CIRCULATION PLANNING

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PART III. MATHEMATICAL APPENDIX

In the following sections, I shall give certain mathematical formulae on which part of the argument in the preceding Sections is built. Some of these formulae are not new, but are here derived in a manner which, in my opinion, is simpler and more systematic than the ones usually found in the literature: this applies in particular to the formulae of Section 23. Other formulae here given I do not recall to have seen before, for instance, those of Section 21.

21. Lemmas on Matrices Consisting of Non-Negative Elements

Consider a square matrix

$$(21.1)\qquad (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

whose elements are non-negative numbers. A request matrix in circulation planning has this property.

The operation which consists in extending a summation to one (or both) of the subscripts on a_{ij} we shall denote simply by replacing the subscript in question by a zero. Thus the sums of rows and columns respectively are denoted.

$$(21.2)\qquad \begin{aligned} a_{i0} &= \sum_k a_{ik} = a_{i1} + \cdots + a_{in} \\ a_{0j} &= \sum_k a_{kj} = a_{1j} + \cdots + a_{nj}. \end{aligned}$$

If the matrix (21.1) is such that

$$(21.3)\qquad a_{i0} = a_{0i} \text{ for all } i = 1, 2 \cdots n$$

we shall say that it has complete row-column equality. A request matrix will in general not have this property.

Let us consider the matrix obtained from (21.1) by subtracting from each diagonal element the sum of all the elements in the corresponding row, i.e. the matrix

$$(21.4)\qquad \begin{pmatrix} a_{11} - a_{10} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - a_{20} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} - a_{n0} \end{pmatrix}.$$

We may generalize the matrix (21.4) by further subtracting from each diagonal element a set of non-negative quantities $t_1, \cdots, t_n$, so that we obtain the matrix

$$(21.5)\qquad \begin{Bmatrix} a_{11} - a_{10} - t_1 & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - a_{20} - t_2 & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} - a_{n0} - t_n \end{Bmatrix}.$$

For the following discussion it is essential that all the original elements and all the quantities t_i be non-negative. It should also be noted that the quantities a_{i0} denote the *complete* sum of the i-th row in the n rowed matrix (21.1). This is essential as we shall later consider certain minors of the above matrices.

The determinant value of (21.4) is equal to zero. This is seen simply by adding to the elements in any given column of (21.4) the elements of all the other columns; this will produce a column consisting exclusively of zeros.

For the more general determinant defined by (21.5) we have the following:

LEMMA I.—*If all the original elements a_{ij} and all the magnitudes t_i are non-negative, the determinant value of (21.5) has the sign $(-)^n$, in other words this determinant is non-negative if n is even and non-positive if n is odd.*

To prove this, let us denote the determinant in question by T, further let $T_{)i(}$ be the $(n-1)$-rowed determinant obtained by omitting the row No. i and the column No. i from T (but all the a_{k0} denoting still the sums of the rows in the complete determinant 21.1), further let $T_{)ij(}$ be the $(n-2)$-rowed determinant obtained by omitting the rows Nos. i and j and the columns Nos. i and j. And so on. Memotechnically we may interpret the inverted parenthesis) (to mean "exclusion of." By convention we shall put any expression of the form $T_{)\alpha\beta\ldots(}$ equal to zero whenever at least two of the subscripts are equal; and if *all* the n subscripts occur in the omission parenthesis we put $T_{)12\ldots n(}=1$. Finally let S, $S_{)i(}$, etc. denote the corresponding determinant and minors obtained from (21.4). Thus S, $S_{)i(}$, etc. depend only on the elements of the original matrix (21.1), while T, $T_{)i(}$, etc. depend also on $t_1 \ldots t_n$.

This notation being adopted, we see that if T is considered as a function of the n variables $t_1 \ldots t_n$, the partial derivatives of T are

$$(21.6)\qquad \frac{\partial T}{\partial t_i} = -T_{)i(} \qquad \frac{\partial^2 T}{\partial t_i \partial t_j} = T_{)ij(} \qquad \frac{\partial^3 T}{\partial t_i \partial t_j \partial t_k} = -T_{)ijk(}, \text{ etc.}$$

This shows that we have the Taylor expansion

$$(21.7)\qquad \begin{aligned} T = S - \sum_i t_i S_{)i(} + \sum_{i<j} t_i t_j S_{)ij(} \\ - \sum_{i<j<k} t_i t_j t_k S_{)ijk(} + \cdots + (-)^n t_1 t_2 \cdots t_n \end{aligned}$$

where i runs through all the numbers $1, 2, \cdots, n$, $i<j$ runs through combinations without repetition of the two affixes (i, j) picked in the set $1, 2, \cdots, n$, etc.

