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RAGNAR FRISCH

On the use of difference equations in the study of frequency distributions

Difference equations and differential equations have occasionally been used in the analysis of frequency distributions. Certain parts of the theories of KARL PEARSON and of CHARLIER are for instance based on the use of such equations.

However, these applications are of a rather special character. It does not seem that a systematic study of frequency distributions has ever been attempted from the differential point of view until a beginning in this direction was made by Professor GULDBERG's beautiful paper "On Discontinuous Frequency-Functions and Statistical Series" *) GULDBERG's paper contains several examples showing that the differential approach furnishes a strikingly simple solution of several classical problems in frequency distributions, for instance the problem of obtaining explicit expressions and recurrence formulae for the moments of the binominal, Poisson, Pascal, and hypergeometric distributions, and the problem of obtaining criteria that can indicate whether a given distribution belongs to a certain type or not. In problems of this kind the differential equation of the distribution seems to be a most natural and powerful tool of analysis. It may therefore be worth while to attempt a more general analysis of the subject, exhibiting the general nature of the principles involved, pointing out some further possibilities and also showing the natural limits of this tool in the study of frequency distributions. In the present paper some remarks on such a general analysis shall be made.

(*) «Skandinavisk Aktuarietidskrift», 1931, p. 167.

1. The (P, Q) class of a frequency distribution.

Let f_x be a finite frequency function given in the enumerable set of points $x = 0, \pm 1, \pm 2 \ldots \pm \infty$. For any given f_x it is always possible to indicate a set of numbers P_x and Q_x such that the equation

$$P_x f_x + Q_{x+1} f_{x+1} = 0 \tag{1.1}$$

holds good identically in x. It is even possible to select the numbers P and Q in such a way that there is no point x where both P_x and Q_{x+1} vanish. A set of numbers P and Q satisfying this condition will be called a (P, Q) set for the frequency function f.

Inversely, if there is given a set of numbers P and Q such that there is no point x where both P_x and Q_{x+1} vanish, these numbers may be taken as defining, by the difference equation (1.1), a frequency function f. Of course the frequency function is not uniquely determined by the numbers P and Q, but its principal characteristics are determined. More precisely expressed, if $x = h$ is any point where f is known, then f is by (1.1) determined upwards of $x = h$ if $Q_{h+1} \neq 0$ and downwards of $x = h$ if $P_h \neq 0$. We shall say that the frequency function f thus determined belongs to the class (P, Q). If (P_x, Q_{x+1}) is a set of numbers defining a class of frequency functions, and if φ_x is any function that is different from zero in all the points x, then the set $(\varphi_x P_x, \varphi_x Q_{x+1})$ obviously defines the same class.

In a region where Q_{x+1} does not vanish, the class may be defined by the single function

$$R_x = \frac{P_x}{Q_{x+1}}$$

and the corresponding difference equation

$$f_{x+1} + R_x f_x = 0 \, . \tag{1.2}$$

And in a region where P_x is different from zero, the class may be defined by the ratio

$$S_x = \frac{Q_{x+1}}{P_x}$$

and the corresponding equation

$$S_x f_{x+1} + f_x = 0 \, . \tag{1.3}$$

But this way of characterizing f breaks down in points where P_x or Q_{x+1} vanish. At the same time we notice that in order to characterize a frequency function that is zero in certain points, say outside of the interval $x = 0,1 \ldots s$, we just need to introduce numbers P_x and Q_{x+1} that vanish in certain points. The frequency function that is zero outside of the interval $x = 0,1 \ldots s$, but different from zero in $x = 0$ and $x = s$ is for instance characterized by $P_x = 0$, $Q_{x+1} \neq 0$ for $x \geq s$ and $Q_{x+1} = 0$, $P_x \neq 0$ for $x < 0$. In order to ensure generality we shall therefore consider the difference equation in the form (1.1) rather than in the form (1.2) or (1.3).

2. The tail equation

Performing on (1.1) a summation over x from t to ω we get

$$\sum_{x=t}^{\omega} (P_x + Q_x) f_x = Q_t f_t - Q_{\omega+1} f_{\omega+1} . \tag{2.1}$$

In the following we shall assume that the numbers Q_x are such that

$$\lim_{x \to \infty} Q_x f_x = 0 . \tag{2.2}$$

For any frequency function that is zero outside of a finite interval, Q may obviously always be selected so as to satisfy (2.2).

If (2.2) is fulfilled, we have

$$\sum_{x=t}^{\infty} (P_x + Q_x) f_x = Q_t f_t . \tag{2.3}$$

This equation we shall call the *tail* equation.

3. A class of incomplete moments that can be easily determined.

Let L_x be a given function of x. The expression

$$[L f]_t = \sum_{x=t}^{\infty} L_x f_x \tag{3.1}$$

is called the *incomplete moment* of the frequency distribution f taken over the function L. If (3.1) converges as $t \to -\infty$, the expression

$$[L f] = \sum_{x=-\infty}^{\infty} L_x f_x \tag{3.2}$$

is called the *complete moment* of f taken over L. There is a class of incomplete moments whose explicit expression can be easily determined by the tail equation. We now proceed to a study of these moments.

If a frequency function f_x is given, one of the corresponding functions P_x and Q_x can be selected arbitrarily, with the proviso that it must be equal to zero in those points where the P function or the Q function respectively, of the given f must vanish in order to define f correctly (if $f_s \neq 0$ and $f_{s+1} = 0$, we must for instance have $P_s = 0$). If a particular function Q_x is selected, the corresponding function P_x is by (1.1) determined in all points x where $f_x \neq 0$. If $f_x = 0$, we may attribute an arbitrary non-zero value to P_x.

For every function Q_x which we select, the function

$$L_x = P_x + Q_x \tag{3.3}$$

may consequently be looked upon as well defined. Over an interval where $Q_{x+1} \neq 0$, so that R_x is finite, the expression for L_x may for instance be written

$$L_x = Q_x + R_x Q_{x+1} \tag{3.4}$$

and over an interval where $P_x \neq 0$, so that S_x is finite, we have

$$L_{x+1} = S_x P_x + P_{x+1}\,. \tag{3.5}$$

Now insert the expression for L_x into the tail equation. This gives

$$\sum_{x=t}^{\infty} L_x f_x = Q_t f_t \tag{3.6}$$

which can also be written

$$\sum_{x=t}^{\infty} (Q_x + R_x Q_{x+1}) f_x = Q_t f_t\,. \tag{3.7}$$

This means that *to every function Q_x which we select, there corresponds an incomplete moment whose explicit expression can be given.* And this holds good no matter what the particular nature of the distribution f_x is.

