goitrous (15·6 %), 309 confirmees out of 1624 (19 %) and 109 soldiers out of 677 (16 %). A more recent survey in 1910 of the district of Husby in the same county showed 15·9 % of males and 33·6 % of females to have goitre. Other important endemic areas in the county were the districts of Bjursås, Sundborn, Stora Tuna and Svärisjo. In the Department of Gefleborg goitre was found to be endemic in a few areas, but for the county as a whole the percentage of recruits affected was only 1·3 % and of children 12·2 %. The most goitrous districts were Ockelbo with 44 % of children and 30 % of adolescents goitrous, and Bjenåker and Alfta with lower rates. A later investigation in the Högbo district showed 5 % of men and 30 % of women to be goitrous.

I have shown in Table II the total cancer deaths, deaths from all causes, deaths from unknown causes and corrected cancer mortality rates for the year 1911 for (i) the whole county of Kopparberg, (ii) goitrous areas of Ockelbo, Bjenåker and Alfta of the county of Gefleborg, and (iii) rural areas of Sweden as a whole.

TABLE II.

| Area | Population 1911 | Cancer deaths 1911 | All deaths 1911 | Deaths from unknown cause 1911 | Corrected cancer death-rate |
|---|---|---|---|---|---|
| Department of Kopparberg ... | 235,427 | 240 | 3067 | 240 | — |
| Goitrous districts of Gefleborg | 19,159 | 20 | 237 | 22 | — |
| Total goitre areas ... ... | 254,586 | 260 | 3304 | 262 | 110·9 |
| Sweden (rural) ... ... | 4,159,216 | — | — | — | 96·7 |

It will be seen that the goitrous areas had apparently a somewhat higher cancer mortality rate than rural Sweden as a whole. It is unfortunately not possible to obtain more extended data.

### III. *Exophthalmic Goitre in England and Wales.*

In view of the interest which is at the present time being aroused on the question of goitre in England and Wales, I have thought it advisable to include with this paper a map (No. 2) showing the standardized mean mortality rates from exophthalmic goitre over the 10 years 1913–1922 in the various counties (excluding county boroughs) which has been compiled from the statistics of the Registrar-General. For explanation of the method used in standardizing reference should be made to Section VIII of the original paper in *Biometrika*, Vol. XVI. pp. 392—398, Dec. 1924.

## Recurrence Formulae for the Moments of the Point Binomial.

By RAGNAR FRISCH, Kristiania.

*Introduction.* In *Biometrika*, May 1924, Professor Pearson has given a very important recurrence formula for the moments of the hypergeometric series. In the special case of the point binomial $(p+q)^r$ Pearson's formula may be written

$$\mu_s = rqp \sum_{i=0}^{s-2} \binom{s-1}{i} \mu_i - q \sum_{i=0}^{s-2} \binom{s-1}{i} \mu_{i+1} \quad \text{.....(1)}$$

or

$$\mu_s + q \binom{s-1}{1} \mu_{s-1} + \sum_{i=2}^{s} \left[ q \binom{s-1}{i} - rqp \binom{s-1}{i-1} \right] \mu_{s-i} = 0 \quad \text{.....(2),}$$

where the moment of order $s$ is defined to be

$$\mu_s = \mu_s(q) = \sum_{\nu=0}^{r} (\nu - rq)^s T_\nu,$$

where

$$T_\nu = \binom{r}{\nu} q^\nu p^{r-\nu}, \quad p+q=1 \quad \text{......(3)},$$

$$\mu_0 = 1,$$

$$\mu_1 = 0.$$

In the following lines I first deduce some general properties of linear equations between the moments of the point binomial. I then proceed to prove a system of recurrence formulae, a special case of which is the formula (1). Lastly I generalize the recurrence formulae of Pearson and Romanovsky* for the case of incomplete moments. The proofs of the last-named generalizations are based on principles entirely different from those originally used by Pearson and Romanovsky. In fact only elementary summation operations are involved. By the help of these operations the proofs may be given an extremely simple form. The formula of Romanovsky in fact arises with hardly any calculation at all.

