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The Relationship Between Primary Investment and Reinvestment*

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The Relationship Between Primary Investment and Reinvestment.

Ragnar Frisch

1. Introduction.

Dr. Schoenheyder asked me some time ago that I provide a mathematical formulation of his new theory of crises. I have not had the opportunity to make a general analysis of the theory. It should not be necessary to do so, either. I have, however, noted a single point, the theoretical content of which lends itself to illustration only with difficulty by simple numerical examples, and the mathematical analysis of which might therefore be of some interest. This refers to a point which several authors have touched upon in their study of cyclical movements in the economic system, i.e., the relationship (connection) between a given primary investment and the reinvestment which is necessary to maintain the concrete capital objects being produced by the given primary investment. The purpose of the following analysis is only to clarify this relationship, not to investigate the consequences for the general theory of crises which can be drawn from it.

Even if the idea and the occasion of the following reflections are Dr. Schoenheyder's, he is not, of course, responsible for the correctness of the results which I give. The framework in which I conduct the analysis, i.e., the distinction between what I have called the numerical theoretical phenomenon, the phenomenon of distribution and the phenomenon of repetition, is, besides, not quite similar to that of Dr. Schoenheyder, according to what I understand from the exposition of the theory given to me by Dr. Schoenheyder.

The problem under consideration can be illustrated by the following simplified example. Let us suppose that at a given point of time a primary investment (new investment) takes place, consisting of the production of a hammer of wood, a hammer of iron and a hammer of steel. The wooden hammer has a durability of one year, the iron hammer two years, and the steel hammer three years. We assume that the hammers are renewed as they are worn out.

The first year after the primary investment, the wooden hammer has to be renewed, the next year the wooden hammer and the iron hammer, the third year the wooden hammer and the steel hammer, the fourth year the wooden hammer and the iron hammer, the fifth year the wooden hammer alone, and the sixth year all three of them have to be renewed.

The annual reinvestment which is occasioned by the given primary investment is thus far from constant. There is a marked fluctuation. The question is what economic significance should be given to this fluctuation. In section 2 I will try to show that no economic significance can be attached to the fluctuation in the annual reinvestment which occurs in this example. This is due to the composition of prime numbers of the numbers 1 - 6 in connection with the arbitrary choice of unit of time in the example. It can in fact be proved that, if the distribution of the primary investment according to the durability of the capital objects has a finer classification, e.g. with class intervals of one quarter or one month, the fluctuations in the annual reinvestment will be damped, and from continuity considerations, they disappear completely. This phenomenon I call the numerical theoretical phenomenon.

There is, however, another kind of fluctuation in the annual reinvestment that does not disappear when viewed continuously and to which economic significance therefore should be given. Suppose that the primary investment does not consist of one wooden hammer, one iron hammer and one steel hammer, but of one wooden hammer, three iron hammers and one steel hammer. The durability distribution of the concrete capital objects is in other words not uniform, but unimodal. The capital objects are distributed around a certain typical durability (two years). In this case the annual reinvestment will show certain (approximately periodical) fluctuations which do not disappear when reviewed continuously. This phenomenon I call the phenomenon of distribution, to indicate that it is due to the statistical durability distribution of the concrete capital objects.

There will also be a damping of the fluctuations in the annual reinvestment by the distribution phenomenon in the sense that the further one moves away from the point of time in which the primary investment took place, the more the fluctuations in the reinvestment will slacken out. After a certain time has elapsed, they will become almost imperceptible. The annual reinvestment will from now on be approximately constant. We can say that the capital objects under consideration have become an integral part of the circulating capital in the economy, of which a certain constant quantity is renewed every year. This phenomenon of damping is in my opinion very interesting. It is treated in more detail in section 3.

It is constructive to compare the distribution phenomenon with the phenomenon of population dynamics which is called Eilert Sundt's law. The comparison is not, however, quite appropriate. Eilert Sundt's law refers to the way in which fluctuations in the number of newborns per annum propagate themselves relative to the composition of population and the mobility of population in subsequent generations. Eilert Sundt's law focuses attention in other words on the fact that the size of a certain influx of newborns (represented by the natality) can vary from one year to another. With the distribution phenomenon we are not, however, concerned about how the influx (here represented by the primary investment) varies from one year to another, because the distribution phenomenon refers to the influx of a single year (the primary investment of a single year). The distribution phenomenon focuses attention on the circumstance that the elements of the influx in the year in question (the individual capital objects) have a typical unimodal distribution according to the time which will elapse until they will be reproduced. The theoretical population phenomenon which would be an analogy to the capitalistic phenomenon of distribution, is the effect upon the composition and mobility of population in subsequent generations which would occur if the current population experienced a certain increase on a certain occasion, and this increase itself was a composed population with a certain age distribution. Conversely: The reinvestment phenomenon which would be an analogy to Eilert Sundt's law is the variation in the annual reinvestment which results when the annual primary investment consists of a certain kind of capital objects (with the same durability), e.g. only steel hammers, and when the size of the primary investment varies according to time, e.g. when 10 steel hammers are invested in 1916, 20 steel hammers in 1917, etc. This last phenomenon I call the repetition phenomenon. It is treated in section 4.

So, under the distribution phenomenon we study the primary investment of a single year, under the assumption that the concrete capital objects have a certain durability distribution. Under the repetition phenomenon we study how the annual reinvestment varies according to time. And the assumption is now that all capital objects of the annual primary investment have the same durability.

The distribution phenomenon is in a certain sense the most general of the two phenomena. The assumption that all physical capital objects have a certain durability is in fact a special assumption which is contained as a limiting case in the more general assumption that the capital objects have a certain durability distribution. In another sense the phenomenon of repetition is the most general. The assumption that a certain primary investment is undertaken at a given point of time is in fact a special assumption which is contained as a limiting case in the more general assumption that the size of the primary investment has to vary according to time in a certain way.

The most special phenomenon is the one which occurs when a primary investment of capital objects is undertaken at a given point of time, and all of them have the same durability v. The reinvestment in this case will simply consist of exactly the same mass of capital being invested anew every v-th year. This phenomenon we might call the pure phenomenon of repetition. It is evidently a limiting case of the above-mentioned general phenomenon of repetition. Moreover it can also be considered as a limiting case of the distribution phenomenon. The difference between the repetition phenomenon on the one hand and the general phenomenon of repetition and the phenomenon of distribution on the other, is that the first one is a simple phenomenon, while the two others (except for certain special cases) are an interference phenomenon or a resultant phenomenon in the sense that the reinvestment at a given point of time will be a sum of partial investments, i.e., the sum of a certain quantity of first-time reinvestment, a certain quantity of second-time reinvestment, etc.

The most general phenomenon is the one that occurs when the size of the primary investment (per annum) as well as its durability distribution vary according to time. This phenomenon I will call the composed phenomenon. This is treated in section 5.

In the following I assume that a common scale of measurement for the quantity of capital objects of various durability is defined by means of prices or standard coefficients, so that the objects can be added. How this is to be done in a given case, e.g. concerning a statistical observation, is a question which it should not be necessary to discuss in this connection.

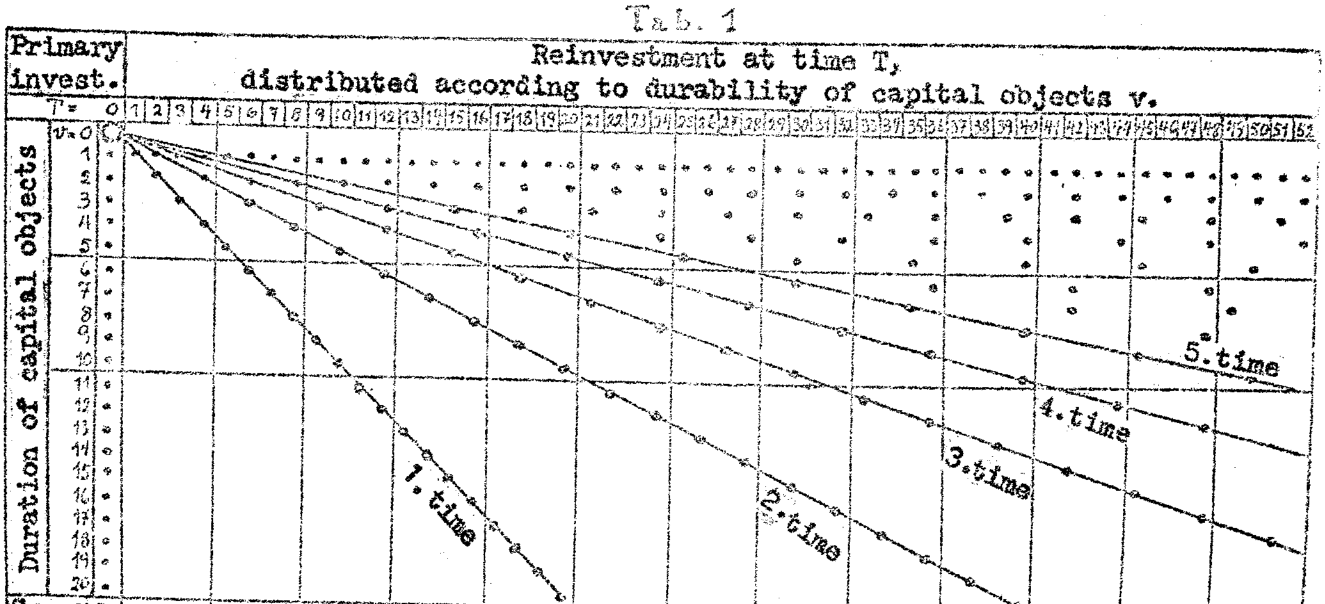
An additional assumption in the following is that every capital object which once has been created by a primary investment, is always effectively renewed, after its lifetime. As a first approximation to the real world we therefore waive the circumstance that some of the capital objects are not renewed when they are worn out. The case when some of the capital objects are not renewed can be treated by considering a negative primary investment, but I will not go into this on this occasion.

