CIRCULATION PLANNING: PROPOSAL FOR A NATIONAL ORGANIZATION OF A COMMODITY AND SERVICE EXCHANGE*

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CONTENTS

* This paper will be supplemented with a mathematical appendix which will appear in the early number of ECONOMETRICA.

THE PROBLEM

THE most striking paradox of great depressions, and particularly of the present one, is the fact that poverty is imposed on us in the midst of a world of plenty. Many kinds of goods are actually present in large quantities, and other kinds could without any difficulty be brought forth in abundance, if only the available enormous productive power was let loose. Yet, in spite of this technical and physical abundance, most of us are forced to cut down consumption. We are compelled to make real sacrifices in order to economize in the use of these very goods and services that *could* easily be produced in abundance if we would only use our resources.

A full recognition of the monstrosity of this situation is the first and basic condition for any intelligent discussion of ways and means to get out of the depression. Of course this implies the conclusion that the cause of great depressions, such as the one we are actually in, is in some way or another connected with the present *form of organization* of industry and trade. The depression is not a real poverty crisis, not due to an actual shortage of real values. This must be admitted by everybody, quite regardless of political color.

One very important aspect of the disastrous effects produced during great depressions by the present organization of industry and trade is what might be called the *incapsulating phenomenon*. It manifests itself in various forms and in different economic fields, but its nature and deplorable consequences are always the same. It is fundamentally connected with the fact that modern economic life has been divided into a number of *regions* or *groups*.

Under the present system, the blind "economic laws" will, under certain circumstances, create a situation where these groups are forced *mutually to undermine each other's position*. Each group is forced to curtail the use of the goods produced and services rendered by the other groups, which, in turn, will cause a still further contraction of the demand for its own products, and so on. This meaningless vicious circle is what I understand by the incapsulating phenomenon.

One of the most spectacular examples of this phenomenon is furnished by the protectionist tendencies in international trade relations during a great depression period.

However, it is not only in the international field that such a tendency is at work. *Its effects within the borders of each nation are probably even greater*. It produces a far-reaching stagnation of the national circulation. The bootmaker finds that he cannot afford to buy a new suit because he is unable to sell any shoes, and at the same time the tailor finds that he cannot afford to buy shoes because he is unable to sell any suits. If the effect of this national incapsulating phenomenon could be calculated in money, it would amount to an enormous sum. It concerns, indeed, all of us, in nearly every phase of our daily life.

The reason for this national circulation stagnation is not, of course, that the parties involved have no desire for the goods and services offered. On the contrary, the desire is in most cases very great. But the reason is that the parties are *unable* to buy or that they do not *dare* to do so. They lack the means or the courage, mainly because *the others* lack the means or the courage.

Unfortunately, the importance of these incapsulating tendencies within the national border is much less clearly recognized by the general public—and even by many professed economists—than the corresponding phenomena in the international field. The reason for this lack of attention to the national aspect of the problem is perhaps that the decisions and reactions by which the innumerably smaller or larger groups in a country yield to the pressure of the forces that tend to contract circulation are less spectacular, and have much less chance of being utilized for sensational front page headlines, than the decisions and happenings in the field of international trade relations. And, as is well known, the attention of the general public is not proportional to the real importance of the phenomena but to the space devoted to them in the newspapers.

Within the national border, the possibilities of doing something constructive to combat the devastating effects of the circulation stagnation are much greater than internationally. Experience has amply proved that very little can at present be obtained by international agreement. Of course this in itself puts certain limits to the possible solution of the circulation problem. A *complete* solution would necessitate international co-operation. But this fact must not make us overlook that great improvements can be achieved by eliminating at least that part of the circulation stagnation which is national. At the moment it seems more realistic to work for this partial solution. There is all the more reason for this attitude as the elimination of the national circulation stagnation will not work counter to a final international solution, but rather prepare the way for it. In the present paper I shall be concerned primarily with the national aspect of the problem.

In the attempt at introducing a certain measure of planning in exchange activity, one is confronted with an optimum problem of great complexity. The prime object of the planning must be to utilize more fully the existing productive capacity, but at the same time the activity must be guided in such a way as to conform as much as possible with the particular desires of the individuals and groups involved. These two objects are in many cases mutually exclusive. The weighing of their relative importance is precisely the fundamental problem of the planning.