§ 1. *Some General Properties of Linear Equations between the Moments of the Point Binomial.* A function $F(x)$ which identically in $x$ satisfies one of the equations

$$F(1-x) = +F(x) \quad \text{......(4a)},$$

$$F(1-x) = -F(x) \quad \text{......(4b)},$$

may be called a *reduced* function of $x$, *positively* reduced if it satisfies (4$a$), *negatively* reduced if it satisfies (4$b$). A positively reduced function obviously is an even function in the variable $t = 1-2x$, and a negatively reduced function is an odd function in the same variable.

Putting $p = 1-q$ for $q$ in (3) we obtain

$$\mu_s(1-q) = \sum_{\nu=0}^{r} (\nu - r + rq)^s \binom{r}{\nu} (1-q)^\nu q^{r-\nu} = (-1)^s \sum_{\nu=0}^{r} ((r-\nu) - rq)^s \binom{r}{r-\nu} q^{r-\nu} (1-q)^{r-(r-\nu)},$$

hence

$$\mu_s(1-q) = (-1)^s \mu_s(q) \quad \text{......(5)}.$$

*The even moments are positively reduced functions of $q$ and the odd moments negatively reduced functions of $q$.* That is: the even moments are even functions and the odd moments odd functions in the variable $\delta = p - q = 1 - 2q$.

Now let us consider a linear relation between the $(a+1)$ moments

$$\mu_s, \mu_{s-1}, \ldots \mu_{s-a},$$

$$\sum_{i=0}^{a} A_i(q) \mu_{s-i} = 0 \quad \text{......(6)},$$

where the coefficients $A_i(q)$ are functions of $q$.

The function $\phi_i(q) = (-1)^i \cdot \dfrac{A_i(1-q)}{A_i(q)}$ may be called the *form-function* of the equation. If the form-function is independent of the subscript $i$, that is to say if a function $\phi(q)$ exists so that identically in $q$ we have: $(-1)^i \cdot \dfrac{A_i(1-q)}{A_i(q)} = \phi(q)$ for all values $i = 0, 1, \ldots a$, then I call (6) a *reduced* equation. If no such function exists, I call (6) a *not-reduced* equation.

The character of an equation as reduced or not-reduced evidently remains unaltered if the equation is multiplied by an arbitrary function of $q$.

* *Biometrika*, Vol. xv. p. 410, 1923. After finishing the present paper I received a letter from Professor Tchouproff who drew my attention to the fact that Romanovsky's formula was first found by Bohlmann. Cf. Bortkiewicz, *Jahresbericht der deutschen Mathematikervereinigung*, Bd. xxvii. (1918), S. 73. [Romanovsky's formula was communicated by him in 1915 to the Society of Naturalists of the University of Warsaw. Ed.]

In the special case $\phi(q)=\pm 1$ I call (6) a *completely reduced* equation, *positively* reduced if $\phi(q)=+1$, *negatively* reduced if $\phi(q)=-1$. A reduced equation which is not completely reduced is obviously transformed into a completely reduced equation when it is divided by any of its coefficients or any other function $F(q)$ which satisfies one of the equations

$$\phi(q).F(q)=\pm F(1-q),$$

where $\phi(q)$ is the form-function of the given equation.

Now suppose that we have given a not-reduced equation of the form (6). Since this equation holds good for every value of the variable between 0 and 1, it holds good for the value $(1-q)$. So we have

$$\sum_{i=0}^{a} A_i(1-q)\,\mu_{s-i}(1-q)=0.$$

Substituting the expression for $\mu_{s-i}(1-q)$ from (5) we get

$$\sum_{i=0}^{a}(-1)^i A_i(1-q)\,\mu_{s-i}=0 \quad\text{......................................(7).}$$

The last equation evidently is a not-reduced equation. Furthermore the two equations (6) and (7) are linearly independent since no function $\phi(q)$ exists, so that for all the values $i=0, 1, \ldots a$ the coefficients $(-1)^i A_i(1-q)$ of the new equation (7) are $\phi(q)$ multiplied by the coefficients of the given equation (6).