As an application of (3.6) consider the binominal distribution

$$f_x = \binom{s}{x} p^x q^{s-x} \qquad q = 1 - p\,.$$

As a (P, Q) set for this frequency function we may select (1)

$$P_x = p\,(x - s) \qquad Q_x = q\,x \tag{3.8}$$

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(1) GULDBERG. loc. cit., p. 168.

Inserting (3.8) into (3.3) we get immediately $L_x = x - sp$ and hence

$$\sum_{x=t}^{\infty} (x - sp)\, f_x = q\, t\, f_t\,.$$

This is the explicit expression for the first order incomplete moment of the point binominal which I gave in the Skandinavisk Aktuarietidskrift, 1924 (2). My original proof was rather an argument ad hoc. The general tail equation exhibits in an interesting manner the underlying reason why it is possible to give an explicit expression for this incomplete moment.

And by (3.7) we see that a whole class of other explicit formulae for incomplete point binominal moments may be derived simply by inserting an arbitrary function Q_x into the formula

$$\sum_{x=t}^{\infty} \left(Q_x + \frac{p\,(x-s)}{q\,(x+1)}\, Q_{x+1}\right) f_x = Q_t f_t\,.$$

The above procedure suggests the following inversion problem: If L_x is a given function, can we determine the corresponding function Q_x, and thus by inserting Q_t in the right member of (3.6) find an explicit expression for the incomplete moment that is written in the left member of (3.6). If so, the function Q_x may be looked upon as a sort of " solving kernel " for the problem.

For simplicity we confine the discussion of the inversion problem to an interval where R_x is finite. The function Q_x in question must then be such that the right member of (3.4) is equal to the given function L_x for all the points x that occur in the summation in (3.6). In other words Q_x must be a solution of the equation (3.4) considered as a difference equation in Q_x. The solution of such a difference equation is as a rule arbitrary to the extent that the value of the solution may be selected in *one* point on the x scale. On the other hand we only need to use one single value of Q_x, namely Q_t. This seems paradoxical: Why not choose Q_t as the arbitrary magnitude? There must obviously be something wrong in this argument for the moment in the left member of (3.6) is a perfectly determinate magnitude when L_x is given. The solution of the puzzle lies in the behaviour of the coefficient R_x in (3.4) when x approaches infinity. If f_x is a frequency function that is zero in a certain finite point, say $x = s + 1$, but different from zero in the preceding point $x = s$, then by (1.2) R_s must

(2) p. 161. See also GULDBERG loc. cit., p. 17.

be equal to zero. Consequently $Q_s = L_s$, which shows that *there is no arbitrariness at all left in* Q_x when L_x is given. On the other hand if R_x does not become rigorously zero in any finite point but approaches a limit (positive or zero) as x approaches infinity, then we have by (3.4)

$$\operatorname{Lim} Q_x = \operatorname{Lim} \frac{L_x}{1 + R_x} \qquad (x \to \infty)$$

which also puts a condition on the solution Q_x which we have to select.

Similarly over an interval where S_x is finite, the solution of the inversion problem would be given by the solution P_x of (3.5).

The inversion problem as here stated can easily be solved by a direct application of the classical formulae for the solution of a linear difference equation.

In fact consider the two difference equations in Y

$$Y_x = A_x Y_{x-1} + B_x \tag{3.9}$$

and

$$Y_x = A_x Y_{x+1} + B_x \tag{3.10}$$

where the coefficients A_x and B_x are finite, possibly zero in one or more points. Let $x = k$ be an initial point with the given initial value Y_k. In the general case where we do not exclude the vanishing of A_x, (3.9) only gives a means of determining Y_x for $x \geqq k$ and (3.10) only gives a means of determining Y_x for $x \leqq k$. In these determinations, the values A_k and B_k are not used. We may therefore attribute arbitrary values to these two numbers. For convenience we put $A_k = 0$ and $B_k =$ the initial value Y_k of Y. With this notation, the solutions of (3.9) and (3.10) are respectively

$$Y_x = \sum_{\varkappa = k}^{x} (A_x A_{x-1} \ldots A_{\varkappa+1}) \cdot B_\varkappa \tag{3.11}$$

and

$$Y_x = \sum_{\varkappa = x}^{k} (A_x A_{x+1} \ldots A_{\varkappa-1}) \cdot B_\varkappa . \tag{3.12}$$

where by convention $A_x A_{x-1} \ldots A_{\varkappa+1} = 1$ for $\varkappa = x$ and $A_x A_{x+1} \ldots A_{\varkappa-1} = 1$ for $\varkappa = x$. The formulae (3.11) and (3.12) are easily verified by insertion in (3.9) and (3.10).

Applying this to the equation (3.4) we get

$$Q_{x+1} = \sum_{\varkappa = x}^{k} (-)^{\varkappa - x} (R_{x+1} R_{x+2} \ldots R_\varkappa) L_{\varkappa+1} \tag{3.13}$$

where by convention $R_{x+1} R_{x+2} \dots R_{\varkappa} = 1$ for $\varkappa = x$. We have seen that if $f_s \neq 0$ and $f_{s+1} = 0$, Q_x must be determined in such a way that $Q_s = L_s$.

This is obtained by putting $k = s - 1$ in (3.13) which gives

$$(3.14) \qquad Q_x = \sum_{\varkappa = x}^{s} (-)^{\varkappa - x} (R_x R_{x+1} \dots R_{\varkappa - 1}) L_{\varkappa}$$

where by convention $R_x R_{x+1} \dots R_{\varkappa - 1} = 1$ for $\varkappa = x$.

In the case where f_x does not vanish in any finite point, it would be necessary to study the convergency of (3.14) under the double limiting process $s \to \infty$, $x \to \infty$ but we shall not enter upon any such discussion here.

The formula (3.14) gives a direct solution of the inversion problem in the case where f_x vanishes outside of a finite interval. But the result obtained by this direct procedure is not particularly useful because the formula to which it leads is not any simpler than the definition of the incomplete moment in question given by (3.1). Indeed, for any $\varkappa \geqq x$ such that $R_x, R_{x+1} \dots R_{\varkappa - 1}$ are finite, we deduce from (1.2)

$$f_{\varkappa} = (-)^{\varkappa - x} (R_x R_{x+1} \dots R_{\varkappa - 1}) f_x$$

so that f_x times the expression (3.14) reduces to

$$Q_x f_x = \sum_{\varkappa = 1}^{s} L_{\varkappa} f_{\varkappa} .$$

The practical usefulness of the inversion process therefore does not lie in the direct determination of the Q_x that corresponds to a given L_x, but lies rather in the possibility of applying the method *indirectly* as a tool in the study of approximate solutions and the like.

