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Propagation Problems and Impulse Problems in Dynamic Economics.

By

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PROPAGATION PROBLEMS AND IMPULSE PROBLEMS IN DYNAMIC ECONOMICS*

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I. INTRODUCTION

THE majority of the economic oscillations which we encounter seem to be explained most plausibly as free oscillations. In many cases they seem to be produced by the fact that certain exterior impulses hit the economic mechanism and thereby initiate more or less regular oscillations.

The most important feature of the free oscillations is that the length of the cycles and the tendency towards dampening are determined by the intrinsic structure of the swinging system, while the intensity (the amplitude) of the fluctuations is determined primarily by the exterior impulse. An important consequence of this is that a more or less regular fluctuation may be produced by a cause which operates irregularly. There need not be any synchronism between the initiating force or forces and the movement of the swinging system. This fact has frequently been overlooked in economic cycle analysis.

If a cyclical variation is analysed from the point of view of a free oscillation, we have to distinguish between two fundamental problems: first, the *propagation* problem; second, the *impulse* problem. The propagation problem is the problem of explaining by the structural properties of the swinging system what the character of the swings would be in case the system was started in some initial situation. This must be done by an essentially dynamic theory, that is to say, by a theory that explains how one situation grows out of the foregoing. In this type of analysis we consider not only a set of magnitudes in a given point of time and study the interrelations between them, but we consider the magnitudes of certain variables in different points of time, and we introduce certain equations which embrace at the same time several of these magnitudes belonging to different instants. This

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is the essential characteristic of a dynamic theory. Only by a theory of this type can we explain how one situation grows out of the foregoing. This type of analysis is basically different from the kind of analysis that is represented by a system of Walrasian equations; indeed in such a system all the variables belong to the same point of time.

In one respect, however, must the dynamic system be similar to the Walrasian: it must be determinate. That is to say, the theory must contain just as many equations as there are unknowns. Only by elaborating a theory that is determinate in this sense can we explain how one situation grows out of the foregoing. This, too, is a fact that has frequently been overlooked in business cycle analysis. Often the business cycle theorists have tried to do something which is equivalent to determining the evolution of a certain number of variables from a number of conditions that is smaller than the number of these variables. It would not be difficult to indicate examples of this from the literature of business cycles.

When we approach the study of business cycle with the intention of carrying through an analysis that is truly dynamic and determinate in the above sense, we are naturally led to distinguish between two types of analyses: the micro-dynamic and the macro-dynamic types. The micro-dynamic analysis is an analysis by which we try to explain in some detail the behaviour of a certain section of the huge economic mechanism, taking for granted that certain general parameters are given. Obviously it may well be that we obtain more or less cyclical fluctuations in such sub-systems, even though the general parameters are given. The essence of this type of analysis is to show the details of the evolution of a given specific market, the behaviour of a given type of consumers, and so on.

The macro-dynamic analysis, on the other hand, tries to give an account of the fluctuations of the whole economic system taken in its entirety. Obviously in this case it is impossible to carry through the analysis in great detail. Of course, it is always possible to give even a macro-dynamic analysis in detail if we confine ourselves to a purely *formal* theory. Indeed, it is always possible by a suitable system of subscripts and superscripts, etc., to introduce practically all factors which we may imagine: all individual commodities, all individual entrepreneurs, all individual consumers, etc., and to write out various kinds of relationships between these magnitudes, taking care that the number of equations is equal to the number of variables. Such a theory, however, would only have a rather limited interest. In such a theory

it would hardly be possible to study such fundamental problems as the *exact time shape* of the solutions, the question of whether one group of phenomena is lagging behind or leading before another group, the question of whether one part of the system will oscillate with higher amplitudes than another part, and so on. But these latter problems are just the essential problems in business cycle analysis. In order to attack these problems on a macro-dynamic basis so as to explain the movement of the system taken in its entirety, we must deliberately disregard a considerable amount of the details of the picture. We may perhaps start by throwing all kinds of production into one variable, all consumption into another, and so on, imagining that the notions "production," "consumption," and so on, can be measured by some sort of total indices.

At present certain examples of micro-dynamic analyses have been worked out, but as far as I know no determinate macro-dynamic analysis is yet to be found in the literature. In particular no attempt seems to have been made to show in an exact way what the relations between the propagation analysis and the impulse analysis are in this field.

In the present paper I propose to offer some remarks on these problems.

2. LE TABLEAU ÉCONOMIQUE

In order to indicate the most important variables entering into the macro-dynamic system we may use a graphical illustration as the one exhibited in Fig. 1.

The system expressed in Fig. 1 is a completely closed system. All economic activity is here represented as a circulation in and out of certain sections of the system. Some of these sections may best be visualized as receptacles (those are the ones indicated in the figure by circles), others may be visualized as machines that receive inputs and deliver outputs (those are the ones indicated in the figure by squares). There are three receptacles, namely, the forces of nature, the stock of capital goods, and the stock of consumer goods. And there are three machines: the human machine, the machine producing capital goods, and the machine producing consumer goods.

The notation is chosen such that capital letters indicate stocks and small letters flows. For instance, R means that part of land (or other forces of nature) which is engaged in the production of consumer goods, r is the services rendered by R per unit time. Similarly V is the stock of capital goods engaged in the production of consumer goods and v the services rendered by this stock per unit time. Further, a is labour (manual or mental) entering into the production of consumer goods, so that the total input in the production of consumer goods is $r + v + a$.

The complete macro-dynamic problem, as I conceive of it, consists in describing as realistically as possible the kind of relations that exist between the various magnitudes in the Tableau Economique exhibited

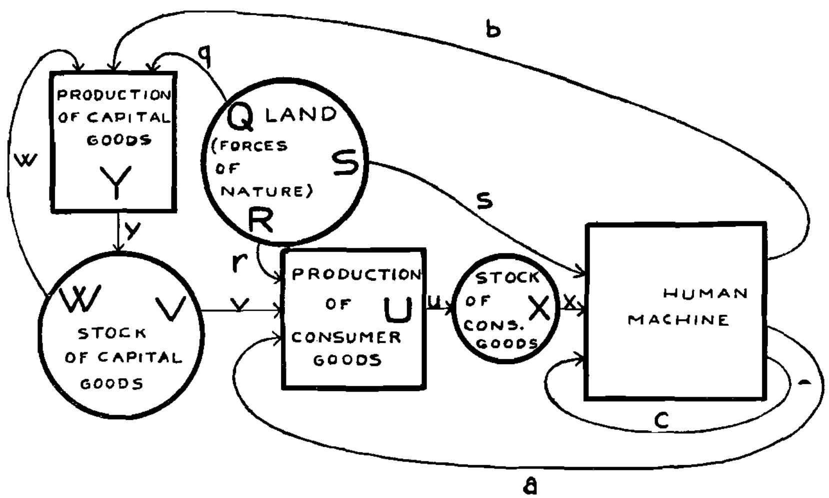


FIG. 1

in Fig. 1, and from the nature of these relations to explain the movements, cyclical or otherwise, of the system. This analysis, in order to be complete, must show exactly what sort of fluctuations are to be expected, how the length of the cycles will be determined from the nature of the dynamic connection between the variables in the Tableau Economique, how the damping exponents, if any, may be derived, etc. In the present paper I shall not make any attempt to solve this problem completely. I shall confine myself to systems that are still more simplified than the one exhibited in Fig. 1. I shall commence by a system that represents, so to speak, the extreme limit of simplification, but which is, however, completely *determinate* in the sense that it contains the same number of variables as conditions. I shall then introduce little by little more complications into the picture, remembering, however, all the time to keep the system determinate. This

procedure has one interesting feature: it enables us to draw some conclusions about those properties of the system that may account for the cyclical character of the variations. Indeed, the most simplified cases are characterized by monotonic evolution without oscillations, and it is only by adding certain complications to the picture that we get systems where the theoretical movement will contain oscillations. It is interesting to note at what stage in this hierarchic order of theoretical set-ups the oscillatory movements come in.

3. SIMPLIFIED SYSTEMS WITHOUT OSCILLATIONS

We shall first consider the following case. Let us assume that the yearly consumption is equal to the yearly production of consumers' goods, so that there is no stock of consumers' goods. But let us take account of the stock of fixed capital goods as an essential element of the analysis. The depreciation on this capital stock will be made up by two terms: a term expressing the depreciation caused by the use of capital goods in the production of consumers' goods, and a term caused by the use of capital goods in the production of other capital goods. For simplicity we shall assume that in both these two fields the depreciation on the fixed capital goods employed are proportional to the intensity with which they are used, this intensity being measured by the volume of the output in the two fields. If h and k are the depreciation coefficients in the capital producing industry and in the consumer goods industry respectively, the total yearly depreciation on the nation's capital stock will be $hx + ky$, where x is the yearly production of consumers' goods and y the yearly production of capital goods. Our assumption amounts to saying that h and k are technically given constants.

What will be the forces determining the annual production of capital goods y? There are two factors exerting an influence on y. First, the need to keep up the existing capital stock, replacing the part of it that is worn out. Second, the need for an increase in total capital stock that may be caused by the fact that the annual consumption is increasing. This latter factor is essentially a progression (or degression) factor, and does not exist when consumption is stationary. I shall consider these two factors in turn.