Thus we have the following fundamental property of the moment equations: *To every not-reduced linear equation between the moments corresponds another not-reduced equation which is linearly independent of the first, and which may be written down immediately when the coefficients of the first equation are known.*

It is easily seen that this property does not belong to reduced equations. From the two not-reduced equations (6) and (7) we may by linear combination deduce an infinity of new equations. Of these however not more than two can be linearly independent. May it in this way be possible to obtain two linearly independent *reduced* equations?

Evidently it does not restrain generality if we suppose the equations to be obtained by adding to the equation (7) as it stands the equation (6) multiplied by some function $\theta(q)$ of $q$. This gives

$$\sum_{i=0}^{a}[\theta(q)A_i(q)+(-1)^i A_i(1-q)]\,\mu_{s-i}=0 \quad\text{..............................(8).}$$

The form-function of this equation is

$$\Phi_i(q)=\frac{\theta(1-q)\,\phi_i(q)+1}{\phi_i(q)+\theta(q)}=\theta(1-q)\,\frac{\phi_i(q)+\dfrac{1}{\theta(1-q)}}{\phi_i(q)+\theta(q)},$$

where $\phi_i(q)$ is the form-function of the given equation (6).

The necessary and sufficient condition that there exists a function $\Phi(q)$ independent of the subscript $i$, so that identically in $q$

$$\Phi_i(q)=\Phi(q)$$

for all the values $i=0, 1, \ldots a$, is that

$$\theta(q)\,\theta(1-q)=1 \quad\text{..........................................(8 b),}$$

for the values of $q$ in the interval $0<q<1$ for which the form-function of (6), $\phi_i(q)$, is not independent of the subscript $i$. For the values of $q$ for which (8 $b$) is satisfied, we have

$$\Phi(q)=\theta(1-q)=\frac{1}{\theta(q)}.$$

Hence the necessary and sufficient condition that (8) is a reduced equation is that $\theta(q)$ is a function with the specified properties. Furthermore it is evident that if we choose two such functions $\theta_1(q)$ and $\theta_2(q)$ so that $\theta_1(q)\neq\theta_2(q)$, the corresponding two equations of the form (8) are linearly independent.

So we have the second property of the linear equations between the moments: *Any not-reduced equation of the form* (6) *may be replaced by two reduced linearly independent equations of the form*

$$\sum_{i=0}^{s}[\theta_1(q)A_i(q)+(-1)^iA_i(1-q)]\mu_{s-i}=0 \qquad (9a),$$

$$\sum_{i=0}^{s}[\theta_2(q)A_i(q)+(-1)^iA_i(1-q)]\mu_{s-i}=0 \qquad (9b),$$

*where* $\theta_1(q)$ *and* $\theta_2(q)$ *are two arbitrary not identical functions of* $q$ *satisfying the conditions specified under* (8 *b*).

In treating linear equations between the moments one will nearly always find it easier to handle the reduced than the not-reduced forms.

*Application.* The equation (2) is a not-reduced equation with coefficients

$$A_0 \quad =1,$$

$$A_1(q)=q\binom{s-1}{1},$$

$$A_i(q)=q\binom{s-1}{i}-rqp\binom{s-1}{i-1}, \quad i=2, 3, \ldots s.$$

Choosing $\theta_1(q)=+1$, $\theta_2(q)=-1$ we may replace the given equation by the following two reduced equations which are written down immediately by the help of (9*a*) and (9*b*),

$$2\mu_s=\sum_{i=1}^{\text{not v.}}\left[-(q+(-1)^ip)\binom{s-1}{i}+(1+(-1)^i)\binom{s-1}{i-1}rqp\right]\mu_{s-i} \qquad (10a),$$

$$\binom{s}{1}\mu_s=\sum_{i=1}^{\text{not v.}}\left[-(q+(-1)^ip)\binom{s}{i+1}+(1+(-1)^i)\binom{s}{i}rqp\right]\mu_{s-i} \qquad (10b),$$

where the summation is to be continued to the last not vanishing ("not v.") term. The equation (10*b*) is the equation obtained from (9*b*) when $(s-1)$ is replaced by $s$.