Certain aspects of this optimum problem are of a theoretical nature. We are indeed here confronted with problems very similar to those we meet in productivity theory, only the corresponding problems in circulation planning are more far-reaching and complicated. Furthermore, most of these problems are quantitative: *how much* should this activity be increased and that other curtailed? All of which means that these problems are essentially *econometric.*

The mathematical aspect of these problems, and the nature of the computation technique involved, is in certain respects similar to that used in actuarial science, but much broader in scope, because the actuaries of national planning must utilize, to a considerable extent, also that type of analysis that is developed in modern economic theory.

Parts I and III of the present paper attempt to make some contribution towards the analysis of these mixed problems where technical-computational and economic-theoretical considerations must concur in determining the final solution. In Part II, I shall make some remarks on the practical aspect of the circulation scheme proposed.

PART I. THEORY

1. Intrinsic Tendencies towards Contraction and Expansion in an Exchange System

I shall first give an analysis of a simplified theoretical system showing the intrinsic tendency to contraction or expansion that may exist in an economy based on *exchange* of commodities and services. This analysis will exhibit the nature of the maladjustment which it is the object of the circulation planning to overcome.

Suppose for simplicity that there are only two persons (or groups), say a shoemaker and a farmer. Let us imagine that they exchange their products at regular intervals, say at fairs held one or twice a year, as was commonly the case in the old days. This assumption is only made in order to simplify the exposition; the essential features of the development will be the same if the exchange activity takes place in a more continuous fashion. For simplicity we shall further assume that the prices are constant. The quantities sold or bought can therefore simply be measures in dollars' worth.

It seems reasonable to assume that the shoemaker will at any given moment buy all the more of the product of the other person, the more he has himself been able to sell at the moment immediately preceding.

And similarly for the farmer. To construct a simple example, we may, for instance, assume that the amount bought by the shoemaker is *proportional* to the amount he sold at the previous moment, and similarly for the farmer. That is, we put

$$(1.1)\qquad \begin{aligned} a_t &= \alpha b_{t-1} \\ b_t &= \beta a_{t-1} \end{aligned}$$

where a_t and b_t are the amounts bought by the shoemaker and the farmer respectively; and α and β are constants; α and β are a kind of "coefficient of optimism" expressing how the previous sales influence each man's further policy.

If the exchange starts at the point of time $t=0$ where a (the shoemaker) buys a_0 and b (the farmer) buys b_0, the further development of the exchange activity will be as follows:

$$\begin{aligned} a_1 &= \alpha b_0 \\ b_1 &= \beta a_0 \\ \hline a_2 &= \alpha\beta a_0 \\ b_2 &= \alpha\beta b_0 \\ \hline a_3 &= \alpha\cdot\alpha\beta b_0 \\ b_3 &= \beta\cdot\alpha\beta a_0 \\ \hline &\text{etc.} \end{aligned}$$

Quite generally, we get

$$(1.2)\qquad \begin{array}{cc} a_t = a_0\gamma^t & a_t = \alpha b_0\gamma^{t-1} \\ b_t = b_0\gamma^t & b_t = \beta a_0\gamma^{t-1} \\ \text{(when } t \text{ is even)} & \text{(when } t \text{ is odd)} \end{array}$$

where

$$(1.3)\qquad \gamma = \sqrt{\alpha\beta}$$

In a single formula this may be written

$$(1.4)\qquad \begin{aligned} a_t &= (A_1 + (-)^t A_2)\gamma^t \\ b_t &= (B_1 + (-)^t B_2)\gamma^t \end{aligned}$$