4. Characteristic multipliers for a given distribution.

Let f_x be a frequency function satisfying (1.1). Further let a be a given real number, and consider the difference equation in H_x

$$(4.1) \qquad P_x H_{x+1} + a\, Q_{x+1} H_x = 0 .$$

Apart from the constant factor a, this equation is the *adjoint equation* of the equation that holds good for the frequency function f_x itself.

A solution H_x of (4.1) we shall call a *characteristic multiplier* for the frequency distribution f_x, H_x will be said to belong to the characteristic number a.

The characteristic multiplier of a given distribution satisfies several interesting formulae. First we see that by multiplying (4.1 by f_{x+1} and (1.1) by aH_x and subtracting the two equations, we get

$$P_x (H_{x+1} f_{x+1} - a H_x f_x) = 0 .$$

Similarly by multiplying (4.1) by f_x and (1.1) by H_{x+1} we obtain

$$Q_{x+1} (H_{x+1} f_{x+1} - a H_x f_x) = 0 .$$

Since by hypothesis at least one of the two numbers P_x and Q_{x+1} is different from zero we have

$$H_{x+1} f_{x+1} = a H_x f_x . \tag{4.2}$$

Extending to this equation a summation over x from t to ω we get the proposition: *If H_x is a characteristic multiplier for the frequency distribution f_x, i. e. if H_x is a function that satisfies* (4.1), then

$$(1 - a) \sum_{x=t}^{\omega} H_x f_x = H_t f_t - H_{\omega+1} f_{\omega+1} . \tag{4.3}$$

This means that the incomplete moment of f taken over a characteristic multiplier for f can always be expressed in a simple form. If

$$\lim_{x \to \infty} H_x f_x = 0$$

(which is certainly fulfilled when H_x is finite and f_x zero for $x > s$), the equation (4.3) can also be written

$$(1 - a) \sum_{x=t}^{\infty} H_x f_x = H_t f_t . \tag{4.4}$$

Let $H_{0,x}, H_{1,x} \ldots H_{n,x}$ be a *set* of functions such that

$$P_x H_{n,x+1} + Q_{x+1} \sum_{\nu=0}^{n} a_{n\nu} H_{\nu,x} = 0 \tag{4.5}$$

where $a_{n\nu}$ is a set of real numbers. Such a set of functions $H_{0,x} \ldots H_{n,x}$ we shall call a set of characteristic multipliers for the frequency distribution f_x. $H_{0,x} \ldots H_{n,x}$ will be said to belong to the characteristic numbers $a_{n\nu}$.

Any frequency distribution possesses a characteristic multiplier. To obtain one we only have to solve (4.2) with respect to H_x. The reason why it is nevertheless of interest to introduce also the notion of a *set* of multipliers as defined by (4.5) is that it may be advantageous to impose certain *conditions* on the nature of the multipliers to suit the particular kind of the problem at hand. And if the multipliers shall belong to a certain class of functions, it may of course happen that we are in a situation where a single function of the class is not a multiplier for a given frequency distribution but where a set of functions from the class does form a set of multipliers according to (4.5).

Consider for instance the case where P_x and Q_x are polynomials in x, Q_x of degree m, this degree being not lower than the degree of P_x. In this case we have the proposition: Let n be any integer $\geqq m$. *Then any set of* $(n+1)$ *linearly independent polynomials of degree not higher than* n, $H_{\nu,x}$ $(\nu = 0, 1 \ldots n)$ *form a set of multipliers provided only that the last of them, namely,* $H_{n,x}$, *contains* Q_x *as a divisor.* Since the degree shall not exceed n and one of the polynomials shall contain Q_x as a divisor, we must obviously have $n \geqq m$.

Indeed if $H_{n,x} = Q_x H'_{n-m,x}$ where H' in any polynomial (of degree zero or positive) we have

$$P_x H_{x+1} + Q_{x+1} \sum_{\nu=0}^{n} a_{n\nu} H_{\nu,x} =$$

$$= Q_{x+1} \left[P_x H'_{n-m,x+1} + \sum_{\nu=0}^{n} a_{n\nu} H_{\nu,x} \right].$$

The bracket in the last expression is a polynomial of degree not higher than n. And the $(n+1)$ coefficients $a_{n\nu}$ can be disposed of in such a way as to make the bracket vanish. In fact, when the bracket is ordered as a polynomial in x, the coefficient of x^k becomes equal to

$$\sum_{\nu=0}^{n} a_{n\nu} H_{\nu k} + C_k \tag{4.6}$$

where $H_{\nu k}$ is the coefficient of x^k in the polynomial $H_{\nu,x}$, and

$$C_k = \sum_{j=0}^{\min[k,n-m]} P_{k-j} H'_{n-m,j}$$

$P_\varkappa$ and $H'_{n-m,\varkappa}$ being the coefficients of x in the polynomial P_x and $H'_{n-m,x}$ respectively. Since the polynomials $H_{\nu k}$ are linearly independent, the determinant $|H_{\nu k}|$ is different from zero. It is consequently always possible to select the coefficients $a_{n\nu}$ in such a way that the expressions (4.6) vanish for $k = 0, 1 \ldots n$.

If P_x and Q_x are polynomials, the degree m of Q_x being not lower than the degree of P_x, we know from the above proposition that it is always possible to select a set of linearly independent polynomial multipliers if we allow the set to contain $m + 1$ such polynomials, none of the polynomials being of degree higher than m and $H_{m,x}$ being a constant times Q_x. And we also see that if no further special conditions are introduced, it will in general not be impossible to restrict the number of polynomials in the set any further.