The first term of the expansion (21.7), namely the determinant S is, as we have seen, equal to zero. This being so, the expansion (21.7) furnishes an easy means of proving Lemma I by complete induction. The lemma is obviously correct for $n=1$ because we have $a_{11}-a_{10}-t_1 = -t_1 =$ non-positive if $t_1 \geqq 0$. Therefore, let us assume that the lemma is correct for all orders up to and including $n-1$. Consider then the determinant $S_{)i(}$. In this determinant we separate from a_{10} the term a_{1i} which we denote for a moment t_1; from a_{20} we separate a_{2i} which we denote t_2, etc. Writing $S_{)i(}$ in this form we see that $S_{)i(}$ is an $(n-1)$-rowed determinant of the form (21.5). Indeed what is left of a_{10} when the term a_{1i} is taken out is just the sum of all the a's that occur in the first row in $S_{)i(}$, and similarly for the other rows. Consequently if Lemma I is correct for $n-1$, $S_{)i(}$ must have the sign $(-)^{n-1}$.

Next consider the determinant $S_{)ij(}$. From a_{10} in this determinant we take out $a_{1i}+a_{1j}$ and denote for a moment this binome t_1. From a_{20} we take out $a_{2i}+a_{2j}$ and denote it t_2, etc. This shows that $S_{)ij(}$ is an $(n-2)$-rowed determinant of the form (21.5). Hence it must have the sign $(-)^{n-2}$ provided Lemma I is correct up to $n-1$.

Quite generally any of the determinants $S_{)ij\ldots k(}$ must have the sign $(-)^{n-\nu}$ where ν is the number of subscripts in $S_{)ij\ldots k(}$. This shows that all the terms in the right member of (21.7) have the sign $(-)^n$, in other words, Lemma I is correct also for n.

LEMMA II.—*In the adjoint of the matrix (21.4) all the rows are equal, in other words, if b_{ik} are the elements of the adjoint, b_{ik} is independent of i.*

The element b_{ik} of the adjoint is defined as $(-)^{i+k}$ times the value of the $(n-1)$-rowed determinant obtained by crossing out the row k and the column i in the original matrix (note the interchange of rows and columns when we pass from the original matrix to the adjoint). For $n=3$ we have for instance

$$(21.8)\qquad \begin{aligned} b_{11} &= (a_{22}-a_{20})(a_{33}-a_{30}) - a_{32}a_{23} &&= a_{21}a_{31}+a_{23}a_{31}+a_{21}a_{32} \\ b_{21} &= -(a_{33}-a_{30})a_{21}+a_{31}a_{23} &&= a_{21}a_{31}+a_{23}a_{31}+a_{21}a_{32}. \\ b_{31} &= a_{21}a_{32}-a_{31}(a_{22}-a_{20}) &&= a_{21}a_{31}+a_{23}a_{31}+a_{21}a_{32} \end{aligned}$$

This shows that for $n=3$ b_{i1} is independent of i.

For the general case Lemma II may be proved thus. Consider the determinant that defines b_{ik}. It contains all the columns of (21.4) except the column i. Let j be an arbitrary affix different from i, this means that the column j actually occurs in the determinant considered. This column j consists of the elements $a_{1j}, a_{2j}, \cdots,)$ except $a_{kj}(\cdots a_{nj}$. To this column we add all the other columns. By this operation the column in question becomes $-a_{1i}, -a_{2i}, \cdots)$ except $-a_{ki}(\cdots, -a_{ni}$. The common factor -1 we take outside the determinant. The result of the operation is thus simply that in the column in question the elements have had their second subscript changed from j to i. We now *move* the column in question to the position it ought to have according to its new second subscript. By this movement the determinant is multiplied by $(-)^{i-j+1}$. This shows that the determinant that defines b_{ik} is equal to $(-)^{i-j}$ times the determinant that defines b_{jk}. The magnitude b_{ik} itself will consequently be equal to b_{jk}, in other words, b_{ik} is independent of i.