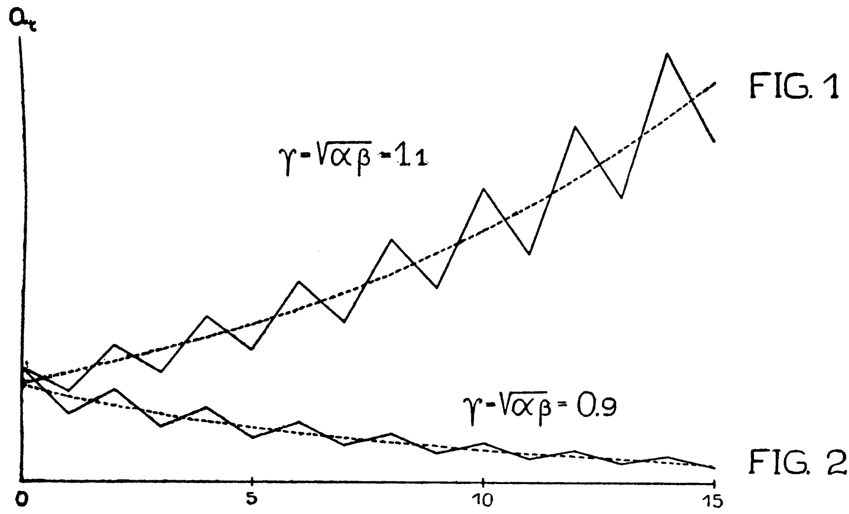
where

$$(1.5)\qquad \begin{aligned} A_1 &= 1 + \frac{1}{2}\left(\sqrt{\frac{\alpha}{\beta}\cdot\frac{b_0}{a_0}} - 1\right) \quad A_2 = -\frac{1}{2}\left(\sqrt{\frac{\alpha}{\beta}\cdot\frac{b_0}{a_0}} - 1\right) \\ B_1 &= 1 + \frac{1}{2}\left(\sqrt{\frac{\beta}{\alpha}\cdot\frac{a_0}{b_0}} - 1\right) \quad B_2 = -\frac{1}{2}\left(\sqrt{\frac{\beta}{\alpha}\cdot\frac{a_0}{b_0}} - 1\right). \end{aligned}$$

The formula (1.4) shows that the development of, say, a_t is described by a time series that has two components: The main component is simply the exponential trend $A_1\gamma^t$, and the second component is a cycle with a period of two time units, in other words, it alternates between a maximum at one moment and a minimum at the other.

The exponential trend that forms the main component will be increasing or decreasing accordingly as $\gamma > 1$ or $\gamma < 1$. But γ as defined by (1.3) can simply be looked upon as an *average* coefficient of optimism. Thus we may say that if the two parties are on the average in the "spending mood," meaning by this that $\gamma > 1$, then the system will expand, but if they are on the average in the "saving mood," i.e., if $\gamma < 1$ *the whole system will gradually dwindle down to nothing.*

The amplitude of the *short* cycle is always proportional to the trend ordinate. This short cycle is simply a reflex of the fact that the exchange started in a way that was not in harmony with the tendency to



FIGURES 1 and 2

further development represented by the coefficients of optimism. More precisely, this maladjustment in the initial conditions can be expressed by saying that

$$\frac{a_0}{\sqrt{\alpha}} \neq \frac{b_0}{\sqrt{\beta}}\,. \tag{1.6}$$

This is obviously the condition that the terms A_2 and B_2 shall be present in (1.3). The short cycle here considered is only an incidental, secondary feature of the development.

Figures 1 and 2 indicate the time shape of the solutions that will follow from two alternative sets of values of the structural constants α and β and of the initial conditions a_0 and b_0.

In one respect the above scheme is too simple: If the parties are in the deflationary mood, expressed by the fact that the average coefficient of optimism γ is less than unity, it is not likely that the ensuing downward movement (as exhibited in Figure 1) will continue straight down to zero. Sooner or later another tendency will come into play: The parties will have a certain minimum of existence and will therefore be forced to *modify* their policy of curtailing purchases in strict proportion to the decrease in their own sales. They will reach a point where they simply *cannot* restrict purchases any more, no matter what the further consequences may be. To express this, we may, for instance, describe their policy by equations of the form

$$
\begin{aligned}
a_t &= a_* + \alpha b_{t-1} \\
b_t &= b_* + \beta a_{t-1}
\end{aligned}
\tag{1.7}
$$

where a_* and b_* are constants depending on the minimum of existence. This leads to

$$
\begin{aligned}
a_t &= a_0\gamma^t + \frac{a_* + \alpha b_*}{1 - \gamma^2}(1 - \gamma^t) \\
b_t &= b_0\gamma^t + \frac{b_* + \beta a_*}{1 - \gamma^2}(1 - \gamma^t)
\end{aligned}
\tag{1.8}
$$

when t is even, and

$$
\begin{aligned}
a_t &= (a_* + \alpha b_0)\gamma^{t-1} + \frac{a_* + \alpha b_*}{1 - \gamma^2}(1 - \gamma^{t-1}) \\
b_t &= (b_* + \beta a_0)\gamma^{t-1} + \frac{b_* + \beta a_*}{1 - \gamma^2}(1 - \gamma^{t-1})
\end{aligned}
\tag{1.9}
$$

when t is odd.