First let us assume that the annual consumption is kept constant at a given level x. How much annual capital production y will this necessitate? This may be expressed in terms of the depreciation coefficients in the following way. Let total capital stock be denoted Z. The rate of increase of this stock will obviously be

$$\dot{Z} = y - (hx + ky) \tag{1}$$

Since the stationary case is characterized by $\dot{Z} = 0$, the stationary levels of x and y must obviously be connected by the relation $y = hx + ky$, i.e.

$$y = mx \tag{2}$$

where

$$m = \frac{h}{1 - k}$$

The constant m represents the *total* depreciation on the capital stock associated with the production of a unit of consumers' goods, when we take account not only of the *direct* depreciation due to the fact that fixed capital is used in the production of consumer goods, but also take account of the fact that fixed capital has to be used in the production of those capital goods that must be produced for replacement purposes. This follows from the way in which (2) was deduced, and it may also be verified by following the depreciation process for an infinite number of steps backwards. Indeed, the production of x causes a direct depreciation of hx. In order to replace these hx units of capital, a further depreciation of khx is caused, and this amount has to be added to the annual capital replacement production. But adding the amount khx to the annual capital production means that the annual depreciation is increased by $k \, . \, (khx) = k^2hx$, which also has to be added to the annual capital production, and so on. Continuing in this way, we find that the total annual capital production needed to maintain the constant consumption x (with no change in the total capital stock) is equal to

$$hx + khx + k^2hx + \ldots = \frac{h}{1 - k}x = mx$$

which is formula (2). For this reason m may be called the *total*, h and k the *partial* depreciation coefficients.

Now let us consider the other factor that effects the annual capital production, namely, the *change* $\dot{x}$ in the annual production of consumption goods.

Let us take a simple example. Suppose that a capital stock of 1,000 units is needed in order to produce a yearly national income (i.e. a yearly national consumption) equal to 100 units, then if the

production of consumer goods rests stationary at a level of 100 units per year, it is only necessary to produce each year enough capital goods to replace the capitals worn out, namely, $m \, . \, 100$, m being the constant in (2).

But if there is in a given year an *increase*, say, of 5 units, in the production of consumer goods, then it is necessary during that year to increase the stock of capital goods. Indeed, in order to maintain a yearly production of consumer goods equal to 105, there is needed a capital stock equal to 1,050. During the year in question it is therefore necessary to produce an additional 50 units of capital goods. We are thus led to assume that the yearly production of capital goods can be expressed in a form where there occurs not only the term (2) but also a term that is proportional to $\dot{x}$, i.e. y will be of the form

$$y = mx + \mu\dot{x} \tag{3}$$

where m and μ are constants. The constant m expresses the wear and tear on capital goods caused directly and indirectly by the production of one unit of consumption, and μ expresses the size of capital stock that is needed directly and indirectly in order to produce one unit of consumption per year. In other words, μ is the total "production coefficient," in the Walrasian sense, for the factor capital.

The two influences expressed by the two terms in (3) have been the object of a certain discussion in the literature which ought to be mentioned here. Professor Wesley C. Mitchell, in one of his studies, observed that the maximum in the production of capital goods preceded the maximum in the production of consumer goods (or, which amounts to the same, the sales of consumer goods if stock variations of consumer goods are disregarded). From this he drew the conclusion that it is rather in production than in consumption we ought to look for the factors that can explain the turning-point of the cycle. Professor J. M. Clark objected to this conclusion. He said that the *rate of increase* of consumption exerts a considerable influence on the production of capital goods, and that the movement of this rate of increase precedes the movement of the absolute value of the consumption. Indeed, during a cyclical movement the rate of increase will be the highest, about one-quarter period before the maximum point is reached in the quantity itself.

The effect which Clark had in mind is obviously the effect which we have expressed by the second term in (3). If we think only of this term, disregarding the first term, we will have the situation where y is simply proportional to the rate of increase of x. If the movement is cyclical, we would consequently have a situation as the one exhibited in Fig. 2.

In Fig. 2 we notice that the peak in consumption comes after the peak in capital production, but if we compare production with the *rate of increase* of consumption, we find that there is synchronism: the maximum rate of increase in consumption occurs at the same moment

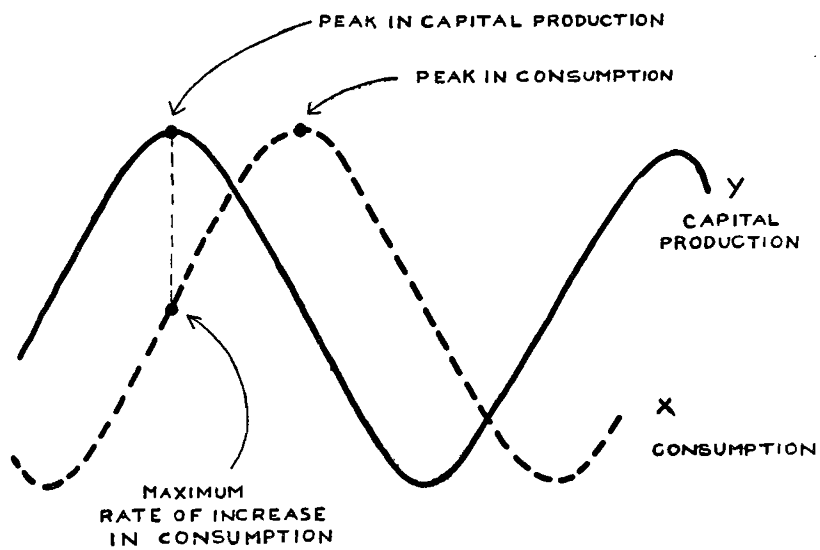


FIG. 2

as the peak in the actual size of capital production. The fact observed by Mitchell can, therefore, just as well be explained—as Clark did—by saying that it is consumption which exerts an influence on production. This is an interesting observation, and it is correct if taken only as an expression for *one* of the factors influencing production.*

In order to have a complete and correct picture we must, however, take account of both terms in (3), and we must also look for some other relation between our variables. So far the problem is not yet determinate.

* Clark, in his discussion of this question, went further than this. He tried to prove that the fact here considered could by itself explain the turning-point of the business cycle, but this is not correct. Indeed, equation (3) is only one equation between two variables. Consequently many types of evolutions are possible. See the discussion between Professor Clark and the present author in the *Journal of Political Economy*, 1931–32.

In order to make the problem determinate we need to introduce an equation expressing the behaviour of the consumers. We shall do this by introducing the Walrasian idea of an *encaisse désirée.* This notion will be introduced here only as a parameter by means of which we express a certain feature of the behaviour of the consumers. The parameter is going to be introduced in the equations and then eliminated. Its introduction, therefore, does not mean that we are actually elaborating a monetary theory of business cycle. It is only an intermediary parameter introduced in order to enable the formulation of a certain simple hypothesis.

The *encaisse désirée*, the need for cash on hand, is made up of two parts: cash needed for the transaction of consumer goods and producer goods respectively. The first of these parts may of course always be written as a certain factor r times the sale of consumer goods, and the second part as a certain factor s times the production of capital goods, provided the factors r and s are properly defined. In other words, the *encaisse désirée* ω may be written

$$\omega = rx + sy \tag{4}$$

As a first approximation one may perhaps consider r and s as constants given by habits and by the nature of existing monetary institutions.

When the economic activity both in consumer goods and in producer goods production increase, as they do during a period of expansion, the *encaisse désirée* will increase, but the total stock of money, or money substitutes, cannot be expanded *ad infinitum* under the present economic system. There are several reasons for this: limitations of gold supply, the artificial rigidity of the monetary systems, psychological factors, and so on. We do not need to discuss in detail the nature of these limiting factors. We simply assume that, as the activity and consequently the need for cash increases, there is created a *tension* which counteracts a further expansion. This tension is measured by the expression (4). It seems plausible that one effect of the tightening of the cash situation, and perhaps the most important one, will be a restriction in consumption. In the boom period when consumption has reached a high level (in many cases it has extended to pure luxuries), consumption is one of the elastic factors in the situation. It is likely that this factor is one that will yield first to the cash pressure. To begin with this will only be expressed by the fact that the rate of increase of consumption is slackened. Later, consumption may perhaps actually decline. Whatever this final development it seems plausible to assume that the *encaisse désirée* ω will enter into the picture as an important factor which, when increasing, will, after a certain point, tend *to diminish the rate of increase* of consumption. Assuming as a first approximation the relationship to be linear, we have

$$\dot{x} = c - \lambda\omega \tag{5}$$

where c and λ are positive constants. The constant c expresses a tendency to maintain and perhaps expand consumption, while λ expresses the reining-in effect of the *encaisse désirée*.

Introducing into (5) the expression for the *encaisse désirée* taken from (4), we get

$$\dot{x} = c - \lambda(rx + sy) \tag{6}$$

This equation we shall call the consumption equation.