From the above proof follows that Lemma II applies, not only to matrices of the special type (21.4), *but more generally to all matrices where the sum of the elements in each row is zero.*

LEMMA III.—*All the n^2 elements in the adjoint of the matrix (21.4) have the same sign, namely $(-)^{n-1}$.*

This follows immediately from the Lemmas I and II. Indeed, from Lemma I follows that all the diagonal elements in the adjoint of (21.4) have the sign $(-)^{n-1}$. But on the other hand we know from Lemma II that all the rows of the adjoint are equal; consequently all the elements in any row must have the sign $(-)^{n-1}$. (21.8) furnishes an example for the case $n=3$.

The above propositions can of course easily be formulated in terms of columns instead of rows. Thus, for instance, if from the diagonal elements of the given matrix we subtract the *column* sums a_{0j} (instead of the row sums a_{i0}), the adjoint will have all its *columns* equal.

22. Solution of Certain Singular Systems of Linear Equations

Consider a system of linear equations of the form

$$(22.1)\qquad \sum_{k=1}^{n} a_{ik}\xi_k = b_i \qquad (i = 1, 2 \cdots n)$$

where ξ_k are unknowns to be determined, and a_{ik} and b_i given constants (the a_{ik} are here taken quite generally as certain constants, not necessarily the elements of the request-matrix).

Suppose that the matrix of the coefficients a_{ik} is of rank $n-1$; the necessary and sufficient condition that the system has a solution is the vanishing of all the n determinants obtained by replacing first the first column, then the second column, etc. in the determinant, a_{ik} by the numbers $b_1 \cdots b_n$. If this condition is fulfilled the solution is obtained by leaving out one equation and solving the remaining $(n-1)$ equation on the assumption that one of the quantities ξ is an arbitrary parameter. The solution thus obtained may be written out explicitly in the following form:

$$\hat{a}_{pq}\cdot\xi_i = \hat{a}_{iq}\cdot\xi_p + \sum_{k=1}^{n} \hat{a}_{ik}^{)qp(}\cdot b_k \tag{22.2}$$

where $\hat{a}_{pq}$ are the elements of the adjoint of the given n rowed matrix a_{ik}, and $\hat{a}_{ik}^{)qp(}$ are the elements of the adjoint of the $(n-1)$ rowed matrix obtained by leaving out the row q and the column p from a_{ik}. By convention $\hat{a}_{ik}^{)qp(}$ is interpreted as equal to zero whenever $i=p$ or $k=q$, or both. With this convention the formula (22.2) holds good if i, p and q independently of each are put equal to any of the numbers $1, 2, \cdots, n$.

If a definite p is selected, all the unknowns are by (22.2) expressed in terms of one of them, namely ξ_p, which may be given an arbitrary value. Instead of thus expressing all the unknowns in terms of one of them, it is for many purposes more convenient to express them in terms of some linear combination $\lambda_1\xi_1 + \cdots + \lambda_n\xi_n$ of the unknowns, $\lambda_1 \cdots \lambda_n$ being certain weights. This solution is immediately obtained by multiplying (22.2) with λ_p and performing a summation over p. The advantage of writing the solution in the form (22.2) is just that we may thus transform the solution to the particular form appropriate for any given case. For instance, if we want to express all the unknowns in terms of their sum $X = x_1 + \cdots + x_n$, we get from (22.2)

$$\hat{a}_{0q}\cdot\xi_i = \hat{a}_{iq}X + \sum_{k=0}^{n}\hat{a}_{ik}^{)q0(}\cdot b_k \tag{22.3}$$

where

$$\hat{a}_{0q} = \sum_{k=1}^{n}\hat{a}_{kq} \qquad \hat{a}_{ik}^{)q0(} = \sum_{h=1}^{n}\hat{a}_{ik}^{)qh(}. \tag{22.4}$$

In this formula q is still arbitrary; if we want to, we may make the solution symmetric also in this affix by extending to (22.3) a summation over q; if desired we could also apply a system of weights depending on q.

The above formulae show that the solutions may be expressed as *linear forms* in the quantities that occur in the second member of the given equations, plus a term involving an arbitrary parameter. If the number of variables is great, say 12 or more, the determination of the coefficients in these linear form involves so much computation that the work becomes virtually prohibitive. We need, therefore, some form of approximation method.

