepts that are not invariant under a general monotonic transformation of the utility indicator.

There Frisch was absolutely on firm ground. As Debreu (1960) showed, and indeed as Samuelson unwittingly showed in his early criticisms of Frisch cited earlier, the property that there exists a utility indicator $U(x)$ such that $\partial U/\partial x_i > 0$ and $\partial^2 U/\partial x_i^2 < 0$ for $i = 1, 2, \ldots, n$, and $\partial U/\partial x_i \partial x_j = 0$ for $i \neq j$, has strong empirical implications (in particular, that all goods are normal, as Pareto had shown as early as 1892), yet this assumption is not invariant with respect to monotone transformation of the utility function.[13] Frisch proceeded to adopt the assumption of *want-independence* as between certain commodities; in the special case in which the cross-elasticities (1.41) vanish for *all* $i \neq j$ (which Frisch did not assume), the Jacobian matrix of the inverse mapping (1.40) – hence that of the original mapping (1.38) – is diagonal, and the cross-utility accelerations (1.39) vanish for $i \neq j$. But the latter assumption is equiv-

[13] For further discussion and references, see Chipman (1977a).

alent to the assumption of independent commodities, $\partial^2 U/\partial x_i \partial x_j = 0$ for $i \neq j$. Combined with the assumption of identical homothetic preferences needed for aggregation, this brings us right back to Frisch's original assumptions analyzed earlier in Section 1.1: Homotheticity plus universal want-independence implies that utility functions are of the Bergson form (1.17).

The foregoing conclusions (which are stronger than warranted by Frisch's actual assumptions) still do not detract from the usefulness of Frisch's 1959 analysis, however, which rests largely on the interrelations developed among his many new concepts. For example, defining the price elasticity of demand by $\pi_{ij} = \partial h_i/\partial p_j \cdot p_j/h_i$, the income elasticity of demand by $\eta_i = \partial h_i/\partial Y \cdot Y/h_i$, the budget share by $\theta_i = p_i x_i/Y$, and the flexibility of the marginal utility of income (or "money flexibility") by

$$\check{\omega} = \frac{\partial \omega}{\partial Y}\frac{Y}{\omega}, \quad \text{where} \quad \omega(Y, p) = \frac{\partial V(Y, p)}{\partial Y}$$

Frisch presented formulas relating the price elasticities to the remaining concepts.[14] Under the assumptions of the Bergson family of utility functions (1.17), the own-utility accelerations and the money flexibility coincide:

$$v_{ii} = \check{\omega} = -(1 + \beta) = -\frac{1}{\sigma} \qquad (i = 1, 2, \ldots, n)$$

hence the own-want elasticities are the reciprocals of these.

2 Estimation of Demand and Supply Functions, and Statistical Confluence Analysis

2.1 *Estimation of Demand and Supply Functions*

In the early part of his career Frisch had been devoting a great portion of his energies to statistical methods, particularly to the study of multicollinearity (Frisch, 1929a). During that decade, considerable progress had been made in developing methods to estimate demand and supply curves, following the seminal paper by Working (1927), who showed by simple geometric arguments that "statistical demand curves" could be fitted to intersections of shifting demand and supply curves and that they could legitimately be interpreted as demand curves only if supply curves shifted much more than demand curves over the sample period. During that period, Schultz (1925, 1928) was the most notable contributor to the literature on statistical estimation of demand and supply curves; his work took advantage of the fact that because foodstuffs entered into international trade, the relevant data needed to estimate a demand function (consumption) were different from the relevant data needed to estimate a supply function (production). However, there was considerable uncertainty as to the proper procedure to follow in the case of a closed economy.

It was during that same period that Leontief (1929) proposed a solution to the problem. He assumed that demand and supply relations were linear in the logarithms, with constant slopes (elasticities) over time, and were subject to random shifts that were independent as between demand and supply relations. His method (1929, p. 29*) was to divide the time series into two periods and perform regressions in each of the two periods, and then solve the resulting equations jointly to obtain two elasticity estimates, one of which would be interpreted as a demand elasticity, and the other as a supply elasticity.

Leontief's article, which had already been criticized by Schultz,[15] provoked Frisch into writing his *Pitfalls* monograph (1933b). He formulated the model as (p. 11)

$$\begin{aligned} x_t &= u_t + \alpha p_t \qquad \text{(demand)} \\ x_t &= v_t + \beta p_t \qquad \text{(supply)} \end{aligned} \tag{2.1}$$

where x_t and p_t stand for the logarithms of the observed quantity and price of a commodity at time t (I have added the time subscripts) and u_t and v_t are unobserved shifts. He then (p. 12) restated equations (2.1) with the variables x_t, p_t, u_t, and v_t expressed as deviations from their sample means; consequently, he took the sums of squares and cross-products of

[14] These were

$$\pi_{ii} = -\eta_i\left(\theta_i - \frac{1 - \theta_i\eta_i}{\check{\omega}}\right) \quad \text{and} \quad \pi_{ij} = -\eta_i\theta_j\left(1 + \frac{\eta_j}{\check{\omega}}\right) \qquad (i \neq j)$$

(assuming want-independence between i and all other goods for π_{ii} and assuming want-independence between i and j for π_{ij}). Unfortunately I have been unable to reconcile these formulas with those that apply in the case of the Bergson family (1.18), which yield, for $i \neq j$,

$$\pi_{ij} = -\frac{\beta}{1 + \beta}\alpha_j^{1/(1+\beta)} p_j^{-1/(1+\beta)} \Big/ \sum_{k=1}^{n} \alpha_k^{1/(1+\beta)} p_k^{\beta/(1+\beta)}$$

[15] Cf. Schultz (1930, app. II), as largely reproduced later (Schultz, 1938, pp. 83–95). Schultz's criticism rested mostly on the *results* of Leontief's procedure. For example, he applied Leontief's method to data on U.S. *consumption* and prices of sugar that he had used earlier (Schultz, 1928) (where he had employed consumption data only for estimation of the demand curve, and production data for the supply curve), saying (p. 87n): "This example is not unfair to Leontief, for in his own examples he derives both coefficients of elasticity sometimes from the statistics of consumption and prices, and sometimes from the data of production and prices, without considering the problems which arise when the economy under consideration is not a self-contained economy." He then ridiculed the resulting estimate of the supply elasticity of 15.0. However, this leaves open the question whether the mistake was to apply the method to a commodity that enters strongly into international trade or whether it was in the statistical method itself.

the deviations of u_t and v_t from their means, expressed as sample moments

$$m_{uu} = \sum_{t=1}^{n}(u_t - \bar{u})^2, \quad m_{vv} = \sum_{t=1}^{n}(v_t - \bar{v})^2, \quad m_{uv} = \sum_{t=1}^{n}(u_t - \bar{u})(v_t - \bar{v})$$

(where n is the sample size) with similar definitions for m_{xx}, m_{pp}, m_{xp} leading to the three equations

$$\begin{aligned} m_{uu} &= m_{xx} - 2\alpha m_{xp} + \alpha^2 m_{pp} \\ m_{vv} &= m_{xx} - 2\beta m_{xp} + \beta^2 m_{pp} \\ m_{uv} &= m_{xx} - (\alpha + \beta)m_{xp} + \alpha\beta m_{pp} \end{aligned} \tag{2.2}$$

Taking the ratios of the first two and the last two equations of (2.2), he obtained the two "fundamental equations"

$$\begin{aligned} (\alpha\beta - h\beta^2) - (\alpha + \beta - 2h\beta)H + (1 - h)K = 0 \\ (\alpha^2 - k\beta^2) - 2(\alpha - k\beta)H + (1 - k)K = 0 \end{aligned} \tag{2.3}$$

where

$$H = \frac{m_{xp}}{m_{pp}}, \qquad K = \frac{m_{xx}}{m_{pp}}, \qquad h = \frac{m_{uv}}{m_{vv}}, \qquad k = \frac{m_{uu}}{m_{vv}} \tag{2.4}$$

He solved these equations for α and β. He then gave precision to Working's conclusions: In particular, if the parameter k expressing the relative variance of u and v (Frisch used the more colorful terminology "relative violence") goes to zero, then the demand elasticity α can be determined uniquely from the moments (p. 15), but the supply elasticity β will be indeterminate. Today we would say that α is "identifiable" and β "unidentifiable." Frisch described this case as one exhibiting a "Cournot effect on the demand side." Likewise, if $k \to \infty$ (a "Cournot effect on the supply side"), β is identifiable, and α not.

Frisch proceeded to spell out his objections to Leontief's procedure. Because Leontief assumed that the u_t and v_t series were uncorrelated (i.e., $h = 0$), the first of Frisch's fundamental equations (2.3) would reduce to (p. 21)

$$\alpha\beta - (\alpha + \beta)H + K = 0 \tag{2.5}$$

Thus, if one of the elasticities is given, the other is determined. Because Leontief's method consisted in dividing the sample period into two subperiods (in Frisch's terminology, two "materials," i.e., samples or data sets), the foregoing equation is replaced by two, where H and K are subscripted according to the data set. These two equations are then solved simultaneously for the two variables $\alpha\beta$ and $\alpha + \beta$, which is possible provided $H_1 \neq H_2$ and $K_1 = K_2$. Frisch carried out an exhaustive classification

of cases, culminating in a table (p. 30). His general conclusion was that there were only three cases in which Leontief's method would give correct results under his assumption of uncorrelated shifts: (1) The two elasticities are known to be equal in magnitude, but of opposite signs; but in that case an ordinary regression would give the elasticities. (2) There is a pronounced Cournot effect on the demand side in one data set, and a pronounced Cournot effect on the supply side in the other; but in that case, too, straightforward regression would give the correct result. (3) Both the "relative violence" and the correlation have significantly different values in the two data sets. Only in the third case would Leontief's method do better than straight regression. But, he reasoned, for Leontief's method to have any raison d'être, it would have to give good results in other cases.

Leontief (1934) vigorously defended himself against those criticisms. He accepted the foregoing case (3) (including uncorrelated shifts) and stated (p. 357) that "these assumptions are essentially identical with the fundamental properties of the supply and demand relations which I have derived from a detailed discussion of the economic aspect of the problem," and because the other cases were "mathematical configurations which do *not* comply with the fundamental economic assumptions, . . . Professor Frisch is tilting at windmills." He then (p. 358) defended the assumption of uncorrelated shifts by the argument that if the shifts were *perfectly* correlated, the concepts of supply and demand functions would make no sense. (Of course, the shifts will generally be correlated, but not perfectly so, and so this argument appears to be irrelevant.) But finally he came to a technical argument that, although arcane, seems worth describing in detail, because he made it his main point.