The two equations (3) and (6) form a determinate system in the two variables x and y. If the parameters μ, m, λ, etc., are constants, the system may easily be solved in explicit form. By doing so we see that the system is too simple to give oscillations. Indeed, by eliminating $\dot{x}$ between (3) and (6) we get a linear relation between x and y. Expressing one of the variables in terms of the others by means of this relation and inserting in one of the two equations (3) or (6) we get a linear differential equation in a single variable. The characteristic equation is consequently of degree one, and has therefore only one single real root. This means that the variables will develop monotonically as exponential functions. In other words, we shall have a secular trend but no oscillations.*

The system considered above is thus too simple to be able to explain developments which we know from observation of the economic world. There are several directions in which one could try to generalize the set-up so as to introduce a possibility of producing oscillations. One idea would be to distinguish between saving and investment. The fact that, in an actual situation, there is a difference between these two factors will tend to produce a depression or an expansion. This is Keynes' point of view. It would be exceedingly interesting to see what sort of evolution would follow if such a set of hypotheses were subject to a truly dynamic and determinate analysis.

Another way of generalizing the set-up would be to introduce the fact that the existence of debts exerts a profound influence on the

* Incidentally this shows that the fact pointed out by Clark does not necessarily lead to a development giving a turning-point.

behaviour of both consumers and producers. This is the leading idea of Irving Fisher's approach to the business cycle problem.

A third direction would be to introduce the Marxian idea of a bias in the distribution of purchasing power. This idea may—with a slight change of emphasis—be expressed by saying that under private capitalism production will not take place unless there is a prospect of profit, and the existence of profits tends to create a situation where those who have the consumption power do not have the purchasing power, and *vice versa*. Thus, under private capitalism, production must more or less periodically kill itself.

A fourth direction would be the introduction of Aftalion's point of view with regard to production. The essence of this consists in making a distinction between the quantity of capital goods whose production is *started* and the activity needed in order to *carry to completion* the production of those capital goods whose production was started at an earlier moment. The essential characteristics of the situation that thus arises are that the activity at a given moment does not depend on the decisions taken at that moment, but on decisions taken at earlier moments. By this we introduce a new element of discrepancy in the economic life that may provoke cyclical oscillations. I do not think that Aftalion's analysis as originally presented by himself can be characterized as a determinate analysis. By putting his arguments into equations one will find that he does not have as many equations as unknowns. But his idea with regard to production is very interesting, and, if properly combined with other ideas, will lead to a determinate system. Not only that, but it may lead to a system giving rise to oscillations. I now proceed to the discussion of such a system.

4. A MACRO-DYNAMIC SYSTEM GIVING RISE TO OSCILLATIONS

Let y_t be the quantity of capital goods whose production is *started* at the point of time t. We shall call y_t the "capital starting" or the "production starting," and we shall assume that this magnitude is determined by an equation of the form (3.3). A capital object whose production is started at a certain moment will necessitate a certain production activity during the following time in order to complete the object. The productive activity needed in the period following the starting of the object will, as a rule, vary in a certain fashion which we may, as a first approximation, consider as given by the technical conditions of the production. Let D_τ be the amount of production activity needed at the point of time $t+\tau$ in order to carry on the production of a unit of capital goods started at the point of time t. The function D_τ we shall call the "advancement function."

This being so, the amount of production work that will be going on at the moment t will be

$$z_t = \int_{\tau=0}^{\infty} D_\tau y_{t-\tau} d\tau \tag{1}$$

The magnitude z_t we shall call "the carry-on-activity" at the point of time t.

In the formula of the *encaisse désirée* it is now z that will occur instead of y, so that the consumption equation will be

$$\dot{x} = c - \lambda(rx + sz) \tag{2}$$

where c, λ, r and s are constants.

The three equations (3.3), (1) and (2) form now a determinate system in the three variables x, y and z.

If the carry-on function D_τ is given, the above system may be solved. If D_τ is given only in numerical form, the system has to be solved numerically, taking for granted a certain set of initial conditions. If D_τ is given as a simple mathematical expression, the system may under certain conditions be solved in explicit form. As an example we shall assume

$$D_\tau = \begin{cases} 1/\epsilon & 0 < \tau < \epsilon \\ 0 & \tau \gtreqless \epsilon \end{cases} \tag{3}$$

where ϵ is a technically given constant. This simply means that a given element of capital starting will cause a certain constant amount of carry-on-activity per unit time over the ϵ units of time following the starting, and that in this point the object is finished, so that no further carry-on-activity is needed. This is obviously a simplified assumption, but may perhaps be taken as a first approximation.

In this case we get from (1) by differentiating with respect to time

$$\epsilon \dot{z}_t = y_t - y_{t-\epsilon} \tag{4}$$

For certain purposes it will be convenient to differentiate also the equation (2) in order to get rid of the constant term, which gives

$$\ddot{x} = -\lambda(r\dot{x} + s\dot{z}) \tag{5}$$

The three equations to be considered now are (2), (3.3) and (4) (or possibly (5) may replace (2)). This is a mixed system of differential and difference equations. It is therefore to be expected that the solution will depend, not only on the initial conditions of the system in a given *point* of time, as initial conditions we shall have to consider the shape of the curve over a whole interval of length ϵ.

We shall in particular investigate whether the system is satisfied if each of the variables is assumed to be made up of a number of *components*, each component being either an exponential or a damped oscillation, i.e. a damped sine curve. It is easier to handle the formulae if each such term is written in the complex exponential form, that is to say in the form

$$a_1 e^{(-\beta+i\alpha)t} + a_2 e^{(-\beta-i\alpha)t} \qquad i = \sqrt{-1} \tag{6}$$

where a_1 and a_2 are constants. For brevity we may write (6)

$$a_1 e^{\rho_1 t} + a_2 e^{\rho_2 t} \tag{7}$$

where ρ_1 and ρ_2 are complex numbers.

This applies to a single component. Considering now several components in each variable, we may express the above assumption by saying that the variables x, y and z considered as time series are of the form

$$\begin{cases} x = a_* + \Sigma_k a_k e^{\rho_k t} \\ y = b_* + \Sigma_k b_k e^{\rho_k t} \\ z = c_* + \Sigma_k c_k e^{\rho_k t} \end{cases} \tag{8}$$

where ρ_k are complex or real constants, and where a, b and c are also constants. By convention we let the numbering in (8) run $k = 0, 1, 2 \ldots$ Does there exist a set of functions of the form (8) which satisfy the system consisting of the equations (3.3), (2) and (4)? Such functions do exist. The exponential characteristics ρ_k of these functions are determined by the structural constants ϵ, λsm, $\lambda s\mu$, λr that enter into the system considered. On the contrary, the coefficients a, b and c in (8) will depend on the initial conditions.

It is easy to verify this and to determine how the exponential characteristics ρ_k depend on the structural constants. This is done by differentiating the various expressions in (8) and inserting the results obtained into the equations of the system. If this is done, one will find that the coefficients a, b and c must satisfy the relations

$$(9) \qquad \begin{cases} \dfrac{c_k}{a_k} = -\dfrac{\lambda r + \rho_k}{\lambda s} \\ \dfrac{b_k}{a_k} = m + \mu\rho_k \\ \dfrac{c_k}{b_k} = \dfrac{1 - e^{-\epsilon\rho_k}}{\epsilon\rho_k} \\ (k = 0, 1, 2 \ldots) \end{cases}$$

This condition entails that all the ρ_k must be roots of the following characteristic equation:

$$(10) \qquad \frac{\epsilon\rho}{1 - e^{-\epsilon\rho}} = -\lambda s \frac{m + \mu\rho}{r\lambda + \rho}$$

This equation may have complex or real roots. For the numerical computation it is therefore convenient to insert into (10)

$$(11) \qquad \rho = -\beta + i\alpha \qquad i = \sqrt{-1}$$

and to separate the real and imaginary parts of the equation after having cleared the equation of fractions. Doing this, we get the following two equations to determine α and β (assuming ϵ and $\lambda s\mu \neq 0$).

$$(12) \qquad 1 + \lambda s\mu e^{\epsilon\beta}\frac{\sin\epsilon\alpha}{\epsilon\alpha} = \frac{m\frac{\epsilon^2}{\mu^2}(m - \lambda r\mu)}{\left(\epsilon\beta - m\frac{\epsilon}{\mu}\right)^2 + (\epsilon\alpha)^2}$$

$$(13) \qquad -\frac{\epsilon\beta - \lambda r\epsilon + m\frac{\epsilon}{\mu}}{\epsilon\beta - m\frac{\epsilon}{\mu}} + \lambda s\mu\frac{1 - e^{\epsilon\beta}\cos\epsilon\alpha}{\epsilon\beta - m\frac{\epsilon}{\mu}} = \frac{m\frac{\epsilon^2}{\mu^2}(m - \lambda r\mu)}{\left(\epsilon\beta - m\frac{\epsilon}{\mu}\right)^2 + (\epsilon\alpha)^2}$$

The terms of these equations have been ordered in the particular form indicated in order to facilitate the numerical solution.

The roots (β, α) of these two equations will determine the shape of the time curves x, y and z. It is obvious from (12) and (13) that if (β, α) is a root, then $(\beta, -\alpha)$ will also be a root. In other words, if complex roots occur they will be conjugate, which means that the

corresponding component of x, y and z will actually be a damped sine curve. If there exists a magnitude β such that $(\beta, 0)$ is a root, then the corresponding component will be a secular trend in the form of an exponential.

