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Changing Harmonics and Other General Types of Components in Empirical Series.

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The method outlined in the present article is an attempt to approach the general problem of components in empirical series by a principle less rigid than the principle of analytical formulae with constant parameters, on which the usual methods of curve fitting, harmonic analysis etc. are based. The main points of the method were first set forth in April 1927 in an article: The Analysis of Statistical Time Series, mimeographed for private circulation. The method is actually worked out with considerably more details than here given. Numerical applications are also under way. I hope that a full account of the method, accompanied by extensive numerical applications, may be published some time next year.

1. Functional Moments.

Consider quite generally a function $w(t)$ which, according to a certain principle, can be expressed as the sum of n other functions

$$w(t)=y_1(t)+\cdots+y_n(t).$$

The functions $y_1 \cdots y_n$ will be called the *components* of w. For convenience the variable t may be considered as time.

Let Φ_h be a functional operation, depending on the parameter h, and having the following two properties:

(I). If u and v are functions of t, and a and b are constants, then

$$\Phi_h(au+bv) \text{ shall be equal to } a\Phi_h u + b\Phi_h v$$

(II) $$\Phi_h y_i \text{ shall be equal to } \varrho_i^h \eta_i \qquad (i=1, 2 \ldots n)$$

where ϱ_i is some function of t (independent of h) and η_i is a known function of y_i (independent of h), such that the inverse function, i. e. y_i as a function of η_i, is uniquely determined, the uniqueness eventually being established by some additional condition as to y_i (for instance the condition that y_i shall be real, or lying between certain limits etc.). The factor ϱ_i will be called the *principal factor* of y_i. As a special case y_i may depend on a certain parameter, *the principal parameter* α_i, in such a way that the principal factor ϱ_i is a known function of α_i (eventually independent of t), and the inverse function, i. e. α_i as a function of ϱ_i, is uniquely determined (eventually through some additional condition as to α_i).

The function of t

$$v_h = \Phi_h w = \varrho_1^h \eta_1 + \cdots + \varrho_n^h \eta_n \tag{1. 1}$$

will be called the hth order *functional moment* of the set of functions $y_1 \ldots y_n$. While the ordinary moments characterize the values assumed by a function in a set of points, the functional moments characterize the nature of a set of functions.

The following are some examples of types of functions y_i and operations Φ_h satisfying the above conditions.

Let the components be of the periodic form

$$y_i = A_i e^{p_i t} + B_i e^{-p_i t} \qquad (i=1, 2 \ldots n) \tag{1. 2}$$

A_i, B_i and p_i being arbitrary constants (no assumption being made as to the commensurability of the p_i). The only reason for considering the form (1. 2) instead of the form

$$y_i = C_i \sin(a_i + \alpha_i t),$$

is to take care of the case where one of the constants A_i and B_i is zero. In any other case nothing is gained in generality by introducing cosine terms beside the sine terms.

If the components are of the form (1. 2), we may for instance take as Φ_h one of the following operations: 1) The derivative D^{2h} of order $2h$, which gives $\eta_i = y_i$, $\varrho_i = p_i^z$. 2) The finite central difference of order $2h$, i. e. Δ^{2h} defined by

$$\Delta^2 u(t) = u(t+\delta) - 2u(t) + u(t-\delta),$$

which gives

$$\eta_i = y_i, \quad \varrho_i = -\left(2 \sin \frac{\delta \alpha_i}{2}\right)^2,$$

where $\alpha_i = p_i \sqrt{-1}$. 3) The symmetric moving total L^h defined by

$$Lu(t) = \int_{t-\lambda}^{t+\lambda} \omega(t-\tau)\, u(\tau)\, d\tau = \int_0^{\lambda} \omega(\xi)\,(u(t+\xi) + u(t-\xi))\, d\xi$$

$\omega(\xi)$ being a weight function satisfying $\omega(\xi) = \omega(-\xi)$. This gives

$$\eta_i = y_i, \quad \varrho_i = 2 \int_0^{\lambda} \omega(\xi) \cos \alpha_i \xi \, d\xi.$$

If

$$2 \int_0^{\lambda} \omega(\xi)\, d\xi = 1$$

we have a moving average. 4) The symmetric moving total Δ^h defined by

$$\Delta u(t) = \omega_0 u(t) + \sum_{j=1}^{\lambda} \omega_j (u(t+j\delta) + u(t-j\delta)),$$

which gives

$$\eta_i = y_i, \quad \varrho_i = \omega_0 + 2 \sum_{j=1}^{\lambda} \omega_j \cos j\delta\alpha_i.$$