A set of multipliers $H_{\nu,x}$ satisfy several interesting formulae. First let us multiply (4.5) by f_{x+1}, and introduce in the expression obtained $-P_x f_x$ for $Q_{x+1} f_{x+1}$. This gives

$$P_x \left[H_{n,x+1} f_{x+1} - f_x \sum_{\nu=0}^{n} a_{n\nu} H_{\nu,x} \right] = 0 .$$

Similarly we multiply (4.5) by f_x and introduce $-Q_{x+1} f_{x+1}$ for $P_x f_x$, which gives

$$Q_{x+1} \left[H_{n,x+1} f_{x+1} - f_x \sum_{\nu=0}^{n} a_{n\nu} H_{\nu,x} \right] = 0 .$$

Since in any point x, at least one of the two numbers P_x and Q_{x+1} is different from zero, we have in any point x

$$(4.7) \qquad H_{n,x+1} f_{x+1} = f_x \sum_{\nu=0}^{n} a_{n\nu} H_{\nu,x} .$$

Further let us us introduce the incomplete moments of f_x taken over the multipliers. We use the notation

$$(4.8) \qquad M_{\nu(t,\omega)} = \sum_{x=t}^{\omega} H_{\nu,x} f_x$$

ω being some conveniently chosen upper limit of the summation. If (4.8) converges as $\omega \rightarrow \infty$ (which is certainly the case for instance when the $H_{\nu,x}$ are finite over any finite range and $f_x = 0$ outside a finite interval), then we may in particular consider the incomplete moments

$$(4.9) \qquad M_{\nu,t} = \sum_{x=t}^{\infty} H_{\nu,x} f_x .$$

The moments (4.8) and (4.9) may be called the *characteristic* moments for the given distribution.

Extending to (4.7) a summation over x from $x = t$ to $x = \omega$ we get

$$(4.10) \quad (1 - a_{nn}) M_{n(t,\omega)} = H_{n,t} f_t - H_{n,\omega+1} f_{\omega+1} + \sum_{\nu=0}^{n-1} a_{n\nu} M_{\nu(t,\omega)} .$$

Further, if the moments converge for $\omega \to \infty$, which entails

$$(4.11) \qquad \lim_{x \to \infty} H_{n,x} f_x = 0 ,$$

we get by extending to (4.7) a summation over x from t to ∞

$$(4.12) \qquad (1 - a_{nn}) M_{n,t} = H_{n,t} f_t + \sum_{\nu=0}^{n-1} a_{n\nu} M_{\nu,t} .$$

Thus: *For any frequency distribution the incomplete moments taken over the characteristic mutipliers satisfy the simple recurrence formulae* (4.10) *and* (4.12).

As an example consider the binomial distribution. $A (P, Q)$ set for this distribution is (3.8). And the functions

$$(4.13) \qquad H_{\nu,n} = x^{\nu},$$

($\nu = 0, 1 \ldots n$ where $n \gtrless 1$), form a set of characteristic multipliers. Indeed if $n = 1$ the polynomial $Q_{x+1} = q (x + 1)$ is a divisor in $H_{n,x+1} = (x + 1)$ so that the left member of (4.5) becomes

$$(4.14) \qquad (x + 1) \left[p (x - s) (x + 1)^{n-1} + q \sum_{\nu=0}^{n} a_{n\nu} x^{\nu} \right].$$

And it is always possible to determine the numbers $a_{\nu n}$ in such a way that the bracket in (4.14) vanish. We only have to put

$$a_{n\nu} = \frac{p}{q} \left(s \binom{n-1}{\nu} - \binom{n-1}{\nu-1} \right).$$

For the binomial distribution the incomplete power moments about the origin therefore satisfies (1)

$$M_{n,t} = q H_{n,t} f_t + p \sum_{\nu=0}^{n-1} \left(s \binom{n-1}{\nu} - \binom{n-1}{\nu-1} \right) M_{\nu,t} .$$

Any set (4.13) where $n \gtrless 1$ forms a set of characteristic multipliers for the binomial distribution. But for $n = 0$, we do *not* get such a set. That is to say a constant *is not* a characteristic multiplier for the binomial distribution. If it had been, we would have been

(1) The recurrence formulae for the incomplete power moments of the point binomial were, so far as I know, first given by me in «Biometrika» 1925 p. 177 (moments about the mean). See also GULDBERG Loc. cit. p. 171. In the formula on the 7th line by GULDBERG there is a misprint. The factor $(1 - p)$ in the left member has dropped out.

able to determine a simple explicit formula for the incomplete zero order moment of the point binomial. This throws some light on the well known fact that the exact value of the incomplete zero order moment of the point binomial is, as it were, surrounded by a Chinese wall in which it has not yet been possible to break any hole. (2).

We have seen that the incomplete characteristic moments of a given distribution satisfy the recurrence formulae (4.10) and (4.12). Inversely : If these recurrence formulae hold good, will the functions $H_{\varkappa,x}$ form a set of characteristic multipliers for f_x? The answer is yes. Indeed, subtracting from (4.10) the same equation for $t+1$ we get

$$(1 - a_{nn})\, H_{n,t}\, f_t = H_{n,t}\, f_t - H_{n,t+1}\, f_{t+1} + \sum_{\nu=0}^{n-1} a_{n\nu}\, H_{\nu,t}\, f_t \,.$$

That is to say

$$H_{n,x+1}\, f_{x+1} - f_x \sum_{\nu=0}^{n} a_{n\nu}\, H_{\nu,x} = 0 \,.$$

Multiplying this equation by P_x and by Q_{x+1} we obtain by (1.1) respectively

$$f_{x+1}\,(P_x\, H_{n,x+1} + Q_{x+1} \sum_{\nu=0}^{n} a_{n\nu}\, H_{\nu,x}) = 0$$

and

$$f_x\,(P_x\, H_{n,x+1} + Q_{x+1} \sum_{\nu=0}^{n} a_{n\nu}\, H_{\nu,x}) = 0 \,.$$

In any point x where either f_x or f_{x+1} or both are different from zero, the set $H_{\nu,x}$ must consequently satisfy (4.5). The equations (4.5) and (4.10) can therefore be looked upon as equivalent ways of defining the characteristic multipliers.

We may generalize (4.5) by considering the case where the right member of the equation is not zero but some function W_x, in other words we consider a set of functions $L_{0,x} \ldots L_{n,x}$ satisfying

$$P_x\, L_{n,x+1} + Q_{x+1} \sum_{\nu=0}^{n} C_{n\nu}\, L_{\nu,x} = W_x \qquad (4.15)$$

where the $C_{\nu n}$ are constants.

(2) Upper and lower limits for the incomplete zero order moment of the point binomial are given in my paper. *Sur les semi-invariants et moments employés dans l'étude des distributions statistiques.* « Det Norske Videnskaps-akademie II », 1926. No. 3.