In order to study the nature of the solutions, I shall now insert for the structural coefficients ϵ, μ, m, etc., numerical values that may in a rough way express the magnitudes which we would expect to find in actual economic life. At present I am only guessing very roughly at these parameters, but I believe that it will be possible by appropriate statistical methods to obtain more exact information about them. I think, indeed, that the statistical determination of such structural parameters will be one of the main objectives of the economic cycle analysis of the future. If we ask for a real *explanation* of the movements, this type of work seems to be the indispensable complement needed in order to co-ordinate and give a significant interpretation to the huge mass of empirical descriptive facts that have been accumulated in cycle analysis for the past ten or twenty years.

Let us first consider the constant ϵ. It expresses the total length of time needed for the completion of big units of fixed capital: big industrial plants, water-power plants, railways, big steamers, etc. This span of time includes not only the actual time needed for the technical construction (the erection of the buildings, etc.) but also time needed for the planning and organization of the work. Indeed, the variable τ in (1) is measured from the moment when the initiative was taken. In many cases the items planning and organization takes more time than the actual technical construction.

It seems that we would strike a fair average if we say that the actual production activity needed in order to complete a typical plant of the above-mentioned kind will be distributed over time in such a way that in general it takes place around three years after the planning began. Some work will of course frequently be done before and some after this time, but three years can, I believe, tentatively be taken as an average. In making this guess I have taken account of an important factor that tends to pull the average up, namely, the fact that in a given individual case the activity will as a rule not be distributed evenly over the period (as assumed in the simplified theoretical set-up) but the peak activity will be concentrated near the *end* of the period. If three years is taken as the *average* lag of the various elements of production activity after the beginning of the planning, we shall have to put $\epsilon = 6$ in (3), and consequently in (4), indeed in the case of equal distribution, as assumed in (3), the *average* lag will be half the *maximum* lag.

Furthermore, let us put $\mu = 10$, which means that the total capital stock is ten times as large as the annual production. Further, let us put $m = 0{\cdot}5$, which means that the direct and indirect yearly depreciation on the capital stock caused by its use in the production of the national income is one-half of that income, i.e. 20 % of the capital stock. Finally, let us put $\lambda = 0{\cdot}05$, $r = 2$, and $s = 1$. These latter constants, which represent the effect of the *encaisse désirée* on the acceleration of consumption, are of course inserted here by a still rougher estimate than the first constants. There is, however, reason to believe that these latter constants will not affect very strongly the length of the cycles obtained (see the computations below).

Inserting these values in the two characteristic equations (12) and (13) we get a numerical determination of the roots. In the actual computation it was found practical to introduce $\epsilon\alpha$ and $\epsilon\beta$ as the unknowns looked for. By so doing, and utilizing an appropriate system of graphical and numerical approximation procedures, the roots may be determined without too much trouble. A good guidance in the search for roots is the fact that the solutions in α are approximately the minimum points of the function

$$\frac{\sin \epsilon\alpha}{\epsilon\alpha} \tag{14}$$

that is to say, a first approximation to the frequencies α will be every other of the roots of the equation

$$tg\epsilon\alpha = \epsilon\alpha \tag{15}$$

The roots of this equation are well known and tabulated.* The results of the computations are given in the first columns of Table 1.

* The above characteristic equation was worked out and the roots numerically determined by Mr. Harald Holme and Mr. Chr. Thorbjörnsen, assistants at the University Institute of Economics. In brief the following procedure was used: The right member of equations (12) and (13) was put equal to a parameter q, and (12) solved with respect to $\varepsilon\beta$, and (13) with respect to $\cos \varepsilon\alpha$. From (15) a first approximation to α was determined and the corresponding β taken as the value determined by putting $q = 0$ in (12). This value of β was as a rule immediately corrected by using (12) with the value of q that followed from the above preliminary determination of α and β. This gave a new value of q that was inserted in (13), thus determining a new value of α. Starting from this new value of α the whole process was iterated. This method gave good results except for very small values of λ, in which case it was found better to start by guessing at the value of β.

TABLE 1

CHARACTERISTIC COEFFICIENTS OF THE COMPONENTS OBTAINED

		Trend ($j = 0$)	Primary Cycle ($j = 1$)	Secondary Cycle ($j = 2$)	Tertiary Cycle ($j = 3$)
Frequency	α	$\rho_0 = -0{\cdot}08045$	0·73355	1·79775	2·8533
Period	$p = \frac{2\pi}{\alpha}$		8·5654	3·4950	2·2021
Damping exponent	β		0·371335	0·5157	0·59105
Damping factor per period	$e^{-2\pi\beta/\alpha}$		0·0416	0·1649	0·2721
Consumption x — Amplitude	A	− 0·32	0·6816	0·27813	0·17524
Consumption x — Phase	ϕ		0	0	0
Production starting y — Amplitude	B	0·09744	5·4585	5·1648	5·0893
Production starting y — Phase	ψ		1·9837	1·8243	1·7582
Carry-on-activity z — Amplitude	C	0·12512	− 10·662	− 10·264	− 10·147
Carry-on-activity z — Phase	θ		1·9251	1·7980	1·7412

The first component ($j = 0$) is a trend, which in all the three variables x, y and z is composed of an additive constant and a damped exponential term. We may write these trends in the form

$$(16) \qquad \begin{cases} x_0(t) = a_* + a_0 e^{\rho_0 t} \\ y_0(t) = b_* + b_0 e^{\rho_0 t} \\ z_0(t) = c_* + c_0 e^{\rho_0 t} \end{cases}$$

These expressions are nothing but the first terms in the composite expressions (8). The damping exponent ρ_0 is the first root of the characteristic equation, it is real and negative $\rho_0 = -0{\cdot}08045$ (see Table 1). The additive constants a_*, b_* and c_* are also determined by the structural coefficients ϵ, λsm, etc. Indeed, if $t \to \infty$ the functions (16) will approach the stationary levels a_*, b_* and c_*. Since the derivatives will vanish in this stationary situation, we get from (3.3), (1) and (2)

$$(17) \qquad b_* = ma_* \qquad c_* = b_* \qquad \lambda ra_* + \lambda sc_* = c$$

putting as an example $c = 0{\cdot}165$ this determines uniquely the three constants $a_* = 1{\cdot}32$, $b_* = 0{\cdot}66$, and $c_* = 0{\cdot}66$.

The coefficients a_0, b_0 and c_0 are not determined uniquely by the structural coefficients, but one initial condition may be imposed on them—for instance, a condition that determines a_0. When a_0 is determined, b_0 and c_0 follow from (9). If, as a numerical example, we impose the condition that x_0 shall be unity at origin, we get the functions x_0, y_0, z_0 in (23*a*).

Besides the secular trend, there will be a primary cycle with a period of 8·57 years, a secondary cycle with a period of 3·50 years, and a tertiary cycle with a period of 2·20 years (see Table 1). These cycles are determined by the first, second and third set of conjugate complex roots of (10). These sets are denoted $j = 1, 2, 3$ in Table 2. There will also be shorter cycles corresponding to further roots of (10), but I shall not discuss them here.

The presence of these cycles in the solution of our theoretical system is of considerable interest. *The primary cycle of 8·57 years corresponds nearly exactly to the well-known long business cycle.* This cycle is seen most distinctly in statistical data from the nineteenth century, but it is present also in certain data from the present

century; in the most recent data it actually seems to come back with greater strength.

Furthermore, the secondary cycle obtained is 3·50 years, which corresponds nearly exactly to the short business cycle. This cycle is seen most distinctly in statistical data from this century, but it is present also in older series. As better monthly data become available back into the nineteenth century the short cycle will become quite evident also here I believe.

The lengths of the cycles here considered depend, of course, on all the structural coefficients; but it is only ϵ that is of great importance. The other coefficients only exert a very small influence on the length of the cycles. Choosing, for instance, $\lambda = 0{\cdot}1$, $s = 1$, $r = 2$, $\mu = 5$, and different values for m, we find

m	Period p	Damping Factor $e^{-2\pi\beta/\alpha}$
0·7	8·53	0·042
0·5	8·43	0·043
0·0	8·20	0·048

In other words, even an extreme variation in the total depreciation factor m leaves both the period and the damping factor nearly unchanged.

A change in the constant λ that expresses the "reining-in" effect of the *encaisse désirée* exerts a considerable influence on the damping factor, but a relatively small influence on the length of the period. For $\lambda = 0{\cdot}001$, $s = 1$, $r = 2$, $\mu = 5$ and $m = 0{\cdot}5$, we find, for instance, $p = 10{\cdot}6$ years, $e^{-2\pi\beta/\alpha} = 0{\cdot}000002$. In other words, the period is still of the same order of magnitude, but the damping is now enormous, the amplitude being brought down to two-millionth in the course of one period. This means that the cycle in question is virtually non-existing.

It is interesting to interpret the last result in the light of the limiting case where $\lambda = 0$, i.e. where the need for cash as a brake on the development of production is eliminated. In this case it follows immediately from (2) that x will evolve as a straight line with *positive* inclination (c being assumed positive). Hence by (3.3) y must also be a straight line, and by (4) $\dot{z}$ must be a constant, hence z linear. The movement of the system will consequently be a steady evolution towards higher levels of consumption and production without the setbacks caused by depressions.