2. Capital Objects with a Discrete Durability Distribution. The Numerical Theoretical Phenomenon.

Both the primary investment and the reinvestment are distributed virtually continuously in time in the real world. Thus for example the annual investment of the nation (primary investment and reinvestment) for 1926 will be distributed over the months, weeks and days of the year. Similarly the durability distribution of the capital objects will be virtually continuous in the real world.

The problem concerning the relationship between primary investment and reinvestment is, however, in a certain respect more surveyable and easier

Tab. 1



Sum re-invest.													
I. v = 1-5	1 2 2 3	2 4 1 3	2 3 1 4	1 2 3 3	1 3 1 4	2 2 1 4	1 2 2 3	1 4 1 3	2 2 2 4	1 2 2 4	1 3 1 3	3 2 1 4	1 3 2 3
II. v = 1-10	3 5	6 6	7 6	4 7	6 6	5 7	4 7	7 5	4 9	3 8	6 4	6 7	6 5
III. v = 1-20	8	12	15	15	16	14	13	14	17	14	12	16	13

to attack when we suppose that the investment of the year takes place concentrated in a moment of time and that the capital objects are discretely distributed according to durability. We shall, therefore, first make this assumption. In the next section the continuous distribution is treated.

We assume that at a given point of time a certain primary investment takes place, consisting of a quantity f_1 of 1-year, f_2 of 2-year ... f_n of n-year capital objects. The question is now what reinvestment this will occasion in the future.

Let us first suppose that there is the same quantity of 1-year, 2-year ... and n-year capital objects in the primary investment under consideration. We have then $f_1 = f_2 = \dots = f_n$. In order to get a view of the reinvestment we will use an illustration as in Table 1.

The head of the table represents the time T, considered from the moment when the primary investment was injected. The primary investment is thus undertaken at time T = 0, i.e. at the beginning of the first year. The first column of the table represents the durability of the capital objects. In the column directly below T = 0 is given the durability distribution of the primary investment by placing a point just opposite v = 1 (durability one year) which represents the 1-year capital objects, just opposite v = 2 (durability two years) is placed a point which represents two-year capital objects, etc.

The reinvestment is shown in the following way. The 1-year capital objects will be renewed at time T = 1, 2, 3, ..., etc., i.e. at the end of the first, second, third years, etc. This is represented by the points on the first row of the table. The 2-year capital objects are renewed at times T = 2, 4, 6, ..., etc. This is represented by the points on the second row in the table. And so on for the following rows. When this construction is done, the reinvestment points can be arranged according to downward sloping lines. First the points (T = 1, v = 1), (T = 2, v = 2) etc. are connected by a line, the slope of which (i.e. the relationship between height and base) equals one. The points along this line represent all first time reinvestments, i.e. the first reinvestment of the 1-year capital objects (the point T = 1, v = 1), the first-time reinvestment of the 2-year capital objects (the point T = 2, v = 2) etc. Then the points (T = 2, v = 1), (T = 4, v = 2), etc. are connected by a line with slope $\frac{1}{2}$. The points along this line represent all second-time reinvestments. Correspondingly for the following lines. It is easy to see that the slope of the k-th line is equal to $\frac{1}{k}$ of the density of the points along the first downward sloping line. It is also easy to see that the density of the points along the v-th horizontal line is $\frac{1}{v}$ per unit of time, i.e. the distance of time between two neighboring points is v. The total reinvestment at time T = 1, 2, 3, ... etc. is represented by the points in the vertical columns corresponding to T = 1, 2, 3, ... etc. These sums are shown at the bottom of the table.

If the durability distribution of the primary investment is not uniform, i.e. if not $f_1 = f_2 = \dots = f_n$, then there must be assigned <u>different weights</u> to the various points in the column of the primary investment (i.e. the column T = 0), according to the size of f_v (v = 1, 2, ..., n). Thus this difference in weights propagates itself from the points of primary investment to the points of reinvestment, as all the points on the v-th horizontal row are to be assigned a weight f_v. This can be illustrated theoretically by constructing a perpendicular on the plane of the table in every single point. The length of each perpendicular equals the size of f_v. Or each of the points of the table can be substituted by a number, and in fact the number which designates the magnitude of f_v. Finally we can give a particularly instinctive illustration of the different weights that are to be assigned to the points, by assuming that every point is assigned a mass equal to the magnitude of f_v. We will primarily use this illustration in the following. The above-mentioned illustrations are, however, equivalent in principle. When the distribution is not uniform, we have to consider the density of the mass along horizontal and downward sloping lines instead of the density of the points. When in the following I use the expression "density", I always mean the density of the mass per unit of time. The expressions "mass" and "quantity of capital objects" are used synonymously.

The average density along the v-th row is $\frac{f_v}{v}$, because there are on the average $\frac{1}{v}$ points per unit of time along the v-th row, and each of these points has a mass equal to f_v. This density along the v-th row is an expression for the average reinvestment per year which is occasioned by the v-year capital objects.

Along the k-th sloped line the average density in an interval of k years (measured along the T-axis) equals $\frac{f_v}{v}$, where v is the durability corresponding to that point of reinvestment which can be found on the k-th sloped line in the interval of time under consideration (there exists one and only one such point). The total mass along the vertical line which corresponds to a certain magnitude of T, equals the total reinvestment at time T.

The density along horizontal lines is related to the concept of the (total) average reinvestment per year.

This is defined in the following way: in the primary investment there is a quantity f_1 of 1-year capital objects. These recur every year; this gives an average per year of $\frac{f_1}{1}$. There is a quantity f_2 of 2-year capital objects. These recur every second year. This gives an average of $\frac{f_2}{2}$ per year etc. The total average annual reinvestment is then

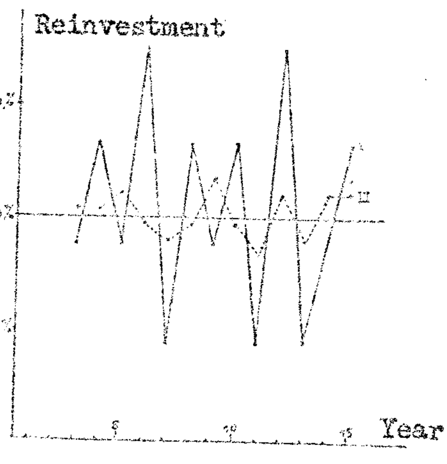
$$(1) \quad a = \frac{f_1}{1} + \frac{f_2}{2} + \dots + \frac{f_n}{n} = \sum_{v=1}^{n} \frac{f_v}{v} \ .$$

If the durability distribution of the primary investment is not divided into intervals of a whole year, but e.g. in intervals of a quarter, a month or generally in intervals, the width of which is a (proper or improper) fraction l of a whole year, and if f_w $(w = 1, 2, \dots, n)$ denotes the quantity of capital objects of durability lw years, the average annual reinvestment will be

$$(2) \quad a = \sum_{w=1}^{n} \frac{f_w}{lw} \ .$$

I have treated the construction of Table 1 and the related concepts of the density along horizontal and descending lines this fully because Table 1 is an important tool in the study of reinvestment.

I return now to the uniform distribution of the primary investment (f_v = constant). I first make the assumption that the durability distribution of the primary investment has 5 classes 1 — 5 years. The development of the annual reinvestment is given in the bottom row (I) in Table 1. There are considerable fluctuations in it. Calculated as a percentage of the average reinvestment $a = \sum_{v=1}^{n} \frac{1}{v} = 2.28$ (as f_v = constant is put equal to 1), the



development is as shown in column (I), Table 2, and the curve (I) in Figure 1. I will now show that these fluctuations (the numerical theoretical phenomenon) are a phenomenon that cannot be given any economic significance.

Suppose that the durability distribution of the primary investment is given with class intervals of half a year ($l = \frac{1}{2}$), the distribution still being assumed to be uniform. Instead of a quantity 1 of capital objects with durability one year etc., there is now a quantity $\frac{1}{2}$ of capital objects with durability half a year, a quantity $\frac{1}{2}$ of capital objects with durability one year etc. The annual reinvestment which is occasioned by means of Table 1 because the scale of measurement for both T and v is now half a year, and every point in the Table counts as $\frac{1}{2}$. We are to consider the upper half of the Table (v = 1 — 10). The sum for T = 1 — 2, i.e. 3 points each with weight $\frac{1}{2}$, gives the total reinvestment for the first year = $1\frac{1}{2}$ etc.

Tab. 2.