In discussing this problem we shall use a notation which is especially appropriate for the application which we want later to make to the request-matrix. Let us consider the system of equations in the form

$$(22.5)\qquad \sum_{k=1}^{n} c_k(A_{ki} - e_{ki}) = s_i.$$

Here c_k are the unknowns to be determined, A_{ki} is any matrix such that $(A_{ki}-e_{ki})$ is of rank $n-1$, and s_i are given constants; e_{ki} are the unit numbers. The difference between the systems (22.1) and (22.5) is thus only formal, (22.5) is just as general as (22.1).

We shall solve (22.5) by an iteration method. This method will lead to a solution that may finally be written out in explicit form by means of a certain matrix whose elements are defined by infinite series.

We first write equation (22.5) in the form

$$(22.6)\qquad \sum_k c_k A_{ki} = c_i + s_i.$$

In this equation we replace for a moment i by x, postmultiply by A_{xi} and perform a summation over x. This gives.

$$\sum_k c_k \sum_x A_{kx} A_{xi} = \sum_x c_x A_{xi} + \sum_x s_x A_{xi}.$$

For the first term in the right member of this equation we insert its expression taken from (22.6), which gives

$$(22.7)\qquad \sum_k c_k \overset{(2)}{A}_{ki} = c_i + \sum_k s_k (e_{ki} + A_{ki}).$$

The numbers

$$\overset{(2)}{A}_{ki} = \sum_x A_{kx} A_{xi}$$

are the elements of the symbolic square of the matrix A_{ki}.

In the equation (22.7) we again write for a moment x instead of i, postmultiply by A_{xi} and utilise the original equation (22.6). This gives

$$\sum_k c_k \overset{(3)}{A}_{ki} = c_i + \sum_k s_k(e_{ki} + A_{ki} + \overset{(2)}{A}_{ki}).$$

The numbers

$$\overset{(3)}{A}_{ki} = \sum_x \overset{(2)}{A}_{kx} A_{xi}$$

are the elements of the symbolic third power of the matrix A_{ki}.

Quite generally we get

$$(22.8)\quad \sum_k c_k A_{ki}^{(N)} = c_i + \sum_k s_k (e_{ki} + A_{ki} + A_{ki}^{(2)} + \cdots + A_{ki}^{(N-1)})$$

where $A_{ki}^{(\nu)}$ are the elements of the ν-th symbolic power of A_{ki}, whose recurrent definition is given by

$$(22.9)\qquad A_{ki}^{(\nu+1)} = \sum_x A_{kx}^{(\nu)} A_{xi}$$

with the initial condition $A_{ki}^{(1)} = A_{ki}$.

Let us assume that the numbers $A_{ki}^{(N)}$ tend towards definite limits as N increases. If this is the case, *these limits for any given k must be proportional to the elements in a row of the adjoint of the matrix in (22.5)*. (Since these rows are themselves proportional, it does not matter which row we consider). Indeed, if $A_{ki}^{(\infty)}$ is the limit of $A_{ki}^{(N)}$ as N increases, we must obviously by (22.9) have

$$A_{ki}^{(\infty)} = \sum_x A_{kx}^{(\infty)} A_{xi}.$$

Hence

$$(22.10)\qquad \sum_x A_{kx}^{(\infty)} (A_{xi} - e_{xi}) = 0 \qquad (i = 1, 2 \cdots n).$$

Let k be a fixed member and consider the n magnitudes $A_{k1}^{(\infty)}, \cdots, A_{kn}^{(\infty)}$. They are solutions of the homogeneous system (22.10) and must hence be proportional to the elements in a row of the adjoint of the matrix $(A_{xi} - e_{xi})$, q.e.d.

From now on we shall make the assumption that the matrix A_{ki} is such that the sum of the elements in each of its rows is equal to unity, in other words

$$(22.11)\qquad A_{k0} = 1.$$

Furthermore, we assume that the sum of the right members is zero, i.e.

$$(22.12)\qquad s_0 = 0.$$

If (22.11) and (22.12) are fulfilled, the system (22.5) certainly has a solution. By complete induction, using (22.9), it is easily seen that all the symbolic powers will also have the property (22.11), i.e.