Of course, the results here obtained with regard to the length of the periods and the intensity of the damping must not be interpreted as giving a final explanation of business cycles; in particular it must be investigated if the same types of cycles can be explained also by other sets of assumption, for instance, by assumptions about the saving-investment discrepancy, or by the indebtedness effect, etc. Anyhow, I believe that the results here obtained, in particular those regarding the length of the primary cycle of $8\frac{1}{2}$ years and the secondary cycle of $3\frac{1}{2}$ years, are not entirely due to coincidence but have a real significance.

I want to go one step further: I want to formulate the hypothesis that if the various statistical production or monetary series that are now usually studied in connection with business cycles are scrutinized more thoroughly, using a more powerful technique of time series analysis, *then we shall probably discover evidence also of the tertiary cycle, i.e. a cycle of a little more than two years.*

Now let us consider the other features of the cycles: phase, etc. We write the various cyclical components

$$(18)\quad \begin{cases} x_j(t) = A_j e^{-\beta_j t} \sin(\phi_j + a_j t) \\ y_j(t) = B_j e^{-\beta_j t} \sin(\psi_j + a_j t) \\ z_j(t) = C_j e^{-\beta_j t} \sin(\theta_j + a_j t) \\ \quad (j = 1, 2 \ldots) \end{cases}$$

$j = 1$ means the primary cycle, $j = 2$ the secondary cycle, etc. The frequencies a and the damping coefficients β are uniquely determined by the characteristic equation, but the phases ϕ, ψ, θ and the amplitudes A, B, C are influenced by the initial conditions. For the primary cycle ($j = 1$) two such conditions may be imposed. We may, for instance, require that $x_1(0) = 0$ and $\dot{x}_1(0) = \frac{1}{2}$. This leads to $\phi_1 = 0$, $A_1 = \frac{1}{2a_1}$. And when the phase and amplitude for the primary cycle in x is thus determined, the phases and amplitudes of the primary cycles in y and z follow by virtue of (9). Similarly, if ϕ_2 and A_2 are determined by two initial conditions imposed on the secondary cycle in x, for instance, by the conditions $x_2(0) = 0$ and $\dot{x}_2(0) = \frac{1}{2}$, the phases and amplitudes of the secondary cycles in y and z are also determined by (9).

When the conditions (9) are formulated in terms of phases and amplitudes, we get

$$(19)\quad \begin{cases} B \sin(\psi - \phi) = A\mu a \\ B \cos(\psi - \phi) = A(m - \mu\beta) \end{cases}$$

$$(20)\quad \begin{cases} C \sin(\theta - \phi) = -\dfrac{Aa}{\lambda s} \\ C \cos(\theta - \phi) = \dfrac{A(\beta - \lambda r)}{\lambda s} \end{cases}$$

These equations hold good for all the cycles, that is, for $j = 1, 2, 3 \ldots$ They show that the lag between x, y and z are independent of the initial conditions and depend only on the structural coefficients of the system. From (18) and (19) we get indeed

$$(21)\quad \begin{cases} tg(\psi - \phi) = \dfrac{\mu a}{m - \mu\beta} \\ tg(\theta - \phi) = -\dfrac{a}{\beta - \lambda r} \end{cases}$$

Similarly the relations between the amplitudes may be reduced to

$$(22)\quad \begin{cases} |B| = \sqrt{(\mu a)^2 + (m - \mu\beta)^2} \cdot |A| \\ |C| = \dfrac{\sqrt{a^2 + (\beta - \lambda r)^2}}{\lambda s} \cdot |A| \end{cases}$$

where the square roots are taken positive. If the amplitudes are taken positive, $\sin(\psi - \phi)$ has the same sign as μa and $\sin(\theta - \phi)$ the same sign as $-a/\lambda s$.

A given set (18) (for a given j) does not—taken by itself—satisfy the dynamic system consisting of (3.3), (2) and (4). It will do so only if the structural constant $c = 0$. If $c \neq 0$ the constant terms a_*, b_* and c_* must be added to (18) in order to get a correct solution. If these constant terms are added, we get functions that satisfy the dynamic system, and that have the property that any linear combination of them (with constant coefficients) satisfy the dynamic system provided only that the sum of the coefficients by which they are linearly combined is equal to unity. This proviso is necessary because any sets of functions that shall satisfy the dynamic system must have the uniquely determined constants a_*, b_* and c_*.

The sets (18), with no constants a_*, b_* and c_* added, are solutions of the system obtained by leaving out c in (2). Or, again, (18) may be looked upon as solutions of the system obtained by letting (5) replace (2).

If we impose on the trends the initial condition that x_0 shall be unity at origin, and on each cycle in x the condition that it shall be zero at origin and with velocity $= \frac{1}{2}$, we get the functions in (23*a*, *b*, *c*, *d*). The corresponding cycles are represented in Figs. 3–5.

$$(23a) \quad \begin{cases} x_0 = 1\cdot 32 - 0\cdot 32e^{-0\cdot 08045t} \\ y_0 = 0\cdot 66 + 0\cdot 09744e^{-0\cdot 08045t} \\ z_0 = 0\cdot 66 + 0\cdot 12512e^{-0\cdot 08045t} \end{cases}$$

$$(23b) \quad \begin{cases} x_1 = 0\cdot 6816e^{-\beta_1 t} \sin \alpha_1 t \\ y_1 = 5\cdot 4585e^{-\beta_1 t} \sin (1\cdot 9837 + \alpha_1 t) \\ z_1 = -10\cdot 662e^{-\beta_1 t} \sin (1\cdot 9251 + \alpha_1 t) \end{cases}$$

$$(23c) \quad \begin{cases} x_2 = 0\cdot 27813e^{-\beta_2 t} \sin \alpha_2 t \\ y_2 = 5\cdot 1648e^{-\beta_2 t} \sin (1\cdot 8243 + \alpha_2 t) \\ z_2 = -10\cdot 264e^{-\beta_2 t} \sin (1\cdot 7980 + \alpha_2 t) \end{cases}$$

$$(23d) \quad \begin{cases} x_3 = 0\cdot 17524e^{-\beta_3 t} \sin \alpha_3 t \\ y_3 = 5\cdot 0893e^{-\beta_3 t} \sin (1\cdot 7582 + \alpha_3 t) \\ z_3 = -10\cdot 147e^{-\beta_3 t} \sin (1\cdot 7412 + \alpha_3 t) \end{cases}$$

(The α and β are given in Table 1.)

From the constants given in Table 1, and from the shape of the curves in Figs. 3–5, we see that the shorter cycles are not so heavily damped as the long cycle. Furthermore, we see that the lead or lag between the variables x, y and z is, roughly speaking, the same in the primary, secondary and tertiary cycles. To study the lag it is therefore sufficient to consider only one of these types—for instance, the tertiary cycle (Fig. 5).

Let us first compare consumption with production starting. Apart from the fact that the cycles in Fig. 5 are damped, the relation between x and y is very much the same as in Fig. 2, i.e. production has its peak nearly at the same time as the *rate of change* of consumption is at its highest. The reason for this is that the constant μ in our example is chosen rather large in comparison to m. This means that our example refers to a highly capitalistic society where the annual depreciation is relatively small. By reducing the size of the capital stock in relation to

the output (i.e. reducing μ) and increasing the annual depreciation (i.e. m) the peak in production starting will advance so as to arrive nearer the peak in consumption.