This leads to the equations

$$P_x \left[L_{n,x+1} f_{x+1} - f_x \sum_{\nu=0}^{n} C_{n\nu} L_{\nu,x} \right] = W_x f_{x+1}$$

$$Q_{x+1} \left[L_{n,x+1} f_{x+1} - f_x \sum_{\nu=0}^{n} C_{n\nu} L_{\nu,x} \right] = - W_x f_x .$$

In other words

$$(4.16) \qquad L_{n,x+1} f_{x+1} = f_x \sum_{\nu=0}^{n} C_{n\nu} L_{\nu,x} + \begin{cases} \dfrac{W_x}{P_x} f_{x+1} \\ -\dfrac{W_x}{Q_{x+1}} f_x . \end{cases}$$

At least one of the numerators in the right member of (4.16) is different from zero, and this expression is chosen. Performing a summation over x on (4.16) we get, assuming convergency

$$(4.17) \qquad (1 - C_{nn}) J_{n,t} = L_{n,t} f_t + \sum_{\nu=0}^{n-1} C_{n\nu} J_{\nu,t} + \sum_{x=t}^{\infty} \begin{cases} \dfrac{W_x}{P_x} f_{x+1} \\ \dfrac{-W_x}{Q_{x+1}} f_x \end{cases}$$

where $J_{\nu,t} = \sum_{x=t}^{\infty} L_{\nu,x} f_x$. For each x in the expression to the extreme right in (4.17) that quantity is chosen for which the numerator (P_x or Q_{x+1}) is different from zero. Of course (4.12) is the special case $W_x = 0$ of (4.17).

5. Improper Multipliers.

Suppose that there is given a set of characteristic multipliers $H_{0,x} \ldots H_{n,x}$ for the frequency distribution f_x. Further let

$$(5.1) \qquad (\alpha_{ij}) = \begin{pmatrix} \alpha_{00} & \cdots & \alpha_{0n} \\ \cdots & \cdots & \cdots \\ \alpha_{n0} & \cdots & \alpha_{nn} \end{pmatrix}$$

be any non singular (n + 1) rowed matrix with constant elements. This matrix has a reciprocal namely

$$(\overset{*}{\alpha}_{ij}) = \begin{pmatrix} \overset{*}{\alpha}_{00} & \cdots & \overset{*}{\alpha}_{0n} \\ \cdots & \cdots & \cdots \\ \overset{*}{\alpha}_{n0} & \cdots & \overset{*}{\alpha}_{nn} \end{pmatrix}$$

where $\overset{*}{\alpha}_{ji}$ is $(-)^{i+j}$ times the expression obtained by leaving out the j-th row and the i-column in (5.1), taking the determinant value of this n rowed matrix and finally dividing by the $(n+1)$ rowed determinant of (5.1). With this notation consider the linear forms in the $H_{j,x}$

$$K_{i,x} = \sum_{j=0}^{n} \overset{*}{\alpha}_{i,j} H_{j,x}.$$

The functions $K_{i,x}$ have the property that the $H_{j,x}$ may be expressed as linear forms in the $K_{i,x}$. We have indeed

$$H_{i,x} = \sum_{j=0}^{n} \alpha_{ij} K_{j,x}. \tag{5.2}$$

Any set of such functions $K_{j,x}$ that are linear forms in the $H_{j,x}$, with constant coefficients forming a non-vanishing determinant, will be called a set of *improper* multipliers for f_x. In distinction to the improper multipliers, the functions $H_{j,x}$ will be called proper multipliers for f_x.

The moments of f_x taken over any set of improper multipliers also satisfy recurrence formulae similar to (4.10) and (4.12), however with different coefficients.

Let

$$N_{\nu,t} = \sum_{x=t}^{\infty} K_{\nu,x} f_x$$

and

$$N_{\nu(t,\omega)} = \sum_{x=j}^{\omega} K_{\nu,x} f_x$$

be the moments of the improper multipliers.

By (5.2) we have

$$M_{i(t,\omega)} = \sum_{j=0}^{n} \alpha_{ij} N_{j(t,\omega)}$$

$$M_{i,t} \quad = \sum_{j=0}^{n} \alpha_{ij} N_{i,t} \tag{5.3}$$

Writing (4.12) in the form

$$M_{n,t} = H_{n,t} f_t + \sum_{i=0}^{n} a_{ni} M_{i,t}$$

and introducing the expression for the $M_{i,t}$ taken from (5.3) we get

$$\Sigma_j \alpha_{nj} N_{j,t} = H_{n,t} f_t + \Sigma_j (\Sigma_i a_{ni} \alpha_{ij}) N_{j,t}$$

that is

$$(5.4) \qquad 0 = H_{n,t} f_t + \sum_{j=0}^{n} b_{nj} N_{j,t}$$

where

$$b_{nj} = \sum_{i=0}^{n} a_{ni} \alpha_{ij} - \alpha_{nj} .$$

Solving (5.4) with respect to $N_{n,t}$ we get

$$(5.5) \qquad (\alpha_{nn} - \sum_{i=0}^{n} a_{ni} \alpha_{in}) N_{n,t} = H_{n,t} f_t + \sum_{\nu=0}^{n-1} b_{nj} N_{\nu,t} .$$

In particular if each $H_{j,t}$ only contains the functions $K_{0,t} \ldots K_{j,t}$ so that $\alpha_{in} = 0$ for $i < n$, the coefficient of $N_{n,t}$ in (5.5) reduces to $\alpha_{nn}(1 - a_{nn})$.

If the α_{ij} are equal to

$$\alpha_{ij} = e_{ij} = \begin{cases} 0 \; (if\, i \neq j) \\ 1 \; (if\, i = j) \end{cases}$$

$N_{i,t}$ reduces to $M_{i,t}$ and the formula (5.5) reduces to (4.12).

As an application of (5.5) consider again the point binomial. Since the powers x^ν ($\nu = 0, 1 \ldots n$) for $n \gtreqless 1$ form a set of proper multipliers for the point binomial, any set of $n + 1$ linearly independent polynomials $K_{0,x} \ldots K_{n,x}$ forms a set of improper multipliers. Any set of more than two polynomial moments in the point binomial (taken over linearly independent polynomials) satisfies therefore a recurrence formula of the kind (5.5).

6. The complete moments.

If the incomplete moments $M_{n,t}$ converge as $t \to -\infty$, we consider the quantities

$$(6.1) \qquad M_\nu = \sum_{x=-\infty}^{+\infty} H_{\nu,x} f_x .$$

These quantities are called the *complete* characteristic moments. The quantities (6.1) converge certainly if the $H_{\nu,x}$ are finite over any finite range, and f_x zero outside of a given finite range.

Similarly we consider the complete moments

$$N_\nu = \sum_{x=-\infty}^{\infty} K_{\nu,x} f_x .$$

The complete moments M_ν and N_ν satisfy the recurrence formulae

$$(1-a_{nn})\,\mathrm{M} = \sum_{\nu=0}^{n=1} a_{a\nu} M_\nu \quad \text{and}$$

$$\left(\alpha_{nn} - \sum_{i=0}^{n} a_{ni}\alpha_{in}\right) N_n = \sum_{\nu=1}^{n-0} b_{n\nu} N_{\nu,t}$$

These formulae are obtained from (4.12) and (5.5) by letting t tend towards $-\infty$.