$$(22.13)\qquad A_{k0}^{(\nu)} = 1 \text{ for any } \nu \text{ and } k.$$

But then the limit $A_{ki}^{(\infty)}$ must be independent of k. Indeed, the numbers $A_{k1}^{(\infty)} \cdots A_{kn}^{(\infty)}$ for two different values of k must be proportional, because both these sets are proportional to the elements in a row of the adjoint of the matrix in (22.5). And by (22.13) the sum of the elements are equal in both sets, namely 1, which is only possible when the elements of the two sets are actually equal. If we let P_i denote the limit towards which $A_{ki}{}^{(N)}$ tends, we consequently have

$$P_i = \text{Lim}\, A_{ki}^{(N)} = \text{independent of } k. \tag{22.14}$$

At the limit the left member of (22.8) can consequently be written CP_i where

$$C = \sum_k c_k = c_1 + \cdots + c_n. \tag{22.15}$$

By (22.11) the sum of the P's is unity, i.e.

$$P_0 = 1. \tag{22.16}$$

If $A_{ki}^{(N)}$ tends towards a limit different from zero, it is clear that the series $e_{ki} + A_{ki} + \cdots + A_{ki}^{(N)}$ in the right member of (22.8) must diverge. This need not mean that the whole expression in the right member diverges. Indeed, for high values of N $A_{ki}^{(N)}$ is nearly independent of k, so that the summation is virtually equal to $P_i(s_1 + \cdots + s_n)$ which by (22.12) is zero. This fact can be utilized to throw the whole expression into a form involving a convergent series. Since the effect of the higher terms are annihilated by the summation over k, because these higher terms become independent of k, it appears that the feature which really determines the total value of the expression is *how the earlier terms deviate from their limiting value.* This suggests to replace the series in the right member of (22.8) by the series

$$(e_{ki} - A_{ki}^{(N)}) + (A_{ki} - A_{ki}^{(N)}) + \cdots + (A_{ki}^{(N-1)} - A_{ki}^{(N)})$$

which may also be written

$$Q_{ki}^{(N)} = 1.D_{ki}^{(1)} + 2.D_{ki}^{(2)} + \cdots + N.D_{ki}^{(N)} \tag{22.17}$$

where

$$D_{ki}^{(\nu)} = A_{ki}^{(\nu)} - A_{ki}^{(\nu-1)} \qquad D_{ki}^{(1)} = A_{ki} - e_{ki}. \tag{22.18}$$

By (22.13) we see that the row sums are zero both in $D_{ki}^{(\nu)}$ and $Q_{ki}^{(N)}$, i.e.

$$D_{k0}^{(\nu)} = 0 \quad Q_{k0}^{(\nu)} = 0 \text{ for any } \nu \text{ and } k. \tag{22.19}$$

Assuming that $Q_{ki}^{(N)}$ converges we put

$$Q_{ki} = \operatorname*{Lim}_{N \to \infty} Q_{ki}^{(N)} \tag{22.20}$$

so that at the limit the equation (22.8) can finally be written

$$c_i = CP_i + \sum_{k=1}^{n} s_k Q_{ki} \tag{22.21}$$

where P_i is defined by (22.14) and Q_{ki} by (22.20); C is an arbitrary parameter that expresses the sum of the c's.

Under assumption of convergency (22.21) furnishes the solution of any system of equations of the form (22.5) where A_{ki} and s_i satisfy (22.11) and (22.12).

23. *Projections in N-Space*

Consider an N dimensional space $(x_1 \cdots x_N)$. The symbol x_K may be looked upon either as the point $(x_1 \cdots x_N)$ or as the vector from origine to this point. The individual numbers $(x_1 \cdots x_N)$ we shall call the *total-space* components of the vector, in order to distinguish from certain other kinds of components later to be considered.

Let L be an N–m dimensional linear subspace through origin of the total space. This means that L is the locus of points x_K that satisfy m equations of the form[4]

$$\sum_K f_{iK} x_K = 0 \qquad (i = 1, 2 \cdots m) \tag{23.1}$$

where f_{iK} is an m-rowed and N-columned matrix of rank m. The equations (23.1) express that any vector belonging to L is orthogonal to all the m vectors f_{iK} $(i = 1, 2 \cdots m)$.