	Reinvestment in T-th year as a percentage of the average reinvestment per year. When the capital objects of the primary investment are distributed with class intervals of		
	one year (I)	half a year (II)	one quarter (III)
T = 3	88 %	102 %	104 %
4	133 »	102	104 »
5	88 »	119 »	111 »
6	175 »	102	97 »
7	14 »	68 »	90 »
8	133 »	119 »	97 »
9	88 »	102 »	118 »
10	133 »	102 »	97 »
11	44 »	85 »	83 »
12	175 »	119 »	111 »
13	44 »	68 »	90 »

Proof: In the limiting case $l \to 0$ we can use the formula (3) in section 3.

$$\varphi(T) = \sum_{k=1}^{\infty} \frac{1}{k} f\left(\frac{T}{k}\right) .$$

[Footnote continued on following page.]

(the bottom row (II) in Table 1). The average annual reinvestment according to formula (2) for $l = \frac{1}{2}$ and $f_w = \frac{1}{2}$ is

$$a = \sum_{w=1}^{10} \frac{f_w}{lw} = \sum_{w=1}^{10} \frac{\frac{1}{2}}{\frac{1}{2}w} = 2.93 \text{ .}$$

The percentage fluctuation in the annual reinvestment is calculated from this. It is shown in column (II) in Table 2.

The annual reinvestment can be calculated in the same way when the durability distribution of the primary investment has class intervals of one

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The distribution will then be continuous with $f(v)$ = constant = c for $0 \leqq v \leqq b$, but $f(v) = 0$ for $v > b$ (in the example $c = 1$, $b = 5$). In order to determine $\frac{\varphi T}{a}$ in this case we can go to the limit and first determine $\frac{\varphi T}{a}$ for the distribution $f(v) = c$ for $\varepsilon \leqq v \leqq b$, but $f(v) = 0$ for $< \varepsilon$ or $v > b$. Then we let $\varepsilon \to 0$.

We have $\varphi(T) = c \sum_{k=M}^{N} \frac{1}{k}$, where M and N are the two positive integers determined by

$$\frac{T}{b} \leqq M < \frac{T}{b} + 1 \qquad\qquad \frac{T}{\varepsilon} - 1 < N \leqq \frac{T}{\varepsilon}$$

and $a = c \int_{\varepsilon}^{b} \frac{dv}{v} = c[\log \frac{b}{T} + \log \frac{T}{\varepsilon}]$

so that,

$$\frac{\varphi T}{a} = \frac{\sum_{k=1}^{N} \frac{1}{k} - \sum_{k=1}^{M-1} \frac{1}{k}}{\log \frac{T}{\varepsilon} + \log \frac{T}{b}}$$

If we let ε take on the values $\frac{T}{v}$ as $v \to \infty$ through whole positive values, we will get, because the second term in the numerator and denominator is finite,

$$\lim_{\varepsilon \to 0} \frac{\varphi T}{a} = \lim_{v \to \infty} \frac{\sum_{k=1}^{v} \frac{1}{k}}{\log v} = \quad \text{for every finite magnitude of T.}$$

In this case we have first let $l \to 0$ and then $\varepsilon \to 0$. If the analysis should be complete, it should be investigated whether the result would be the same when the sequence, in which we have taken the limits, is reversed, but there is hardly any reason to examine this further here.

quarter of a year (the bottom row in Table 1). In this case we have $l = \frac{1}{4}$, so that the average reinvestment per annum is $\sum_{w=1}^{20} \frac{\frac{1}{4}}{\frac{T_w}{4}} = 3.60$. The corresponding percentage variation in the annual reinvestment is given in column (III), Table 2 and by the curve (III) in Figure 1. It is evident from both Table 2 and Figure 1 that the fluctuations in the annual reinvestment are damped considerably when smaller class intervals are adopted for the durability distribution of the primary investment. And it is possible to prove exactly,[1] that when the class interval approaches zero, the damping becomes absolute: The relationship between the annual reinvestment and the average reinvestment becomes constant. Also in another respect is the picture of the fluctuations of the reinvestment quite different for the different class intervals. The year which is a maximum year for one class interval may be a minimum year for another class interval. We may for example compare the fifth year (T = 5) for class intervals of one year and half a year, or the ninth year (T = 9) for class intervals of one year and one quarter of a year.

The phenomenon of damping when the class interval gets smaller, can also be analyzed in another way, and in fact by a probabilistic theoretical consideration. This analysis gives an interesting insight into the way in which the damping occurs.

I reproduce the reinvestment schedule which was used in Table 1, simplified as in Figure 2.

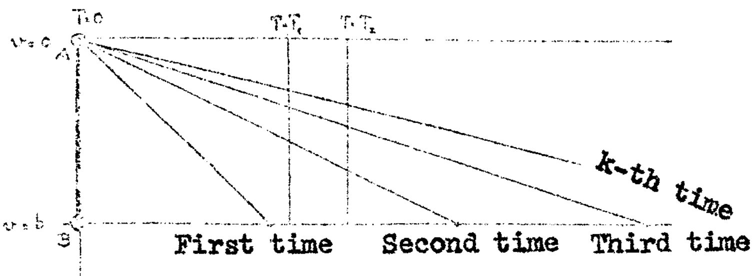


Fig 2.

The line AB represents the primary investment, b is the longest durability. The durability distribution can, as shown before, be illustrated by a distribution of mass along AB. I have not marked this distribution by points as in Table 1, because the distribution is going to vary in the following analysis. If the capital objects are distributed uniformly in a certain number, e.g. 5 classes of one whole year, the representative mass along AB will be concentrated in 5 discrete points along A, with the same quantity of mass c in each point.

The more smoothly the primary investment is distributed, i.e. the smaller the class intervals, the more representative mass points there will be along AB, and the smaller quantity of mass will be contained in each point. If the class interval is l years (e.g. $l = \frac{1}{4}$ which corresponds to class intervals of one quarter of a year), the distance between two neighboring points on AB equals l. There is a total of $\frac{b}{l}$ points along AB, and each point has a mass equal to lc. In the limiting case $l \rightarrow 0$ the number of points becomes infinite, and at the same time the mass in each point becomes 0. The distribution has become continuous.

The reinvestment will be represented by analogous mass distributions along the respective descending lines. If l is the distance of durability between two neighboring points along AB, then the distance of time between two neighboring points on the k-th sloped line is kl (the distance of time being measured in the direction of the T-axis). There are then on the average $\frac{1}{kl}$ points per unit of time (years). As each point has a mass cl, the average mass per unit of time (the average density) along the k-th sloped line is equal to $\frac{c}{k}$.

Let us consider the mass which falls inside a vertical segment $T = T_2$ for $T = T_2$ (see Figure 2). This mass represents the total reinvestment between the points of time T_1 and T_2. If the k-th descending line cuts this segment and <u>if the mass distribution is continuous,</u> the sloped line inside the segment must have a total mass equal to the density times the width of the segment. Let us call the width of the segment $\delta = T_2 - T_1$, and the k-th sloped line inside the segment T_1 to T_2 will then have a mass equal to $\frac{c}{k}\delta$ in a continuous mass distribution. When the mass is not distributed continuously, but concentrated on certain discrete points along the sloped line, $\frac{c}{k}\delta$ represents only the <u>probable</u> mass which falls in the segment between T_1 and T_2 on the k-th sloped line. This means that if we choose at random a number of segments of width δ from different places along the T-axis, there will probably on the average per segment fall a mass $\frac{c}{k}\delta$ from the k-th sloped line. The more smoothly the primary investment is distributed according to durability, the greater is the probability that a randomly chosen segment (of width δ) will have a mass on the k-th sloped line approximately equal to $\frac{c}{k}\delta$.

The same kind of smoothing which results from the law of large numbers when we consider a certain sloped line (the k-th) and a number of different segments of width δ, will also result when we consider a certain segment of width δ and a number of different sloped lines which cut this segment. And the smoothing will be the better, the smaller are the class intervals of the distribution of the primary investment. The total mass inside a segment of width δ is approximately $\delta < \Sigma \frac{1}{k}$ where Σ is extended to all k's corresponding to the numbers of the sloped lines which cut the segment. If we therefore compare two different segments of width δ, the difference between the total masses of the segments can only be explained by $\Sigma \frac{1}{k}$ being different for the two segments. As the class interval gets smaller and smaller, however, the relative difference between $\Sigma \frac{1}{k}$ for two different segments will disappear.

For an arbitrarily chosen segment T_1 to T_2 we have in fact $\Sigma \frac{1}{k} = \sum_{k=M}^{N} \frac{1}{k}$ where M and N are two positive integers which are determined by

$$\frac{T_1}{b} \leqq M < \frac{T_1}{b} + 1 \qquad\qquad \frac{T_2}{l} - 1 < N \leqq \frac{T_2}{l} .$$

l is the class interval and b the greatest durability which occurs, both designated in years.