In this case the contraction of the exchange, which takes place if $\gamma < 1$, will not completely eliminate all activity, but will bring it down to the stationary levels.

$$
a_\infty = \frac{a_* + \alpha b_*}{1 - \gamma^2}
\tag{1.10}
$$

$$
b_\infty = \frac{b_* + \beta a_*}{1 - \gamma^2}\,.
\tag{1.11}
$$

2. The Indebtedness Factor in Exchange Contraction and Expansion

So far nothing has been said about the monetary and credit relations between the two parties. The preceding analysis assumes of course that some form of credit is used, otherwise we would always have $a_t = b_t$ so that the theoretical set-up would be reduced to a perfectly static one. The simplest assumption about the credit transaction is that a owes to b, or vice versa, the total difference between the purchases and sales up to the instance considered. It is easy to see that, in the contraction case illustrated in Figure 1, this total indebtedness will converge towards a definite magnitude, while in the expansion case it will increase towards infinity.

The latter consequence shows that the theoretical scheme of Section 1 is too simple to describe the long-time effects that take place in the system. In order to make the scheme more realistic, we ought to allow the magnitude of the existing debt to exert an influence on the buying policy of the parties at a given instant. Professor Irving Fisher has recently drawn attention to the rôle played by debts in the dynamics of the business cycle.[1] He emphasizes in particular the fact that attempts at paying off debts during a depression period will generally *increase* the real debt burden, because the paying off of nominal debts will, through its deflationary effect, increase the value ot the monetary unit more rapidly than the nominal debts are brought down. Undoubtedly this is a very important aspect of the problem. In the present connection we shall, however, not follow up particularly the effect of changing prices, but rather look into the effect on the buying activity that will in general be produced by the indebtedness factor quite irrespective of any price changes. These effects are of particular importance for our problem. They will be *magnified* by the fact pointed out by Professor Fisher.

Let us again study the evolution that will take place in a certain simplified system. Let us assume that the buying policies of the parties are defined by equations of the form

$$
\begin{aligned}
a_t &= \alpha b_{t-1} - \lambda G_{t-1} \\
b_t &= \beta a_{t-1} + \mu G_{t-1}
\end{aligned}
\tag{2.1}
$$

where α, β, λ and μ are non-negative constants, and

$$
G_t = (a_0 + a_1 + \cdots + a_t) - (b_0 + b_1 + \cdots + b_t) \tag{2.2}
$$

is the total sum which a owes to b. In this case an uninterrupted long-time increase in the volume of transactions is not likely to take place.

[1] See his book *Booms and Depressions*, New York, 1933, and his paper in ECONOMETRICA, October 1933.

Indeed, any difference between the two parties' ways of acting that results in an increasing indebtedness from one to the other will increase the second terms in (2.1) and this will probably after a while put a brake on further expansion. A full understanding of the forces at work in this case, and of the time shape of the resulting curves, cannot, however, be obtained by a verbal discussion; we have to analyse the situation with the appropriate mathematical tools.

We first notice that it is possible to deduce from the equations (2.1) a certain higher order difference-equation in the single variable a_t. Writing down the equations (2.1) both for t and $t-1$, and subtracting, we get

$$
\begin{aligned}
a_t - a_{t-1} &= \alpha(b_{t-1} - b_{t-2}) - \lambda(a_{t-1} - b_{t-1}) \\
b_t - b_{t-1} &= \beta(a_{t-1} - a_{t-2}) + \mu(a_{t-1} - b_{t-1}).
\end{aligned}
$$

Replacing t by $t+1$ in the first of these equations and bringing the terms with b over to the left side, we get

$$
\begin{aligned}
(\alpha + \lambda)b_t - \alpha b_{t-1} &= a_{t+1} - (1 - \lambda)a_t \\
b_t - (1 - \mu)b_{t-1} &= (\beta + \mu)a_{t-1} - \beta a_{t-2}.
\end{aligned}
$$

If the determinant

$$
\begin{vmatrix} \alpha + \lambda & \alpha \\ 1 & 1 - \mu \end{vmatrix}
$$

is $\neq 0$, this system may be solved with respect to b_t and b_{t-1}. Both in the expression thus obtained for b_t and in that for b_{t-1} there will occur certain a's. Consequently, if in the subscripts on the a's in the latter expression we replace $t-1$ by t, we ought to obtain the former expression. This entails a certain consequence on the a's. The condition turns out to be that a_t must satisfy the fourth order difference equation

$$
\begin{aligned}
(2.3)\quad a_{t+4} - (2 - \lambda - \mu)a_{t+3} - \big[(\alpha + \lambda)(\beta + \mu) - (1 - \lambda)(1 - \mu)\big]a_{t+2} \\
+ \big[(\alpha + \lambda)\beta + \alpha(\beta + \mu)\big]a_{t+1} - \alpha\beta a_t = 0.
\end{aligned}
$$

Since the coefficients of this equation do not change if the symbols α and β, λ and μ are interchanged, we conclude that b_t must satisfy the same equation.

