NECESSARY AND SUFFICIENT CONDITIONS REGARDING THE FORM OF AN INDEX NUMBER WHICH SHALL MEET CERTAIN OF FISHER'S TESTS

By Ragnar Frisch

In his fundamental work on index numbers, Professor Irving Fisher has introduced a certain set of tests that has become more or less basic in the theory of index numbers. Through the work of Professor Fisher and others, a great number of formulae have been scrutinized and various examples of sufficient conditions for the fulfillment of the tests in question have been found. The purpose of the present article is to establish certain conditions which are both necessary and sufficient.

Let p'_t, p''_t . . . be a set of prices quoted at the time t, and let x'_t, x''_t . . . be the corresponding quantities traded or consumed or produced, etc., as the case may be. The time series representing the prices p and the quantities x are supposed given. We are here concerned only with the logic of the properties of index numbers, and shall, therefore, not go into the practical questions which arise out of the fact that price series are so much more easily obtained than the corresponding quantity series. The general logical problem of index number construction may be formulated in one of the following two ways:

(I) t being a point of time, how can we transform the aggregate

$$p'_t x'_t + p''_t x''_t + \ . \ . \ . \ = \Sigma p_t x_t$$

into a product

$$P_t X_t = \Sigma p_t x_t$$

in such a way that P_t and X_t may be considered as some sort of an average of prices, respectively, of quantities, and further such that P_t and X_t satisfy certain tests?

(II) t and s being two points of time, how can we transform in a similar way the ratio

$$\Sigma p_t x_t / \Sigma p_s x_s$$

into a product of two factors

$$P_{ts} X_{ts} = \Sigma p_t x_t / \Sigma p_s x_s$$

where P_{ts} and X_{ts} also satisfy certain tests?

The formulation (I) leads to index numbers P_t and X_t of a type which might be called *absolute index numbers*. They depend on one time-subscript only. The formulation (II) leads to *relative index numbers* giving a comparison between the situation on two points of time and consequently depending on two time-subscripts.[1] A relative index number which is of the form $P_{ts}=P_t/P_s$ (or $X_{ts}=X_t/X_s$) where P_t (or X_t) is an absolute index number, might be called an index number of the *ratio type.* Since an index number of the ratio type can be considered as a special case of a relative index number, we shall confine our attention to tests for relative index numbers. It is sufficient to formulate the test for the P's. The application of the same principles to the X's by analogy is obvious.

The tests which shall be considered in the following are:

(1) *The Identity Test.* Let t be any point of time. The identity test requires that P_{tt} shall be equal to unity.

(2) *The Time Reversal Test.* Let t and s be two arbitrary points of time. The time reversal test requires that $P_{st}.P_{ts}$ shall be equal to unity. If the index number considered is assumed real and positive, the fulfillment of the time reversal test entails the fulfillment of the identity test. In fact, by putting $s=t$ in the time reversal condition, we get $P^2_{tt}=1$.

(3) *The Base Test.* Let 1, 2 and s be three arbitrary points of time. The base test requires that the ratio $R_{21}=P_{2s}/P_{1s}$ shall be independent of s; that is, the comparison between the points of time 2 and 1 via a third point s shall be unaffected by a change of this third point.

(4) *The Circular Test.* Let 1, 2 and s be three points of time. The circular test requires that the ratio $R_{21}=P_{2s}/P_{1s}$ shall be not only independent of s (as required by the base test) but even equal to P_{21}, i.e., we shall have

$$P_{2s}/P_{1s}=P_{21}. \tag{1}$$

Thus, the fulfillment of the circular test entails the fulfillment of the base test, but the inverse is not true in general. It is true, however, if the identity test is satisfied. In this case the fulfillment of the base test entails the fulfillment of the circular test. In fact if $R_{21}=P_{2s}/P_{1s}$ is independent of s, and P satisfies the identity test, we get, by putting $s=1$, $R_{21}=P_{21}$. That is to say, R and P are equal, and consequently, $P_{2s}/P_{1s}=P_{21}$, which is equation (1). The fulfillment of the base test for an index number P_{ts} does not imply the fulfillment of the identity test or the time reversal test for P_{ts} (but it does entail the fulfillment of these two tests for the derived index number $R_{21}=P_{2s}/P_{1s}$). On the other hand, the circular test always entails the fulfillment of the

[1] In the notation here adopted, the first subscript refers to the "given" point of time and the second subscript to the "base" point of time. Professor Fisher uses the opposite rule.

identity test and the time reversal test. In fact, putting $s=1$ in (1) we get $P_{11}=1$, and therefore, by putting $s=2$ in (1) $P_{21}.P_{12}=1$.