The relationship between $\Sigma \frac{1}{k}$ for two different segments T_1' to T_2' and T_1'' to T_2'' is thus

$$\frac{\sum_{k=M'}^{N'} \frac{1}{k}}{\sum_{k=M''}^{N''} \frac{1}{k}}$$

where the limits of summation are determined by inequalities analogous to those given for M and N. When now $l \to 0$, this relationship becomes

$$\text{Lim} \frac{\log N'}{\log N''} = \text{Lim} \frac{\log T_2' - \log l}{\log T_2'' - \log l} = 1 .$$

As the class intervals of the distribution of the primary investment become smaller and smaller, the relative difference between the total masses inside two arbitrary segments of equal width will therefore become smaller and smaller and at last it will disappear. Consequently the relationship between the annual reinvestment and the average reinvestment is constant (= 1), when the primary investment is distributed sufficiently in the bottom rows of Table 1 (reproduced as percentages in Table 2 and Figure 1) are therefore a phenomenon which cannot be assigned any economic significance.

I have treated this phenomenon in such great detail because numerical examples are a much used means of demonstration in theoretical economic analyses, and because it could lead to incorrect conclusions if this method is applied uncritically to the problem under consideration.

If the distribution is not uniform, there is a certain fluctuation that does not disappear even if the class interval of the primary investment becomes small. This phenomenon (the phenomenon of distribution) is best analyzed by assuming a continuous distribution of the primary investment.

3. Capital Objects with a Continuous Durability Distribution. The Phenomenon of Distribution.

Let $f(v)$ be the distribution function for the continuous distribution of the primary investment. There is then a quantity of capital objects between the durabilities v and $v + dv$ equal to $f(v)\,dv$.

The distribution $f(v)$ can be interpreted as the density in point v for a continuous mass distribution along the line AB in Figure 2. We must, however, think of the line AB as extended ad infinitum for it to represent any distribution. If, in a special case, the durability of the capital objects has an upper limit b, this can be interpreted by putting $f(v) = 0$ for $v > b$.

The primary investment occasions reinvestments which are represented by mass distributions along the sloped lines in Figure 2. (The sloped lines are to be extended ad infinitum together with AB.) The question is now what the density is (per unit of time) along these sloped lines.

Let us consider durabilities between two arbitrary limits v_1 and v_2. These durabilities are represented by a horizontal segment of width $(v_2 - v_1)$. It is easy to see that the total mass between the durability limits v_1 and v_2 is the same along all sloped lines, and in fact equal to the total mass which is contained in the primary investment between these durability limits. The sum of all first-time reinvestment of capital objects between v_1 and v_2 must be equal to the sum of all capital objects which are contained in the primary investment between v_1 and v_2. And this is also the case for the second-time reinvestment, etc.

This mass, which is equal for all sloped lines, is distributed along the k-th sloped line over an interval of time of length $(v_2 - v_1)k$. In the primary investment this same mass is distributed over an interval of durability of length $(v_2 - v_1)$. It is then obvious that the average density in the interval of durability v_1 to v_2 along the k-th sloped line is equal to $\frac{1}{k}$ of the average density in the same interval of durability in the primary investment. This holds for arbitrary limits of durability. It also holds if we let the length of the interval of durability, i.e. the difference $(v_2 - v_1)$ decrease towards zero. Then the average density in the interval of durability will be the density of a point, i.e. the point v towards which v_1 and v_2 converge. The density (per unit of time) in the point with durability v on the k-th sloped line is then $\frac{1}{k}$ of the density in the corresponding point (i.e. for the same durability) in the primary investment, i.e. equal to $\frac{f(v)}{k}$. This formula is the analogous formula for discrete distributions.

The capital objects which are on the k-th sloped line at time T have a durability $v = \frac{T}{k}$. The density (per unit of time) at time T on the k-th sloped line is then equal to $\frac{1}{k} f(\frac{T}{k})$. <u>I.e. the quantity of capital objects which is reinvested between the points of time T and T + dT for the k-th line equals $\frac{1}{k} f(\frac{T}{k})\, dT$</u>.

Let now φT denote the total reinvestment (reckoned per year) at time T, so that the total reinvestment between T and T + dT is $\varphi(T)dT$. This reinvestment is evidently the sum of the reinvestments between T and T + dT for all of the sloped lines (k = 1, 2, ... etc)

i.e.

(3) $\varphi(T) = \sum_{k=1}^{\infty} \frac{1}{k} f(\frac{T}{k})$.

This expression shows how the reinvestment (reckoned per year) at a given point of time T is derived from the durability distribution f(v) of the primary investment.

In Figure 3 a geometric illustration is given of the relationship which exists according to (3) between the reinvestment $\varphi(T)$ and the durability distribution f(v) of the primary investment.

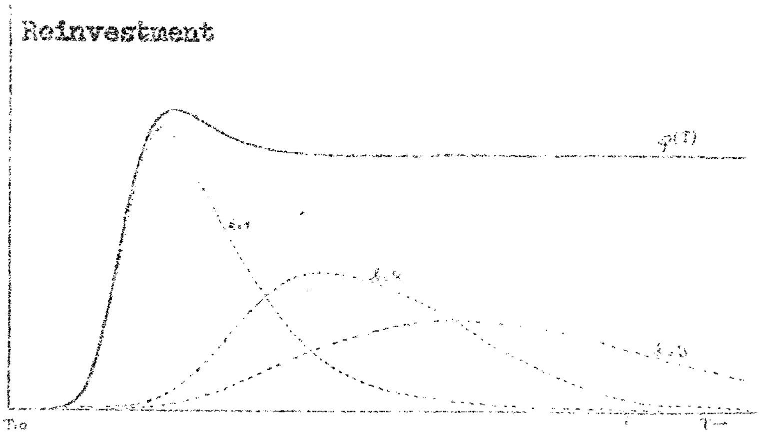


Fig. 3

We have first drawn the distribution curve f(T), T now being used as abscissa instead of v. This gives the element corresponding to k = 1 in formula (3) (the element corresponding to first-time reinvestment). Then another curve is drawn which is derived from the first one by halving the ordinate and doubling the abscissa. This gives the element corresponding to k = 2 (second-time reinvestment). Then a third curve is drawn which is derived from the first one by reducing the ordinate to one third and making the abscissa three times as large. This gives the element corresponding to k = 3, and so on ad infinitum. The curves for k = 1, 2, 3, ..., etc, we may call the partial reinvestment curves or the <u>partial curves</u>.

We now construct the resultant curve for the partial curves. By this we mean the curve, the ordinate of which for every point of time T is the sum of the ordinates of the partial curves. The resultant curve being constructed in this way, is then the reinvestment curve $\varphi(T)$.

We can now see that the course of $f(v)$ near $v = 0$ is of considerable significance for the course of $\varphi(T)$, and this is not only the case for points of time close to $T = 0$, but also for subsequent ones. Even if T is large, subsequent elements in the formula for $\varphi(T)$, i.e. the elements for large k's, will depend upon the magnitude of $f(v)$ for small v's. The farther out in the number of the elements we get, the smaller is the v on which the magnitude of the element depends, because subsequent elements in the formula (subsequent partial curves) are derived from $f(v)$ by stretching the abscissa of $f(v)$ more and more. Because the number of these subsequent elements is infinitely large, they will have a diminishing influence on the magnitude of $\varphi(T)$ provided $f(v)$ does not decrease very quickly as $v \to 0$. We can also express the relationship in this way: The quantity of short-lasting capital objects in the primary investment is of considerable importance even for the more distant reinvestment, since the short-lasting capital objects recur frequently, and the more frequently the more short-lasting they are.

The influence of the short-lasting capital objects on the reinvestment is evident when we consider the average reinvestment. For discrete distributions (section 2) the average annual reinvestment was defined as

$a = \sum_{v=1}^{n} \frac{f_v}{v}$, set where n is the highest class of durability. The expression can also be written as $a = \sum_{v=1}^{\infty} \frac{f_v}{v}$, when we put $f_v = 0$ for $v > n$. By analogy we define the average reinvestment per annum when the primary investment has a continuous durability distribution f(v) as

$$(4) \quad a = \int_0^\infty \frac{f(v)\,dv}{v} \quad .$$

We can see that we must have f(0) = 0 for the average reinvestment to be finite. And this is not the only condition, but as $v \to 0$, f(v) must also decline so much that the integral converges. The durability distribution of the primary investment must in other words be such that the quantity of capital objects of a certain durability declines strongly as we consider smaller and smaller durabilities. In the transition to the continuous durability distribution which was analyzed in section 2, the durability distribution was such that the integral did not converge. This did not, however, prevent the relative fluctuations in the annual reinvestment to become finite.

The future reinvestment[1] is another concept which plays a certain part in the analysis of the continuously distributed primary investment. By this we mean the annual reinvestment which will result when a longer time has elapsed ($T \to \infty$) from the point of time when the primary investment was injected. If $\lim_{T \to \infty} \varphi(T)$ exists, the future reinvestment equals this limit. One of the main purposes of the analysis in this section is in fact to show that this limit exists. I will show that as time elapses, the fluctuations in the annual reinvestment are damped. The reinvestment approaches a certain

[1] The adjective "future" cannot be claimed to belong to common usage, but I have not been able to find a commonly used word which rendered the exact meaning. By using the expression the asymptotic reinvestment we would a priori have indicated that the future reinvestment really has a definite limit, which is obviously unjustified. Under the phenomenon of distribution the reinvestment certainly has a definite limit (see the proof in the text) but this is not always the case with the more general phenomenon treated in section 5.

constant normal level. And this level is equal to the magnitude of which is defined above. The future reinvestment is thus equal to the average reinvestment. This statement we will call the law of distribution.