Frisch (1933b, p. 12) had expressed the parameters H and K of (2.4) as

$$H = rl, \qquad K = l^2, \quad \text{where } r = \frac{m_{xp}}{(m_{xx}m_{pp})^{1/2}} \quad \text{and} \quad l = \left(\frac{m_{xx}}{m_{pp}}\right)^{1/2} \tag{2.6}$$

That is, r is the correlation between x and p, and l is the "relative violence" of x over p, that is, "the intensity (the amplitude) of fluctuations in x as compared with the intensity of fluctuations in p" (p. 11). Accordingly, under Leontief's assumption $h = 0$, the fundamental equations (2.5) for the two data sets become

$$\begin{aligned} \alpha\beta - (\alpha + \beta)r_1l_1 + l_1^2 = 0 \\ \alpha\beta - (\alpha + \beta)r_2l_2 + l_2^2 = 0 \end{aligned} \tag{2.7}$$

which, when summed, give, in Leontief's notation (1934, p. 360),

$$-(\alpha + \beta)r_1l_1 + l_1^2 = -(\alpha + \beta)r_2l_2 + l_2^2 \tag{2.8}$$

from which Leontief concluded the following: "Now it is evident that if $l_1 = l_2$ it follows that $r_1 = r_2$ and, on the other hand, if $r_1 \neq r_2, l_1 \neq l_2$." He went on to say that "any judgment concerning the 'significance' of the numerical inequality (or equality) $r_1 \neq r_2 \ldots$ necessarily implies a judgment about the 'significance' of the corresponding inequality (or equality) $l_1 \neq l_2$."

Frisch (1934c) began his reply by arguing that the whole intent of his monograph had been to show that Leontief's assumptions concerning independent shifts, however compelling they might be from the economic point of view, were disconfirmed by Leontief's own data. He pointed out, rewriting (2.8) as

$$l_1^2 - l_2^2 = (\alpha + \beta)(r_1 l_1 - r_2 l_2) \tag{2.9}$$

that while it implies, as Leontief noted, that $l_1 = l_2$ and $\alpha + \beta \neq 0$ imply $r_1 = r_2$, it does not show the converse. Setting $r_1 = r_2 = r$ yields $l_1^2 - l_2^2 = (\alpha + \beta)(l_1 - l_2)r$, so that either $l_1 = l_2$ or $l_1 + l_2 = (\alpha + \beta)r$, and the second of these equations can be satisfied in an infinite number of ways; he proceeded to illustrate this with a detailed numerical example. Frisch interpreted the first implication to mean that if uncorrelated shifts are assumed, and data sets are found with $r_1 \neq r_2$ and (approximately) $l_1 = l_2$, then the assumption must be accepted that $\alpha = -\beta$; otherwise the assumption of uncorrelated shifts must be rejected. But as for the converse, Frisch ended his discussion with the following statement:

> One cannot help feeling that the prestige of economics as a science must suffer when papers containing such mistakes and oversights as Dr. Leontief's last paper, appear in a journal of high international standing.

Leontief's rejoinder (1934) was unrepentant. He reiterated the argument that the alternative to perfect independence was perfect correlation, and, as for his slip, he stated that Frisch only "proves, with the help of an elaborate numerical example, that a quadratic equation has more than one solution."

Presumably at the editor's behest, Marschak (1934), who had previously discussed Leontief's work in his own study (1931, pp. 23–8), was brought in as an arbiter. He pointed out that both demand and supply were related to other common variables such as population and price level. He suggested introducing other variables [anticipating the method of instrumental variables introduced by Reiersøl (1945a) and the methods introduced by Koopmans (1949) to overcome the identification problem], as well as choosing the data sets in such a way as to make the independence assumption more likely to be satisfied, rather than arbitrarily.

Frisch's *Pitfalls* monograph (1933b) not only had tremendous influence on the development of the simultaneous-equations approach

in econometrics but also affected many subsequent developments that did not employ this approach, such as the consumer-demand studies of Wold and Juréen (1953) and Stone (1954), which relied basically on the type of reasoning introduced by Frisch.

Finally, it may be supposed that the interchange with Leontief, who had appealed to Marshall (1890) to support his assumptions, had something to do with Frisch's penetrating lectures during that period on Marshall's theory of value, which were later made available in English translation (1950). Frisch was not one to leave any stone unturned.

Almost 50 years after the Leontief–Frisch controversy, Leamer (1981) returned to the subject with a fresh look. Instead of concentrating on consistency of estimates, he approached the problem from the point of view of maximum-likelihood estimation subject to inequality constraints. He assumed the disturbances to be independently (serially and contemporaneously) normally distributed. The inequality constraints were that the demand curve have negative slope, and the supply curve positive slope. He showed that the set of maximum-likelihood estimates of the two elasticities consisted of a hyperbola whose two branches were separated by a horizontal line and a vertical line going through the points $(b, 0)$ and $(0, b)$, respectively, where b is the least-squares estimate. Leamer showed – where I use Frisch's notation α and β in place of Leamer's notation for the demand and supply elasticities – that under the assumptions $\alpha < 0$ and $\beta > 0$,

$$b > 0 \quad \text{implies} \quad \hat{\alpha} < 0, \qquad 0 < b < \hat{\beta} < b_r$$

where b_r is the reverse least-squares estimate (obtained by regressing price on quantity and taking the reciprocal of the regression coefficient), and that, alternatively,

$$b < 0 \quad \text{implies} \quad b_r < \hat{\alpha} < b < 0, \qquad 0 < \hat{\beta}$$

Leamer concluded as follows:

> The method . . . rests on the unlikely assumption that the slopes α and β are constant over time but the variances are not. Still, Leontief did have the hyperbola properly defined,[16] which is only one short step from the results of this paper. It is therefore surprising that Leontief's contribution has been so completely ignored by the post-1940 econometrics literature. The fault seems to me to lie with excessive attention to

[16] I have not been able to find this hyperbola in Leontief (1929), though a similar hyperbola can be found in Allen (1939). Leontief did not assume normality; rather, as Schultz (1930, app. II; 1938, p. 84) explained, Leontief (1929, p. 24*) fitted the demand curve (a straight line on double-logarithmic scale) by minimizing the sum of squares of deviations measured parallel to the (unknown) supply curve, and *vice versa*. See also Schultz (1939b).

> asymptotic properties of estimators and insufficient interest in the shapes of likelihood functions.

It would seem appropriate to add that the independence between demand and supply disturbances is crucial to this result, as Frisch stressed (and Leamer acknowledged).

2.2 *The Frisch-Waugh Theorem*

In the course of his investigations of the relationship between sugar consumption and sugar prices, Frisch had to face up to the problem that, as was contended by a number of authors at that time, if there were a strong upward trend in consumption accompanied by a strong downward trend in price, it would be a mistake to de-trend the data series and then perform a regression on the de-trended series [called the "individual trend method" by Frisch and Waugh (1933)], as opposed to including time explicitly as one of the explanatory variables (called the "partial time regression method"). It was the accomplishment of the paper by Frisch and Waugh (1933) to show that so long as the relations are assumed to be linear, both methods give exactly the same result. The Frisch-Waugh theorem was generalized by Reiersøl (1945b) to any instrumental set of variables and was applied by Lowell (1963) to seasonal adjustment. It has been given prominence in a recent text (Davidson and MacKinnon, 1993, pp. 19–24) and has been further developed and generalized by Fiebig, Bartels, and Krämer (1996), who point out that the result has also been implicitly derived and used by a number of authors.

The problem can be formulated in terms of the simple regression model

$$y = X\beta + \varepsilon = [X_1\ X_2]\begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix} + \varepsilon = X_1\beta_1 + X_2\beta_2 + \varepsilon$$

where the $n \times k$ matrix X (assumed of rank k) is partitioned into $n \times k_1$ and $n \times k_2$ matrices. According to the individual-trend method, one first regresses both y and X_1 on the trend term X_2 to obtain

$$b_2^* = (X_2'X_2)^{-1}X_2'y \quad\text{and}\quad B_2^* = (X_2'X_2)^{-1}X_2'X_1$$

The deviations of y and X_1 from their trends (the de-trended series) are then

$$y* = y - X_2b_2^* = (I - H_2)y \quad\text{and}\quad X_1^* = X_1 - X_2B_2^* = (I - H_2)X_1 \tag{2.10}$$

respectively, where we define $H_i = X_i(X_i'X_i)^{-1}X_i'$ for $i = 1, 2$. The estimate of β_1 by the individual-trend method is then

$$b_1^* = \left(X_1^{*\prime}X_1^*\right)^{-1}X_1^{*\prime}y* \tag{2.11}$$

The estimate of β_1 by the partial-time-regression method is b_1, where

$$b = \begin{bmatrix}b_1\\ b_2\end{bmatrix} = \begin{bmatrix}X_1'X_1 & X_1'X_2\\ X_2'X_1 & X_2'X_2\end{bmatrix}^{-1}\begin{bmatrix}X_1'\\ X_2'\end{bmatrix}y = (X'X)^{-1}X'y \tag{2.12}$$

The expression for b_1 can be obtained from standard formulas for inverses of partitioned matrices, but Frisch and Waugh showed that there is a simpler, direct approach. Defining $H = X(X'X)^{-1}X'$, and observing that

$$X_1 = X\Phi_1 = [X_1\ X_2]\begin{bmatrix}I_{k_1}\\ 0\end{bmatrix} \quad\text{and}\quad X_2 = X\Phi_2 = [X_1\ X_2]\begin{bmatrix}0\\ I_{k_2}\end{bmatrix} \tag{2.13}$$

we see easily that $H_iH = HH_i = H_i$ for $i = 1, 2$; hence

$$(I - H_i)(I - H) = (I - H)(I - H_i) = I - H \qquad (i = 1, 2) \tag{2.14}$$

consequently, from (2.13) and the definition of H,

$$(I - H)X_i = (I - H)X\Phi_i = 0 \quad\text{and}\quad (I - H_i)X_i = 0 \qquad (i = 1, 2) \tag{2.15}$$

Therefore, denoting the residual from the regression (2.12) by $e = y - Xb = (I - H)y$, we have

$$y = Xb + e = X_1b_1 + X_2b_2 + (I - H)y \tag{2.16}$$

Now we observe from (2.14) and (2.13) that premultiplication of (2.16) by $X_1'(I - H_2)$ annihilates the last two terms on the right, leaving

$$X_1'(I - H_2)y = X_1'(I - H_2)X_1b_1$$

hence

$$b_1 = [X_1'(I - H_2)X_1]^{-1}X_1'(I - H_2)y \tag{2.17}$$

Given the definitions (2.10), and using the idempotency and symmetry of H_2, it follows that the estimators (2.11) and (2.17) of β_1 are precisely the same. This is the Frisch-Waugh theorem.