In the case of equal weights, $\omega_j = 1$, we get

$$\varrho_i = \sin(2\lambda+1)\frac{\delta\alpha_i}{2} \Big/ \sin\frac{\delta\alpha_i}{2}.$$

5) The composite operation $\Phi_h = D^{2(a+bh)} L^{a'+b'h}$ where a, a', b, b' are non negative integers. This gives

$$\eta_i = p_i^{2a}\varrho_L^{a'} y_i, \quad \varrho_i = p_i^{2b}\varrho_L^{b'},$$

where ϱ_L designates the expression for ϱ_i in the example (3). 6) The composite operation $\Phi_h = \Delta^{2(a+bh)} \Delta^{a'+b'h}$ which gives

$$\eta_i = \varrho_J^{a}\varrho_\Delta^{a'} y_i, \quad \varrho_i = \varrho_J^{b}\varrho_\Delta^{b'},$$

where ϱ_J and ϱ_Δ designate the expressions for ϱ_i in the examples (2) and (4) respectively. 7) $\Phi_h = \Delta_1^{a_1+b_1h}\Delta_2^{a_2+b_2h}\ldots\Delta_\varkappa^{a_\varkappa+b_\varkappa h}$ where $\Delta_1 \ldots \Delta_\varkappa$ are $\varkappa$ different types of moving totals. This operation contains as a special case the operation (6), for Δ^2 (and more generally Δ^{2h}) can be looked upon as a special case of Δ (namely a Δ with some of the weights negative). In all the cases 1—7) the p_i (or the α_i) may be considered as the principal parameters.

If $y_i = A_i t^{p_i}$, we might take $\Phi_h = \theta^h$, where θ is the operation $\theta = t\frac{d}{dt}$. This gives $\eta_i = y_i$, $\varrho_i = p_i$.

If the y_i are normal distributions

$$y_i = \frac{A_i}{\sigma_i\sqrt{2\pi}} e^{-(t-m)^2/2\sigma_i}$$

with the same mean m, we might take as Φ_h the operation of replacing t by $t+\delta$, where $\delta = -(t-m) \pm \sqrt{(t-m)^2 + A + Bh}$. A and B being two arbitrary functions of t (independent of h). This gives $\eta_i = e^{-A/2\sigma^2 i} y_i$, $\varrho_i = e^{-B/2\sigma^2 i}$. In the last example ϱ_i is a function of t, while in the other examples ϱ_i is independent of t.

Let $p, q \ldots r$ and $p', q' \ldots r'$ be two sets of ν arbitrary non negative integers. And let $i, j \ldots k$ be a set of ν integers picked from the set $1, 2 \ldots n$. Let $p, q \ldots r, s$ and $i, j \ldots k, l$ be similar sets of $\nu+1$ integers. We introduce the following notations.

$$(1.3) \qquad \Delta_{\binom{p' \ldots r'}{p \ldots r}} = \begin{vmatrix} v_{p+p'} \ldots v_{p+r'} \\ \ldots\ldots\ldots \\ v_{r+p'} \ldots v_{r+r'} \end{vmatrix} \qquad \Delta_{(p \ldots r)} = \Delta_{\binom{0\,1 \ldots \nu-1}{p\,q \ldots r}}$$

$$(1.4) \qquad D_{\binom{p \ldots r}{i \ldots k}} = \begin{vmatrix} \varrho_i^p \varrho_i^q \ldots \varrho_i^r \\ \ldots\ldots \\ \varrho_k^p \varrho_k^q \ldots \varrho_k^r \end{vmatrix}$$

$$(1.5) \qquad Q_\delta = Q_{\delta\binom{p \ldots r\,s}{i \ldots k_*}} = \sum_{l=1}^{n} \varrho_l^\delta \begin{vmatrix} \varrho_i^p \ldots \varrho_i^r \varrho_i^s \\ \ldots\ldots \\ \varrho_k^p \ldots \varrho_k^r \varrho_k^s \\ \varrho_l^p \ldots \varrho_l^r \varrho_l^s \end{vmatrix} \eta_l .$$

The quantity Q_δ is a residuum in the sense that it does not contain any of the terms $\eta_i \ldots \eta_k$. More precisely: it does not contain any term whose principal factor is equal to one of the quantities $\varrho_i \ldots \varrho_k$. This holds good for arbitrary non negative integers $p, q \ldots r, s$. If all the different values ϱ which occur in the set $\varrho_1 \ldots \varrho_n$ are represented in the set $\varrho_i \ldots \varrho_k$, then $Q_\delta = 0$ for all non negative $p \ldots r, s$. Putting $\nu = n$, we see in particular that we always have $Q_{\delta\binom{p\,q \ldots r\,s}{1\,2 \ldots n_*}} = 0$ identically in t.