As is well known from the theory of difference equations, the time shape of the solution a_t of an equation of the form (2.3) depends on the roots of the corresponding characteristic equation, i.e., the algebraic equation obtained by replacing a_{t+4}, a_{t+3}, etc., by z^4, $z^3 \cdots$ etc., z being the unknown to be determined from the equation. It is easily seen that $z=1$ is a root of the equation thus obtained. Dividing out by $(z-1)$, we thus find that the characteristic equation may be written

$$
(2.4)\qquad z^3 + c_1 z^2 + c_2 z + c_3 = 0
$$

where

$$
(2.5)\qquad \begin{aligned} c_1 &= -\ (1 - \lambda - \mu) \\ c_2 &= -\ (\lambda\beta + \alpha\mu + \alpha\beta) \\ c_3 &= \alpha\beta . \end{aligned}
$$

The root $z = 1$ means that a_t will contain as one of its components a *constant* $\bar{a}$. The nature of the other components is determined by the roots of (2.4).

Before we proceed to a discussion of these in the general case, let us apply the equation to the special case where the buying policy is given by (1.1). In this case we have $\lambda = \mu = 0$, which by (2.4) gives the characteristic equation.

$$
z^3 - z^2 - \alpha\beta z + \alpha\beta = 0
$$

that is

$$
(2.6)\qquad (z - 1)(z^2 - \alpha\beta) = 0.
$$

Thus in this case one more root becomes equal to 1, and the two other roots are $z = +\gamma$ and $z = -\gamma$, γ denoting as before the constant $\sqrt{\alpha\beta}$. This shows that, apart from a linear term in t (which here vanishes since by the initial conditions both its constants are determined to be zero), the time function a_t will contain the two components γ^t and $(-\gamma)^t$ each multiplied by a constant to be determined by the initial conditions. This leads to the solution (1.4) which we originally found by a direct elementary method.

Now as to the general equation (2.4). Its discriminant is

$$
(2.7)\qquad D = c_1^2 c_2^2 + 18 c_1 c_2 c_3 - 4 c_2^3 - 4 c_1^3 c_3 - 27 c_3^2 .
$$

The time shape of the solution will depend primarily on the sign of this discriminant; if it is positive, the characteristic equation will have three real roots and the system consequently evolve exponentially. But if the discriminant is negative, cycles will occur. The accompaning table indicates the value of D for $\alpha = 0.9$, $\beta = 0.8$, $\mu = 0.1$ and different values of λ

(2.8)

	Discriminant D
$\lambda = 0.2$	$-\ 0.097$
0.3	$-\ 0.182$
0.4	$-\ 0.224$
0.6	$-\ 0.167$
0.7	$-\ 0.062$
0.8	$+\ 0.127$
0.9	$+\ 0.330.$

This shows that both cyclical and exponential solutions are possible within the range of values which it is plausible to attribute to the structural parameter α, β, λ and μ.

If the discriminant is negative, that is, if the equation (2.4) has one real and two conjugate imaginary roots, the real root must be negative. This follows from the fact that in our case, we must always have $c_3>0$ since both α and β must be assumed positive.

Let $z=-\kappa$ be the negative root, and let $z=e^{-\theta}(\cos\omega\pm i\sin\omega)$ be the two conjugate imaginary roots. In other words, $-\theta$ is the natural logarithm of the modulus of the complex roots, and ω their phase angles in the Gaussian plane. From the classical theory of linear difference equations, it follows that the time shape of a_t and b_t will then be of the form.

$$a_t = \bar{a} + e^{-\theta t}(A_1 \sin\omega t + A_2\cos\omega t) + (-)^t A_3\kappa^t \tag{2.9}$$

$$b_t = \bar{b} + e^{-\theta t}(B_1 \sin\omega t + B_2\cos\omega t) + (-)^t B_3\kappa^t \tag{2.10}$$

where $\bar{a}$ and $\bar{b}$ as well as the A's and B's are constants to be determined by the initial conditions.