The damping which we consider now (which occurs as time elapses) is evidently a phenomenon of a completely different kind than the one which was treated in section 2, (which occurred by diminishing the class interval in the durability distribution of the primary investment).

The correctness of the law of distribution can be seen in the following way:

Let $T = \frac{1}{\delta}$, $x_k = k\delta = \frac{k}{T}$ $(k = 1, 2, 3, \ldots, \infty)$

thus

$$x_{k+1} - x_k = \frac{1}{T}.$$

Then we get

$$(5) \quad \varphi(T) = \sum_{k=1}^{\infty} \frac{1}{k\delta} f\left(\frac{1}{k\delta}\right) \cdot \delta = \sum_{k=1}^{\infty} \left(\frac{1}{x} f\left(\frac{1}{x}\right)\right)_{x=x_k} (x_{k+1} - x_k).$$

When $T \to \infty$, so that $\delta \to 0$ and consequently $(x_{k+} - x_k) \to 0$ then the last expression becomes

$$\int_0^{\infty} \frac{1}{x} f\left(\frac{1}{x}\right) dx = \int_0^{\infty} \frac{f(v)}{v} dv$$

This follows directly from the definition of the integral.[1]

We have thus

$$(6) \quad \lim_{T \to \infty} \varphi(T) = \int_0^{\infty} \frac{f(v)}{v} dv = a.$$

It follows not only from (5) that the annual reinvestment $\varphi(T)$ ultimately approaches a constant level (on the assumption that $\int_0^{\infty} \frac{f(v)}{v} dv$ converges), but also an important consequence concerning the way in which the fluctuations in $\varphi(T)$ are damped. We can immediately see from the formula that the deviation of the annual reinvestment at time T from the

[1] When the definition of the integral is taken in classical form of Riemann.

average reinvestment per year is simply equal to the remaining magnitude which we get when the integral of the function $\frac{1}{x}\, f(\frac{1}{x})$ between 0 and ∞ is approximated by mechanical quadrature according to the rectangle method with interval equal to $\frac{1}{T}$.

This illuminates in an interesting way the nature of the damping of the fluctuations of the annual reinvestment which emerges as time elapses. The damping is of the same nature as the decline in the remaining magnitude by a mechanical quadrature.

According to this we will expect that, if the durability distribution of the primary investment is unimodal (as in the example in Figure 3), the deviation of the primary investment from the average is large when T has a magnitude near the magnitude of v, for which the distribution function f(v) has a maximum. And the effect must be the stronger, the more "peaked" is the distribution i.e. the more densely the capital objects in the primary investment are clustered around the typical durability. This can also be seen from Figure 3. If the durability distribution f(v) which represents the first partial curve (k = 1) is pronounced "peaked", this must express itself in the resultant curve $\varphi(T)$.

To show how different degrees of "peakedness" in the distribution function f(v) result in different degrees of fluctuation in the resultant curve $\varphi(T)$, I have given three different examples in Table 3 and Figure 4.

These examples need an explanation. On account of the above mentioned circumstance concerning the effect of the course of f(v) for small v's, it is impossible to draw an arbitrary curve and then derive graphically the resultant curve $\varphi(T)$ from this. We must as an example start from a distribution function f(v), the analytical expression of which we know, and then derive $\varphi(T)$ by formula (3). We must choose as distribution curve f(v) a curve which is truncated to the left, because no capital objects have negative durability. f(v) must also be a function for which the sum according to formula (3) can be given in closed form. Finally the function should have parameters to vary its "peakedness". A distribution function which satisfies these requirements is:

$$(7) \qquad f(v) = \frac{\left(\frac{\beta}{v}\right)^{\alpha+1} e^{-\frac{\beta}{v}}}{\beta\Gamma(\alpha)}$$

where α and β are parameters which determine the shape of the curve, $\Gamma(\alpha)$ is the ordinary gamma function. The property of the function as a distribution function is characterized by $\int_0^\infty f(v)\, dv = 1$. Furthermore $f(0) = f(\infty) = 0$, f(v) increases monotonicially from v = 0 to a maximum for $v = \frac{\beta}{\alpha + 1}$ and decreases from there monotonically to $v = \infty$. The curve has points of inflection at $v = \frac{\beta}{\alpha + 1} (1 \pm \frac{1}{\sqrt{\alpha + 2}})$ i.e. in equal distance before and after the maximum point. If $\beta = \alpha + 1$ is chosen, the maximum (i.e. the typical durability) is at v = 1. This has been done in the examples. It will now be easier to compare the curves in the three examples. As the unit of time can be chosen arbitrarily, the choice of $\beta = \alpha + 1$ does not mean that the typical durability is set equal to 1 year. This would compare badly to the real circumstances. It only means that, if the type of curve which we consider here is to be applied to statistical data, the unit of time must be chosen equal to the typical durability. For $\beta = \alpha + 1$, the maximum of f(v) is approximately equal to $\frac{\alpha}{\sqrt{2\pi\,(\alpha + 1)}}$ provided α is a positive integer. The distribution curve will thus be the more "peaked" the greater is α. The average reinvestment (consequently also the future one) will be equal to $\frac{\alpha}{\beta}$.

When f(v) is chosen according to formula (7) and α is a positive integer, the expression for $\varphi(T)$ can be given in closed form.

We have

$$\varphi(T) = \sum_{k=1}^{\infty} \frac{\left(\frac{\beta k}{T}\right)^{\alpha+1} e^{-\frac{\beta k}{T}}}{k \cdot \beta\Gamma(\alpha)} = \frac{\left(\frac{\beta}{T}\right)^{\alpha+1}}{\beta\Gamma(\alpha)} \sum_{k=1}^{\infty} k^{\alpha} x^{k}$$

where $x = e^{-\frac{\beta}{T}}$. For $\frac{\beta}{T}$ positive, $|x| < 1$, and the series is thus convergent.

For $\alpha > 0$ the lower limit of summation can be extended to k = 0, i.e.

$$(8) \qquad \varphi(T) = \frac{f(T)}{x^2} \sum_{k=0}^{\infty} k^{\alpha} x^{k+1}$$

As α is now assumed to be a positive integer, we have

$$k^{\alpha} = \sum_{i=0}^{\alpha} \Delta^{i}\, 0^{\alpha} \binom{k}{i}$$

where $\Delta^{i}\, 0^{\alpha}$ are the differences of zero.

Now, however

$$\sum_{k=0}^{\infty} \binom{k}{i} x^{k+1} = \left(\frac{x}{1-x}\right)^{i+1} \qquad |x| < 1$$

thus

$$\varphi(T) = \frac{f(T)}{x(1-x)} \sum_{i=0}^{\alpha} \Delta^{i} 0^{\alpha} \left(\frac{x}{1-x}\right)^{i}$$

which is the expression which we seek.

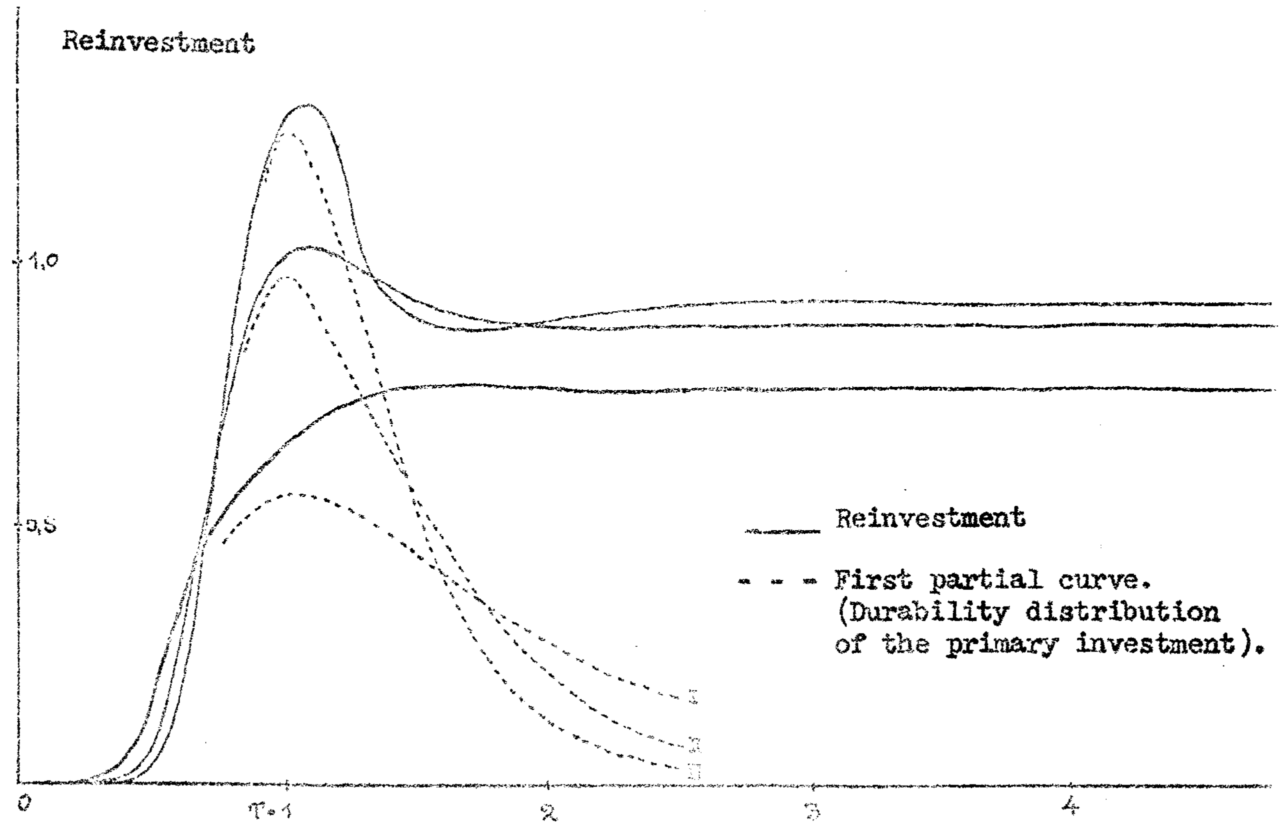
I have chosen the following three examples

(I) $\alpha = 3, \quad \beta = 4$

(II) $\alpha = 7, \quad \beta = 8$

(III) $\alpha = 11, \quad \beta = 12$

It is these examples which are given in Table 3 and Figure 4. The numerical calculations have been performed by actuary Andersen. I will take this opportunity to express my best thanks to him for this work.