The expansion of (1.3) in terms of $\eta_1 \ldots \eta_n$ is

$$(1.6) \qquad \Delta_{\binom{p' \ldots r'}{p \ldots r}} = \Sigma_{i \ldots k} D_{\binom{p \ldots r}{i \ldots k}} D_{\binom{p' \ldots r'}{i \ldots k}} \eta_i \ldots \eta_k$$

the summation being extended to combinations without repetition of the ν integers $i \ldots k$ picked from the set $1, 2 \ldots n$. For $\nu = n$ we get in particular

$$(1.7) \qquad \Delta_{\binom{p' \ldots r'}{p \ldots r}} = D_{\binom{p \ldots r}{1 \ldots n}} D_{\binom{p' \ldots r'}{1 \ldots n}} \eta_1 \ldots \eta_n .$$

Formula (1. 6) is proved thus: Introducing the expression (1. 1) for the quantities v_h in the first row of $\Delta_{\binom{p' \dots r'}{p \dots r}}$ and using the ordinary rule for summation of determinants which are identical except for the elements of one row, we get

$$\Delta_{\binom{p' \dots r'}{p \dots r}} = \sum_{i=1}^{n} \begin{vmatrix} \varrho_i^{p+p'} & \varrho_i^{p+q'} & \dots & \varrho_i^{p+r'} \\ v_{q+p'} & v_{q+q'} & \dots & v_{q+r'} \\ \dots & \dots & \dots & \dots \\ v_{r+p'} & v_{r+q'} & \dots & v_{r+r'} \end{vmatrix} \eta_i .$$

Repeating the process on the second row in the last determinant, and so on, we finally get

$$\Delta_{\binom{p' \dots r'}{p \dots r}} = \sum_{i=1}^{n} \dots \sum_{k=1}^{n} \begin{vmatrix} \varrho_i^{p+p'} & \dots & \varrho_i^{p+r'} \\ \dots & \dots & \dots \\ \varrho_k^{r+p'} & \dots & \varrho_k^{r+r'} \end{vmatrix} \eta_i \dots \eta_k .$$

If at least two of the ν integers $i \dots k$ are equal, the coefficient of $\eta_i \dots \eta_k$ in the last formula vanishes. The determinant $\Delta_{\binom{p' \dots r'}{p \dots r}}$ can therefore be written in the form $\Sigma_{i \dots k} A_{i \dots k} \eta_i \dots \eta_k$ where the summation is extended to combinations without repetition, and the coefficients A are independent of the quantities η.

This being so, we can determine one particular of the coefficients A, say $A_{i \dots k}$, by putting $\eta_i = \dots = \eta_k = 1$ and all the other quantities η equal to zero. The determinant obtained from $\Delta_{\binom{p' \dots r'}{p \dots r}}$ by this specialization is exactly the value of $A_{i \dots k}$. Hence, by the ordinary rule of determinant multiplication, $A_{i \dots k} = D_{\binom{p \dots r}{i \dots k}} D_{\binom{p' \dots r'}{i \dots k}}$.

We further have the formula

$$\begin{vmatrix} \varrho_i^p & \dots & \varrho_i^s \\ \dots & \dots & \dots \\ \varrho_k^p & \dots & \varrho_k^s \\ v_p & \dots & v_s \end{vmatrix} = \sum_{l=1}^{n} \begin{vmatrix} \varrho_i^p & \dots & \varrho_i^s \\ \dots & \dots & \dots \\ \varrho_k^p & \dots & \varrho_k^s \\ \varrho_l^p & \dots & \varrho_l^s \end{vmatrix} \eta_l$$

which is readily proved either by conclusion from ν to $\nu+1$ or directly by using the rule for summation of determinants which are identical except for the elements of one row.