If the circular test is fulfilled, we have the equation

$$P_{01}P_{12}P_{20}=1. \tag{2}$$

This simply follows from the fact that the fulfillment of the circular test entails the fulfillment of the time reversal test. Inversely, if equation (2) is satisfied and the index number is assumed real, the circular test must be fulfilled. In fact, from (2) follows $P^3_{tt}=1$, hence $P_{tt}=1$. Therefore, by now putting 0 instead of 2 in (2), $P_{01}.P_{10}=1$, and consequently $P_{01}.P_{12}=P_{02}$, which is equation (1). The equations (1) and (2) are, therefore, equivalent ways of expressing the circular condition, provided the index number is assumed real.

(5) *The Commensurability Test.* This test requires that the index number shall be unaffected by a change in units of measurement; i.e., if for any commodity p is replaced by λp and at the same time x is replaced by x/λ both at the time 0 and at the time 1, then P_{10} shall remain unchanged, regardless of the value of λ.

(6) *The Determinateness Test.* This test requires that the index number shall not be rendered zero, infinite, or indeterminate by an individual price or quantity becoming zero.

(7) *The Factor Reversal Test.* Let P_{ts} be a given index number of prices. The quantity index obtained from P_{ts} by interchanging the letters p and x might be called the *transposed* of P_{ts} and the quantity index which is such that its product by P_{ts} is equal to $\Sigma p_t x_t/\Sigma p_s x_s$ might be called the *cofactor* of P_{ts}.

The cofactor of the transposed of P_{ts} is again a price index which is called the *factor antithesis* of P_{ts}. The given index number of prices P_{ts} is said to satisfy the factor reversal test if the factor antithesis of P_{ts} is identical with P_{ts} itself; that is, if the cofactor of P_{ts} is equal to the transposed of P_{ts}.

The factor reversal test should not be confused with the following test: Let P_{ts} and X_{ts} be index numbers of prices and quantities respectively. If P_{ts} and X_{ts} are considered simultaneously, we may request that their product should be equal to the value ratio $\Sigma p_t x_t/\Sigma p_s x_s$. This test might be called the *product test.* The product test only requires that if an index number of prices P_{ts} is given, and we want simultaneously to consider an index number of quantities X_{ts}, then we should choose as X_{ts} the cofactor of P_{ts}. The factor reversal test requires something more, namely that this cofactor of P_{ts} shall be at the same time the transposed of P_{ts}. This latter requirement evidently involves a condition as to the nature of P_{ts} while the product test involves no such condition.

I now proceed to derive the general form which it is necessary and sufficient that an index number shall have in order that certain combinations of the above test shall be fulfilled, and the index number satisfy certain conditions regarding continuity and existence of derivatives with respect to the variables involved. This analysis will, for instance, show that the three tests (3), (5), (6) are incompatible.

Condition for the Fulfillment of the Commensurability Test. Consider for a moment P_{ts} as a function of the data for one particular of the commodities, i.e., we consider for a moment P_{ts} has a function

$$P_{ts}=f(p_t,\ p_s;\ x_t,\ x_s) \tag{3}$$

of four variables, namely the price p and the quantity x of the commodity in question, at the points of time t and s. If the commensurability test shall be fulfilled, it is necessary and sufficient that the function f satisfy the equation

$$f(\lambda p_t,\ \lambda p_s;\ x_t/\lambda,\ x_s/\lambda)=f(p_t,\ p_s;\ x_t,\ x_s)$$

for all values of λ. Assuming the partial derivatives of f to exist, differentiating the last equation with respect to λ, and putting $\lambda=1$, we get

$$p_tf_1+p_sf_2=x_tf_3+x_sf_4$$

where $f_1, f_2\ \ldots$ designate the partial derivatives of f with respect to its four variables. The last equation is a partial differential equation, the general solution of which is of the form

$$f(p_t,\ p_s,\ x_t,\ x_s)=\Phi\left(\frac{p_t}{p_s},\ a_t,\ a_s\right), \tag{4}$$

where $a_t=p_tx_t$, and $\Phi(z_1,\ z_2,\ z_3)$ designates an arbitrary function of three variables. *The real nature of the commensurability test is thus that it restricts the number of independent variables.*

Since (4) must hold good for any of the commodities included in the index we see that the general form of an index number satisfying the commensurability test (and the specified conditions of continuity) is

$$P_{ts}=\Phi\left(\frac{p'_t}{p'_s},\ \frac{p''_t}{p''_s}\ \ldots\ ;\ a'_t,\ a''_t\ \ldots\ ;\ a'_s,\ a''_s\ \ldots\right), \tag{5}$$

Φ being a function of the three sets of variables indicated.