Reinvestment
1,0
0,5
0
T=1
2
3
4
Reinvestment
First partial curve.
(Durability distribution
of the primary investment).

Fig.4.

For each of the three resultant curves $\varphi(T)$ in Figure 4 there is given the corresponding distribution curve $f(T)$. The figure illustrates plainly the fact that, when the "peakedness" of the distribution curve is small (as with (I)), the damping in the fluctuations of the reinvestment occurs very quickly. Already the first wave in the reinvestment is small. By (I) for example the fluctuations in the reinvestment are so small that they become almost imperceptible on the scale in which the curve is drawn. The course is practically that the reinvestment curve ascends monotonically to its average level, from then on to follow this. We have to look at the figures in Table 3 to get an impression of the wave-like motion by (I). There is also a small first wave with maximum near T = 1.74.

Tab. 3.

Time T	Reinvestment $\varphi(T)$		
	I	II	III
0.27	0.0015	[illegible]	0.0006
0.40	0.0267	[illegible]	0.0011
0.50	[illegible]	[illegible]	0.0317
0.75	0.5113	[illegible]	[illegible]
1.00	0.6420	[illegible]	1.274
1.25	0.7341	[illegible]	1.074
1.48	0.7509	0.9208	0.9108
1.74	0.7562	[illegible]	[illegible]
2.00	0.7652	0.8791	[illegible]
2.22	0.7545	[illegible]	0.9100
2.50	[illegible]	0.8724	0.9166
2.64	0.7526	0.8787	0.9178
3.07	0.7519	[illegible]	0.9175
3.33	0.7515	0.8745	0.9170
4.00	0.7509	0.8746	0.9167
4.45	0.7506	[illegible]	0.9167
5.00	0.7502	0.8749	0.9767
∞	[illegible]	0.8750	0.9167

If on the other hand, "peakedness" in the distribution curve is more pronounced, as by (II) and still more by (III), the waves in the reinvestment will also become more pronounced. By (III) the first wave is very marked. For the first reinvestment wave in (III)/a maximum is reached after a time (T = approx. 1.1) which is somewhat greater than the typical durability. This is a general phenomenon which will always occur, provided the distribution curve of the primary investment increases monotonically up to the typical durability.

For all three cases there is a damping of the wave-motion as time elapses. By III the second wave (the maximum of which is reached at T = approx. 2.9) is far more pronounced than the first one. Compared with the first wave, the second wave appears only as a slack and lengthened elevation of the curve. The subsequent waves by (III) are imperceptible. By (I) and (II) already the second wave is imperceptible. By (I) there is even no minimum after the first maximum. The reinvestment declines from the first maximum point monotonically down towards the normal level (0.75) which is given by the average reinvestment. By II there is certainly a minimum after the first wave (at approx. 2.3), but there is no complete wave after the minimum at 2.3. The reinvestment increases monotonically from this minimum point up towards the normal level (0.875) given by the average reinvestment. These examples illustrate the various alternatives which there can be. If the "peakedness" in the durability distribution of the primary investment is not particularly pronounced, the reinvestment will only show one, at most two waves, before the damping occurs. For the distribution curves which we will find in the real economic world we will probably not commit a large error by substituting the exact expression for the annual reinvestment (formula (3)) with the expression for the average reinvestment determined by the simple formula (4), after a time equal to $1\frac{1}{2}$ to 2 times the typical durability.

This phenomenon of damping does not deprive the reinvestment fluctuations which we consider here of their economic significance. This is particularly the difference between the distribution phenomenon and the numerical theoretical phenomenon. The decisive fact from a theoretical viewpoint of crises is evidently whether there exists a pronounced first wave or not.

A more exact analysis of the consequences of the theory of crises which can be deduced from the phenomenon which we have demonstrated here, would therefore require a statistical investigation of the degree of "peakedness" in the durability distribution of newly invested capital objects which really exists in economic life. We mentioned in the introduction, however, that it is outside the framework of this article to enter into a discussion of the consequences of the theory of crises. I will therefore only mention this circumstance.

4. Capital Objects with Equal Durability. The Repetition Phenomenon.

In the preceding sections I have analyzed the effects of a single primary investment having a certain durability distribution and being injected at a certain point of time. In this section I am going to treat the case where a continuous primary investment takes place, and all capital objects have one and the same durability v.

Let t denote the point of time for the primary investment and T the point of time for the reinvestment. We assume the course of the primary investment to be given by a continuous function g(t) which designates the primary investment (per unit of time) at time t. Between time t and t + dt there will then be injected a primary investment g(t) dt.

A primary investment g(t) dt between t and t + dt must evidently occasion a reinvestment of equal magnitude g(t) dt, in the first place between time t + v and t + v + dt and in the second place between time t + 2v and t + 2v + dt etc.

Conversely: Between time T and T + dT there is a first-time reinvestment of the capital objects which were initially invested between T - v and T - v + dT; the quantity of these is g(T - v) dT. Between T and T + dT there is moreover a second-time reinvestment of the capital objects which were initially invested between T - 2v and T - 2v + dT; the quantity of which is g(T - 2v) dT etc. There is then a total reinvestment between T and T + dT equal to

$$\sum_{k=1}^{\infty} g(T - kv)\, dT.$$

If the reinvestment (reckoned per unit of time) is denoted $\psi(T)$, so that there is a reinvestment between T and T + dT of magnitude $\psi(T)$ dT, then we have

$$(10) \qquad \psi(T) = \sum_{k=1}^{\infty} g(T - kv).$$

If we now consider the reinvestment which is due to primary investments injected after a time t_0, we have only to put $g(t) = 0$ for $t < t_0$. The summation over k in (10) will then have a finite limit. In this case the relationship between primary investment and reinvestment which is expressed in formula (10) can be illustrated geometrically in the way which is indicated in Figure 5.

In Figure 5, t_0T is the time axis (compare Figure 2), t_0 is the point of time from which the primary investment starts. From t_0 we draw a downward sloping line at an angle of 45°. This line is cut by horizontal lines at distances v, 2v, 3v, etc. from the time axis. The first-time reinvestment

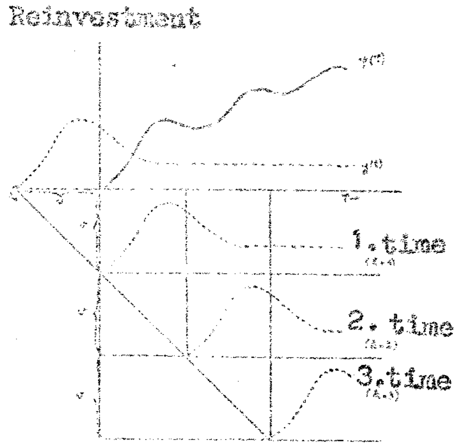


Fig. 5.

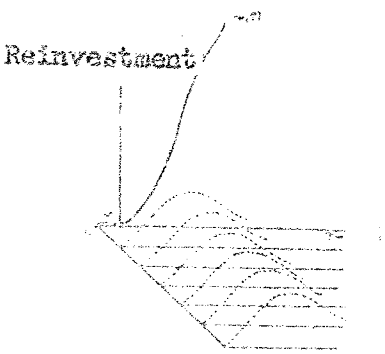


Fig. 6.

is given by a curve which is identical with the primary investment curve (which starts in t_0), but displaced a time interval equal to v to the right (so that it starts in $t_0 + v$). The second-time reinvestment will be given by a curve which is also identical with the primary investment curve, but displaced a time interval equal to 2v to the right etc. The partial reinvestment curves thus appear by repeating the primary investment curve every v-th year in the future. Therefore the name repetition phenomenon. There is a difference between the partial curves of the repetition phenomenon and those of the distribution phenomenon: that the last ones become successively more and more deformed, but not the first ones. Compare Figure 5 with Figure 3.