By the classical rules of linear equations there exists a set of $n = N$-m linearly independent vectors g_{jK} $(j = 1, 2 \cdots n)$ belonging to L and such that *any* vector u_K belonging to L can be written as a linear form in the g_{jK}, in other words such that we have

$$u_K = \sum_j \eta_j g_{jK} \qquad (K = 1, 2 \cdots N) \tag{23.2}$$

where the η_j form a set of numbers called the components of u_K in the reference system g_{jK}. The vectors g_{jK} are said to *unfold* L. Since every g_{jK} belongs to L we have the orthogonality relation

$$\sum_K f_{iK} g_{jK} = 0 \quad \left(\begin{array}{c} i = 1, 2 \cdots m \\ j = 1, 2 \cdots n \\ m + n = N \end{array} \right). \tag{23.3}$$

[4] I do not express the following formulae in tensor notation because I shall here make no use of the "up-or-down" character of the affixes, that is of whether the affixes are covariant or contravariant.

In the same way as the g_{jK} unfold L, the f_{iK} unfold a certain linear manifold that is *normal* to L. The vectors g_{jK} may be called the *unfolding* vectors and the f_{iK} the *normality* vectors of L. The matrix g_{jK} may be called the exterior unfolding matrix and f_{jK} the exterior normality matrix L. The expression "exterior" is used to indicate that these matrices depend, not only on the intrinsic properties of the vector bunches g_{jK} and f_{iK} respectively, but also on their position relative to the reference system of the total space.

Consider the square matrices

$$\phi_{ij} = \sum_K f_{iK} f_{jK} \qquad (i, j = 1, 2 \cdots m) \tag{23.4}$$

$$\gamma_{ij} = \sum_K g_{iK} g_{jK} \qquad (i, j = 1, 2 \cdots n) \tag{23.5}$$

ϕ and γ are called the *metric* matrices of L. They are symmetric and may obviously be looked on as moment matrices, hence they are by the Gramian determinant formula positive definite.

Since we assume the matrix $\|f_{iK}\|$ to be of rank m, ϕ_{ij} is non singular, and the same applies to γ_{ij}. We may therefore consider their reciprocals

$$\overset{*}{\phi}_{ij} = \text{the reciprocal of } \phi_{ij} \tag{23.6}$$

$$\overset{*}{\gamma}_{ij} = \text{the reciprocal of } \gamma_{ij} \tag{23.7}$$

By means of the reciprocal metric matrices we form the *reciprocal exterior* matrices

$$\overset{*}{f}_{iK} = \sum_\kappa \overset{*}{\phi}_{i\kappa} f_{\kappa K} \qquad \left(\begin{matrix} i = 1, 2 \cdots m \\ K = 1, 2 \cdots N \end{matrix} \right) \tag{23.8}$$

$$\overset{*}{g}_{jK} = \sum_\kappa \overset{*}{\gamma}_{j\kappa} g_{\kappa K} \qquad \left(\begin{matrix} j = 1, 2 \cdots n \\ K = 1, 2 \cdots N \end{matrix} \right). \tag{23.9}$$

By inverting these equations for a given K we obtain

$$f_{iK} = \sum_\kappa \phi_{i\kappa} \overset{*}{f}_{\kappa K} \tag{23.10}$$

$$g_{jK} = \sum_\kappa \gamma_{j\kappa} \overset{*}{g}_{\kappa K}. \tag{23.11}$$

These relations show that between the exterior matrices and their reciprocals there exist the following relations

$$\sum_K \overset{*}{f}_{iK} f_{jK} = \sum_K f_{iK} \overset{*}{f}_{jK} = e_{ij} \tag{23.12}$$

$$\sum_K \overset{*}{g}_{iK} g_{jK} = \sum_K g_{iK} \overset{*}{g}_{jK} = e_{ij} \tag{23.13}$$

where e_{ij} are the unit numbers $=0$ or 1 accordingly as $i\neq j$ or $i=j$.

Now let us consider the set of $N=m+n$ vectors $f_{1K}\cdots f_{mK}$ $g_{1K}\cdots g_{nK}$. They form a set of linearly independent vectors. Indeed the Gramian of this total set is by (23.3).

$$\begin{vmatrix} \phi_{11} & \cdots & \phi_{1m} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \phi_{m1} & \cdots & \phi_{mm} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \gamma_{11} & \cdots & \gamma_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & 0 & \gamma_{1n} & \cdots & \gamma_{nn} \end{vmatrix} = |\phi_{ij}|\cdot|\gamma_{ij}| \mp 0 \tag{23.14}$$

Therefore *any* vector x_K in total space can be written in the form

$$x_K = \sum_{i=1}^{m}\xi_i f_{iK} + \sum_{j=1}^{n}\eta_j g_{jkK}. \tag{23.15}$$

The numbers ξ_i are the components of x_K in the manifold that is normal to L, and the numbers η_j are the components of x_K in L itself.