The last terms in (2.9) and (2.10) represent a short cycle which alternates between a maximum at one moment and a minimum at the next. This oscillation is of the same nature as the short cycle we discussed in the simple case (1.1). It is not essential in the present connection.

The first terms in (2.9) and (2.10) represent the constant levels around which the fluctuations take place; these terms need not occupy us very much either.

The interesting feature of the movement which the present theoretical set-up will give rise to is the cyclical movement represented by the middle terms of (2.9) and (2.10).

In all the cases listed in (2.8) the *damping* of this cycle is very heavy, it is indeed so heavy that if the curve is drawn on a usual scale no marked cycles will appear. Is this a general property necessarily connected with such values of the structural parameters as are plausible in view of the concrete meaning of the parameter? To investigate this question we may first notice that it would seem quite unreasonable to assume that the behavior of the parties is such that any of the structural parameters are negative. Consider then for instance the condition $\mu\geqq 0$; we find that this condition entails[2]

$$\cos\omega \geqq \frac{\alpha+\kappa}{\alpha+e^{-2}}\cdot\frac{1+\kappa}{2\kappa}e^{-\theta}. \tag{2.11}$$

[2] These limits as well as the numerical solutions of the characteristic equation were worked out by Mr. Harald Holme, assistant in my laboratory in the University Institute of Economics, Oslo.

Putting as a limiting case $\mu=0$, and utilizing the further condition $\beta \geqq 0$ we get

$$\kappa(2 \cos \omega - e^{-\theta}) \geqq e^{-\theta}. \tag{2.12}$$

Assuming here κ to be unity (which means that the secondary feature of the development, namely the *short* cycle, shall neither dampen away nor "explode," but just go on with the same intensity) the inequality (2.12) reduces to

$$\cos \omega \geqq e^{-\theta}. \tag{2.13}$$

By means of these inequalities it is easy to find examples of sets of values of the structural parameters leading to cycles *that are not heavily damped.* In Table II are given some such examples.

TABLE I

Example No.	If the structural parameters are				The time shape characteristics will be		
	α	β	λ	μ	Length of cycle $p=2\pi/\omega$	Damping factor pr. cycle $e^{-\theta p}$	Real root $-\kappa$
1	2.00	0.50	0.56	0.04	20	0.10	−1.3
2	1.50	0.584	0.274	0.0262	30	0.05	−1.07
3	2.00	0.376	0.28	0.013	22	0.05	−0.98
4	2.00	0.36	0.25	0.05	30	0.01	−0.95
5	3.00	0.29	0.29	0.003	30	0.05	−0.95
6	2.048	0.433	0.158	0	30	0.165	−1.0
7	6.428	0.1465	0.102	0	30	0.407	−1.0
8	2.734	0.364	0.0088	0	120	0.698	−1.0
9	1.350	0.739	0.0023	0	360	0.698	−1.0

Consider example No. 5. Here *a* (the shoemaker) is in a very expansive mood ($\alpha=3.0$), while *b* (the farmer) is very much for saving ($\beta=0.29$); at the same time *a* is rather sensitive to changes in his debt situation ($\lambda=0.29$), while *b* is not ($\mu=0.003$). The result is a cycle that has a good amount of damping, the damping is, however, far less extreme than in the examples of (2.8). After the lapse of a length of time equal to one cycle (30 "years") there is still 5 per cent of the amplitude left.

In the next example, No. 6, there is less difference between the "expansiveness" of the two parties, and they are both less sensitive to changes in their debt situation. This turns out to produce a situation where the damping is less heavy; after one cycle there is now left 16.5 per cent of the amplitude. In example No. 9, this tendency is still more pronounced. In this example the difference in the expansiveness of the parties is not great ($\alpha = 1.35$, $\beta = 0.739$) and none of them is very sensitive to changes in the debt situation. After all, the last example seems to be the most plausible. *And in this case there appears a cycle that is very lightly damped,* in the course of each cycle the curve only loses about 30 per cent of its amplitude, so that the swings will continue for a considerable span of time. The development of the system in this case is indicated in Figure 3.