Next, comparing consumption with the carry-on-activity, we see

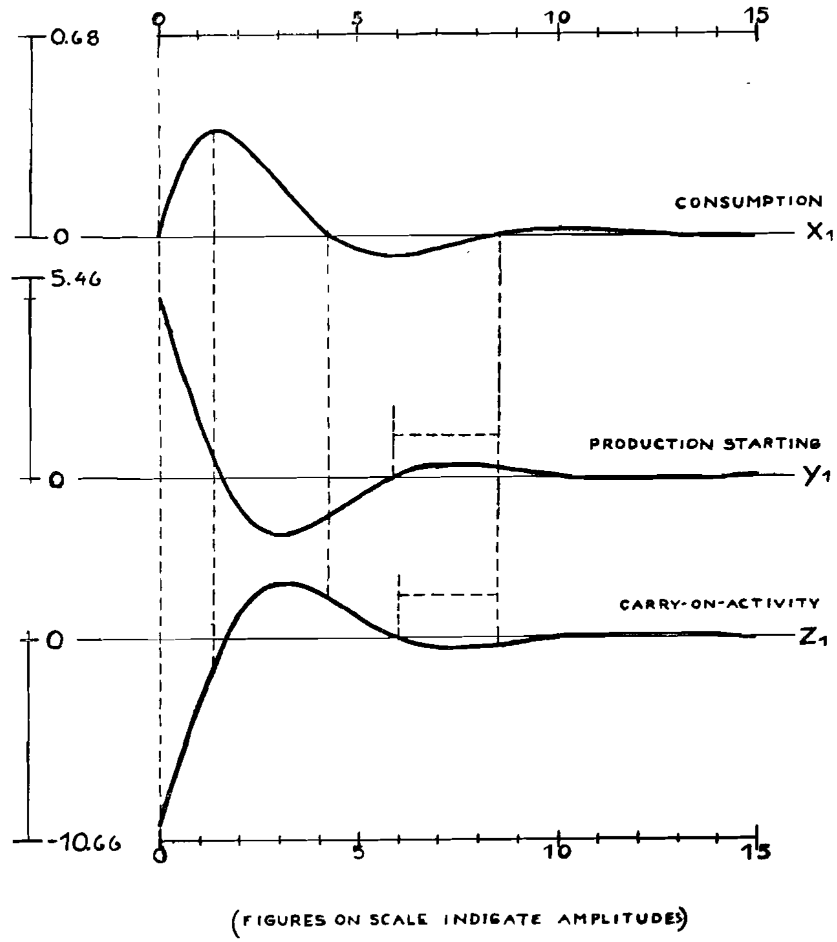
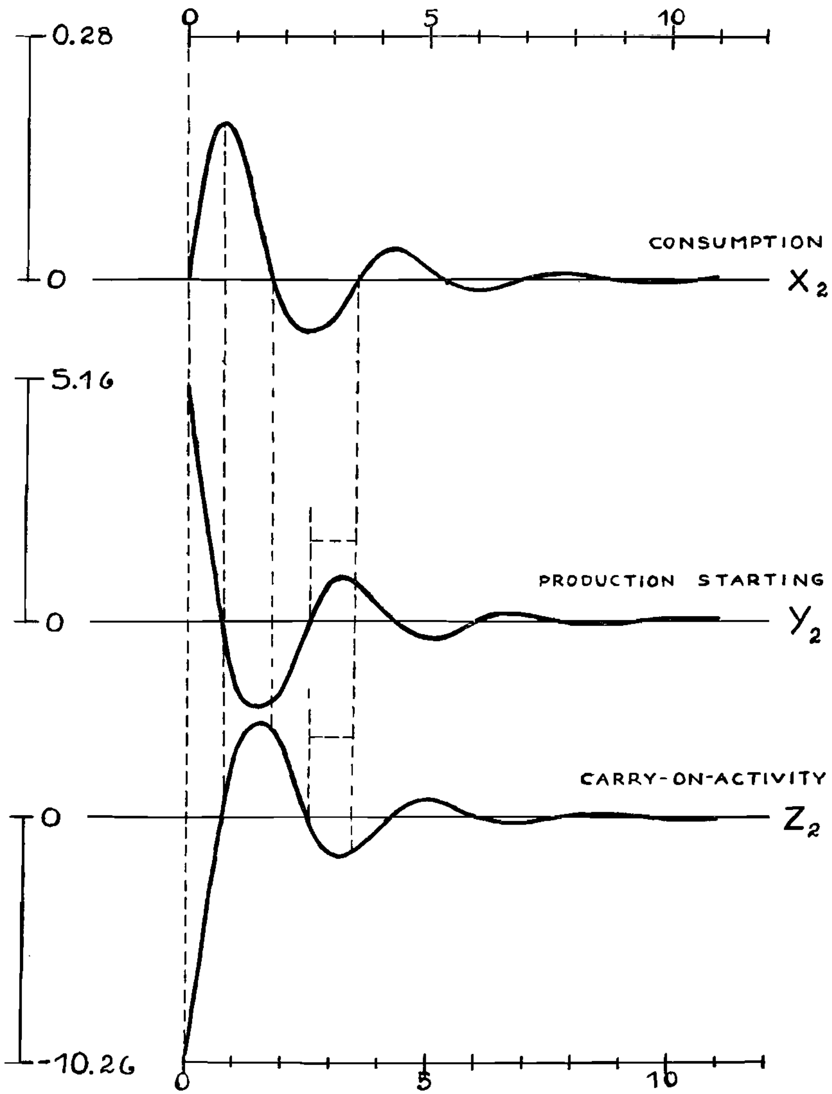


Fig. 3

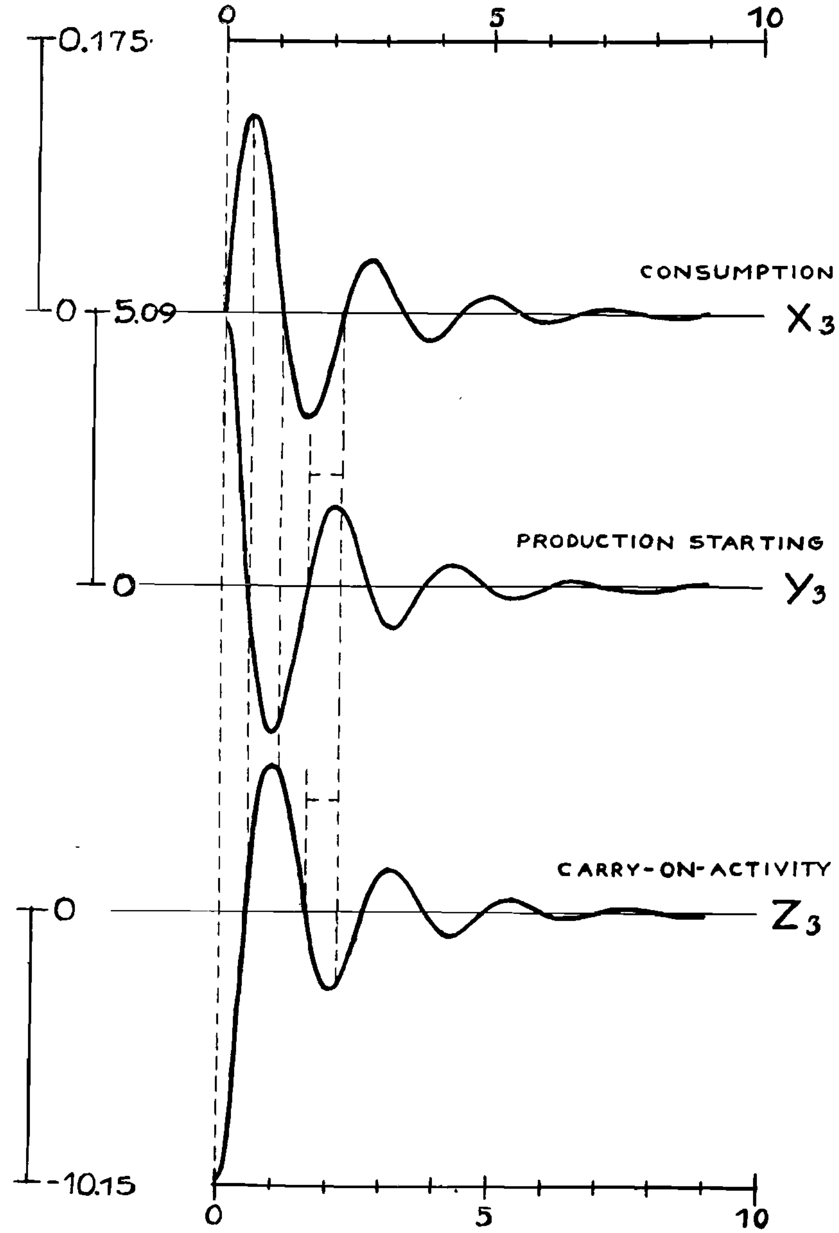
that the former is leading by a considerable span of time, i.e. in the depression the carry-on-activity starts to increase only when the upswing in consumption is well under way, and the carry-on-activity continues to increase even after consumption has started to decline.

The way in which the structural relations determine the time shape of the solutions may perhaps be rendered more intuitively by a method of successive numerical approximation.



(FIGURES ON SCALE INDICATE AMPLITUDES)

FIG. 4



(FIGURES ON SCALE INDICATE AMPLITUDES)

FIG. 5

In order to show this, we take for granted that the time shape of one of the curves—for instance, y_1—is known in the interval $-6 < t < 0$. That is to say, in this interval we simply consider the values of y_1 as

TABLE 2

STEP-BY-STEP COMPUTATION OF THE PRIMARY CYCLE

t	x	y	z	$\dot{x}$	$\dot{z}$	y_{t-6}
0	0	5·0000	− 10·000	0·5000	6·4264	− 33·5581
0·16667	0·08333	4·4229	− 8·929	0·4381	6·6811	− 35·6634
0·33333	0·15635	3·8297	− 7·8155	0·3752	6·7865	− 36·8894
0·50000	0·21887	3·2329	− 6·6844	0·3124	6·7592	− 37·3221
0·66667	0·27093	2·6435	− 5·5579	0·2508	6·6153	− 37·0480

TABLE 3

PRIMARY CYCLE COMPUTED DIRECTLY BY FORMULA (23*b*)

t	x	y	z
0	0	5·0000	− 10·0000
0·16667	0·07814	4·4138	− 8·9058
0·33333	0·14581	3·8179	− 7·7817

given by the expression (23*b*). Then we want by the dynamic equations to determine the solutions numerically from the point $t = 0$ and onwards.