Let δ be an arbitrary non negative integer. Changing p to $p+\delta$, q to $q+\delta$ etc. in the last formula, we get

$$D_p v_{p+\delta} + \cdots + D_s v_{s+\delta} = Q_\delta \tag{1. 8}$$

where Q_δ is defined by (1. 5), and $D_p \ldots D_s$ designate the coefficients of the polynomial in ϱ

$$D(\varrho) = \begin{vmatrix} \varrho_i^p \ldots \varrho_i^s \\ \cdots\cdots \\ \varrho_k^p \ldots \varrho_k^s \\ \varrho^p \ldots \varrho^s \end{vmatrix} = D_p \varrho^p + \cdots + D_s \varrho^s. \tag{1. 9}$$

In particular we have $D_s = D_{\binom{p \ldots r}{i \ldots k}}$. For $\nu = n$ the right hand side of (1. 8) vanishes identically in t, for all non negative values of δ.

2. The Principal Equation and the Determination of the Individual Components.

Suppose that the function w and the number n of components are known. Furthermore suppose that we know operations Φ_h and the corresponding functions $v_h = \Phi_h w$ at least for $h = \alpha, \alpha+1 \ldots \alpha+n;\ \beta, \beta+1 \ldots \beta+n; \ldots \gamma, \gamma+1 \ldots \gamma+n$, where $\alpha, \beta \ldots \gamma$ are n given non negative integers such that $\Delta_{(\alpha \ldots \gamma)}$ vanishes at most in discrete points. The following argument refers to points where $\Delta_{(\alpha \ldots \gamma)} \neq 0$. From (1. 7) is seen that in such a point all the principal factors ϱ must be different, and none of them can be equal to zero unless $\alpha = 0$.

Let $p, q \ldots r, s,\ p', q' \ldots r', s'$ and $i, j \ldots k$ have the same significance as before. Supposing $D_{\binom{p \ldots r}{i \ldots k}} \neq 0$ we shall consider the polynomial in ϱ

$$(2.1)\qquad \Delta(\varrho)=\begin{vmatrix} v_{p+p'} & v_{p+q'} & \dots & v_{p+r'} & \varrho^p \\ v_{q+p'} & v_{q+q'} & \dots & v_{q+r'} & \varrho^q \\ \dots & \dots & \dots & \dots & \dots \\ v_{r+p'} & v_{r+q'} & \dots & v_{r+r'} & \varrho^r \\ v_{s+p'}-R_{p'} & v_{s+q'}-R_{q'} & \dots & v_{s+r'}-R_{r'} & \varrho^s \end{vmatrix}$$

where $R_\delta = Q_\delta / D_{\binom{p\dots r}{i\dots k}}$, Q_δ being defined by (1. 5). For $\nu = n$ the quantities R in the last row of (2. 1) vanishes identically in t.

A part from a factor which is independent of ϱ the polynomial $\Delta(\varrho)$ is identical with the polynomial $D(\varrho)$ defined by (1. 9). More precicely we have

$$(2.2)\qquad D_{\binom{p\dots r}{i\dots k}} \cdot \Delta(\varrho) = \Delta_{\binom{p'\dots r'}{p\dots r}} \cdot D(\varrho).$$

In fact, multiply the last row of $\Delta(\varrho)$ by D_s. This being done, add to the last row the first row multiplied by D_p, the second row multiplied by D_q etc. In the determinant thus obtained all the elements in the last row are zero by virtue of (1. 8), except the last element in the last row, which by (1. 9) is equal to $D(\varrho)$.

Since $D(\varrho)$ vanishes for $\varrho_i \dots \varrho_k$ and (2. 2) holds good identically in $\varrho_i \dots \varrho_k$, we see that *the polynomial* $\Delta(\varrho)$ *defined by* (2. 1) *vanishes for* $\varrho = \varrho_i \dots \varrho_k$.

If $|\varrho_i| \dots |\varrho_k|$ are great as compared with the other $|\varrho|$, and the smallest of the integers $p' \dots r'$ is >0, the quantities R may as an approximation be dropped in the last row of (2. 1). Therefore, if we put $p = 0$, $q = 1, \dots s = \nu$ and, for convenience, drop the primes on $p', q' \dots$, we see that, if $\Delta_{(p\dots r)} \neq 0$, the ν roots of the equation

$$(2.3)\qquad \begin{vmatrix} v_p & v_{p+1} & \dots & v_{p+\nu} \\ v_q & v_{q+1} & \dots & v_{q+\nu} \\ \dots & \dots & \dots & \dots \\ v_r & v_{r+1} & \dots & v_{r+\nu} \\ 1 & \varrho & \dots & \varrho^\nu \end{vmatrix} = 0$$

may be taken as an approximation to the ν principal factors which have the largest modules. For $\nu=1, 2 \ldots$ (2. 3) gives successive approximations. For $\nu = n$ the principal factors are rigorously determined as the n roots of (2. 3), (even if the smallest of the non negative integers $p \ldots r$ is equal to zero). The equation (2. 3) may be called *the principal equation of degree* ν.