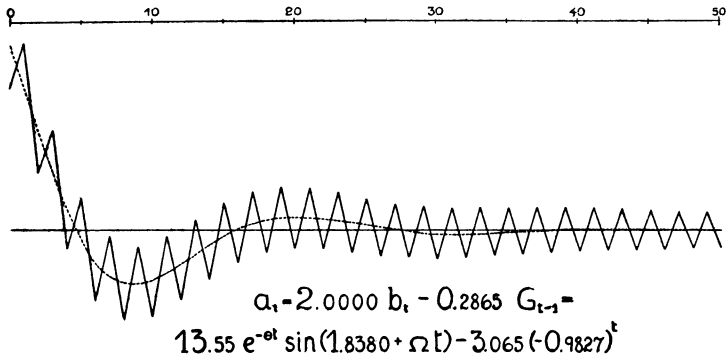


FIG 1

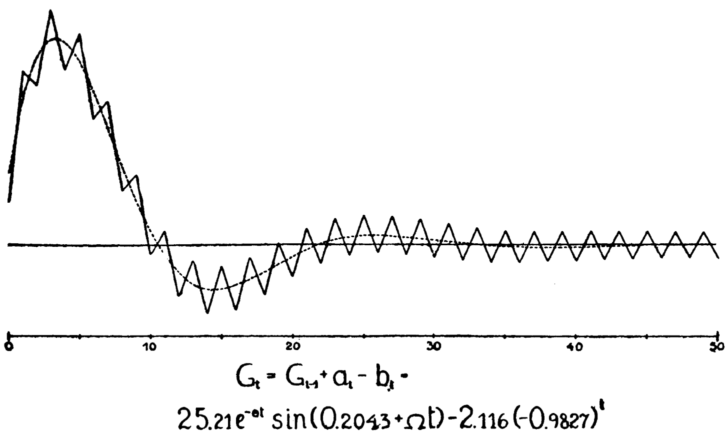


FIG 2

Figures 1 and 2 represent examples Nos. 2 and 3 respectively. A comparison between these two figures exhibits the rôle played by the real root: In Figure 1, where κ is larger than unity, the short fluctuations are increasing, while in Figure 2, where κ is less than unity, they dampen out.

In examples Nos. 8 and 9 the periods appear much longer than in the other examples. Since the unit of measurement for time in the examples is arbitrary, the long periods in the examples in question can simply be looked upon as expressing the fact that the exchange acts here take place *more frequently*. Also in this respect are these latter examples more realistic than the others.

All the cycles here obtained dampen out if we consider a sufficiently long span of time, but as I have pointed out elsewhere,[3] when a system obeying such dynamic laws is hit by erratic impulses of various sorts, for instance accidental disturbances which cause the parties in-

$$a_t = 1.3503\, b_{t-1} - 0.0023\, G_{t-1} - 9.7308\, e^{-\alpha t} \sin(1.5891 + \Omega t) + 0.27086\,(-1)^t$$

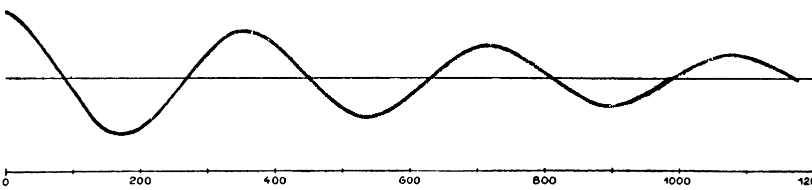


FIGURE 3

volved to *deviate* every so often a little from the underlying dynamic laws of the system, then a *maintained* oscillation will be created whose period is somewhat changing but on the average equal to the period of the original undisturbed solution. Figure 4 indicates the result obtained if the system in example No. 9, is maintained by a stream of erratic shocks. The "shocks" by means of which Figure 4 was constructed were taken from end-digits in the Norwegian State lottery.

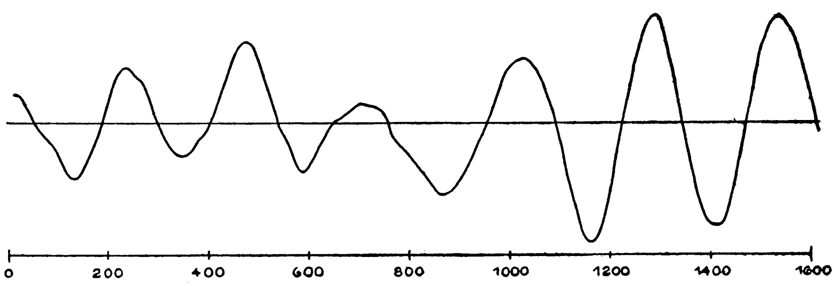


FIGURE 4

This analysis leading up to the result exhibited in Figure 4, shows that violent depressions may be caused by the mere fact that the parties involved have a certain typical *behavior* in regard to buying and loaning. These cycles are pure circulation-cycles. They have nothing