Between the total space components x_K and the components ξ_i and η_j there exists a unique correspondence. By (23.15) x_K is determined in terms of the ξ_i and η_j. And multiplying (23.15) by $f^*_{\kappa K}$ and $g^*_{\delta K}$ respectively and performing a summation over K we get

$$\xi_i = \sum_K \overset{*}{f}_{iK} x_K \qquad (i=1,2\cdots m) \tag{23.16}$$

$$\eta_j = \sum_K \overset{*}{g}_{jK} x_K \qquad (j=1,2\cdots n). \tag{23.17}$$

By means of the exterior matrices and their reciprocals we finally form the *projection* matrices

$$F_{HK} = \sum_i f_{iH}\overset{*}{f}_{iK} = \sum_i \overset{*}{f}_{iH} f_{iK} \tag{23.18}$$

$$G_{HK} = \sum_j g_{jH}\overset{*}{g}_{jK} = \sum_j \overset{*}{g}_{jH} g_{jK}. \tag{23.19}$$

The expression for these matrices may also be written

$$F_{HK} = \sum_{ij}\phi_{ij}\overset{*}{f}_{iH}\overset{*}{f}_{jK} = \sum_{ij}\overset{*}{\phi}_{ij} f_{iH} f_{jK} \tag{23.20}$$

$$G_{HK} = \sum_i \gamma_{ij}\overset{*}{g}_{iH}\overset{*}{g}_{jK} = \sum_i \overset{*}{\gamma}_{ij} g_{iH} g_{jK} \tag{23.21}$$

Between the two projection matrices there exists the fundamental relation

$$F_{HK} + G_{HK} = e_{HK}. \tag{23.22}$$

To prove this consider the N dimensional linear form $\sum_K F_{HK} x_K$. Inserting here the expression for x_K taken from (23.15) ,and the expression for F_{HK} taken from the second and third member respectively in (23.20) we get

$$\sum_\kappa \xi_\kappa \sum_{ijK} \phi_{ij} \overset{*}{f}_{iH} \overset{*}{f}_{jK} f_{\kappa K} + \sum_\delta \eta_\delta \sum_{ijK} \overset{*}{\phi}_{ij} f_{iH} f_{jK} g_{\delta K}.$$

By the summation over K the last term in this expression vanishes on account of (23.3), and the first term reduces to

$$\sum_{ij} \xi_i \phi_{ij} \overset{*}{f}_{jH}.$$

If this expression is multiplied by x_H and summed over H we get by again using (23.15)

$$\sum_{HK} F_{HK} x_H x_K = \sum_{ij} \xi_i \phi_{ij} \sum_H \overset{*}{f}_{jH} \left\{ \sum_\kappa \xi_\kappa f_{\kappa H} + \sum_\delta \eta_\delta g_{\delta H} \right\}.$$

By the summation over H the second term vanishes and the first reduces so that we finally get

$$\sum_{HK} F_{HK} x_H x_K = \sum_{ij} \phi_{ij} \xi_i \xi_j \tag{23.23}$$

similarly we get

$$\sum_{HK} F_{HK} x_H x_K = \sum_{ij} \gamma_{ij} \eta_i \eta_j \tag{23.24}$$

so that

$$\sum_{HK} (F_{HK} + G_{HK}) x_H x_K = \sum_{ij} \phi_{ij} \xi_i \xi_j + \sum_{ij} \gamma_{ij} \eta_i \eta_j. \tag{23.25}$$

This being so, let us consider the N dimensional quadratic form $\sum_{HK} e_{HK} x_H x_K$. Inserting here for x_H and x_K we find easily that the form reduces to the same expression as the one we have in the second member of (23.25). Hence we must have for *any* choice of the x's

$$\sum_{HK} (F_{HK} + G_{HK}) x_H x_K = \sum_{HK} e_{HK} x_H x_K \tag{23.26}$$

which is only possible when (23.22) is fulfilled.