7. The determination of the characteristic numbers.

The characteristic numbers can be expressed in different ways. Amongst others they can be expressed in terms of certain types of complete moments of the distribution.

Let

$$\varphi_{i,x}\,(i = 0, 1 \ldots n)$$

be any set of $n + 1$ functions, such that the complete moment

$$U_{ij} = \sum_{x=-\infty}^{\infty} \varphi_{i,x} H_{j,x} f_x \begin{pmatrix} i = 0, 1 \ldots n \\ j = 0, 1 \ldots n \end{pmatrix} \tag{7.1}$$

$$V_i = \sum_{x=-\infty}^{\infty} \varphi_{i,x-1} H_{n,x} f_x \,(i = 0, 1 \ldots n) \tag{7.2}$$

converge, and further such that the determinant

$$|U_{ij}| = \begin{vmatrix} U_{00} & \ldots & U_{0n} \\ U_{n0} & \ldots & U_{nn} \end{vmatrix}$$

is different from zero.

Multiplying (4.7) by $\varphi_{i,x}$ and performing a complete summation over x we obtain

$$\sum_{j=0}^{n} U_{ij} a_{nj} = V_i \qquad (i = 0, 1 \ldots n) \tag{7.3}$$

Since $|U_{ij}| \neq 0$ the system (7.3) may be solved with respect to the a_{nj} which gives

$$a_{nj} = \sum_{j=0}^{n} \overset{*}{U}_{ij} V_j \tag{7.4}$$

where the $\overset{*}{U}_{ij}$ are the elements of the reciprocal of the matrix U_{ij}. We may also write the solution of (7.3) in another form which is more convenient for the application we shall later make of the characteristic numbers. Let $B_o \ldots B_n$ be any set of numbers. Then we have

$$\sum_{i=0}^{n} a_{ni} B_i = - \begin{vmatrix} o & B_o \ldots B_n \\ V_o & U_{oo} \ldots U_{on} \\ \ldots & \ldots \\ V_n & U_{no} \ldots U_{nn} \end{vmatrix} : \begin{vmatrix} U_{oo} \ldots U_{on} \\ \ldots \\ U_{no} \ldots U_{nu} \end{vmatrix} \tag{7.5}$$

If we put all the coefficients B in (7.5) equal to zero, except the special coefficient B_i, we get back to the formula (7.4).

It is also possible to express the numbers a_{ni} in terms of the moments

$$X_{ij} = \sum_{x=-\infty}^{\infty} \varphi_{i,x} K_{j,x} f_x$$

and

$$Y_{ij} = \sum_{x=-\infty}^{+\infty} \varphi_{i,x-1} K_{j,x} f_x$$

Inserting from (5.2) into (7.1) and (7.2) we get indeed

$$U_{ij} = \sum_{k=0}^{n} \alpha_{jk} X_{ik}$$

$$V_i = \sum_{k=0}^{n} \alpha_{nk} Y_{ik}$$

8. Local criteria for the nature of the distribution.

Let (P, Q) be a set of functions defining a class of frequency distributions. Let F_x be a numerically given frequency distribution. We want a criteria expressing if F_x can be looked upon as being *approximately* of the class (P, Q). We may consider two sorts of criteria: Local and total criteria. The local criteria are expressed in terms of the values of F_x in the vicinity of a given point x, and the total criteria are expressed by certain parameters depending on the totality of the values of F_x as expressed by certain complete moments of F_x.

In order to develop such criteria we shall again make use of a set of characteristic multipliers for the class (P, Q). Let $H_{\nu,n}$ $(\nu = 0, 1 \ldots n)$ be such a set. These multipliers can be determined, provided only that the set of functions P_x and Q_{x+1} is given, that is to say the $H_{\nu,x}$ may be constructed once for all quite independently of the particular numerically given distribution F_x which it is wanted to study.

If the function F_x belongs rigorously to the class (P, Q), then the function

$$\Psi_x = \frac{F_x}{F_{x+1}} \cdot \frac{\sum_{\nu=0}^{n} a_{n\nu} H_{\nu,x}}{H_{x,n+1}}$$

shall by (4.7) be identically equal to unity. It therefore seems plausible to adopt the closeness with which Ψ_x actually fluctuates around unity as a criterion of how close F_x belongs to the class (P, Q). This is a generalization of the criterion which Professor Guldberg has given for the special distributions considered by him. (1).

In order to construct Ψ_x we must fit the constants $a_{n\nu}$ to the given distribution F_x. A plausible fitting procedure is to require that the general moments (7.1) and (7.2) shall coincide when computed for the ideal distribution f_x that belongs to the class (P,Q) and for the given distribution F_x. By (7.5) this leads to the following expression for Ψ_x

$$\Psi_x = -\frac{F_x}{F_{x+1} H_{n,x+1}} \cdot \begin{vmatrix} 0 & H_{0,x} & \ldots & H_{n,x} \\ V_0 & U_{00} & \ldots & U_{0n} \\ \cdot & \cdot & \cdots & \cdot \\ V_n & U_{n0} & \ldots & U_{nn} \end{vmatrix} : \begin{vmatrix} U_{00} & \ldots & U_{0n} \\ \cdot & \cdots & \cdot \\ U_{n0} & \ldots & U_{nn} \end{vmatrix} \tag{8.1}$$

The local criterion for the fact that F_x belongs approximately to the class (P_x, Q_{x+1}) may then be taken as expressed by the function (8.1) oscillating closely around unity. The functions $H_{0,x} \ldots H_{n,x}$ in (8.1) are a set of characteristic multipliers for the class $(P_x Q_{x+1})$, and V_i and U_{ji} are the complete moments defined by (7.1) and (7.2), where $\varphi_{i,x}$ $(i = 0,1 \ldots n)$ is a set of functions that may be selected

(1) Loc. cit. 172.

arbitrarily with the only proviso that the determinant $|U_{ij}|$ shall be different from zero.