Let α be any of the ν integers $i \ldots k$, and consider the formula obtained from (1. 8) by leaving the integer α out of the set $i \ldots k$ and the integer s out of the set $p \ldots r, s$.

If $|\varrho_i| \ldots |\varrho_k|$ are great as compared with the other $|\varrho|$, and the smallest of the integers $p \ldots r$ is > 0, the term containing η_α will be the principal term on the right hand side of the formula. We may therefore put as an approximation

$$(2.4) \qquad \varrho_\alpha^\vartheta \begin{vmatrix} \varrho_i^p \ldots \varrho_i^r \\ \ldots\ldots \\ \varrho_k^p \ldots \varrho_k^r \end{vmatrix} \eta_\alpha = \begin{vmatrix} \varrho_i^p & \ldots \varrho_i^r \\ \ldots\ldots\ldots\ldots \\ \{v_{p+\vartheta}\} & \cdots \{v_{r+\vartheta}\} \\ \ldots\ldots\ldots\ldots \\ \varrho_k^p & \ldots \varrho_k^r \end{vmatrix},$$

where $\{\}$ designates that the row $\varrho_\alpha^p \ldots \varrho_\alpha^r$ has been replaced by the row $v_{p+\vartheta} \ldots v_{r+\vartheta}$. If $\nu = n$, i. e., if the set $i \ldots k$ is the set $1, 2 \ldots n$, the formula (2. 4) is rigorous (even if the smallest of the integers $p \ldots r$ is equal to zero).

3. Criteria for the Number of Components.

In a point t where the set of principal factors $\varrho_1 \ldots \varrho_n$ contains at most m distinct values ($m \gtreqless n$), all $m+1$ and higher rowed determinants of the type $\Delta_{\binom{p' \ldots r'}{p \ldots r}}$ must vanish. This follows immediately from (1. 6), the determinants D now being $(m+1)$ rowed and consequently vanishing. In particular all $n+1$ and higher rowed determinants of the type $\Delta_{\binom{p' \ldots r'}{p \ldots r}}$ are always $= 0$ identically in t.

If $m < n$ at most in discrete points, the components $y_1 \ldots y_n$ might be called distinguishable (under the operation Φ_h). This terminology is justified by the following two facts: First, if two (or more) of the components, say y_i and y_j, have principal factors which are identical over a certain finite interval, the sum $y = y_i + y_j$ behaves in this interval as a single component under the operation Φ_h. In fact, putting $\varrho = \varrho_i = \varrho_j$ and $\eta = \eta_i + \eta_j$, we have $\Phi_h y = \varrho^h \eta$.

Second, consider the n rowed determinant

$$\Delta_{pn} = \begin{vmatrix} v_p & v_{p+1} \ldots v_{p+n-1} \\ v_{p+1} & v_{p+2} \ldots v_{p+n} \\ \ldots\ldots\ldots\ldots \\ v_{p+n-1} & v_{p+n} \ldots v_{p+2n-2} \end{vmatrix}.$$

By virtue of (1. 7) we have

$$\Delta_{pn} = (\varrho_1 \ldots \varrho_n)^p \prod_{\beta>\alpha} (\varrho_\beta - \varrho_\alpha)^2 \cdot \eta_1 \ldots \eta_n \tag{3. 1}$$

where β runs through $\beta = 2, 3 \ldots n$ and α through

$$\alpha = 1, 2 \ldots \beta - 1.$$

The last formula shows that Δ_{pn} can at most vanish in discrete points if the components are distinguishable and the quantities $\eta_1 \ldots \eta_n$ are at most vanishing in discrete points. The only exception is when one of the principal factors say ϱ_i is equal to zero identically in t, and $p > 0$. In this case y_i is equivalent under the operation Φ_h with a non existing component. This is also revealed by the fact that now $\Delta_{p,n-1}$ by (1. 6) is equal to

$$(\varrho_1 \ldots [\varrho_i] \ldots \varrho_n)^p \prod_{\beta>\alpha} (\varrho_\beta - \varrho_\alpha)^2 \eta_1 \ldots [\eta_i] \ldots \eta_n$$

([ ] designating »exclusion of»), and consequently at most vanishing in discrete points. The preceeding propositions furnish a lower limit for the number of components. In

certain important special cases it is possible to develop exact criteria.