Note on a Different Formulation of the Commensurability Test. Instead of defining an index number by a formula expressing direct comparison between the points of time t and s, i.e., by a formula involving only the quantities p_t, p_s, x_t, x_s we might define more generally an index number by a formula which expresses the comparison between the points of time t and s *via a third point* of time T; i.e., by a formula involving all the quantities p_t, p_s, p_T, x_t, x_s, x_T. In this case, the index number would be a magnitude depending on three subscripts t, s, and T, T serving as base. Instead of the index number of prices P_{st} we would have to consider an index number of the form P_{stT}. Instead of equation (3), we should have $P_{tsT}=f(p_t,\ p_s,\ p_T,\ x_t,\ x_s,\ x_T)$. The commensurability test would now be that f should satisfy

$$f(\lambda p_t,\ \lambda p_s,\ \lambda p_T;\ x_t/\lambda,\ x_s/\lambda,\ x_T/\lambda)=f(p_t,\ p_s,\ p_T,\ x_t,\ x_s,\ x_T)$$

which leads to

$$f(p_t,\ p_s,\ p_T,\ x_t,\ x_s,\ x_T)=\Phi\left(\frac{p_t}{p_T},\ \frac{p_s}{p_T},\ a_t,\ a_s,\ a_T\right) \tag{6}$$

Φ being an arbitrary function of five variables.

If the comparison between the points of time t and s shall be independent of T, i.e., if $\frac{\partial}{\partial_T}P_{tsT}=0$, Φ defined by (6), reduces to $\Phi\left(\frac{p_t}{p_s},\ a_t,\ a_s\right)$ so that we get back to equation as (4). Similar reductions take place regarding the other tests. No real advantage is, therefore, attained by using the index number definition P_{tsT} instead of the definition P_{ts}.

Condition for the Fulfillment of the Base Test. The tests should hold good generally, and therefore also in the case where the time series of prices and quantities possess continuous derivatives with respect to time. Further, if the price index number has continuous derivatives with respect to the variables involved, the necessary and sufficient condition for the fulfillment of the base test is that $\frac{\partial}{\partial_s}(P_{2s}/P_{1s})=0$ identically in s and in the points of time 2 and 1. The sign $\frac{\partial}{\partial_s}$ means a derivation with respect to s, keeping 1 and 2 constant. Applying this to equation (3) we get

$$[g_2\dot{p}_s+g_4\dot{x}_s]_{t=1}^{t=2}=0$$

where $p_s=dp_s/ds$ and $\dot{x}_s=dx_s/ds$, g_2 and g_4 designating the partial derivatives of the function $g=\log f$ with respect to its second and fourth variable. The last equation shows that we must have

$$\frac{\partial}{\partial_t}(g_2\dot{p}_s+g_4\dot{x}_s)=0$$

identically in t and s. That is,

$$\dot{p}_t\dot{p}_s g_{12}+\dot{p}_s\dot{x}_t g_{23}+\dot{x}_t\dot{x}_s g_{34}+\dot{p}_t\dot{x}_s g_{41}=0 \tag{7}$$

where g_{ij} designates the second order partial derivative of g with respect to its i-th and j-th variable. Since the time curves representing $\dot{p}_t$ and $\dot{x}_t$ are arbitrary and the circular test shall be fulfilled regardless of the shape of $\dot{p}_t$ and $\dot{x}_t$, we see from (7) that we must have separately

$$g_{12}=g_{23}=g_{34}=g_{41}=0.$$

The equation $g_{34}=0$ shows that our function of four variables $g(y_1, y_2, y_3, y_4)$ must be of the form

$$g(y_1, y_2, y_3, y_4)=h(y_1, y_2, y_3)+k(y_1, y_2, y_4)$$

where h and k are functions only of the variables indicated. Introducing this expression for g into the equation $g_{23}=0$, we see that h must be of the form

$$h(y_1, y_2, y_3)=u(y_1, y_3)+r(y_1, y_2).$$

It does not restrict generally if we assume the function $r(y_1, y_2)$ to be included in $k(y_1, y_2, y_4)$; g can, therefore, be written in the form

$$g(y_1, y_2, y_3, y_4)=u(y_1, y_3)+k(y_1, y_2, y_4).$$

If this expression is introduced into the equation $g_{41}=0$, we see that k must be of the form

$$k(y_1, y_2, y_4)=s(y_1, y_2)+v(y_2, y_4).$$

Finally introducing the expression for g thus obtained into the equation $g_{12}=0$, we see that s must be of the form $s(y_1, y_2)=s_1(y_1)+s_2(y_2)$. But s_1 can be included in u and s_2 in v, so that finally g can be written in the form

$$g(y_1, y_2, y_3, y_4)=u(y_1, y_3)+v(y_2, y_4) \tag{8}$$

u and v being two functions of two variables.