The reinvestment curve $\psi(T)$ of the repetition phenomenon will be the resultant curve of the partial curves in Figure 5, i.e. the curve the ordinate of which at any time T is the sum of the ordinates of the partial curves. An analogous geometrical representation applies when the primary investment is not restricted to the time after t_0, but the number of partial curves will then become infinite.

With the repetition phenomenon the reinvestment curve will not approach a constant level as with the distribution phenomenon as time passes. On the contrary the reinvestment curve will largely increase with time. The reinvestment will contain a secular movement which constantly elevates the level around which the possibly occuring periodical fluctuations take place. This level is the analogous concept to the average reinvestment which was defined under the analysis of the distribution phenomenon. While the average reinvestment is a constant with the distribution phenomenon, it is a function of time with the repetition phenomenon.

The precise definition of the average reinvestment of the repetition phenomenon can be stated in the following way: The primary investment which is injected between t and t + dt occasions every v-th year a reinvestment of magnitude g(t)dt, i.e. $\frac{g(t)\,dt}{v}$ per year. This expression integrated up to time T gives the average reinvestment at time T. Denote this by b(T), and we will have

$$b(T) = \frac{1}{v}\int_{t_0}^{T} g(t)\;dt \;=\; \frac{1}{v}\int_{-\infty}^{T} g(t)\;dt$$

because $g(t) = 0$ for $t < t_0$.

Suppose there is a single wave in the course of the primary investment and that the primary investment is fairly constant before and after this wave (as in Figure 5). In this case there will also be a damping with the repetition phenomenon in the sense that the relative fluctuation in the reinvestment is smoothed with time, as we now have

$$\lim_{T \to \infty} \frac{\psi(T)}{b(T)} = 1.$$

The damping will take place the quicker, the flatter is the reinvestment wave and the smaller is the durability v of the capital objects in relation to the width of the primary investment wave. By the width of the primary investment wave is meant the distance in time between the point before and after the wave from which the primary investment is fairly constant. This is obviously not a precise definition of the term wave-width. It is only a geometrical illustration of the concept. I think, however, that it is sufficiently exact for the following analysis.

Figures 5 and 6 give an impression of the importance of the wave-width for the fluctuations in the reinvestment and for the damping of these fluctuations.

In Figure 5 the wave-width is only a fraction of the durability v of the capital objects. The result is a marked periodicity in the reinvestment. The secular movement appears also distinctly. As time passes, the relative fluctuations, i.e. the fluctuations in the ratio $\frac{\psi(T)}{b(T)}$ will be damped. This would be easier to see if the reinvestment curve in Figure 5 had been drawn on logarithmic scales.

In Figure 6 the wave-width is several times the durability of the capital objects. Besides the wave is slacker. The consequence is that the damping of the fluctuations of the reinvestment sets in practically immediately. In return the secular movement is the more pronounced.

If the wave-width decreases towards zero at the same time as the primary investment becomes zero outside the area of the wave, there will result as a limiting case the phenomenon which we called the pure repetition phenomenon in the introduction.

5. The General Problem

In the preceding sections I have investigated on the one hand the effect of a single primary investment taking place at a certain time and having a certain durability distribution, and on the other hand the effect of a continuous primary investment of capital objects of one and the same durability. The general problem is to investigate the effects of a continuous primary investment having in each moment of time a certain durability distribution, which may change with time.

In this case the variations in the reinvestment are a composed phenomenon, where both the distribution phenomenon and the repetition phenomenon assert themselves.

I am not going to analyze in detail the fluctuations in the reinvestment in this general case. I will be content to give the basic formulas for the relationship between primary investment, reinvestment and capital mass. It will hardly be possible to go deeper into the general problem, theoretically or statistically, without taking these formulas as the starting point.

With the distribution phenomenon where we were only concerned about illustrating the durability distribution of a single primary investment being injected at a certain point of time, the primary investment could be represented by a mass distribution along the straight line AB in Figure 2 (as we could imagine AB being extended ad infinitum). With the composed phenomenon the primary investment is to be represented by a mass distribution in the plane.

This is to be understood in the following way: For every point (tv) in the plane (Figure 7) there is a certain density P(tv). We assume that P(tv) is continuous so that the mass in the plane segment dt dv is P(tv) dt dv. This means that between time t and t + dt there is injected a primary investment of capital objects P(tv) dt dv, which has a durability between v and v + dv.

This primary investment will occasion a first-time reinvestment of the same magnitude P(tv) dt dv, but distributed between time t + v and t + v + dt + dv, and a k-th time reinvestment of the same magnitude, but distributed between

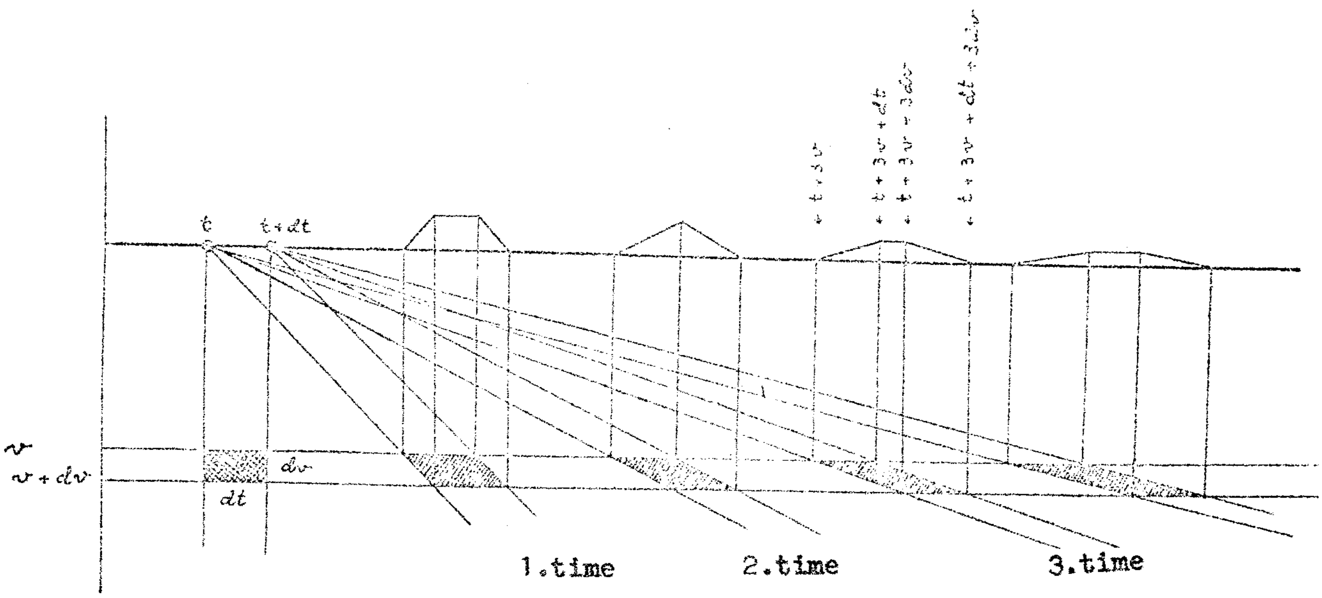


Fig. 7.

time t + kv and t + kv + dt + kdv. See Figure 7 where the primary investment is double shaded, the reinvestment single shaded. The way in which the reinvestment is distributed between t + kv and t + kv + dt + kdv is represented by the trapeziums in the part of the Figure which is above the time axis. This part of the plane is to be imagined as a vertical plane clapped down in the plane of the Figure. In this vertical plane the reinvestment curve is to be represented in the same way as in Figures 3, 5, and 6. If we for a moment consider dt and dv not as infinitesimal but as finite magnitudes and assume that P(tv) is constant inside the rectangle dtdv, but zero outside of this, the trapeziums above the time axis will correspond to the partial curves in Figures 3, 5, and 6; Figure 7 gives then a representation of the way in which the reinvestment behaves when there is injected between t and t + dt a constant primary investment which in every moment is continuous and equally distributed over the durabilities between v and v + dv. This is the most simple form of the composed phenomenon.[1] At the beginning the reinvestment will show periodic fluctuations represented by the trapeziums. The resultant curve will only contain one partial curve to begin with (one trapezium). As time passes the amplitude of the oscillations is damped: the trapeziums become flatter and more lengthened. At last the trapeziums will begin to work into each other (the resultant curve is going to contain several partial curves), which in addition contributes to dampen the fluctuation in the annual reinvestment. The trapeziums start to work into each other the sooner, the greater dv is in relation to v.

It is interesting to compare the composed phenomenon with the distribution phenomenon and the repetition phenomenon. The trapeziums resemble the partial

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[1] This example is due to Dr. Schoenheyder. Dr. Schoenheyder has for one thing emphasized the rhombic shape of the reinvestment figures.

curves of the distribution phenomenon in the sense that they gradually become more flattened, but there is a difference in so far as the partial curves of the distribution phenomenon are rectangle contours when the primary investment is uniformly distributed between the durabilities v and v + dv. The trapeziums in Figure 7 resemble the partial curves of the repetition phenomenon only in the sense that there is a certain accordance in the distance of time between two succeeding partial curves (trapeziums). In other respects there are differences. Thus with the repetition the partial curves would be rectangle contours, and these would gradually not become flattened. We may therefore say that the distribution phenomenon contributes to a greater extent than the repetition phenomenon to mark the composed phenomenon.