If there exists no linear relation with constant coefficients (not all zero) between the $\binom{n}{\nu}$ functions obtained by forming all products of ν factors picked from the set $\eta_1 \ldots \eta_n$, the set $\eta_1 \ldots \eta_n$ will be called linearly independent of order ν. If the principal factors are constants independent of t, and the quantities $\eta_1 \ldots \eta_n$ are linearly independent of all orders $\nu = 1, 2 \ldots$, then by (1. 6) none of the lower rowed determinants $\Delta_{\binom{\nu' \ldots r'}{p \ldots r}}$ $(\nu \gtrless n)$ can vanish identically in t. In this case the knowledge of any sequence of determinants $\Delta_{\binom{\nu' q'}{p q}}$, $\Delta_{\binom{\nu' q' r'}{p q r}}$ etc. furnish a necessary and sufficient criterion for the number of components, the number being equal to n when and only when the first determinant which vanishes identically in t is $n+1$ rowed. This criterion is for instance applicable in the case where the components are sine functions with different (but otherwise arbitrary) periods, and the operation Φ_h is any of the operations mentioned in Section 1 in connection with periodic components.

4. The Principle of Moving Fit. Curvefitting without Parameters.

Let $w(t)$ be an empirically known function. The usual methods of curvefitting, such as least squares, method of moments, harmonic analysis etc. are all *total* methods in the sense that the totality of the values assumed by w is taken into account in order to determine the values of the parameters which occur in the analytic formula adopted. The introduction of these parameters and the assumption that they are constants, is the very essence of the total methods of curvefitting.

The facts indicated in Sections 2 and 3 suggest a principle of a different kind, namely the principle of specializing the nature of the fitted components (the fitted curve itself if there is only one component), not by assuming some type of

rigid formula with constant parameters, but by assuming that the components *behave approximately in a certain way under a certain type of operations* Φ_h.

If analytical formulae with constant parameters intervene in this type of curvefitting, it is only in order that their properties under the operations Φ_h may serve as conceptual patterns to suggest plausible assumptions regarding the approximate effect of Φ_h on the components of the given curve in the vicinity of a point. From this point of view the method here considered may also be characterized as a differential method or a method of moving fit.

The following example will illustrate the principle. Suppose we have a chain of n pendula: To a long pendulum with a great mass is attached a much shorter pendulum with a much smaller mass, and so on. Suppose the whole system is in movement in a field of gravitation whose intensity is slowly changing. The length of the individual pendula may also be slowly changing. The fluctuations of the lowest pendulum measured from the vertical through the point of suspension of the system, is given. The problem is to determine the individual components, i. e., determine the fluctuations of each pendulum measured from the vertical through its own point of suspension.

If the interval of observation is long enough to cover a considerable total change in the intensity of the field or in the length of the pendula, no kind of curvefitting with constant period sine functions would be successful. In particular the harmonic components determined by ordinary harmonic analysis will have no real significence. But in the vicinity of a point of time the components y_i will approximately satisfy a relation of the form $\Phi_h y_i = \varrho_i^h y_i$ *and this is sufficient to determine approximately the ordinates of the respective components in the point considered.*

Since the process may be repeated in any point (where $\Delta_{(\alpha \ldots \gamma)} \neq 0$), the time series representing the individual components may be traced approximately. Components of this type might be called *changing harmonics.*

The great generality of the operation Φ_h makes it possible to try different assumptions and procedures in different cases. In practice it will be necessary to adopt an operation Φ_h which do not amplify but rather smoothes out the erratic fluctuations of the given series w. For instance instead of using iterated differences alone, it will be better to use such an operation superposed on a moving average. Or one might simply use iterated moving averages alone. The choice of operation Φ_h, in particular the length of the moving average which eventually enters into Φ_h, has to be decided upon in each case. The choice will depend on one side on the amount of erratic element present and on the other side on the rapidity with which the underlying »harmonic» components are changing.