If we further require the identity test to be fulfilled, we get, putting $y_1=y_2$ and $y_3=y_4$ in (8), $u(y_1, y_3)=-v(y_1, y_3)$, and therefore

$$g(y_1, y_2, y_3, y_4)=u(y_1, y_3)-u(y_2, y_4) \tag{9}$$

u being a function of two variables.

The formulae (8) and (9) must hold good with respect to any of the commodities entering into the index. *In order that the index number* P_{ts} (with the continuity properties here assumed) *shall fulfill the base test it is, therefore, necessary and sufficient that it be of the form*

$$P_{ts}=P_tQ_s \tag{10}$$

where P_t is a function only of the prices p_t and the quantities x_t at the point of time t, and Q_s is a function only of the prices p_s and the quantities x_s at the point of time s.

If further P_{ts} shall satisfy the identity test, Q_t must be equal to $1/P_t$. *If an index number* P_{ts} *shall meet the circular test it is, therefore, necessary and sufficient that it be of the ratio type,* i.e., that it can be written in the form

$$P_{ts}=\frac{P_t}{P_s} \tag{11}$$

where P_t is an absolute index number.

The Simultaneous Fulfillment of the Base Test and the Commensurability Test. The condition that the base test and the commensurability test shall be fulfilled simultaneously is obtained by applying the criterion

$$\frac{\partial}{\partial_s}(P_{2s}/P_{1s})=0$$

to the equation (4), which gives

$$[z_1\psi_1\dot{\pi}_s-\psi_3\dot{a}_s]_{t=1}^{t=2}=0 \tag{12}$$

where $z_1=p_t/p_s$, $\dot{\pi}_s=d\log p_s/ds$ and $\dot{a}_s=da_s/ds$, ψ_1 and ψ_3 designating the partial derivatives of the function $\psi(z_1, z_2, z_3)=\log\phi(z_1, z_2, z_3)$ with respect to its first and third variable respectively.

Equation (12) shows that we must have

$$\frac{\partial}{\partial_t}(z_1\psi_1\dot{\pi}_s-\psi_3\dot{a}_s)=0$$

identically in t and s. That is

$$\dot{\pi}_t\dot{\pi}_s(z^2{}_1\psi_{11}+z_1\psi_1)+\dot{a}_t\dot{\pi}_sz_1\psi_{12}-\dot{\pi}_t\dot{a}_sz_1\psi_{13}-\dot{a}_t\dot{a}_s\psi_{23}=0$$

ψ_{ij} designating the second order partial derivatives of ψ with respect to its i-th and j-th variable. Since the time curves representing $\dot{\pi}_t$ and $\dot{a}_t$ are arbitrary we must have separately

$$z_1\psi_{11}+\psi_1=0. \tag{13}$$

$$\psi_{12}=0. \tag{14}$$

$$\psi_{13}=0. \tag{15}$$

$$\psi_{23}=0. \tag{16}$$

Equation (13) shows that ψ must be of the form

$$\psi(z_1, z_2, z_3) = H(z_2, z_3) + K(z_2, z_3).\ \log z_1$$

H and K being functions of z_2 and z_3 only. Introducing this expression into (14) and (15) we get

$$\frac{\partial K}{\partial z_2} = \frac{\partial K}{\partial z_3} = 0.$$

This, together with (16), shows that ψ must be of the form

$$\psi(z_1, z_2, z_3) = U(z_2) + V(z_3) + c \log z_1 \tag{17}$$

U and V being two functions of a single variable and c being a constant.

If, further, the identity test is fulfilled, we get, putting $z_1 = 1$, $z_2 = z_3$ in (17), $U(z) = -V(z)$, and therefore finally

$$\psi(z_1, z_2, z_3) = U(z_2) - U(z_3) + c \log z_1 \tag{18}$$

where U is a function of a single variable and c a constant. The formulae (17) and (18) must hold good with respect to any of the commodities entering into the index.

We consequently have the two propositions: *In order that an index number* P_{ts} (with the assumed continuity properties) *shall fulfill at the same time the commensurability test and the base test it is necessary and sufficient that it be of the form*

$$P_{ts} = F(a'_t, a''_t \ldots) \cdot G(a'_s, a''_s \ldots) \cdot \left(\frac{p'_t}{p'_s}\right)^{c'} \cdot \left(\frac{p''_t}{p''_s}\right)^{c''} \ldots \tag{19}$$

where $a_t^{(i)} = p_t^{(i)} x_t^{(i)}$ is the money value of the quantity of commodity number i at the time t, and F and G are two functions only of the n variables indicated (n being the number of commodities). The weights c', c'' . . . are constants independent of t and s and also independent of the other variables occurring in the formula.