Writing out the formula (10*a*) we get

$$\mu_s=\left(\frac{p-q}{2}\right)\left[\binom{s-1}{1}\mu_{s-1}+\binom{s-1}{3}\mu_{s-3}+\ldots\right]$$

$$+rqp\left[\binom{s-1}{1}\mu_{s-2}+\binom{s-1}{3}\mu_{s-4}+\ldots\right]-\frac{1}{2}\left[\binom{s-1}{2}\mu_{s-2}+\binom{s-1}{4}\mu_{s-4}+\ldots\right] \qquad (11),$$

where each of the sequences is to be continued to the last not vanishing term.

The formula (11) may be slightly more convenient for numerical calculations than the formula (1). Other formulae involving still less calculation may however be obtained as I shall show in the next paragraph.

§ 2. *A general System of Recurrence Formulae.* Let us denote by $H_s$ the left-hand side of Pearson's equation (2),

$$H_s=\mu_s+q\binom{s-1}{1}\mu_{s-1}+\sum_{i=2}^{s}\left[q\binom{s-1}{i}-rqp\binom{s-1}{i-1}\right]\mu_{s-i}.$$

Calculating the first few differences of $H_s$,

$$\Delta H_s=H_{s+1}-H_s,$$

$$\Delta^2 H_s=\Delta H_{s+1}-\Delta H_s,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

the following formula for $\Delta^k H_s$ is suggested,

$$\Delta^k H_s=-\sum_{i=0}^{\text{not v.}}\left[(-1)^{i-1}\binom{k}{i}p-\binom{s-1}{i}q+\left(\binom{s-1}{i-1}+(-1)^i\binom{k}{i-1}\right)rqp\right]\mu_{s+k-i} \qquad (12).$$

The formula may easily be proved by complete induction. It obviously holds good for $k=0$. Supposing it to hold good for $k$ we have

$$\begin{aligned}
\Delta^{k+1} H_s &= \Delta^k H_{s+1} - \Delta^k H_s \\
&= -\sum_{i=0}^{\text{not v.}} \left[ (-1)^i \binom{k}{i} p - \binom{s}{i} q + \left( \binom{s}{i-1} + (-1)^i \binom{k}{i-1} \right) rqp \right] \mu_{s+1+k-i} \\
&\quad + \sum_{i=0}^{\text{not v.}} \left[ (-1)^{i-2} \binom{k}{i-1} p - \binom{s-1}{i-1} q + \left( \binom{s-1}{i-2} + (-1)^{i-1} \binom{k}{i-2} \right) rqp \right] \mu_{s+k-(i-1)} \\
&= -\sum_{i=0}^{\text{not v.}} \left\{ \left[ (-1)^{i-1} \left( \binom{k}{i} + \binom{k}{i-1} \right) p - \left( \binom{s}{i} - \binom{s-1}{i-1} \right) q \right] \right. \\
&\qquad \left. + \left[ \binom{s}{i-1} - \binom{s-1}{i-2} + (-1)^i \left( \binom{k}{i-1} + \binom{k}{i-2} \right) \right] rqp \right\} \mu_{s+(k+1)-i} \\
&= -\sum_{i=0}^{\text{not v.}} \left[ (-1)^{i-1} \binom{k+1}{i} p - \binom{s-1}{i} q + \left( \binom{s-1}{i-1} + (-1)^i \binom{k+1}{i-1} \right) rqp \right] \mu_{s+(k+1)-i},
\end{aligned}$$

which is the expression for $\Delta^{k+1} H_s$ obtained from (12).