As is well known, practically *any* function with »observable» properties (such as a finite number of maxima, minima and discontinuities over a finite interval etc.) can be developped in a convergent Fourier series. The procedure of computing successive harmonic components and considering them as terms of an infinite series, will therefore often lead away from that which is the essential point in the analysis of empirical time series, namely to discover if there should be one or a limited number of components which are really significant for the phenomenon at hand.

A definite answer to the question of knowing wether a component is significant or not, can evidently not be given by formal operations on the empirical data. Some informations in this respect may, however, be obtained, I believe, from the results of Section 3. In practice where a more or less erratic element is present, none of the determinants Δ can be expected to be rigorously equal to zero for all values of t. The important question in practice is if the deviation of the Δ's from zero is significant or not.

Consider the sequence of determinants $\Delta_{\binom{p'\,q'}{p\,q}}$, $\Delta_{\binom{p'\,q'\,r'}{p\,q\,r}}$ etc. Suppose that the $(n+1)$ rowed determinant of this sequence changes sign irregularily and frequently, say in points between which there is only one or some few points of observation. Further suppose that there can be traced quite distinctly cer-

tain intervals where the n rowed determinant of the sequence is essentially positive, and other intervals where it is essentially negative. If this is the case, the number of significant components may be taken equal to n. Anything over and above this should be considered as non significant. The number of the components being given, the determination of the individual components might be performed by the method of Section 2.

The principle of moving fit has also an application in mechanical smoothing. In ordinary mechanical smoothing with a central moving average a great difficulty is encountered by the fact that such an average has a tendency not only to eliminate the erratic element, but also to »cut corners», that is, to dampen the fluctuations of the underlying curve, the true shape of which it is precisely the object to bring forth.

Let v_h be a h times iterated moving average of the given curve. The elimination of the erractic element is most satisfactory in the moving averages of high order (i. e., with a large h). But these averages of high order are exactly the ones where the troublesom damping effect is heaviest. The problem is therefore to combine the quantities v_h (for comparatively large values of h) into an expression which is itself not subject to the damping effect.

Let $p \ldots r$ and $p' \ldots r'$ be two sets of ν positive integers. If the number of underlying components in the given curve is ν, and each component is of the changing harmonic type, we have approximately $\Delta_{\binom{0\ p' \ldots r'}{0\ p \ldots r}} = 0$ identically in t. By this equation the original, undamped curve, i. e. v_0, is expressed in terms of the higher order moving averages. This expression, namely

$$(4.1) \qquad v_0 = -\begin{vmatrix} 0 & v_{p'} & \ldots & v_{r'} \\ v_p & v_{p+p'} & \ldots & v_{p+r'} \\ \cdot & \cdot & \cdot & \cdot \\ v_r & v_{r+p'} & \ldots & v_{r+r'} \end{vmatrix} : \begin{vmatrix} v_{p+p'} & \ldots & v_{p+r'} \\ \cdot & \cdot & \cdot & \cdot \\ v_{r+p'} & \ldots & v_{r+r'} \end{vmatrix}$$

may therefore be adopted as a smoothing formula.

A consequence of our theoretical assumption is that if there exist zeros of the denominator in (4. 1), these points will also be zeros of the numerator. In practice this might not hold good rigorously, so that the computation of v_0 according to (4. 1) will be uncertain in the vicinity of the zeros of the denominator.

This difficulty might be overcome by using short interval interpolation instead of direct computation in the vicinity of the zeros of the denominator. Or one might use the following procedure: Instead of a single set of integers $p \ldots r$ and $p' \ldots r'$, let us consider several sets. Let $N_1 N_2 \ldots$ and $D_1 D_2 \ldots$ be the corresponding values of the numerator and denominator respectively in (4. 1). The absolute value of the ordinate of the smoothed curve may then be taken as the square root of the expression $(N_1^2 + N_2^2 + \cdots)/(D_1^2 + D_2^2 + \cdots)$. Or one might adopt any other average of the ratios N_i/D_i which is such that the denominator of the average can only vanish if all the D_i vanish separately.

5. Graphic Approximation to Changing Harmonics. The Method of Normal Points.

Consider a $2h$ times iterated finite difference operation superposed on a moving average. Let v_h be the result of this operation performed on the given curve. Suppose that the components are of the nature of changing harmonics. And suppose that $|\varrho_1|$ is large as compared with all the other principal factors, over the entire t interval considered. This means that the curvature of the other components is negligible as compared with the curvature of y_1 (except in the vicinity of the points where y_1 changes curvature). The other components will therefore be practically eliminated in v_1 and the higher v_h. *The zeros of* v_h $(h \geqq 1)$ *will consequently coincide approximately with the zeros of* y_1.