Frisch and Waugh also proved the identity of the residuals

$$e = y - Xb = y - X_1b_1 - X_2b_2 \quad\text{and}\quad e* = y* - X_1^*b_1^*$$

This follows from (2.10) and the fact that

$$
\begin{aligned}
e^* &= (I - H_2)(y - X_1 b_1) && (\text{since } b_1^* = b_1) \\
&= (I - H_2)(y - X_1 b_1 - X_2 b_2) && [\text{since } (I - H_2)X_2 = 0] \\
&= (I - H_2)(y - Xb) \\
&= (I - H_2)(I - H)y \\
&= (I - H)y = e && [\text{from } (2.14)]
\end{aligned}
$$

2.3 *Statistical Confluence Analysis*

Frisch's work on what he called "confluence analysis" stemmed from the anomalous results obtained when statistical methods such as multiple regression were extended from experimental applications to applications involving nonexperimental observations of the type studied by economists. That work consists largely of two main publications (Frisch, 1929a, 1934a). The Introduction to the second of these states (p. 9): "The present study has been undertaken as an indispensable preliminary step for certain projects, namely statistical productivity studies and statistical construction of econometric functions (demand and supply curve and the like). . . ."

Frisch formulated the regression problem in econometrics in terms of a model $y = X\beta$, in which both the $n \times 1$ vector y and the $n \times k$ matrix X of observations on explanatory variables consisted of sums of two terms: a systematic part and a disturbance. "Multicollinearity" refers to linear dependence among the systematic parts of the columns of X; owing to the disturbances, the observed matrix X can always be assumed to have full rank k even if the underlying systematic part of X has rank less than k, which leads to the danger that an investigator may erroneously infer a causal or structural relationship between X and y when there is none. Specifically, he laid down the hypothesis (1934a, p. 85) that the addition of a new explanatory variate was likely to exacerbate the multicollinearity problem, increasing the likelihood that there would be linear dependence among the systematic parts of the explanatory variates. To guard against this, Frisch suggested that it would be preferable to deliberately omit some explanatory variables that *a priori* would be considered relevant. In a passage that is strikingly prophetic of recent developments in statistics that emphasize sacrificing unbiasedness in order to improve mean-square error, Frisch stated (1934a, pp. 86–7) that

> in target shooting the result depends, not only on the correct aiming but just as much on the steadiness with which one pulls the trigger. If for some particular reason it is impossible to pull the trigger steadily when one aims *exactly* at the target, it is quite conceivable that it would be better deliberately to aim a little on the side of the target. And so in statistical analysis it may be found safer deliberately to leave some bias in the regression coefficients by not including a certain variate in the analysis.

Frisch (1934a, p. 60) based his approach on his theorem – which had previously been proved by Gini (1921) and later was generalized by Koopmans (1937, pp. 98–115), Reiersøl (1941), and Willassen (1987) – that the "true" linear-regression coefficient between two variates lies between the two elementary regressions (the regression coefficient of x_1 on x_2 and the reciprocal of the regression coefficient of x_2 on x_1). In order to draw inferences concerning multicollinearity among the systematic parts of the variates, Frisch (1934a) took pairs, triples, and so forth, of these variates and considered the regressions within each l-tuple in all possible directions. If adding a third variate to a pair, say, increased the stability of the regressions in the sense that the regression coefficient of x_1 on x_2 and the inverse of the regression coefficient of x_2 on x_1 were closer together when x_3 was an additional explanatory variable, that was taken as an indication that the disturbances were major parts of these variates; if adding a third variate had the opposite effect, that was taken as an indication of multicollinearity. This is the basic idea of Frisch's "bunch analysis" (1934a, pp. 86–106), closely related to his method of "optimum regression" (1931d) and the theory of "cluster types" in Frisch and Mudgett (1931); a good exposition of his "bunch map" technique was provided by Haavelmo and Staehle (1951, pp. 16–21).

Frisch's work had a strong influence on Koopmans (1937), who provided a probabilistic formulation of Frisch's model, as well as on Reiersøl (1941, 1945a), who introduced the method of instrumental variables. And his formulation of the regression model in terms of errors in both dependent and independent variables led to a flurry of contributions culminating in the surprising finding of Wald (1940) that a consistent estimator of the slope of a straight line $y = \alpha + \beta x$ in a sample of even size n when both variables are subject to error is given by

$$b = \frac{\sum_{t=1}^{m} y_t - \sum_{t=m+1}^{n} y_t}{\sum_{t=1}^{m} x_t - \sum_{t=m+1}^{n} x_t}$$

where $m = n/2$.

It is natural to ask what influence Frisch's confluence analysis had on the development of simultaneous-equations models. Because the stimulus for the latter was mainly due to Haavelmo (1943, 1944), although Frisch (1993b) certainly played a part, as did Marschak and Andrews (1944), it is instructive to consider Haavelmo's retrospective assessment (1950). His basic point was that there was no reason to believe that a "true" structural relationship would hold exactly, as Frisch posited; rather, it would hold only stochastically. Thus, in a negative sense (but very positive for the progress of econometrics), Haavelmo's dissatisfaction with Frisch's formulation probably was the most important factor in the development of the simultaneous-equations approach.

3 Capital Theory and Dynamic Economics

Frisch's earliest work on capital theory was his 1927 article on primary investment and reinvestment. The problem he considered was this: Suppose that at an initial time an investment is made in a number of different goods with different durabilities, and assume that each of these goods has a definite durability and is replaced as soon as it wears out. What will be the subsequent pattern of total reinvestment over time? He gave examples (such as the production of a wooden hammer, iron hammer, and steel hammer with respective durabilities of one, two, and three years) that would give rise to a very marked subsequent limit cycle, but showed that this was atypical (resulting from the distribution of prime numbers from 1 to 6); with finer class intervals, he showed that fluctuations in reinvestment will be damped, so that a smooth flow of reinvestment will be approached asymptotically. This approach to production theory was first made known to English-speaking readers with the 1965 translation of his book *Theory of Production* – a work that has not received the attention that it deserves (cf. Frisch 1965, ch. 16–19, pp. 293–345).

A few years later, Frisch presented his methodological approach to static and dynamic economics in a penetrating Norwegian paper (1929b), of which unfortunately only the introductory sections are available in English.

Shortly thereafter, during a visit to the University of Minnesota, a lively discussion in the Campus Club with Alvin Hansen prompted Frisch to make his first well-known contribution to business-cycle theory (1931e). Not surprisingly, in view of his previous investigation, he again turned his attention to the cyclical nature of replacement investment. The occasion for the discussion was Hansen's treatment (1927, p. 113) building upon the "acceleration principle" introduced by Clark (1917, 1923), among others.[17] Clark had stated (1917, p. 220) that "the demand for maintenance and replacement of existing capital varies with the amount of the demand for finished products, while the demand for new construction or enlargement of stocks depends upon whether or not the

[17] Including Aftalion (1909a, pp. 219–20; 1909b, pp. 71–2; 1913, II, pp. 371–3) and Bickerdike (1914). Earlier hints of the acceleration principle are contained in Cassel (1904, pp. 76–7; 1918, §70, pp. 510–11; 1927, §70, pp. 527–9; 1932a, §70, pp. 528–30; 1932b, II, §69, pp. 596–8) and Bouniatian (1908, pp. 109–10; 1915, p. 172; 1922, p. 236; 1930, p. 266). Curiously, Frisch (1931e, p. 646n) cited Bickerdike's article and attributed it to Clark, evidently mistaking it for Clark (1917); but it seems that his criticisms of Clark were based entirely on Clark (1923).

sales of the finished product are growing." Thus, "if demand be treated as a rate of speed . . . , maintenance varies roughly with the speed, but new construction depends upon the acceleration." From this he had concluded (pp. 222–3) that "in order to bring about an absolute shrinkage in the demand for the intermediate product, all that may [*sic*] be needed is that the final demand should slacken its rate of growth." However, he subsequently (1923, p. 390) stated (less cautiously) that "the makers of capital equipment are bound, in the nature of the case, to suffer an absolute decline in the demand for their products . . . whenever ultimate demand slackens its rate of growth," and further, "once demand for finished products starts growing it cannot pause or else the derived demand for means of production will [*sic*] shrink. . . ."

Frisch (1931e) set out to correct that formulation by introducing the model

$$I = \delta K + \dot{K} = \varkappa(\delta C + \dot{C}); \quad \text{hence} \quad \dot{I} = \varkappa(\delta \dot{C} + \ddot{C}) \tag{3.1}$$

where C is the rate of consumption, K is the aggregate capital stock, and I is the rate of gross investment. It assumes that capital must maintain a fixed ratio $\varkappa$ to consumption (i.e., $K = \varkappa C$), so that net investment must retain the corresponding relation to the rate of change of consumption (i.e., $\dot{K} = \varkappa\dot{C}$), and replacement investment is a fixed proportion of the capital stock (i.e., $R = \delta K$). Thus, a slowdown in the rate of growth of consumption, though it necessarily entails a fall in *net* investment $\dot{K}$, need not lead to a fall in *gross* investment I. Moreover, Frisch observed (p. 649) that

> the system is, so far, quite indeterminate. In the reduced form [(3.1)] of the relationship, we have two variables but only one equation. It would be attempting the logically impossible if, from the conditions here considered, we should try to demonstrate that the system must after a while turn into depression.

He followed with numerous numerical examples illustrating this point.

Clark, in his reply (1931), essentially admitted his error, but without missing the opportunity to retort that while Frisch's point was mathematically correct, his own treatment still had "the legitimacy of sound formulations adapted to the thinking of the majority who are laymen in mathematics" – evidently because replacement investment was, in his opinion, a small proportion of total investment. In his rejoinder, Frisch (1932a, p. 254) pointed out that if consumption is assumed to move cyclically, there is a small interval of time after the point of fastest increase in consumption during which investment continues to rise:

> This little interval of time around the turning-point in capital production is the critical interval in the business cycle. It is here that the enigma of business cycles lies. And in this critical interval capital production

> for expansion purposes *is not the dominating element in total capital production.*

That is, replacement is not a small proportion of investment as Clark claimed. In his "Further Word," Clark (1932, p. 692) agreed that

> for a full explanation . . . one must take account of factors acting in the reverse direction, namely, the fact that actual movements of consumer demand depend on the movements of purchasing power; and these in turn are governed by the rate of production in general. . . .