And in order that P_{ts} *shall meet at the same time the commensurability test and the circular test, it is necessary and sufficient that* P_{ts} *be of the form*

$$P_{ts} = \frac{F(a'_t, a''_t \ldots)}{F(a'_s, a''_s \ldots)} \cdot \left(\frac{p'_t}{p'_s}\right)^{c'} \cdot \left(\frac{p''_t}{p''_s}\right)^{c''} \quad \ldots \tag{20}$$

where F is a function only of the n variables indicated.

In the deduction of the formulae (19) and (20) we did not make any assumption regarding the constants c', c'' . . . But if the quantity P_{ts} shall really be some sort of an average of prices, it will be plausible to make the assumption that the constants c', c'' . . . are *positive* and satisfy the equation

$$c' + c'' + \ldots = 1.$$

Since the weights $c', c'' \dots$ are constants, (20) can be looked upon as an index number of the form $P_{ts} = P_t/P_s$.

Simultaneous Fulfillment of the Base Test, the Commensurability Test and the Factor Reversal Test. We shall now proceed one step further: Determine F in (20) by imposing upon P_{ts} the factor reversal test. The factor antithesis of (20) is

$$\frac{F(a'_s, a''_s \dots)}{F(a'_t, a''_t \dots)} \cdot \frac{\Sigma p_t x_t / \Sigma p_s x_s}{\left(\frac{x'_t}{x'_s}\right)^{c'} \cdot \left(\frac{x''_t}{x''_s}\right)^{c''} \dots}$$

Putting this equal to (20), we obtain

$$P_{ts} = \left(\frac{p'_t}{p'_s}\right)^{c'} \cdot \left(\frac{p''_t}{p''_s}\right)^{c''} \dots \sqrt{\frac{\Sigma p_t x_t / \Sigma p_s x_s}{\left(\frac{p'_t x'_t}{p'_s x'_s}\right)^{c'} \cdot \left(\frac{p''_t x''_t}{p''_s x''_s}\right)^{c''} \dots}} \qquad (21)$$

This is consequently the only existing index number formula (possessing partial derivatives) *which satisfies at the same time the commensurability test, the circular test, and the factor reversal test.*

It is not possible to choose the functions F and G in (19) in such a way that the determinateness test is fulfilled. In fact, if one of the prices, say p'_t, tends toward zero, P_{ts} will also tend toward zero unless F has a pole of the order c' in the point $a'_t = 0$. But if F is chosen with a pole in the point $a'_t = 0$, P_{ts} will tend toward infinity if x'_t tends towards zero while p'_t remains finite. *The base test, the commensurability test and the determinateness test, therefore, cannot be fulfilled at the same time* (if the index number possesses partial derivatives). It should be noticed that we are here concerned only with *direct* index numbers, that is, index numbers which depend only on the data referring to the two points of time compared. It is well known that a *chain* index will always fulfill the circular test and consequently, a fortiori, fulfill the base test.

If it is required that the base test shall be fulfilled, we have to choose between the determinateness test and the commensurability test. Personally, I feel a great repugnance against any index which does not satisfy the determinateness test. For instance, the withdrawal or entry of any commodity will often have to be performed as a limiting case when either the quantity x_t or the money value $a_t = p_t x_t$ of the commodity in question decreases toward zero, respectively increases from zero. I would, therefore, rather drop the commensurability test than the determinateness test. The following is a modification of the geometric mean which avoids the indeterminateness of the geometric mean and at the same time avoids the arbitrariness connected with the assumption of constant weights. Let us put

$$P_t = p'^{\alpha'_t}_t \cdot p''^{\alpha''_t}_t \ldots \tag{22}$$

where

$$\alpha^{(i)}_t = \frac{p^{(i)}_t x^{(i)}_t}{\Sigma p_t x_t}$$

is the fraction of the total value which at the point of time t is represented by the commodity number i. This index might be called *the instantaneous geometric mean,* because the weights α', α'' . . . are determined by the data in the same point of time as the prices themselves. This formula meets the determinateness test because $\lim_{x \to 0} x^x = 1$. As a theoretical construction, this formula is, therefore, rather interesting. The main practical difficulty in its application would be that the quantity data as a rule are not available at the same regular time intervals as the price data. They would, therefore, in practice have to be evaluated or interpolated.