This suggests the following graphic approximation to y_1: Pick out the inflexion points of the smoothed curve w. These points may be called be *normal points* (because it is approximately in the vicinity of these points that y_1 passes its nor-

mal). Join every two consecutive normal points by a straight line, or draw a kth degree parabola through $k+1$ consecutive normal points, or use any other plausible method of interpolation. The series S_1 thus obtained represents approximately the residuum of w after elimination of y_1, and $w - S_1$ represents approximately y_1. It should be noticed that the error in locating the zeros of y_1 by this method is always in the direction of damping the oscillations of the residuum S_1.

If it is further plausible to assume that $|\varrho_2|$ is large as compared with the rest of the principle factors, the series S_1 may be treated in the same way, thus eliminating y_2 and so forth. This method might be called the method of normal points.

The normal points for y_1 may also be located by iterated moving averages. Let v_h be a h times iterated moving average of the empirically given curve. If the curvature of the other components is negligible as compared with the curvature of y_1 (except in the vicinity of the points where y_1 changes curvature), then y_1 *passes zero approximately in the points where* v_0 *and* v_1 *intersect.*

The essential point in the methods here considered, is the fact that the operation Φ_h considered (iterated differences or iterated moving averages) intensifies the relative importance of one particular term of the sum $y_1 + y_2 + \cdots$. That is to say, one particular of these terms is multiplied by a factor which is large as compared with the factors with which the other terms are multiplied. The idea therefore naturally presents itself to generalize the method of normal points in the following way: Let $\psi_\varkappa$ be a linear, functional operation depending on the parameter $\varkappa$ and being performed, not with respect to t, as the operation Φ_h, but with respect to h. Let $w_\varkappa$ be the effect of $\psi_\varkappa$ performed on v_h. $\psi_\varkappa$ being linear, we have

$$w_\varkappa = \Sigma_i \sigma_{i\varkappa} \eta_i$$

where

$$\sigma_{i\varkappa} = \psi_\varkappa \varrho_i^h .$$

By a suitable choice of $\psi_\varkappa$ we might secure a very great ratio between $\sigma_{1\varkappa}$ and the other $\sigma_{i\varkappa}$, even if the ratio between ϱ_1^h

and the other ϱ_i^h is not particularly great. If, for instance $\psi_\varkappa$ is a finite difference operation of order $\varkappa$ and interval δ, i. e., $\psi_\varkappa = \Delta^\varkappa$, $\Delta^\varkappa$ being defined by $\Delta u_h = u_{h+\delta} - u_h$, we have

$$\sigma_{i\varkappa} = \varrho_i^h (\varrho_i^\delta - 1)^\varkappa .$$

If the components are of the nature of changing harmonics, and Φ_h is an iterated moving average, ϱ_i will be all the closer to unity, the longer the waves of the component in question. The relative importance of the component with the shortest wave, say y_1, will therefore be strongly intensified by the operation $\psi_\varkappa \Phi_h$ provided we choose h just large enough to eliminate the erratic element (eventually $h=0$) and choose $\varkappa$ as a relatively large positive integer; δ might be put $=1$ (or larger). If h, $\varkappa$ and δ are disposed of in this way, the function of time

$$w_\varkappa = \Delta^\varkappa v_h = \sum_{k=0}^{\varkappa} (-)^{\varkappa+k} \binom{\varkappa}{k} v_{h+\delta k}$$

will pass zero approximately in the same points as y_1. For increasing $\varkappa$ we get successive approximations. If the location of one particular normal point of y_1 is practically unaffected by a further increase in $\varkappa$, the approximation may be taken as satisfactory. The approximation will of course be all the better the closer the length of the moving average is to covering just one wave of y_1.

A final correction may be obtained by considering the points where an n-rowed determinant of the type (1. 3) vanishes. By (1. 7) these points are exactly the points where at least one of the components vanishes.

The method of normal points may be refined, in another direction, as follows. Consider any of the operations Φ_h and let $\varrho_i \ldots \varrho_k$ be the ν principal factors which have the greatest modules. If the set $\varrho_i \ldots \varrho_k$ is determined approximately as the solution of (2. 3) and then introduced in the right hand side of (2. 4), we get an expression which passes zero approximately in the same points as η_α.