If $\Psi_x = 1$, then we also have $\Delta_x = 0$, where

$$\Delta_x = \begin{vmatrix} \Phi_x & H_{0,x} & \dots & H_{n,x} \\ V_0 & U_{00} & \dots & U_{0n} \\ \dots & \dots & \dots & \dots \\ V_n & U_{n0} & \dots & U_{nn} \end{vmatrix} \tag{8.2}$$

Φx being equal to

$$\Phi_x = \frac{F_{x+1} H_{n,x+1}}{F_x}. \tag{8.3}$$

The criterion in question may therefore also be expressed by saying that Δ_x *defined by* (8.2) *shall be close to zero for any x.* Since x only occurs in the first row of (8.2) we may formulate the criterion by saying that the function Φ_x defined by (8.3) shall be a linear form in the $H_{0,x} \dots H_{n,x}$ with constant coefficients. The last formulation of the criterion is of course contained already in the formula (4.7). What is obtained by (8.2) is that we have here an expression for the coefficients of the form.

As an example, consider again the binomial distribution. Here we may select

$$n = 1 \qquad H_{0,x} = 1 \qquad H_{1,x} = x$$

$$\varphi_{0x} = x^g \qquad \varphi_{1x} = x^h$$

g and h being two non negative integers.

This gives

$$M_0 = \sum_{x=-\infty}^{\infty} f_x \qquad M_1 = \sum_{x=-\infty}^{\infty} x \, . \, f_x$$

The matrix (U_{ij}) is now two rowed and equal to

$$(U_{ij}) = \begin{pmatrix} M_g & M_{g+1} \\ M_h & M_{h+1} \end{pmatrix}$$

where we have put for brevity

$$M_i = \sum_{x=-\infty}^{\infty} x^i \, . \, f_x \, .$$

Further the moments V_0 and V_1 are

$$V_0 = \sum_{i=0}^{g} (-)^{g-i} \binom{g}{i} M_{i+1}$$

$$V_1 = \sum_{i=0}^{h} (-)^{h-i} \binom{h}{i} M_{i+1}$$

So that

$$\Delta_x = \begin{vmatrix} \Phi_x & 1 & x \\ V_0 & M_g & M_{g+1} \\ V_1 & M_h & M_{h+1} \end{vmatrix} \qquad (8.4)$$

where

$$\Phi_x = \frac{(x+1) F_{x+1}}{F_x}. \qquad (8.5)$$

The quantities in the second and third row of (8.4) are constants independent of x. Quite generally we may therefore say that *the criterion for a binomial distribution is that the function Φ_x defined by (8.5) is approximately a straight line.* The coefficients of this straight line are determined by

$$\Phi_x = \left(\frac{V_0 M_{h+1} - V_1 M_{g+1}}{d}\right) - \left(\frac{V_0 M_h - V_1 M_g}{d}\right) x \qquad (8.6)$$

where

$$d = \begin{vmatrix} M_g & M_{g+1} \\ M_h & M_{h+1} \end{vmatrix}.$$

The two expressions in parenthesis in the right member of (8.6) ought to be independent of g and h. For $g = 0$, $h = 1$ we get $V_0 = M_1$, $V_1 = -M_1 + M_2$, and hence

$$\Phi_x = \frac{M_1^2}{\mu_2} + \frac{\mu_2 - M_1}{\mu_2} x \qquad (8.7)$$

where

$$\mu_2 = M_2 - M_1^2$$

μ_2 is the second power moment about the mean.

The formula (8.7) is the local criterion given by Guldberg (1) for the case of the point binomial.

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(1) Loc. cit. p. 172, the formula given at the bottom of the page.

9. Total criteria for the nature of the distribution.

A total criterion for the fact that a numerically given frequency distribution F_x belongs approximately to a given class may be obtained from the formula

$$M_n = \sum_{\nu=o}^{n} a_{n\nu} M_\nu$$

simply by introducing the expression for the characteristic numbers taken from (7.5). This gives the general criterion that the number

$$D = \begin{vmatrix} M_n & M_o & \ldots & M_n \\ V_o & U_{oo} & \ldots & U_{on} \\ \ldots & \ldots & \ldots & \ldots \\ V_n & U_{no} & \ldots & U_{nn} \end{vmatrix}$$

ought to be close to zero.

In the case of the point binomial we have with the notation of the preceding section

(9.1) $$D = \begin{vmatrix} M_1 & M_o & M_1 \\ V_o & M_g & M_{g+1} \\ V_1 & M_h & M_{h+1} \end{vmatrix}$$

For $g = 0$ and h arbitrary the two first rows in (9.1) become equal. The selection $g = 0$ does therefore not lead to any condition on F_x.

For $g = 1$ and h arbitrary we get

$$D = \begin{vmatrix} M_1 & M_o & M_1 \\ (-M_1 + M_2) & M_1 & M_2 \\ V_1 & M_h & M_{h+1} \end{vmatrix}$$

Subtracting here the last row from the first we get

$$D = \begin{vmatrix} o & M_o & M_1 \\ -M_1 & M_1 & M_2 \\ V_1 - M_{h+1} & M_h & M_{h+1} \end{vmatrix}$$

that is to say

$$D = M_1 (M_0 M_{h+1} - M_1 M_h) + (V_1 - M_{h+1}) (M_0 M_2 - M_1^2) .$$

For $h = 1$ this does not give any condition. For $h = 2$ we get, since $M_0 = 1$

(9.2) $$D = M_1 (M_3 - M_1 M_2) + (M_1 - 2 M_2) (M_2 - M_1^2) .$$

The condition that this expression shall be equal to zero is the same as Guldberg's first total criterion (1) namely

$$M_1 \mu_3 = 2\mu_2^2 - M_1 \mu_2 . \tag{9.3}$$

Indeed introducing in (9.3)

$$\mu_2 = M_2 - M_1^2$$

$$\mu_3 = M_3 - 3 M_1 M_2 + 3 M_1^3$$

we get the condition that the right member of (9.2) shall be equal to zero.

10. Frequency functions defined by a differential equation.

The main ideas of the preceding analysis can be applied also to the case where the frequency function is defined by a differential equation, instead of by a difference equation. Since in the case of a discrete frequency function, the difference equation (1.1) can be written in the form

$$(P_x + Q_{x+1}) f_x + Q_{x+1} \triangle f_x = 0$$

where

$$\triangle f_x = f_{x+1} - f_x,$$

it seems natural to consider, in the continuous case, the differential equation

$$(P_x + Q_x) f_x + Q_x f'_x = o \tag{10.1}$$

where

$$f'_x = \frac{d f_x}{d x} .$$

By integrating (10.1) over x from t to ω and noticing that

$$\int_t^\omega Q_x f'_x \, d x = [Q_x f_x]_t^\omega - \int_t^\omega Q'_x f_x \, d x, \quad \text{we get}$$

$$\int_t^\omega (P_x + Q_x - Q'_x) f_x \, d x = Q_t f_t - Q_\omega f_\omega$$

(1) Loc. cit. p. 173.