From Pearson's equation we know that $H_s = 0$ identically in $s$, consequently $\Delta^k H_s = 0$.

Hence replacing $(s+k)$ by $s$ in (12) we have the following equation,

$$\sum_{i=0}^{\text{not v.}} \left[ (-1)^{i-1} \binom{k}{i} p - \binom{s-k-1}{i} q + \left( \binom{s-k-1}{i-1} + (-1)^i \binom{k}{i-1} \right) rqp \right] \mu_{s-i} = 0 \quad \text{......(13}a\text{)},$$

where $k$ denotes an arbitrary not negative integer. This is a not-reduced equation, which is easily seen by an inspection of the form-function of the equation*. Using the result of the preceding paragraph we may immediately write down the following not-reduced equation,

$$\sum_{i=0}^{\text{not v.}} \left[ (-1)^{i-1} \binom{s-k-1}{i} p - \binom{k}{i} q + \left( (-1)^i \binom{s-k-1}{i-1} + \binom{k}{i-1} \right) rqp \right] \mu_{s-i} = 0 \quad \text{...(13}b\text{)}.$$

Taking first the sum and then the difference of the last two equations, which corresponds to choosing the functions $\theta_1(q)$ and $\theta_2(q)$ from the preceding paragraph equal to $+1$ and $-1$ respectively, we obtain the following two *reduced* equations,

$$\sum_{i=0}^{\text{not v.}} \left[ \left( \binom{s-k-1}{i} + \binom{k}{i} \right) ((-1)^{i-1} p - q) + (1 + (-1)^i) \left( \binom{s-k-1}{i-1} + \binom{k}{i-1} \right) rqp \right] \mu_{s-i} = 0$$
$$\text{...............(14}a\text{)},$$

$$\sum_{i=0}^{\text{not v.}} \left[ \left( \binom{s-k}{i+1} - \binom{k}{i+1} \right) ((-1)^{i-1} p - q) + (1 + (-1)^i) \left( \binom{s-k}{i} - \binom{k}{i} \right) rqp \right] \mu_{s-i} = 0 \quad \text{...(14}b\text{)},$$

where $k$ again denotes an arbitrary not negative integer, not necessarily the same in (14$a$) as in (14$b$). The equation (14$b$) is the equation obtained by subtracting (13$a$) from (13$b$) and replacing $(s-1)$ by $s$.

To every not negative value of $k$ correspond two recurrence formulae (14$a$) and (14$b$). Putting $k=0$ we obtain the formulae (10$a$) and (10$b$). In this special case the formulae contain all the moments from $\mu_0$ to $\mu_s$. By choosing $k$ conveniently we may reduce the number of moments occurring in the formulae. In fact if we choose any particular value $k_0 \left( < \frac{s}{2} \right)$ of $k$, the coefficients of all the moments from order zero to order $\left\{ \begin{matrix} k_0 - 1 \text{ (if } (s-k_0) \text{ be even)} \\ k_0 \text{ (if } (s-k_0) \text{ be odd)} \end{matrix} \right\}$ vanish in the formula. If $s$ be even and $k = \frac{s}{2}$, the coefficients in (14$a$) vanish for the moments from order zero to order $\left\{ \begin{matrix} k-1 \text{ (if } \frac{1}{2}s \text{ be even)} \\ k-2 \text{ (if } \frac{1}{2}s \text{ be odd)} \end{matrix} \right\}$. If $s$ be odd we may for instance choose $k = \frac{1}{2}(s-1)$†. Formula

* The only case in which (13$a$) might be a reduced equation is the case $k = \frac{1}{2}(s-1)$.

† $k = \frac{1}{2}(s-1)$ being the case in which (13$a$) is a reduced equation, the special formula (15), p. 170, may of course also be derived directly from (13$a$) or from (13$b$).