And Clark expressed the "hope that this discussion may stimulate some mathematical economist to produce a solution" (p. 693). As we shall see, his hope was soon fulfilled; and for his part, most of his later (1935) treatise on the business cycle may be regarded as his own attempt to meet Frisch's challenge.

Prior to developing his business-cycle model, Frisch (1928) had developed a method to extract cyclical and trend components from economic time series in such a way as to allow for variable amplitudes and phases, and he announced (1931b) a forthcoming monograph in which these methods would be further developed and applied to business-cycle research; unfortunately that never appeared. Subsequently he presented his vision (1931c) of future business-cycle research at a conference in Stockholm, including even a proposed model containing some 38 variables and equations. He also carried out a detailed empirical study of price and quantity fluctuations in several countries (1932d), with data in some cases going back as far as the fourteenth century.

Frisch (1933c) proceeded to develop a self-contained model of the business cycle in a brilliant contribution incorporating ideas from his early 1927 work, as well as his controversy with Clark. His first task was to generalize (3.1), and the next was to add enough suitable equations to make the model self-contained and determinate. The generalization consisted mainly in taking account of the fact that capital is needed to produce not only consumer goods but also more capital goods. In the following exposition I shall fill in some details that Frisch left to the reader and replace Frisch's notation with the more common Keynesian type of notation that has become customary in the macroeconomic literature.

Let K_C and K_I denote the stocks of capital in the consumer-good and investment-good industries, respectively. Overlooking, at first, the length of time needed to produce capital (or, alternatively, interpreting K_C and K_I as the desired stocks of capital in the respective industries), these are related to production (and consumption, because there are no inventories) of consumer and producer goods by

$$K_C = \varkappa_C C \quad \text{and} \quad K_I = \varkappa_I I$$

where $\varkappa_C$ and $\varkappa_I$ are fixed coefficients. Next, denoting by δ_C and δ_I the rates of depreciation of capital in the respective industries (the reciprocals of their durability), replacement investment is given by

$$R = \delta_C K_C + \delta_I K_I = \delta_C \varkappa_C C + \delta_I \varkappa_I I = \delta_C \varkappa_C C + \delta_I \varkappa_I (R + \dot{K}) \tag{3.2}$$

which is equal to gross investment I when net investment $\dot{K}$ is zero. Thus, under stationary conditions (cf. Frisch 1933c, pp. 176–7) replacement investment is[18]

$$R = \frac{\delta_C \varkappa_C}{1 - \delta_I \varkappa_I} C \tag{3.3}$$

Now imagine that the level of consumption moves from one stationary level C to another $C + \Delta C$; then formula (3.3) shows that replacement investment must rise from the indicated level to the one in which the factor on the right is $C + \Delta C$. Thus, net investment is

$$\begin{aligned} \dot{K} &= \dot{K}_C + \dot{K}_I = \varkappa_C \dot{C} + \varkappa_I \dot{I} \\ &= \varkappa_C \dot{C} + \varkappa_I \frac{\delta_C \varkappa_C}{1 - \delta_I \varkappa_I} \dot{C} \\ &= \left(1 + \frac{\delta_C \varkappa_I}{1 - \delta_I \varkappa_I}\right) \varkappa_C \dot{C} \end{aligned} \tag{3.4}$$

It follows from (3.3) and (3.4) that gross investment is

$$I = R + \dot{K} = \frac{\varkappa_C}{1 - \delta_I \varkappa_I} \{\delta_C C + [1 + \varkappa_I(\delta_C - \delta_I)]\dot{C}\} = mC + \mu\dot{C} \tag{3.5}$$

where m and μ are the symbols used by Frisch to denote the more complicated coefficients of C and $\dot{C}$ shown in (3.5). Note that if the durability of capital is the same in the two industries ($\delta_C = \delta_I$), then (3.5) reduces formally to (3.1).

Frisch next took into account the production lag in construction of capital. The actual capital formation at time t is the result of activities that have taken place in the past and that continue up to time t; these activities Frisch called "production starting" or "capital starting," as opposed to the capital formation itself, which he called the "carry-on activity." The latter is determined from the former by the convolution operation

$$I_t = \int_0^\infty D(\tau) J_{t-\tau} d\tau \tag{3.6}$$

[18] More generally, one would have to add to the right side of (3.3) the term $[\delta_I \varkappa_I/(1 - \delta_I \varkappa_I)]\dot{K}$, introducing an unpleasant complication into formula (3.4).

where J_t is the investment starting at time t, and $D(t)$ is the delay function or, in Frisch's terminology, the "advancement function." Frisch chose, for simplicity, the function

$$D(t) = \begin{cases} 1/\theta & \text{for } 0 < t < \theta \\ 0 & \text{for } t \geqq \theta \end{cases} \tag{3.7}$$

Defining the average period of production as $\int_0^\infty D(\tau)\tau/d\tau$,[19] this becomes $\theta/2$ for the foregoing special function. Differentiating (3.6) with respect to time, and employing (3.7), we obtain

$$\dot{I}_t = \frac{1}{\theta}\int_0^\theta \dot{J}_{t-\tau}d\tau = \frac{1}{\theta}[-J_{t-\tau}]_0^\theta = \frac{1}{\theta}(J_t - J_{t-\theta}) \tag{3.8}$$

Frisch attributed this idea to Aftalion.[20]

Formula (3.5) now needs to be adjusted to take account of production lags. Frisch did that by replacing the actual capital formation or "carry-on activity" I on the left by the "starting investment" J:

$$J = mC + \mu\dot{C} \tag{3.9}$$

Equations (3.9) and (3.8) are two of the fundamental equations of Frisch's system – one a differential equation and the other a difference equation – involving the three variables I, J, and C. One more equation is needed to close the system.

For that, Frisch got his idea (or at least the terminology) from Walras (1926, ch. 29, §275, p. 305; 1954, p. 321); the desired cash balance (*encaisse désirée*), $\alpha C + \beta I$, where α and β are constants, consists in amounts of money needed for both consumption and investment purposes, which are assumed to rise during the boom faster than the money supply M. An excess of money demand over money supply is assumed to have its greatest impact on consumption (Frisch, 1933c, p. 179), causing it to contract; thus, the third equation of the system is[21]

[19] This appears to be Frisch's implicit definition, and it is suitable for the case of a stationary (i.e., nongrowing) economy. In the case of an economy growing at the rate r, one would want to include the factor e^{-rt} in the integrand; cf. Chipman (1977b, p. 301).

[20] Cf. Aftalion (1908, p. 703; 1909b, p. 11; 1913, II, ch. VII, pp. 113ff.), who in turn attributed the idea to Böhm-Bawerk and Jevons. This formulation had already been anticipated in Frisch (1931e, p. 652): "In Aftalion's manner we could distinguish between capital goods *ordered* and capital goods *delivered*." Frisch apparently was unaware that formula (3.1) also goes back to Aftalion; see footnote 17.

[21] Frisch multiplies the *encaisse désirée* by a factor λ, which he describes (p. 189) as its "reining-in effect," but it seems to me simpler to absorb this factor in the coefficients α and β. Further, Frisch, in place of M in (3.10), writes c (a constant); but it seems easier to understand the model if this is interpreted as the money supply. However, it is treated as a (constant) parameter to be estimated, rather than as a variable.

$$\dot{C} = M - (\alpha C + \beta I) \tag{3.10}$$

Frisch made the important observation (p. 180) in connection with his previous controversy with Clark that if there is no production lag (3.6), so that J is replaced by I in (3.9), as in (3.5), then equations (3.9) and (3.10) together imply a linear relation between C and I; substitution of this relation in either (3.9) or (3.10) yields a first-order linear differential equation in a single variable, whose solution is a path of steady growth or contraction.[22] This proved, he observed, that contrary to Clark's claim, (3.1) was not sufficient by itself to explain turning points in the business cycle.

Frisch's model of the "propagation" process thus consists of the three equations (3.8), (3.9), and (3.10). He assumed that its solution would be of the form[23]

$$i_t = a_{i0} + \sum_{k=1}^{\infty} a_{ik}e^{\rho_k t} \quad \text{for } i = C, I, J$$

where the ρ_k are complex numbers, and found that the ρ_k must be roots of the characteristic equation

$$\frac{\theta\rho}{1 - e^{-\theta\rho}} = -\beta\frac{m + \mu\rho}{\alpha + \rho}$$

He studied the solutions for the choices $\theta = 6, m = 0.5, \mu = 10, \alpha = 0.1$, and $\beta = 0.05$ (p. 186). One curious but extremely interesting feature of the model, which might have been better clarified if prices and incomes

[22] Specifically, it yields the differential equation

$$(1 - \beta\mu)\dot{C} + (\alpha - \beta m)C - M = 0$$

which has the solution

$$C = \frac{M}{\alpha - \beta m} + ke^{\lambda t}$$

(for some arbitrary constant k), where $\lambda = -(\alpha - \beta m)/(1 - \beta\mu)$. If $\lambda > 0$, this gives a path of steady growth; for the numerical coefficients supposed by Frisch (see the later text), $\lambda < 0$, and the solution converges to $C = M/(\alpha - \beta m)$.

[23] It seems to be an open question whether or not this gives the correct general solution for Frisch's system. It is based on the Herz–Herglotz method adopted by Alfred Lotka in the 1930s for solving the integral equation of renewal theory. Following an acrimonious controversy between him and Gabriel Preinreich, the dispute was finally settled by Feller (1941) in favor of Preinreich's criticism (but not in favor of Preinreich's own method). For instance, Frisch (1965, p. 323) derived the renewal equation and tried to solve it by Lotka's method, with mixed success, whereas the solution by Laplace transforms is known from Feller (1941). On this, see Chipman (1977b), as well as the comments by Samuelson (1974, pp. 8–10).

had been incorporated in the analysis, is that during an upswing workers are being paid for production of capital that will be completed in time to produce more consumer goods only several years hence; because inventory accumulation or decumulation is not allowed, equation (3.10) must bring about *forced saving* (likewise, forced dissaving during a downswing).