With the numerical constants ϵ, μ, m, etc., inserted, the dynamic system (where c in (2) is left out) will now be

$$(24)\qquad \begin{cases} y = 0\cdot5x + 10\dot{x} \\ \dot{x} = -0\cdot1x - 0\cdot05z \\ -6\dot{z}_t = y_{t-6} + 0\cdot5(x_t + z_t) \end{cases}$$

We shall use (24) for a step-by-step computation. Since $x = 0$ and $\dot{x} = 0\cdot5$ are given at the origin, $z = -10$ may be determined from the second equation in (24). Furthermore, since y_{t-6} is given, we may compute $\dot{z}$ in origin by means of the third equation in (24), and finally

y may be computed by the first equation in (24). Thus we have all the items in the first line of Table 2. Since we know x and $\dot{x}$ in origin, we may by a straight linear extrapolation determine x and z in the next point of time, that is to say, in the second line of Table 2. And knowing x and z in this point, we may from the second equation of (24) compute $\dot{x}$. Further, taking the value of y_{t-6} as given also in the next line we can compute $\dot{z}$, etc. In this way we may continue from line to line and determine the development of all the three variables x, y and z. A comparison between the values in Table 2 for x, y and z with the values in Table 3 (determined by the explicit formulae (23*b*) and represented in Fig. 3) will give an idea of the closeness of the approximation obtained by the numerical step-by-step solution.

5. ERRATIC SHOCKS AS A SOURCE OF ENERGY IN MAINTAINING OSCILLATIONS

The examples we have discussed in the preceding sections, and many other examples of a similar sort that may be constructed, show that when an economic system gives rise to oscillations, these will most frequently be damped. But in reality the cycles we have occasion to observe are generally not damped. How can the maintenance of the swings be explained? Have these dynamic laws deduced from theory and showing damped oscillations no value in explaining the real phenomena, or in what respect do the dynamic laws need to be completed in order to explain the real happenings? I believe that the theoretical dynamic laws do have a meaning—much of the reasoning on which they are based are on *a priori* grounds so plausible that it is too improbable that they will have no significance. But they must not be taken as an immediate explanation of the oscillating phenomena we observe. They only form *one* element of the explanation: they solve the propagation problem. But the impulse problem remains.

There are several alternative ways in which one may approach the impulse problem and try to reconcile the results of the determinate dynamic analysis with the facts. One way which I believe is particularly fruitful and promising is to study what would become of the solution of a determinate dynamic system if it were exposed to a stream of erratic shocks that constantly upsets the continuous evolution, and by so doing introduces into the system the energy necessary to maintain the swings. If fully worked out, I believe that this idea will give an interesting synthesis between the stochastical point of view and the point of view of rigidly determined dynamical laws. In the present section I shall discuss this type of impulse phenomena. In the next I shall consider another type which exhibits another—and perhaps equally important—aspect of the swings we observe in reality.

Knut Wicksell seems to be the first who has been definitely aware of the two types of problems in economic cycle analysis—the propagation problem and the impulse problem—and also the first who has formulated explicitly the theory that the source of energy which maintains the economic cycles are erratic shocks.* He conceived more or less definitely of the economic system as being pushed along irregularly, jerkingly. New innovations and exploitations do not come regularly he says. But, on the other hand, these irregular jerks may cause more or less regular cyclical movements. He illustrates it by one of those perfectly simple and yet profound illustrations: "If you hit a wooden rocking-horse with a club, the movement of the horse will be very different to that of the club."

Wicksell's idea on this matter was later taken up by Johan Åkerman, who in his doctorial dissertation† discussed the fact that small fluctuations may be able to generate larger ones. He used, among others, the analogy of a stream of water flowing in an uneven river bed. The irregularities of the river bed will cause waves on the surface. The irregularities of the river bed illustrate in Åkerman's theory the seasonal fluctuations; these seasonals, he maintains, create the longer cycles. Unfortunately Åkerman combined these ideas with the idea of a *synchronism* between the shorter fluctuations and the longer ones. He tried, for instance—in my opinion in vain—to prove that there always goes an exact number of seasonal fluctuations to each minor business cycle. This latter idea is, to my mind, very misleading. It is also, as one will readily recognize, in direct opposition to Wicksell's profound remark about the rocking-horse.

Neither Wicksell nor Åkerman had taken up to a closer mathematical study the *mechanism* by which such irregular fluctuations may be transformed into cycles. This problem was attacked independently of each other by Eugen Slutsky‡ and G. Udny Yule.§

* See, for instance, his address, "Krisernas gåta," delivered to the Norwegian Economic Society, 1907, *Statsøkonomisk Tidsskrift*, Oslo, 1907, pp. 255–86.

† *Det ekonomiska livets rytmik*, submitted 1925, published Lund, 1928.

‡ *The Summation of Random Causes as the Source of Cyclic Processes*, vol. iii, no. 1, Conjuncture Institute of Moscow, 1927. (Russian with English summary.)

§ *On a Method of Investigating Periodicity in Disturbed Series*, *Trans. Royal Society*, London, A, vol. 226, 1927.

In this connection may also be mentioned a paper by Harold Hotelling.*

Slutsky studied experimentally the series obtained by performing iterated differences and summations on random drawings (lottery drawings, etc.). Yule only used second order differences, but tried to interpret the random impulses concretely as shocks hitting an oscillating pendulum. By the experimental numerical work done by these authors, particularly by Slutsky, it was definitely established that some sort of swings will be produced by the accumulation of erratic influences, but the exact and general law telling us what *sort* of cycles that a given kind of accumulation will create was not discovered.

Later certain mathematical results which are of interest in connection with this problem were given by Norbert Wiener.†

But still the main problem remained, both with regard to the mechanism by which the *time shapes* of the resulting curves are determined and with regard to the concrete economic interpretation. In the present section I shall offer some remarks on these questions. For a more detailed mathematical analysis the reader is referred to a paper to appear in one of the early numbers of *Econometrica*.

Consider for simplicity an oscillating pendulum whose movement is hampered by friction. If y denotes the deviation of the pendulum from its vertical position, the equation governing the movement of the pendulum will be

$$\ddot{y} + 2\beta\dot{y} + (\alpha^2 + \beta^2)y = 0 \qquad (1)$$

where $\dot{y}$ and $\ddot{y}$ are the first and second derivatives of y with respect to time, and β and α are positive constants, β expressing the strength of the friction. The equation expresses the fact that the net force acting on the pendulum (and being expressed by the acceleration $\ddot{y}$) is made up of two factors. First a factor which tends to make the force proportional to the deviation y (and of opposite sign). This gives the gross force expressed by the last term of the equation. From this gross force must be subtracted the effect of the friction, and this effect is proportional to the *velocity* $\dot{y}$ and is expressed by the second term of the equation.

It is easily verified that the solution of (1) is a function of the form

$$He^{-\beta t} \sin (\phi + \alpha t)$$

where α and β are the constants occurring in (1).

* *Differential Equations Subject to Error*, *Journal of the American Statistical Association*, 1927, pp. 283–314.

† *Generalized Harmonic Analysis*, *Acta Mathematica*, 1930.

The amplitude H and the phase ϕ are determined by the initial conditions. For our present purpose it is convenient to write the solution in such a way that we can see immediately how the initial conditions determine the curve. If y_0 and $\dot{y}_0$ are the values of y and $\dot{y}$ respectively at the point of time $t = t_0$ the solution may be written in the form

$$y(t) = P(t - t_0) \cdot y_0 + Q(t - t_0) \cdot \dot{y}_0 \tag{2}$$

where $P(\tau)$ and $Q(\tau)$ are two functions independent of the initial conditions and defined by

$$P(\tau) = \frac{\sqrt{\alpha^2 + \beta^2}}{\alpha} e^{-\beta\tau} \sin (v + \alpha\tau) \tag{3}$$

$$Q(\tau) = \frac{1}{\alpha} e^{-\beta\tau} \sin \alpha\tau \tag{4}$$

where

$$\sin v = \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}} \qquad \cos v = \frac{\beta}{\sqrt{\alpha^2 + \beta^2}} \tag{5}$$

By convention the square root in (3) and (5) may be chosen positive. By insertion into the equation (1) it is easily verified that (2) is a function that satisfies the equation and the specified initial condition. $P(\tau)$ may be looked upon as the solution of (1), which is equal to unity and whose derivative is equal to zero at the origin $\tau = 0$, and $Q(\tau)$ may be looked upon as the solution which is equal to zero and whose derivative is equal to unity at the origin. These functions satisfy indeed the equation, and we have

$$\begin{cases} P(0) = 1 & Q(0) = 0 \\ \dot{P}(0) = 0 & \dot{Q}(0) = 1 \end{cases} \tag{6}$$

Suppose that the pendulum starts with the specified initial conditions at the point of time t_0 and that it is hit at the points of time $t_1, t_2 \ldots t_n$ by shocks which may be directed either in the positive or in the negative sense and that may have arbitrary strengths. Let y_k and $\dot{y}_k$ be the ordinate and the velocity of the pendulum immediately before it is hit by the shock number k. The ordinate y_k is not changed by the shock, but the velocity is suddenly changed from $\dot{y}_k$ to $\dot{y}_k + e_k$, where e_k is the strength of the shock; mechanically expressed it is the quantity of motion divided by the mass of the pendulum. The concrete

interpretation of the shock e_k does not interest us for the moment. The essential thing to notice is that at the point of time t_k the only thing that happens is that the velocity is increased by a constant e_k. Let us consider separately the effects produced by the two terms $\dot{y}_k$ and e_k. From (2) we see that the initial conditions enter linearly. Consequently we can consider $\dot{y}_k$ and e_k as two independent contributions to the later ordinates of the variable. In other words, the fact of the shock may simply be represented by letting the original pendulum move on undisturbed but letting a *new* pendulum start at the point of time t_k with an ordinate equal to zero and a velocity equal to e_k. This argument may be applied to all the points of time. We simply have to start in each of the points of time $t_1, t_2 \ldots t_n$ a new pendulum with an ordinate equal to zero and a velocity equal to the strength of the shock occurring at that moment, and then let all these pendulums continue their undisturbed motion into the future. The sum of the ordinates of all these pendulums at any given point of time t will then be the same as the ordinate $y(t)$ of a single pendulum which has been subject to all the shocks. In other words, the ordinate $y(t)$ will simply be

$$y(t) = P(t - t_0) \cdot y_0 + Q(t - t_0)\dot{y}_0 + \sum_{k=1}^{n} Q(t - t_k)e_k \tag{7}$$