By means of the projection matrices and the fundamental formula (23.22) it is now easy to express the coordinates x_K' of the point obtained by taking an arbitrary point x_K and *projecting it orthogonally to* L. Indeed this projection is defined as the point whose L components are the *same* as the L components of x_K, but whose components in the manifold normal to L *are all zero*. The ξ and η components of the projection considered are thus immediately given by the very definition of the projection. The thing that interests us is, however, how the total-space coordinates of the projection can be expressed directly in terms of the total-space coordinates x_K. This is obtained by expressing the total-space coordinates of x_K' by means of its ξ and η coordintes using (23.15), then putting here the ξ coordinates equal to zero and the η coordinates equal to those of x_K, and finally by (23.17) to express the η coordinates of x_K by means of its total space coordinates. Doing this we find

$$x_H' = \sum_K G_{HK} x_K. \tag{23.27}$$

By means of the fundamental relation (23.22) this is equivalent to

$$x_H' = \sum_K (e_{HK} - F_{HK}) x_K \tag{23.28}$$

which can also be written

$$x_H' = x_H - \sum_K F_{HK} x_K. \tag{23.29}$$

This explains the name projection matrices given to F_{HK} and G_{HK}.

As an example consider the case $m=1$, that is $n=N-1$. In this case L is simply a plane defined by an equation of the form

$$f_1 x_1 + \cdots + f_N x_N = 0. \tag{23.30}$$

We now get

$$\phi_{11} = f_1^2 + \cdots + f_N^2$$

$$\overset{*}{\phi}_{11} = \frac{1}{\phi_{11}}$$

$$\overset{*}{f}_K = \frac{f_K}{f_1^2 + \cdots + f_N^2}$$

$$F_{HK} = \frac{f_H}{\sqrt{f_1^2 + \cdots + f_N^2}} \cdot \frac{f_K}{\sqrt{f_1^2 + \cdots + f_N^2}}$$

and therefore

$$x_H' = x_H - \frac{f_H}{f_1^2 + \cdots + f_N^2} \sum_K f_K x_K. \tag{23.31}$$

Finally let us determine the projection of x_K, not on L, but on the manifold M that is defined by the *inhomogeneous* equations

$$\sum_K f_{iK} x_K = u_i \qquad (i = 1, 2 \cdots m) \tag{23.32}$$

u_i being a set of given numbers.

The two manifolds L and M are parallel because they have the same normality manifold, namely the one unfolded by the vectors f_{iK} $(i=1,2 \cdots m)$. If x_K'' is the projection of x_K on M and x_K' its projection on L, x_K'' will consequently have the same η coordinates as x_K', that is, as x_K itself. And the ξ coordinates of x_K'' must obviously be just large enough to satisfy the equations (23.22). In other words we must have

$$\sum_K f_{iK}\left\{ \sum_\kappa \xi_\kappa'' f_{\kappa K} + \sum_\delta \eta_\delta g_{\delta K} \right\} = u_i$$

where ξ_κ'' are the ξ components of x_K''. The summation over K in the last of the above terms give 0, so that the equation reduces to

$$\sum_j \phi_{ij}\xi_j'' = u_i \tag{23.33}$$

hence

$$\xi_i'' = \sum_j \overset{*}{\phi}_{ij} u_j . \tag{23.34}$$

In the expression for the total-space components of x_K'', namely

$$x_H'' = \sum_i \xi_i'' f_{iH} + \sum_j \eta_j g_{jH}$$

we notice that the last term just gives the previously calculated coordinate x_K', i.e. the expression (23.28). And the first term in the above expression reduces by (23.34) to

$$\sum_{ij} u_i \overset{*}{\phi}_{ij} f_{jH} = \sum_i u_i \overset{*}{f}_{iH} .$$

Hence: *If we take an arbitrary point* x_K *and project it on to the linear manifold* (23.32) *we get a point* x_K'' *whose total-space coordinates are*

$$x_H'' = \sum_{i=1}^{m} u_i \overset{*}{f}_{iH} + \sum_{K=1}^{N} (e_{HK} - F_{HK}) x_K \tag{23.35}$$

where $\overset{*}{f}_{iH}$ is the reciprocal exterior matrix and F_{HK} the projection matrix defined above.

In the special case $m=1$ we get

$$x_H'' = x_H + \frac{f_H}{f_1^2 + \cdots + f_N^2}\left(u - \sum_K f_K x_K\right) \tag{23.36}$$

where u is the constant in the second member of the equation defining the plane on to which the projection is made.

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