Frisch's business-cycle model was an extraordinary achievement. It was the first self-contained model of the cycle. Moreover, he found three cycles, of durations 8.57, 3.50, and 2.20 years, in conformity with much statistical data. Some observers have criticized it for exhibiting contrafactual symmetry between upswings and downswings (Blatt, 1980), and no doubt it has many other faults that Frisch would readily have conceded; but it was a first, and it still compares very favorably with other models that have been developed. Kalecki's (1935) model, for which Frisch and Holme (1935) provided a solution, was followed by those of Samuelson (1939a, 1939b), Kaldor (1940), Hicks (1950), and Goodwin (1951). Except for Kalecki's model, none of those allowed for replacement investment, which Frisch considered of such importance. By allowing for nonlinearities such as capacity constraints and nonnegativity constraints, Hicks and Goodwin were able to generate persistent limit cycles. But Frisch accomplished the same thing by his idea, stimulated by Wicksell (1907, 1918), of subjecting his model – which he regarded as a model of the "propagation" process, leading to damped cycles – to random shocks or "impulses" so as to obtain persistent cycles – an approach that found justification in the work of Slutsky (1927, 1937), Yule (1927), and Hotelling (1927). His strong belief that economic cycles were basically damped may have been formed by the findings of his early 1927 article on replacement cycles. In Frisch and Holme (1935), Frisch criticized Kalecki for forcing the roots of his system to lie on the unit circle in order to obtain persistent cycles, a procedure that was further criticized by Haavelmo (1940).

Frisch (1933c), toward the end of his paper, illustrated his method for a model of the pendulum. He showed by means of simulation that application of random shocks to a pendulum would give rise to what he called a "changing harmonic" (already defined in his 1928 paper), namely "a curve that is moving more or less regularly in cycles, the length of the period and also the amplitude being to some extent variable, these variations taking place, however, within such limits that it is reasonable to speak of an *average* period and an *average* amplitude" (p. 202). For the case of his business-cycle model (for which simulations would not have been possible in 1933), he identified the random shocks with innovations in Schumpeter's (1926) theory. Subsequently, in replying to a criticism of Tintner (1938), Frisch (1939c) elaborated by explaining that "a shock is any event which contradicts the assumptions of some pure economic theory and thus prevents the variables from following the exact course

implied by that theory." He agreed with Tintner that Slutsky had not explained "in terms of economic theory how the effects of the shocks are 'summed'," but went on to point out that his paper was precisely intended to fill that gap:

> I showed that if a set of variables are defined by a linear system . . . , the time shape of one of the variables, *when hit by shocks*, is obtained by extending to the shock series a moving summation whose weight system is exactly the same sort of curve as that which would have given the time evolution of this variable, if no shocks had occurred. Thus economic theory furnishes the weight system, statistical theory does the rest.

In his article on Wicksell, Frisch (1952, pp. 698–9) quoted a very revealing passage from Wicksell outlining this approach to business-cycle theory, in which the shocks were identified with technical progress. In fact, this article reveals the profound influence that Wicksell's work must have had on Frisch, and in particular there is probably more of Wicksell behind (3.10) than Walras.

Not until 1990 did it occur to anybody to take the logical step of using a computer to simulate Frisch's model – which of course was impossible in 1933. Thalberg (1990) carried out such simulations and found that fluctuations were persistent and that "the amplitude of the fluctuations increases with the disturbances [which he added to (3.9) and (3.10)], while the length of the fluctuations is more or less strongly tied to the propagation mechanism" (p. 108). But "the generated fluctuations become irregular and unpredictable," precluding forecasting more than two years ahead. Furthermore, the three variables of the model did not move in tandem, which he attributed to the omission from the model of any feedback from investment to consumption. He therefore added another independent variable $\dot{J}_{t-\xi}$ to (3.10) so as to approximate the Keynesian multiplier.[24] He also replaced the advancement function by $D(\tau) = \omega^2 \tau e^{-\omega\tau}$. Although the modified system exhibited instability for some values of the parameters, in the contrary case it appeared to reproduce observed cycles in a satisfactory way.

4 Depression and Circulation Planning

One of Frisch's most striking contributions (1934b) was born out of the experience of the Great Depression; it might have had greater impact if it had not been overshadowed by Keynes's *General Theory*, which

[24] Another choice, which would bring the model closer to that of Samuelson (1939a, 1939b), would be $\dot{C}_{t-\xi} + \dot{J}_{t-\xi}$ (the rate of increase of gross national product at factor cost), whose coefficient would be the marginal propensity to consume. The pre-Keynesian nature of Frisch's model is brought out by the fact that Frisch never thought to sum consumption and investment to obtain national income.

appeared two years later.[25] Frisch opened his book-length article with the following passage:

> The most striking paradox of great depressions, and particularly of the present one, is the fact that poverty is imposed on us in the midst of a world of plenty. Many kinds of goods are actually present in large quantities, and other kinds could without any difficulty be brought forth in abundance, if only the available enormous productive power was let loose. Yet, in spite of this technical and physical abundance, most of us are forced to cut down consumption. We are compelled to make real sacrifices in order to economize in the use of these very goods and services that *could* easily be produced in abundance if we would only use our resources.

He attributed the problem to a defect in the form of organization and to a situation in which groups "are forced *mutually to undermine each other's position*. . . . This meaningless vicious circle is what I understand by the incapsulating phenomenon." The picture he painted was of a game that ended up in a sub-Pareto-optimal situation and even a prisoner's dilemma. But in some respects it also is a picture of *unstable equilibrium*.

The problem was well expressed in a subsequent publication (1963, p. 2) as follows:

> To illustrate . . . we may think of the tailor and the shoemaker who were standing looking at each other with sorrowful faces. The sorrow stemmed from the fact that the tailor did not *dare* to order the shoes he needed because he was not sure that he would be able to sell any suit to the shoemaker, while the shoemaker did not *dare* to order the suit he needed because he was not sure that he would be able to sell any shoes to the tailor. If we include also the baker, the fisherman and a few others in the picture we have a good illustration of what is meant by the multilateral balancing problem, and in particular we get an illustration of the need for a system which can *assure* everybody that he is taking part in a game where a multilateral balancing is *automatically provided for*.

One way to interpret these passages is that Frisch is showing the failure of Walras's law to hold in a dynamic economy. The worker, according to this law, will effectively demand the goods which he would purchase if he were to get the job he is looking for. The essential asymmetry between the buying and selling sides of the transaction is what leads to the breakdown; the selling must take place first. And all proofs of stability of competitive equilibrium assume Walras's law.[26]

[25] It was preceded by an equally striking little book, *Saving and Circulation Control* (1933a), in which he traced the mechanism by which a rise in saving, instead of leading to increased investment, would lead to decreases in output and employment.

[26] I have discussed this question in greater detail elsewhere (Chipman, 1965).

In his 1934 paper, Frisch set forth a schematic model of the tailor-shoemaker problem, with "coefficients of optimism" in the difference equations. He showed in this schematic model how fluctuations could result merely from the nature of the trading game. He then introduced a model of "planned exchange" in which warrants were issued in order to reestablish equilibrium, then destroyed when no longer needed. The scheme is, of course, very close to the idea of the role of a central bank as lender of last resort.

In later years Frisch extended these ideas to the problem of international equilibrium (1947, 1948b). He extended his 1934 idea of a "request matrix" to that of a "trade matrix" whose typical element a_{ij} represented the value of country i's exports to country j, where the term "exports" is used in a very broad sense to include long-term capital transactions; thus, a sustained capital movement from i to j would be included among the exports from j to i. Frisch's object was to devise a scheme to prevent a vicious circle from developing in which, starting from unbalanced trade, each country would successively reduce its imports to the level of its exports until an equilibrium would be reached with a very low level of trade. He distinguished (1947, p. 537) between the "specialization effect" of trade (the mutual advantage of international specialization and trade) and the "payment effect" referred to earlier, which he attributed to the failure of Say's law (which I have always considered to be the proposition that a full-employment competitive equilibrium exists, whereas the failure of Walras's law does not preclude the existence of such an equilibrium, but results in its being dynamically unstable). He took as his "rough indicator of the welfare created by international trade" the *total* volume of international transactions in goods and services (p. 545); however, because this is later identified with the sum of all the elements of the trade matrix, it is really the total *value* of international transactions, capital as well as current.[27] The other concept Frisch introduces is the *skewness* of the matrix: If the ith-row sum (country i's total exports) exceeds the ith-column sum (country i's total imports), the amount is entered to the right of the ith row as a surplus; in the contrary case the amount is entered at the bottom of the ith column as a deficit. The sum of all the surpluses (which is, of course, equal to the sum of all the deficits) is the skewness. The mathematical policy problem is to reduce the skewness to zero with minimum reduction in the total value of transactions.

Frisch noted correctly that a one-country-at-a-time policy of reduction of deficit countries' imports to the levels of their exports, by *pro-*

[27] Frisch subsequently insisted (1963, p. 5) that he did mean volumes, and that "this can be made tangible through the medium of value figures expressed in base year prices." However, very few countries have price indices of imports and exports, and the available unit-value data are regarded with much suspicion by many experts.

rata proportional reduction of imports from all countries, was a pessimal solution to this problem, that a balancing of accounts with much less drastic reduction in total transactions could be accomplished by discriminatory reduction in imports as between different countries. The algorithm used by Frisch to obtain the optimal solution was that developed in his earlier paper (1934b).

Frisch's 1947 paper was greeted with respectful skepticism on the part of Polak (1948) and Meier (1948). Polak pointed out that the beneficial effect of discrimination rested on the assumption of asymmetry in the behaviors of surplus and deficit countries, deficit countries reducing their imports to the levels of their exports, but surplus countries not increasing their imports to the levels of their exports. This asymmetry, he thought, was realistic in 1947, but not in the period of the Great Depression with reference to which Frisch made his case. In that period, he stated, countries' imports depended on their national income and relative prices, and there was no reason to believe that the marginal propensities to import for different countries were different. His second main point was that if countries were forced to switch their sources of imports to other countries, "they would have to accept a commodity structure of imports which might be quite incompatible with full national output" unless there were considerable multilateral negotiations; but then if one is to have negotiation, why not use it to increase world trade rather than mitigate its decline? Finally, he pointed out that

> if serious disequilibria in international trade and payments were dealt with, not by the necessary fundamental adjustments, but by successive doses of discrimination, the "specialization" advantages of the remaining international trade would continually decline and the volume of international trade would lose all value as an indicator of national welfare.