(14 *a*) then gives

$$\mu_s=(p-q)\left[\binom{k}{1}\mu_{s-1}+\binom{k}{3}\mu_{s-3}+\dots\right]+2rqp\left[\binom{k}{1}\mu_{s-2}+\binom{k}{3}\mu_{s-4}+\dots\right]$$
$$-\left[\binom{k}{2}\mu_{s-2}+\binom{k}{4}\mu_{s-4}+\dots\right] \quad\text{......(15).}$$

This formula involves considerably less calculation than the analogous formula (11). For instance to calculate $\mu_{13}$ by (11) we must use the values of all the moments from $\mu_0$ to $\mu_{12}$ with numerical coefficients such as 495, 792, 924, etc. To calculate $\mu_{13}$ by (15) we only use the moments from $\mu_7$ to $\mu_{12}$ with numerical coefficients 6, 15, and 20.

I think it should be possible by further transformation of Pearson's equation, using the results from § 1, to obtain an equation only containing four consecutive moments, but I have not yet been able to carry out this transformation.

§ 3. *Generalisation of Pearson's and Romanovsky's Formulae to incomplete Moments.* The incomplete moment of order $s$ may be defined as

$$\mu_s=\sum_{\nu=\rho}^{r}(\nu-rq)^s T_\nu,$$

where

$$T_\nu=\binom{r}{\nu}q^\nu p^{r-\nu} \quad\text{......(16).}$$

If it be desired to emphasize the lower limit of summation we may use the notation ${}_\rho\mu_s$.

Differentiating the equation (16) with respect to $q$ we obtain

$$\mu_s'=\sum_{\nu=\rho}^{r}\left\{\frac{(\nu-rq)^{s+1}}{qp}T_\nu-rs(\nu-rq)^{s-1}T_\nu\right\},$$

hence

$$\mu_{s+1}=qp\left[rs\mu_{s-1}+\mu_s'\right] \quad\text{......(17).}$$

This I think is the simplest proof of which Romanovsky's formula is capable. It is a very interesting fact that the formula holds good without any alteration, even for the general case of incomplete moments.

For the incomplete moment of the first order $\mu_1=\sum_{\nu=\rho}^{r}(\nu-rq)T_\nu$ I have given the explicit expression in the *Skandinavisk Aktuarietidsskrift* *,

$$\mu_1=\rho pT_\rho=\rho p\binom{r}{\rho}q^\rho p^{r-\rho} \quad\text{......(18),}$$

Introducing this expression in the equation obtained from (17) by putting $s=0$, we get

$$\mu_0'=\frac{\rho}{q}T_\rho.$$

Integrating between 0 and $q$ we obtain in the case $0<\rho$,

$$\mu_0=\rho\binom{r}{\rho}\int_0^q x^{\rho-1}(1-x)^{r-\rho}dx \quad\text{......(19).}$$

This is the expression for the zero moment of the point binomial first found by Professor Pearson †. $r$ and $\rho$ being positive integers, the integral (19) may of course be calculated directly, but the expression for $\mu_0$ obtained would not be any simpler than the definition

$$\mu_0=\sum_{\nu=\rho}^{r}T_\nu \ddagger.$$

The formula (19) may still be useful in numerical applications as shown by Dr Camp §. Now

* No. 3, 1924, p. 161. † *Biometrika*, May 1924, Vol. XVI. p. 202, and note by Dr Camp, p. 171.

‡ The most interesting results of the direct integration of (19) are obtained by equating coefficients of equal powers of $q$. In this way some not unimportant identities satisfied by the binomial coefficients may be obtained.

§ *Biometrika*, May 1924, Vol. XVI. p. 157.

assuming the numerical value of $\mu_0$ to be calculated, I proceed to prove a recurrence formula by which the values of the higher incomplete moments may be calculated successively.