If the point t is very far from the initial point t_0, and if β is positive so that there is actually a dampening, then the influence of the initial situation y_0 and $\dot{y}_0$ on the ordinate $y(t)$ will be negligible, that is, the ordinate will be

$$y(t) = \sum_{k=1}^{n} Q(t - t_k) \cdot e_k \tag{8}$$

This means that the ordinate $y(t)$ of the pendulum at a given moment will simply be the *cumulation* of the effects of the shocks, the cumulation being made according to a system of weights. And these weights are simply the shape of the function $Q(\tau)$. That is to say, $y(t)$ is the result of applying a linear operator to the shocks, and *the system of weights in the operator will simply be given by the shape of the time curve that would have been the solution of the determinate dynamic system in case the movement had been allowed to go on undisturbed.*

The fundamental question which arises is, therefore: If we perform a cumulation where the weights have the form $Q(\tau)$, what sort of time

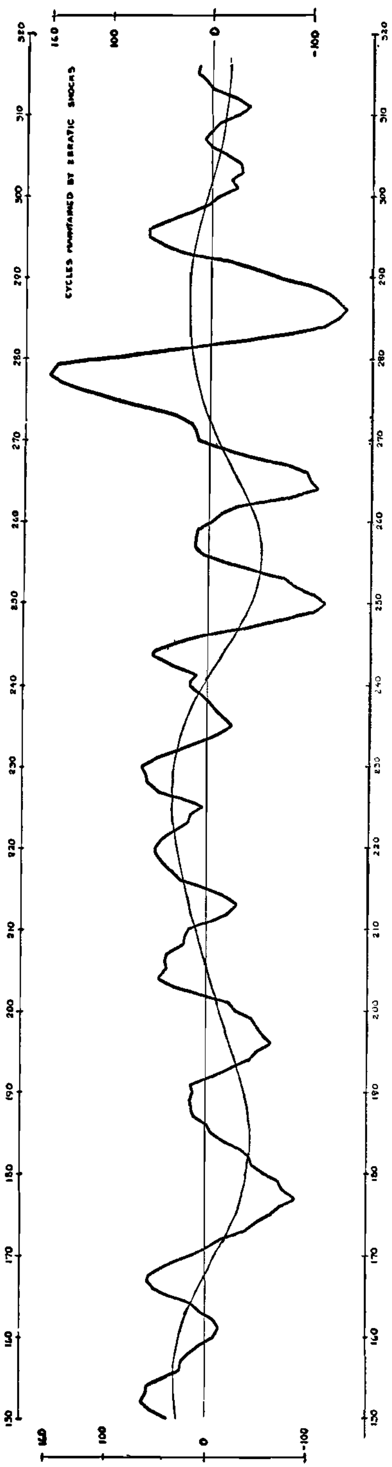


Fig. 6

shape will the function $y(t)$ get? The answer to this question is given by studying the effects of linear operators on erratic shocks. The result of this analysis is that the time shape of the curve will be a *changing harmonic* with the same frequency α as the one occurring in $Q(\tau)$. By a changing harmonic I understand a curve that is moving more or less regularly in cycles, the length of the period and also the amplitude being to some extent variable, these variations taking place, however, within such limits that it is reasonable to speak of an *average* period and an *average* amplitude. In other words, there is created just the kind of curves which we know from actual statistical observation. I shall not attempt to give any formal proof of these facts here. A detailed proof, together with extensive numerical computations, will be given in the above-mentioned paper to appear in *Econometrica*. Here I shall confine myself to reproducing the graph (see Fig. 6) of a changing harmonic produced experimentally as the cumulation of erratic impulses, the weight function being of the form (4).

Thus, by connecting the two ideas: (1) the continuous solution of a determinate dynamic system and (2) the discontinuous shocks intervening and supplying the energy that may maintain the swings—we get a theoretical set-up which seems to furnish a

rational interpretation of those movements which we have been accustomed to see in our statistical time data. The solution of the determinate dynamic system only furnishes a part of the explanation: it determines the *weight system* to be used in the cumulation of the erratic shocks. The other and equally important part of the explanation lies in the elucidation of the general laws governing the effect produced by linear operations performed on erratic shocks.

6. THE INNOVATIONS AS A FACTOR IN MAINTAINING OSCILLATIONS

The idea of erratic shocks represents one very essential aspect of the impulse problem in economic cycle analysis, but probably it does not contain the whole explanation. There is also present another source of energy operating in a more continuous fashion and being more intimately connected with the permanent evolution in human societies. The nature of this influence may perhaps be best exhibited by interpreting it in the light of Schumpeter's theory of the innovations and their role in the cyclical movement of economic life. Schumpeter has emphasized the influence of new ideas, new initiatives, the discovery of new technical procedures, new financial organizations, etc., on the course of the cycle. He insists in particular on the fact that these new ideas accumulate in a more or less continuous fashion, but are put into practical application on a larger scale only during certain phases of the cycle. It is like a force that is released during these phases, and this force is the source of energy that maintains the oscillations. This idea is also very similar to an idea developed by the Norwegian economist, Einar Einarsen.* In mathematical language one could perhaps say that one introduces here the idea of an auto-maintained oscillation.

Schumpeter's idea may perhaps be best explained by a mechanical analogy. Personally, I have found this illustration very useful. Indeed it is only after I had constructed this analogy that I really succeeded in understanding Schumpeter's idea. After long conversations and correspondence with Professor Schumpeter I believe the analogy may be taken as a fair representation of his point of view.

Suppose that we have a pendulum freely suspended to a pivot. Above the pendulum is fixed a receptacle where there is water. A small pipe descends all along the pendulum, and at the lower end of the pendulum the pipe opens with a valve which has a peculiar way of

* *Gode og daarlige tider*, Oslo, 1904.

functioning. The opening of the valve points towards the left and is larger when the pendulum moves towards the right than when it moves towards the left. Concretely one may, for example, assume that the valve is influenced by the air resistance or by some other factor that determines the opening of the valve as a function of the velocity of the pendulum. Finally we assume that the water in the receptacle is fed from a constantly running stream which is given as a function of time. The stream may, for instance, be a constant.

Now, if the instrument is let loose it is easy to see what will happen. The water will descend through the pipe, and the force of reaction at the lower end of the pendulum created by the fact that the water is emptying through the valve will push the pendulum towards the right, and this movement will continue until the force of gravitation has become large enough to pull the pendulum back again towards its equilibrium point. During the return the opening of the valve and consequently the force that tends to push the pendulum towards the right will diminish, and thus the movement back towards the central position will be accelerated. The pendulum which is now returning with considerable speed will work up an amount of inertia that will push it behind its equilibrium point away over to a position at the left, but here again the gravitation will start to pull it back towards the centre, and now the valve will widen, and by doing so will increase the force which accelerates the movement towards the right. In this way the movement will continue, and it will continue even although friction is present. One could even imagine that the movement would be more than maintained, i.e. that the oscillations would become wilder and wilder until the instrument breaks down. In order to avoid such a catastrophe one may of course, if necessary, add a dampening mechanism which would tend to stabilize the movement so that the amplitude did not go beyond a certain limit.

The meaning of the various features of this instrument as an illustration of economic life is obvious. The water accumulating in the receptacle above the pendulum are the Schumpeterian innovations. To begin with they are kept a certain time without being utilized. Some of them will perhaps never be utilized, which is illustrated by the fact that some of the atoms in the receptacle will rest there indefinitely. But some others will descend down the pipe, which illustrates that these new ideas are utilized in economic life. This utilization constitutes the new energy which maintains the oscillations.

The instrument as thus conceived will give a picture of the

oscillations but not of the secular or perhaps supersecular tendency of evolutions. This tendency seems to us to be irreversible because we have not yet lived long enough to see the decline. It is not difficult to complete the instrument in such a way that it will express also this secular or supersecular movement. We may, for instance, imagine that the pivot to which the pendulum is suspended is not fixed but slides in a crack in the wall, the crack ascending towards the right. This being so, the whole instrument will move by jumps, and each jump will carry it to a higher position than before. We only have to feed the instrument by a constantly running stream of water. The impulsion which the water creates as it leaves the valve will maintain the oscillations, and these oscillations will constitute the jumps which carry the instrument steadily to higher levels. Thus there will be an intimate connection between the oscillations and the irreversible evolution.

It would be possible to put the functioning of this whole instrument into equations under more or less simplified assumptions about the construction and functioning of the valve, etc. I even think this will be a useful task for a further analysis of economic oscillations, but I do not intend to take up this mathematical formulation here.

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