Consequently if $\lim_{x \to \infty} Q_x f_x = 0$

(10.2) $$\int_t^\infty (P_x + Q_x - Q'_x) f_x \, dx = Q_t f_t$$

This is the tail equation in the continuous case. Any function Q_x inserted in (10.2) furnishes the explicit expression for an incomplete moment. As an example let us consider the Pearson class of frequency functions.

This is the following special case of (10.1)

$$P_x + Q_x = a_0 + a_1 x$$

$$Q_x = -(b_0 + b_1 x + b_2 x^2)$$

where the a's and b's are constants. Consequently, for *any* frequency function in the Pearson class there exists an incomplete power-moment of the first order whose explicit expression can be immediately given namely

$$\int_t^\infty (a_0 + b_1 + (a_1 + 2 b_2) x) f_x \, dx = -(b_0 + b_1 t + b_2 t^2) f_t .$$

And, more generally, if Q_x is an arbitrary function we have for any frequency function of the Pearson class

$$\int_t^\infty \left(\frac{a_0 + a_1 x}{b_0 + b_1 x + b_2 x^2} Q_x + Q'_x \right) f_x \, dx = -Q_t f_t .$$

Let us put

$$R_x = \frac{P_x}{Q_x}$$

The ratio R_x is determined by the nature of the frequency distribtion. Introducing this ratio we see that the problem of determining the incomplete moment

$$\int_t^\infty L_x f_x \, dx = Q_t f_t$$

is equivalent with the problem of solving the differential equation in Q_x

$$Q'_x = (1 + R_x) Q_x - L_x .$$

In the continuous case we define a characteristic multiplier belonging to the constant a, as a function H_x satisfying

$$Q_x H'_x - (P_x + a Q_x) H_x = 0$$

and we define a set of characteristic multipliers $H_{\nu,x}$ ($\nu = o, 1 \ldots n$) belonging to the constants $a_{n\nu}$, and a set of functions satisfying

$$(10.3) \qquad Q_x H'_{n,x} - (P_x + a_{nn} Q_x) H_{n,x} = Q_x \sum_{\nu=o}^{n-1} a_{n\nu} H_{\nu,x} .$$

Let us for brevity denote by Θ the left member of the equation obtained from (10.3) by carrying all the terms over on the left side. Then we have

$$(10.4) \quad Q_x \left[\frac{d}{dx} (H_{n,x} f_x) + (1 - a_{nn}) H_{n,x} f_x - f_x \sum_{\nu=o}^{n-1} a_{n\nu} H_{\nu,x} \right] = f_x \Theta$$

and $(P_x + Q_x)$ times the bracket in (10.4) is equal to $-f'_x \Theta$.

Hence: In any point x where either $Q_x \neq 0$ and f_x finite or $P_x + Q_x \neq o$ and f'_x finite, we have

$$(10.5) \qquad \frac{d}{dx} (H_{n,x} f_x) + (1 - a_{nn}) H_{n,x} f_x = f_x \sum_{\nu=o}^{n-1} a_{n\nu} H_{\nu,x}$$

Let

$$M_{\nu,t} = \int_t^\infty H_{\nu,x} f_x \, dx$$

be the incomplete moments taken over a set of characteristic multipliers. If $\lim_{x \to \infty} H_{n,x} f_x = 0$, the integration $\int_t^\infty dx$ extended to (10.5) gives

$$(1 - a_{nn}) M_{\nu,t} = H_{n,t} f_t + \sum_{\nu=o}^{n-1} a_{n\nu} M_{\nu,t}$$

This formula is analagous to (4.12).

In order to obtain an expression for the characteristic numbers $a_{n\nu}$ in terms of the integral properties of the frequency function, we introduce the complete moments

$$(10.6) \qquad U_{ij} = \int_{-\infty}^{+\infty} \varphi_{i,x} H_{j,x} f_x \, dx$$

$$(10.7) \qquad V_i = \int_{-\infty}^{\infty} (\varphi_{i,x} - \varphi'_{i,x}) H_{n,x} f_x \, dx$$

where $\varphi_{i,x}$ ($i = 0, 1 \ldots n$) is any set of functions such that the determinant $|U_{ij}|$ is different from zero. Let us multiply (10.5) by $\varphi_{i,x}$

and perform an integration over x between t and ω. If we use the partial integration

$$\int_t^\omega \varphi_{i,x} \frac{d}{dx}(H_{n,x} f_x)\, dx = [\varphi_{i,x} H_{n,x} f_x]_t^\omega - \int_t^\omega \varphi'_{i,x} H_{n,x} f_x\, dx$$

and then let $t \to -\infty$ and $\omega \to \infty$ we get, on the assumption that.

$$\text{Lim}\ \varphi_{i,x} H_{n,x} f_x = 0 \quad \text{when } x \to \pm \infty$$

(10.8) $$\sum_{j=0}^{n} U_{ij} a_{nj} = V_i \quad (i = 0, 1 \ldots n) .$$

The system of linear equations (10.8) is the same as (7.3). The whole analysis of Section 7 therefore applies also to the present continuous case, provided only that the moments U_{ij} and V_i are defined by (10.6) and (10.7) instead of by (7.1) and (7.2).

If F_x is a numerically given frequency distribution, the local criterion for the fact that F_x belongs approximately to a given (P, Q) class defined by the differential equation (10.1), can be formulated thus: Consider the function

$$\Psi_x = \frac{-1}{(1+\theta_x) H_{n,x}} \cdot \begin{vmatrix} 0 & H_{0,x} \ldots H_{n,x} \\ V_0 & U_{00} \ldots U_{0n} \\ \ldots & \ldots \\ V_n & U_{n0} \ldots U_{nn} \end{vmatrix} : \begin{vmatrix} U_{00} \ldots U_{0n} \\ \ldots \\ U_{n0} \ldots U_{nn} \end{vmatrix}$$

where

$$\Theta_x = \frac{d \log (H_{n,x} F_x)}{dx} = \frac{d \log H_{n,x}}{dx} + \frac{d \log F_x}{dx}$$

The criterion is that Ψ_x shall fluctuate closely around unity. Indeed, if F_x belongs rigorously to the class (P,Q), Ψ_x must be equal to unity. This is seen by expressing in (10.5) the constants $a_{n\nu}$ in terms of the complete moments U_{ij} and V_i by the same formulae as those used in Section 7.

The total criteria takes on exactly the same form as in the case of a discrete distribution, provided only that we consider, in the continuous case, U_{ij} and V_i as being defined by (10.6) and (10.7).