Putting
$$f_\nu = (\nu - rq)^{s-1},$$
$$g_\nu = (\nu - rq)\, T_\nu,$$

we have, by partial summation of (16),

$$\mu_s = \sum_{\nu=\rho}^{r} f_\nu g_\nu = f_r \sum_{\nu=\rho}^{r} g_\nu - \sum_{\nu=\rho}^{r-1} \Delta f_\nu \sum_{i=\rho}^{\nu} g_i.$$

Introducing the following expressions taken from (18),

$$\sum_{\nu=\rho}^{r} g_\nu = \rho p T_\rho,$$

$$\sum_{i=\rho}^{\nu} g_i = \rho p T_\rho - (\nu+1) p T_{\nu+1},$$

we get

$$\mu_s = \rho p T_\rho f_r - \rho p T_\rho \sum_{\nu=\rho}^{r-1} \Delta f_\nu + \sum_{\nu=\rho}^{r-1} p(\nu+1) T_{\nu+1} \Delta f_\nu = \rho p T_\rho f_\rho + q \sum_{\nu=\rho}^{r} (r-\nu)\, T_\nu \Delta f_\nu$$

$$= \rho p T_\rho f_\rho + q \sum_{\nu=\rho}^{r} (rq - \nu + rp)\, T_\nu \Delta f_\nu = \rho p T_\rho f_\rho + rqp \sum_{\nu=\rho}^{r} T_\nu \Delta f_\nu - q \sum_{\nu=\rho}^{r} (\nu - rq)\, T_\nu \Delta f_\nu.$$

Introducing $\quad \Delta f_\nu = \binom{s-1}{1} (\nu - rq)^{s-2} + \binom{s-1}{2} (\nu - rq)^{s-3} + \ldots + \binom{s-1}{s-1}$

we finally obtain

$$\mu_s = \rho p T_\rho (\rho - rq)^{s-1} + rqp \sum_{i=0}^{s-2} \binom{s-1}{i} \mu_i - q \sum_{i=0}^{s-2} \binom{s-1}{i} \mu_{i+1} \quad \ldots\ldots\ldots\ldots\ldots(20).$$

This is Pearson's formula generalized to the case of incomplete moments. The formula only differs from the complete-moment formula (1) by the additional term $\rho p T_\rho (\rho - rq)^{s-1}$. Putting $\rho = 0$ in (20) we get of course (1). The incomplete moments do not satisfy any simple relation such as (5), so the equation (20) cannot be replaced by such simple reduced equations as (14*a*) and (14*b*).

The first few incomplete moments as calculated from (20) are

$$\mu_1 = \rho p T_\rho,$$
$$\mu_2 = \rho p T_\rho [\rho - (r+1) q] + rqp \cdot \mu_0,$$
$$\mu_3 = \rho p T_\rho [(\rho - (r+1) q)^2 + qp (2r-1)] + rqp (p - q) \cdot \mu_0,$$

........................................................

## Review: The Elements of Vital Statistics in their bearing on Social and Public Health Problems. By Sir ARTHUR NEWSHOLME, K.C.B., M.D., F.R.C.P. New Edition, George Allen and Unwin, Ltd.

THIS new edition of *The Elements of Vital Statistics* contains much fresh matter. Sir Arthur Newsholme discusses in an early chapter the different methods of estimating population. We think that he has some doubt as to the advantage of substituting "age in years and months" for "age last birthday" in the 1921 census. Personally we doubt whether it will conduce to accuracy. If anyone knows his age "age last birthday" seems to us the simplest form in which to state that age and one wonders how recorders deal with a portion of a month when they have to enter "age in years and months." In considering "age at marriage" Sir Arthur Newsholme shows that between 1896 and 1920 there has been very little postponement of marriage, ·88 of a year for bachelor bridegrooms and ·40 of a year for spinster brides, and one concludes that such a slight postponement can have little to do with the falling birth-rate.

The birth-rate is considered in Chapters VII, VIII, and IX. In connection with the registration of births we should like to point out that though the maiden surname of the mother is