Meier (1948) made the additional point that Frisch's analysis

> makes the further assumption that when the active balance of a surplus country is reduced by the import restrictions of the deficit countries, the former country will retain its imports from all countries at the initial level in spite of the fact that its national income will be falling due to the decline in its export balance.

Frisch offered no response to those criticisms. In his second formulation (1948b) he broke each country's trade accounts down into 10 categories, numbered from 0 to 9. However, those numbers represented not SITC categories, but rather the "export priority numbers" fixed by the authorities in exporting countries. An importer in country A would contact an exporter in country B, who would file an application with its export authority, which would assign the export priority number and return the application to the importer, who would file an application with his own country's import authority, which would in turn assign an import priority number to the application. Finally, an international "bureau of compensation" would consider all these applications, each with two priority numbers, to decide which ones to accept. It would compute the mean of the export priorities by weighting them by the total amount of exports applied for, and similarly for the mean of the import priorities. For each country, the import average would have to exceed the export average. The bureau of compensation would then maximize the "global priority surplus" (the sum of countries' average import priorities minus the sum of countries' average export priorities) subject to balanced trade for each country.

Hinshaw's (1948) discussion, following Frisch's, consisted largely in a very useful exposition of Frisch's earlier (1947) article, but with respect to the new formulation he pointed out that it would require utmost cooperation among countries and that "if this almost utopian degree of international cooperation were to be attained, it would be just as easy to follow a more rational criterion of trade policy than the essentially restrictive goal of minimum-contraction-via-discrimination" (p. 274). Furthermore, "the scheme involves a much more comprehensive supervision and control of international transactions than is the case even at the present time."

With the Mexican debacle so fresh in our memories,[28] we can appreciate the need for new approaches to the problem of attaining international equilibrium. Frisch's scheme had definite drawbacks, but the problem it addressed is very real, one that was even addressed by a very classic economist, the young John Stuart Mill (1844). There can be no doubt that the problems to which Frisch drew attention urgently need the continued attention of theoretical economists.

References

Afriat, S. N. (1972). The theory of international comparisons of real income and prices. In: *International Comparisons of Prices and Output*, ed. D. J. Daly, pp. 13–69. New York: Columbia University Press.

(1977). *The Price Index*. Cambridge University Press.

Aftalion, A. (1908). La réalité des surproductions générales. *Revue d'économie politique* 22:696–706.

(1909a). La réalité des surproductions générales. *Revue d'économie politique*. 23:81–117, 201–29, 241–59.

(1909b). *Essai d'une théorie des crises périodiques de surproduction. La réalité des surproductions générales*. Paris: Librairie de la Société du Recueil J.-B. Sirey et du Journal du Palais, Ancien Maison L. Larose & Forcel.

(1913). *Les crises périodiques de surproduction*, 2 vols. Paris: Marcel Rivière et Cie.

Allen, R. G. D. (1933). On the marginal utility of money and its application. *Economica* 13:186–209.

[28] These words were written in 1995, but they apply with still greater force to the 1997–8 crisis in Southeast Asia.

(1935). Some observations on the theory and practice of price index numbers. *Review of Economic Studies* 3:57–66.

(1939). The assumptions of linear regression. *Economica, N.S.* 6:191–201.

(1949). The economic theory of index numbers. *Economica, N.S.* 16:197–203.

(1975). *Index Numbers in Theory and Practice*. Chicago: Aldine.

Alt, F. (1936). Über die Meßbarkeit des Nutzens. *Zeitschrift für Nationalökonomie* 7:161–9. English translation (1971): On the measurability of utility. In: *Preferences, Utility, and Demand*, ed. J. S. Chipman, L. Hurwicz, M. K. Richter, and H. F. Sonnenschein, pp. 429–31. New York: Harcourt Brace Jovanovich.

Antonelli, G. B. (1886). *Sulla teoria matematica della economia politica*. Pisa: Tipografia del Folchetto. English translation (1971): On the mathematical theory of political economy. In: *Preferences, Utility, and Demand*, ed. J. S. Chipman, L. Hurwicz, M. K. Richter, and H. F. Sonnenschein, pp. 332–64. New York: Harcourt Brace Jovanovich.

Arrow, K. J. (1960). The work of Ragnar Frisch, econometrician. *Econometrica* 28:175–92.

Arrow, K. J., Chenery, H. B., Minhas, B. S., and Solow, R. M. (1961). Capital–labor substitution and economic efficiency. *Review of Economics and Statistics* 43:225–50.

Balk, B. M. (1995). Axiomatic price index theory: a survey. *International Statistical Review* 63:69–93.

Bergson (Burk), A. (1936). Real income, expenditure proportionality, and Frisch's "New methods of measuring marginal utility." *Review of Economic Studies* 4:33–52. Reprinted in Bergson, A. (1966). *Essays in Normative Economics*, pp. 93–122. Cambridge, MA: Harvard University Press (Belknap Press).

Bernoulli, D. (1738). Specimen theoriae novae de mensura sortis. *Commentarii academiae scientiarum imperialis Petropolitanae* 5:175–92. English translation (1954): Exposition of a new theory on the measurement of risk. *Econometrica* 22:23–6.

Bickerdike, C. F. (1914). A non-monetary cause of fluctuations in employment. *Economic Journal* 24:357–70.

Birck, L. V. (1922). *The Theory of Marginal Value* (translated from the Danish *Læren om Grænseværdien*). London: George Routledge & Sons.

Blatt, J. M. (1980). On the Frisch model of business cycles. *Oxford Economic Papers, N.S.* 32:467–79.

Bortkiewicz, L. von (1928). Review of *Der Sinn der Indexzahlen* by Gottfried Haberler. *Magazin der Wirtschaft* 4:427–9.

(1932). Die Kaufkraft des Geldes und ihre Messung. *Nordic Statistical Journal* 4:1–68.

Bouniatian, M. (1908). *Wirtschaftskrisen und Ueberkapitalisation*. Munich: Ernst Reinhardt.

(1915). *Ekonomicheskie krizisy*, 2nd ed. Moscow: Tip. Mysl'.

(1922). *Les crises économiques* (French translation of 2nd edition). Paris: Marcel Giard.

(1930). *Les crises économiques*, 2nd French edition. Paris: Marcel Giard.

Bowley, A. L. (1928). Notes on index numbers. *Economic Journal* 38:216–37.

(1932). Review of *New Methods of Measuring Marginal Utility* by Ragnar Frisch. *Economic Journal* 42:252–6.

(1938). Note on Professor Frisch's "The problem of index numbers." *Econometrica* 6:83–4.

Buscheguennce (Byushgens), S. S. (1925). Sur une classe des hypersurfaces (a propos de "l'index idéal" de M. Irv. Fischer.). *Matematicheskii sbornik (Recueil mathématique de la Société Mathématique de Moscou)* 32:625–31.

Cassel, G. (1904). Om kriser och dåliga tider (On crises and bad times). *Ekonomisk Tidskrift* 6:21–35, 51–81.

(1918). *Theoretische Sozialökonomie*. Leipzig: C. F. Wintersche Verlagshandlung.

(1927). *Theoretische Sozialökonomie*, 4th ed. Leipzig: A. Deichertsche Verlagsbuchhandlung Dr. Werner Scholl.

(1932a). *Theoretische Sozialökonomie*, 5th ed. Leipzig: A. Deichertsche Verlagsbuchhandlung Dr. Werner Scholl.

(1932b). *The Theory of Social Economy*, 2 vols. (English translation by S. L. Barron from the manuscript of the 5th edition). London: Ernest Benn.

Chipman, J. S. (1953). Linear programming. *Review of Economics and Statistics* 35:101–17.

(1965). The nature and meaning of equilibrium in economic theory. In: *Functionalism in the Social Sciences*, pp. 35–64. Monograph no. 5. Philadelphia: American Academy of Political and Social Science.

(1977a). An empirical implication of Auspitz-Lieben-Edgeworth-Pareto complementarity. *Journal of Economic Theory* 14:228–31.

(1977b). A renewal model of economic growth: the continuous case. *Econometrica* 45:295–316.

Chipman, J. S., Hurwicz, L., Richter, M. K., and Sonnenschein, H. F. (eds.) (1971). *Preferences, Utility, and Demand*. New York: Harcourt Brace Jovanovich.

Chipman, J. S., and Moore, J. C. (1980). Compensating variation, consumer's surplus, and welfare. *American Economic Review* 70:933–49.

Clark, J. M. (1917). Business acceleration and the law of demand: a technical factor in economic cycles. *Journal of Political Economy* 25:217–35.

(1923). *Studies in the Economics of Overhead Costs*. University of Chicago Press.

(1931). Capital production and consumer-taking – a reply. *Journal of Political Economy* 39:814–16.

(1932). A further word. *Journal of Political Economy* 40:691–3.

(1935). *Strategic Factors in Business Cycles*. Washington, DC: National Bureau of Economic Research.

Davidson, R., and MacKinnon, J. G. (1993). *Estimation and Inference in Econometrics*. Oxford University Press.

Debreu, G. (1960). Topological methods in cardinal utility theory. In: *Mathematical Methods in the Social Sciences, 1959*, ed. K. J. Arrow, S. Karlin, and P. Suppes, pp. 16–26. Stanford University Press.

Diewert, W. E. (1976). Exact and superlative index numbers. *Journal of Econometrics* 4:115–45.

(1981). The economic theory of index numbers: a survey. In: *Essays in the Theory and Measurement of Consumer Behaviour: In Honour of Richard Stone*, ed. A. Deaton, pp. 163–208. Cambridge University Press.

(1983). The theory of the cost-of-living index and the measurement of welfare change. In: *Price Level Measurement*, ed. W. E. Diewert and C. Montmarquette, pp. 163–233. Ottawa: Ministry of Supply and Services Canada. Reprinted (1990) in *Price Level Measurement*, ed. W. E. Diewert, pp. 79–147. Amsterdam: North Holland.

Edgeworth, F. Y. (1881). *Mathematical Psychics*. London: C. Kegan Paul & Co.

Eichhorn, W. (1976). Fisher's tests revisited. *Econometrica* 44:247–56.

Eichhorn, W., and Voeller, J. (1976). *Theory of the Price Index*. Berlin: Springer-Verlag.

Feller, W. (1941). On the integral equation of renewal theory. *Annals of Mathematical Statistics* 12:243–67.

Fiebig, D. G., Bartels, R., and Krämer, W. (1996). The Frisch-Waugh theorem and generalized least squares. *Econometric Reviews* 15:431–43.