I return now to consider dt and dv as infinitesimal magnitudes. Above (Figure 7) I investigated how the effect of an element of primary investment propagated itself forward in time. To derive the general formula for the relationship between primary investment and reinvestment we must make the converse consideration (Figure 8).

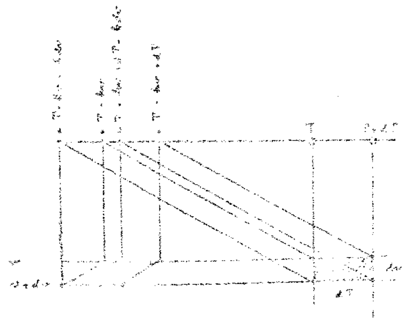

Fig. 8.

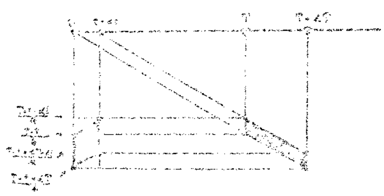

Fig. 9.

Between T and T + dT there is reinvested for the k-th time a certain quantity of capital objects of durability between v and v + dv. These capital objects were initially invested between T - kv - kdv and T - kv + dT. See Figure 8 where the primary investment is still double shaded, and the reinvestment single shaded. The magnitude of the reinvestment is consequently equal to $P(T - kv, v)\, dT\, dv$. Let $R(Tv)$ denote the reinvestment of v-year capital objects at time T, so that there is reinvested between T and T + dT a quantity $R(Tv)\, dT\, dv$ of capital objects of durability between v and v + dv, and hence

$$(11) \quad R(Tv) = \sum_{k=1}^{\infty} P(T - kv, v) .$$

If the primary investment starts at time t_0, we must put $P(tv) = 0$ for $t < t_0$.

The total reinvestment at time T of capital objects of all durabilities is

$$(12) \quad R(T) = \int_0^{\infty} R(Tv)\, dv = \sum_{k=1}^{\infty} \int_0^{\infty} P(T - kv, v)\, dv .$$

The same expression can also be derived by another method of reasoning. (Figure 7). The capital objects which are initially invested between t and t + dt and being reinvested for the k-th time between T and T + dT, have a durability between $\frac{T - t - dt}{k}$ and $\frac{T - t + dT}{k}$. The quantity of these capital objects is $P(t, \frac{T - t}{k})$ times the content of the shaded area in Figure 9. This content is $\frac{t + dT}{k}$. This is obviously the same for the area element to the left (the primary investment element) as for the one to the right (the reinvestment element). The quantity of capital which we consider, is therefore equal to

$$\frac{1}{k} P(t, \frac{T - t}{k})\, dt\, dT.$$

The total quantity of capital which is initially invested between t and t + dt and being reinvested between T and T + dt is thus equal to $dt\, dT \sum_{k=1}^{\infty} \frac{1}{k} P(t, \frac{T - t}{k})$ i.e. the total reinvestment at time T,

$$(13) \quad R(T) = \sum_{k=1}^{\infty} \int_{-\infty}^{T} \frac{1}{k} P(t, \frac{T - t}{k})\, dt = \sum_{k=1}^{\infty} \int_0^{\infty} P(T - kv, v)\, dv$$

which is the expression we have derived earlier.

The average reinvestment is defined by a line of argument analogous to the one used above. The primary investment element $P(tv)\, dt\, dv$ recurs every

v-th year, which gives an average annual reinvestment of $\frac{P(tv)\ d + dv}{v}$.
This expression, integrated over all occurring durabilities v and for all points of time t up to T of primary investment, gives the average reinvestment per year at time T. Let this be denoted A(T) and we have

$$A(T) = \int_{-\infty}^{T} dt \int_{0}^{\infty} dv \frac{P(tv)}{v} .$$

The total capital mass existing at time T can be determined in the following way. Let C(Tv) dv denote the quantity of capital objects existing at time T having a durability between v and v + dv. The quantity of v-year capital objects at time T is evidently the sum of all primary investment of v-year capital objects which has taken place up to time T, because the capital objects that once have entered into the capital mass will be maintained according to our assumption. We have then

$$(14) \qquad C(Tv) = \int_{-\infty}^{T} P(tv)\ dt.$$

The quantity of capital objects being built as v-year ones and whose <u>age</u> at time T is τ must on the other hand be equal to the total investment (primary investment plus reinvestment) of v-year capital objects that took place at time T - τ, provided $\tau \leqq v$.

If total investment is denoted $Q(tv) = P(tv) + R(tv)$, then there is at time T a quantity $Q(T - \tau, v)\ dv\ dt$ of capital objects which are built with a durability between v and v + dv and whose age at time T is between τ and $\tau + d\tau$ ($\tau \leqq v$). At time T there is then a total quantity of capital objects being built with a durability between v and v + dv equal to

$$dv \int_{0}^{v} Q\ (T - \tau,\ v)\ d\tau \ = dv \int_{T - v}^{T} Q(tv)\ dt.$$

Hence we have

(15) $$C(Tv) = \int_{T-v}^{T} Q(tv)\, dt = \int_{T-v}^{T} P(tv)\, dt + \int_{T-v}^{T} R(tv)\, dt.$$

If this expression is compared with (14) we see that we will have

$$\int_{-\infty}^{T-v} P(tv)\, dt = \int_{T-v}^{T} R(tv)\, dt.$$

Or when T (which is arbitrary here) is substituted by T + v

(16) $$\int_{-\infty}^{T} P(tv)\, dt = \int_{T}^{T+v} R(tv)\, dt.$$

The total primary investment of v-year capital objects that has taken place up to an arbitrary point of time T is thus equal to the total reinvestment of v-year capital objects which takes place from T to T + v. That expression (16) is correct can also be seen directly from the fact that every element of v-year capital objects which is initially invested up to time T must recur once and only once between T and T + v. The expression follows also from (11), as

$$\int_{T}^{T+v} R(tv)\, dt = \sum_{k=1}^{\infty} \int_{T}^{T+v} P(t - kv,\ v)\, dt = \sum_{k=1}^{\infty} \int_{T-kv}^{T-(k-1)v} P(tv)\, dt = \int_{-\infty}^{T} P(tv)\, dt$$

It follows then from (16) that the capital mass C(Tv) must be equal to

(17) $$b(Tv) = \int_{T}^{T+v} R(tv)\, dt.$$

The quantity of v-year capital objects which exists at time T can thus be expressed in three different ways: either by the primary investment, or by the reinvestment or by the total investment. First it is equal to the sum of all primary investment of v-year capital objects which has taken place up to time T (formula (14)), secondly it is equal to the sum of the reinvestment of v-year capital objects which will take place between time T and T + v (formula (17)), thirdly it is equal to the sum of the total investment of v-year capital objects which has taken place between T - v and T (formula (15)).

The total quantity of capital objects of all durabilities which exists at time T can analogously be expressed in the following three ways:

$$(18) \qquad C(T) = \int_{-\infty}^{T} dt \int_{0}^{\infty} dv\ P(tv) = \int_{T}^{T+v} dt \int_{0}^{\infty} dv\ R(tv) = \int_{T-v}^{T} dt \int_{0}^{\infty} dv\ Q(tv) .$$

In conclusion I will indicate how the distribution phenomenon and the repetition phenomenon each in their way can be thought of as a limiting case of the composed phenomenon.

The mass distribution in the (tv) plane represents the distribution of the primary investment according to time (horizontally) and according to durability (vertically). Imagine that the whole mass distribution is compressed horizontally (retaining the vertical distribution), so that the whole mass is concentrated in the segment t_0 to t_0 + dt. We can choose the zero point of time so that $t_0 = 0$. The segment which we consider, is then the segment from 0 to dt. When the mass is compressed, the density inside the segment becomes inversely proportional to the width of the segment. Inside the segment (i.e. for $0 \leqq t \leqq dt$) we have then,

$$P(tv) \frac{f(v)}{dt} \quad \text{where } f(v) = \int_{-\infty}^{+\infty} \bar{P}\ (tv)\ dt$$

and $\bar{P}(tv)$ is the original density distribution. Outside the segment we have $P(tv) = 0$.

If this is inserted in (13) we get

$$R(T) = \sum_{k=1}^{\infty} \int_{0}^{dt} \frac{1}{k} \frac{f(\frac{T-\xi}{k})}{dt}\ d\xi = \sum_{k=1}^{\infty} (\frac{T}{K})$$

which is just the formula (2) of the phenomenon of distribution.

If we on the other hand imagine the mass distribution in the (tv) plane being compressed vertically (retaining the horizontal distribution), so that all mass is concentrated in the segment v to v + dv, then P(tw) inside the segment (i.e. for $v \leqq w \leqq v + dv$) is equal to $\frac{g(t)}{dv}$ where $g(t) = \int_0^\infty \bar{P}(tv)\, dv$. Outside the segment P(tw) = 0.

Insert this in (13) to get

$$R(T) = \sum_{k=1}^{\infty} \int_{v}^{v+dv} \frac{g(T - kw)}{dv} = \sum_{k=1}^{\infty} g(T - kv)$$

which is just the formula (10) of the repetition phenomenon.