Fisher, I. (1892). Mathematical investigations in the theory of value and prices. *Transactions of the Connecticut Academy* 9:1–124. Reprinted (1925) by Yale University Press.

(1922). *The Making of Index Numbers.* Boston: Houghton Mifflin.

(1927). A statistical method for measuring marginal utility and testing the justice of a progressive income tax. In: *Economic Essays, Contributed in Honor of John Bates Clark*, ed. J. H. Hollander, pp. 157–93. New York: Macmillan.

Frisch, R. (1926a). Sur un problème d'économie pure. *Norsk Matematisk Forenings Skrifter* serie I, no. 16, pp. 1–40. Reprinted (1957) in *Metroeconomica* 9:79–111. English translation (1971): On a problem in pure economics. In: *Preferences, Utility, and Demand*, ed. J. S. Chipman, L. Hurwicz, M. K. Richter, and H. F. Sonnenschein, pp. 386–423. New York: Harcourt Brace Jovanovich.

(1926b). Kvantitativ formulering av den teoretiske økonomikks lover. *Statsøkonomisk Tidsskrift* 40:299–334.

(1927). Sammenhengen mellem primærinvestering og reinvestering. *Statsøkonomisk Tidsskrift* 41:117–52. English translation by Einar Hope (1965): The relationship between primary investment and reinvestment (mimeographed). Economics Translation Series, no. 8, ed. J. S. Chipman. University of Minnesota.

(1928). Changing harmonics and other general types of components in empirical series. *Skandinavisk Aktuarietidskrift* 11:220–36.

(1929a). Correlation and scatter in statistical variables. *Nordic Statistical Journal* 8:36–102.

(1929b). Statistikk og dynamikk i den økonomiske teori. *Nationaløkonomisk Tidsskrift* 67:321–79. English translation of sections 1-3 (1992): Statics and dynamics in economic theory. *Structural Change and Economic Dynamics* 3:391–401.

(1930). Necessary and sufficient conditions regarding the form of an index number which shall meet certain of Fisher's tests. *Journal of the American Statistical Association* 25:397–406.

(1931a). Der Einfluß von Veränderungen des Preisniveaus auf den Grenznutzen des Geldes. *Zeitschrift für Nationalökonomie* 2:625–31.

(1931b). A method of decomposing an empirical series into its cyclical and progressive components. *Journal of the American Statistical Association* (*Suppl.*) 26:73–8.

(1931c). Konjunkturbevegelsen som statistik og som teoretisk problem. In: *Förhandlingar vid Nordiska Nationalekonomiska Mötet i Stockholm 15–17 juni 1931*, pp. 127–68, 224–34. Stockholm: Ivar Hæggströms Boktryckeri och Bokförlags A. B.

(1931d). The optimum regression (mimeographed). Department of Economics, University of Minnesota.

(1931e). The interrelation between capital production and consumer-taking. *Journal of Political Economy* 39:646–54.

(1932a). A rejoinder. *Journal of Political Economy* 40:253–5.

(1932b). A final word. *Journal of Political Economy* 40:694.

(1932c). *New Methods of Measuring Marginal Utility*. Tübingen: Verlag von J. C. B. Mohr (Paul Siebeck).

(1932d). Konjunkturene. In: *Verdensøkonomien i efterkrigstiden*, ed. W. Keilhau, I. Wedervang, and R. Frisch, pp. 79–139. Universitetets Radioforedrag, serie B, no. 6. Oslo: Forlagt av H. Aschehoug & Co. (W. Nygaard).

(1933a). *Sparing og circulasjons regulering*. Oslo: Fabritius & Sønners Forlag.

(1933b). *Pitfalls in the Statistical Construction of Demand and Supply Curves*. Veröffentlichen der Frankfurter Gesellschaft für Konjunkturforschung, Herausgegeben von Dr. Eugen Altschul, Neue Folge, Heft 5 (16. Heft der ganzen Reihe). Leipzig: Hans Buske Verlag.

(1933c). Propagation problems and impulse problems in dynamic economics. In: *Economic Essays in Honour of Gustav Cassel*, pp. 171–205. London: George Allen & Unwin.

(1934a). *Statistical Confluence Analysis by Means of Complete Regression Systems*. Publikasjon no. 5. Oslo: Universitetets Økonomisk Institutt.

(1934b). Circulation planning: proposal for a national organization of a commodity and service exchange. *Econometrica* 2:258–336, 422–35.

(1934c). More pitfalls in demand and supply curve analysis. *Quarterly Journal of Economics* 48:749–55.

(1934d). Reply to Mr. Subramanian's note. *Journal of the American Statistical Association* 29:317.

(1936a). Annual survey of general economic theory: the problem of index numbers. *Econometrica* 4:1–38.

(1936b). Errata. *Econometrica* 4:192.

(1936c). Note on the term "econometrics." *Econometrica* 4:95.

(1936d). On the notion of equilibrium and disequilibrium. *Review of Economic Studies* 3:100–5.

(1937). Price index comparisons between structurally different markets. In: *Comptes rendus du Congrès International des Mathématiciens Oslo 1936*, tome II (Conférence de Sections), pp. 220–1. Oslo: A. W. Brøggers Boktrykkeri.

(1938). The double-expenditure method. *Econometrica* 6:85–90.

(1939a). The Dupuit taxation theorem. *Econometrica* 7:145–50.

(1939b). A further note on the Dupuit taxation theorem. *Econometrica* 7:156–7.

(1939c). A note on errors in time series. *Quarterly Journal of Economics* 53:639–40.

(1947). On the need for forecasting a multilateral balance of payments. *American Economic Review* 37:535–51.

(1948a). Overdeterminateness and optimum equilibrium. *Nordisk Tidsskrift for Teknisk Økonomie* 12:95–105.
(1948b). Outline of a system of multicompensatory trade. *Review of Economics and Statistics* 30:265–71.
(1950). Alfred Marshall's theory of value. *Quarterly Journal of Economics* 64:495–524.
(1951). Knut Wicksell. A cornerstone in modern economic theory. Memorandum from the Institute of Economics, University of Oslo. Reprinted (1952): Frisch on Wicksell. In: *The Development of Economic Thought. Great Economists in Perspective*, ed. H. W. Spiegel, pp. 652–99. New York: Wiley.
(1959). A complete scheme for computing all direct and cross demand elasticities in a model with many sectors. *Econometrica* 27:177–96.
(1963). A multilateral trade clearing agency. *Statsøkonomisk Tidsskrift* 77:1–10.
(1964). Dynamic utility. *Econometrica* 32:418–24.
(1965). *Theory of Production*, translated from the 9th Norwegian edition of *Innledning til Produksjonsteorien*. Chicago: Rand McNally.
Frisch, R. and Holme, H. (1935). The characteristic solutions of a mixed difference and differential equation occurring in economic dynamics. *Econometrica* 3:225–39.
Frisch, R., and Mudgett, B. D. (1931). Statistical correlation and the theory of cluster types. *Journal of the American Statistical Association* 26:375–92.
Frisch, R., and Mudgett, B. D. (1932). A correction. *Journal of the American Statistical Association* 27:187.
Frisch, R., and Waugh, F. V. (1933). Partial time regressions as compared with individual trends. *Econometrica* 4:387–401.
Gale, D., and Nikaido, H. (1965). The Jacobian matrix and global univalence of mappings. *Mathematische Annalen* 159:81–93.
Gini, C. (1921). Sull'interpolazione di una retta quando i valori della variabile indipendente sono affeti da errori accidentali. *Metron* 1:63–82.
Goodwin, R. M. (1951). The nonlinear accelerator and the persistence of business cycles. *Econometrica* 19:1–17.
Haavelmo, T. (1940). The inadequacy of testing dynamic theory by comparing theoretical solutions and observed cycles. *Econometrica* 8:312–21.
(1943). The statistical implications of a system of simultaneous equations. *Econometrica* 11:1–12.
(1944). The probability approach in econometrics. *Econometrica* (*Suppl.*) 12:1–118.
(1950). Remarks on Frisch's confluence analysis and its use in econometrics. In: *Statistical Inference in Dynamic Economic Models*, ed. T. C. Koopmans, pp. 258–65. New York: Wiley.
Haavelmo, T., and Staehle, H. (1951). The elements of Frisch's confluence analysis. Memorandum fra Universitetets Socialøkonomiske Institutt, Oslo, serie 5, no. 6.
Haberler, G. (1927). *Der Sinn der Indexzahlen. Eine Untersuchung über den Begriff des Preisniveaus und die Methoden seiner Messung.* Tübingen: Verlag von J. C. B. Mohr (Paul Siebeck).
(1929). Der volkswirtschaftliche Geldwert und die Preisindexziffern. Eine Erwiderung. *Weltwirtschaftliches Archiv* 30:6**–14**.
Hansen, A. H. (1927). *Business-Cycle Theory*. Boston: Ginn & Co.

Hardy, G. H., Littlewood, J. E., and Pólya, G. (1934). *Inequalities*. Cambridge University Press.
Hicks, J. R. (1950) *A Contribution to the Theory of the Trade Cycle*. Oxford: Clarendon Press.
(1969). Direct and indirect additivity. *Econometrica* 37:353–4.
Hinshaw, R. (1948). Professor Frisch on discrimination and multilateral trade. *Review of Economics and Statistics* 30:271–5.
Hotelling, H. (1927). Differential equations subject to error, and population estimates. *Journal of the American Statistical Association* 22:283–314.
(1938). The general welfare in relation to problems of taxation and of railway and utility rates. *Econometrica* 6:242–69.
(1939). The relation of prices to marginal costs in an optimum system. *Econometrica* 7:151–5, 158–9.
Houthakker, H. S. (1960). Additive preferences. *Econometrica* 28:244–57.
Jevons, W. S. (1871). *The Theory of Political Economy*. New York: Macmillan. Second edition 1897, third 1888, fourth 1911.
Johansen, L. (1969). Ragnar Frisch's contributions to economics. *Swedish Journal of Economics* 71:302–24.
Jordan, C. (1924). On Daniel Bernoulli's "moral expectation" and on a new conception of expectation. *American Mathematical Monthly* 31:183–90.
Kaldor, N. (1940). A model of the trade cycle. *Economic Journal* 50:78–92.
Kalecki, M. (1935). A macrodynamic theory of business cycles. *Econometrica* 3:327–44.
Keynes, J. M. (1930). *A Treatise on Money*, 2 vols. London: Macmillan.
(1936). *The General Theory of Employment, Interest and Money*. London: Macmillan.
Klein, L. R., and Rubin, H. (1948). A constant-utility index of the cost of living. *Review of Economic Studies* 15:84–7.
Konüs (Konyus), A. A. (1924). Problema istinnovo indeksa stoimosti zhizni *Konyunkturnyi institut. Ekonomicheskii byulleten' (Economic Bulletin of the Conjuncture Institute)* (*Moscow*) 9–10:64–71. English translation (1939): The problem of the true index of the cost of living. *Econometrica* 7:10–29.
Konüs (Konyus), A. A., and Buscheguennce (Byushgens), S. S. (1926). K probleme pokupatelnoi cili deneg (On the problem of the purchasing power of money). *Voprosi Konyunkturi* (*Problems of Economic Conditions*) (supplement to the *Economic Bulletin of the Conjuncture Institute*) 2:151–72.
Koopmans, T. (1937). *Linear Regression Analysis of Economic Time Series*. Haarlem: De Erven F. Bohn N. V.
(1949). Identification problems in economic model construction. *Econometrica* 19:125–44. Reprinted (1953) in *Studies in Econometric Method*, ed. W. C. Hood and T. C. Koopmans, pp. 27–48. New York: Wiley.
Leamer, E. E. (1981). Is it a demand curve, or is it a supply curve? Partial identification through inequality constraints. *Review of Economics and Statistics* 63:319–27.
Leontief, W. W. (1929). Ein Versuch zur statistischen Analyse von Angebot und Nachfrage (with an appendix, Mathematische Ableitung der Berechningsformeln, by Robert Schmidt). *Weltwirtschaftliches Archiv* 30:1*–53*.
(1934). Pitfalls in the construction of demand and supply curves: a reply. *Quarterly Journal of Economics* 48:355–61, 755–9.

Lovell, M. C. (1963). Seasonal adjustment of economic time series and multiple regression analysis. *Journal of the American Statistical Association* 58:993–1010.

McFadden, D. (1963). Further results on CES production functions. *Review of Economic Studies* 30:73–83.

Malmquist, S. (1953). Index numbers and indifference surfaces. *Trabajos de Estadistica* 4:209–42.

Marschak, J. (1931). *Elastizität der Nachfrage.* Tübingen: Verlag von J. C. B. Mohr (Paul Siebeck).

(1934). More pitfalls in demand and supply curve analysis: some comments. *Quarterly Journal of Economics* 48:759–66.

Marschak, J., and Andrews, W. H., Jr. (1944). Random simultaneous equations and the theory of production. *Econometrica* 12:143–205.

Marshall, A. (1890). *Principles of Economics.* London: Macmillan.

Meier, G. M. (1948). The trade matrix: a further comment on Professor Frisch's paper. *American Economic Review* 38:624–6.

Menderhausen, H. (1938). The definition of "equal well-being" in Frisch's double-expenditure method. *Econometrica* 6:285–6.

Mill, J. S. (1844). On the influence of consumption on production. Essay II in: *Essays on Some Unsettled Questions of Political Economy*, pp. 47–74. London: John W. Parker.

Pareto, V. (1892). Considerazioni sui principi fondamentali dell'economia politica pura. *Giornale degli Economisti* 5:119–57.

(1906). *Manuale di economia politica, con una introduzione alla scienza sociale.* Milan: Società Editrice Libraria. French translation (1909): *Manuel d'Économie politique.* Paris: V. Giard et E. Brière.

Pigou, A. C. (1912). *Wealth and Welfare.* London: Macmillan.

(1926). *The Economics of Welfare.* London: Macmillan. Second edition 1924, third 1929, fourth 1932.

Polak, J. J. (1948). Balancing international trade: a comment on Professor Frisch's paper. *American Economic Review* 38:139–42.

Pollak, R. A. (1971). The theory of the cost-of-living index. Research discussion paper no. 11, Office of Prices and Living Conditions, U.S. Bureau of Labor Statistics. First published (1983) in *Price Level Measurement*, ed. W. E. Diewert and C. Montmarquette, pp. 87–161. Ottawa: Ministry of Supply and Services Canada. Also published (1990) in *Price Level Measurement*, ed. W. E. Diewert, pp. 5–77. Amsterdam: North Holland.

Reiersøl, O. (1941). Confluence analysis by means of lag moments and other methods of confluence analysis. *Econometrica* 9:1–24.

(1945a). Confluence analysis by means of instrumental sets of variables. *Arkiv för Matematik, Astronomi och Fysik*, 32A:1–119.

(1945b). Residual variables in regression and confluence analysis. *Skandinavisk Aktuarietidsskrift* 28:201–17.

Roy, R. (1942). *De l'utilité. Contribution à la théorie des choix.* Paris: Hermann & Cie.

Samuelson, P. A. (1939a). Interactions between the multiplier analysis and the principle of acceleration. *Review of Economics and Statistics* 21:75–8.

(1939b). A synthesis of the principle of acceleration and the multiplier. *Journal of Political Economy* 47:786–97.

(1942). Constancy of the marginal utility of income. In: *Studies in Mathematical Economics and Econometrics, in Memory of Henry Schultz*, ed. O. Lange, F. McIntyre, and T. O. Yntema, pp. 75–91. University of Chicago Press.

(1947). *Foundations of Economic Analysis.* Cambridge, MA: Harvard University Press.

(1951). Abstract of a theorem concerning substitutability in open Leontief models. In: *Activity Analysis of Production and Allocation*, ed. T. C. Koopmans, pp. 142–6. New York: Wiley.

(1965). Using full duality to show that simultaneously additive direct and indirect utilities implies unitary price elasticity of demand. *Econometrica* 33:781–96.

(1969). Corrected formulation of direct and indirect additivity. *Econometrica* 37:355–9.

(1974). Remembrances of Frisch. *European Economic Review* 5:7–23.

Samuelson, P. A., and Swamy, S. (1974). Invariant economic index numbers and canonical duality: survey and synthesis. *American Economic Review* 64:942–52.

Schultz, H. (1925). The statistical law of demand as illustrated by the demand for sugar. *Journal of Political Economy* 33:481–504, 577–637.

(1928). *The Statistical Laws of Demand and Supply, with Special Application to Sugar.* University of Chicago Press.

(1930). *Der Sinn der statistischen Nachfragekurven.* Veröffentlichen der Frankfurter Gesellschaft für Konjunkturforschung, Herausgegeben von Dr. Eugen Altschul, Heft 10. Bonn: Kurt Schroeder Verlag. 1930. [German translation of The meaning of statistical demand curves (mimeographed), University of Chicago, 1930].

(1933). Frisch on the measurement of utility. *Journal of Political Economy* 41:95–116.

(1938). *The Theory and Measurement of Demand.* University of Chicago Press.

(1939a). A misunderstanding in index-number theory: the true Konüs condition on cost-of-living index numbers and its limitations. *Econometrica* 7:1–9.

(1939b). The assumptions of linear regression: comments. *Economica, N.S.* 6:202–4.

Schumpeter, J. A. (1926). *Theorie der wirstchaftlichen Entwicklung. Eine Untersuchung über Unternehmergewinn, Kapital, Kredit, Zins und der Konjunkturzyklus*, 2nd ed. Munich: Duncker & Humblot. English translation (1934): *The Theory of Economic Development: An Inquiry into Profits, Capital, Credit, Interest, and the Business Cycle.* Cambridge, MA: Harvard University Press.

Shephard, R. W. (1953). *Cost and Production Functions*, Princeton, N.J., Princeton University Press.

Slutsky, E. (1927). The summation of random causes as the source of cyclic processes (in Russian). *Voprosi ekonomiky* 3. English translation, revised by the author (1937): *Econometrica* 5:105–46.

Staehle, H. (1934). International comparison of food costs, part I. In: *International Comparisons of Cost of Living.* Studies and reports of the International Labour Office, series N (statistics), no. 20. Geneva: International Labour Office.

(1935). A development of the economic theory of price index numbers. *Review of Economic Studies* 2:163–88.
(1936). Further notes on index numbers. *Review of Economic Studies* 3:153–5.
Stone, R. (1954). *The Measurement of Consumer Expenditure and Behaviour in the United Kingdom 1920–1938*, vol. I. Cambridge University Press.
Subramanian, S. (1934). On a certain conclusion of Frisch's. *Journal of the American Statistical Association* 29:316–17.
Swamy, S. (1965). Consistency of Fisher's tests. *Econometrica* 33:619–23.
Thalberg, B. (1990). A reconsideration of Frisch's original cycle model. In: *Nonlinear and Multisectoral Macrodynamics. Essays in Honour of Richard Goodwin*, ed. K. Velupillai, pp. 96–117. New York University Press.
Tintner, G. (1938). A note on economic aspects of the theory of errors in time series. *Quarterly Journal of Economics* 53:141–9.
Uzawa, H. (1962). Production functions with constant elasticities of substitution. *Review of Economic Studies* 29:291–9.
Wald, A. (1937). Zur Theorie der Preisindexziffern. *Zeitschrift für Nationalökonomie* 8:179–219.
(1939). A new formula for the index of cost of living. *Econometrica* 7:319–31.
(1940). The fitting of straight lines if both variables are subject to error. *Annals of Mathematical Statistics* 11:284–300.
Walras, L. (1926). *Élements d'économie politique pure, ou Théorie de la richesse sociale.* Paris: R. Pichon et R. Durand-Auzias. English translation by W. Jaffé (1954): *Elements of Pure Economics, or The Theory of Social Wealth.* London: George Allen & Unwin.
Wicksell, K. (1907). Krisernas gåta. *Statsøkonomisk Tidsskrift* 21:255–86.
(1918). Ett bidrag till krisernas teori (review of *Goda och dårliga tider* by Karl Petander). *Ekonomisk Tidskrift* 20:66–75.
Willassen, Y. (1987). A simple alternative derivation of a useful theorem in linear "errors-in-variables" regression models together with some clarifications. *Journal of Multivariate Analysis* 2:296–311.
Wold, H., and Juréen, L. (1953). *Demand Analysis*. New York: Wiley.
Working, E. J. (1927). What do statistical "demand curves" show? *Quarterly Journal of Economics* 41:212–35.
Yule, G. U. (1927). On a method of investigating periodicities in disturbed series, with special reference to Wolfer's sunspot numbers. *Philosophical Transactions of the Royal Society of London* [*A*] 226:267–98.

PART TWO

UTILITY MEASUREMENT