[3] "Propagation Problems and Impulse Problems in Dynamic Economics," in the volume published in honor of Gustav Cassel, London, 1933. A more extensive analysis is given in my forthcoming book *Changing Harmonics* to be published as a monograph of the Cowles Commission for Research in Economics.

to do with the various technical features of the production process, or with real investment activity. Indeed, no such considerations have been introduced in the preceding analysis.

Of course I do not maintain that the technical features do not in reality play some rôle in the course of business cycles (my paper in the Cassel volume was devoted to cycles produced by certain peculiarities of capitalistic production); but I do maintain that there exist also certain other important phenomena that are connected solely with the circulation and exchange activity and act more or less independently to *aggravate* the complex fluctuations which we know from experience. It is a regulation of these circulation factors that is the aim of the plan developed in the subsequent Sections.

3. A Simplified Example of a Planned Exchange

Consider three persons (or groups): a shoemaker, a tailor and a farmer. Suppose that they are in a heavy depression period of the sort described in the preceding Section. This means these three persons are mutually waiting for each other's orders, none of them daring to make a purchase, because they do not feel assured that they will be able to sell their own products.

Now suppose that an organizer goes to the shoemaker and puts the following question up to him: How much will you buy from the tailor and the farmer respectively if you are *assured* an increase in the sales of your own products equal to the total sum of what you buy from the others. The shoemaker answers, for instance, that if he is assured an increase in his own sales of $150 he will buy $50 worth from the tailor and $100 worth from the farmer, making a total of $150. The condition for the shoemaker's choice is thus that the total sum by which his sales are guaranteed to increase will be used by him for purchases; in other words, he will not have an opportunity to *save* any part of the increase in sales.

Our organizer will put similar questions to the tailor and the farmer. Suppose that the answers are as indicated in (3.1). Such a table may perhaps be called a *request-table* or a *request-matrix*. In (3.1) it is

(3.1)	Wants to buy from			Total
	Shoemaker	Tailor	Farmer	
Shoemaker	0	50	100	150
Tailor	40	0	180	220
Farmer	110	170	0	280
Total	150	220	280	650

assumed that the demand for shoes is equal to the supply, and similarly for the tailor's products and for farm products. Of course, in reality, this condition will hardly ever be exactly fulfilled; one of the big problems of circulation planning is just *how* such a balance shall be brought about: the greater part of this paper is devoted to this question. For a moment let us assume, however, that the demand-supply balance exists.

If this is so, it is, of course, an easy matter for our organizer to *force* the circulation to move on. He only has to print $150 of warrants ("commodity notes") on shoes, of which he gives $40 to the tailor and $110 to the farmer. Further, he prints $220 warrants on clothing, of which he gives the shoemaker $50 and the farmer $170. Finally, he prints $280 warrants on farm products, of which he gives the shoemaker $100 and the tailor $180. Each of the parties has thus got warrants on just the goods, and for exactly the amounts, they asked for.

If the organizer announces that these warrants are valid for a certain short period, say a month, it is clear that all three parties concerned will be almost certain to use the warrants. The probability that some of the warrants will not be used will hardly be larger than the probability that a lottery-stake will remain uncalled for.

Further, it is obvious that, during the process, each of the three parties will see his sales increase by just the amount intended, so that, when the process is completed, each person will have acquired from his customers a sum in warrants just equal to the sum he received at the beginning from the organizer. When the process is completed the organizer may, therefore, again visit all the three parties and collect from each an amount in warrants equal to the amount which this person originally received. The organizer will bring all the warrants home and *burn them.* The cycle is closed, the "money," or the warrants, have done their service and may be destroyed. But the economic *reality* is obtained: the various parties have used their productive faculty and have thus satisfied their own and the other parties' wants.

This indicates in all its simplicity the underlying idea of a system of circulation planning. When this idea is to be worked out, various special problems arise, the most important of which will be discussed in the following Sections.

4. The Correction of the Request Matrix by the Principle of Partakers' Percentages

The circulation scheme the underlying idea of which is indicated in Section 3 is built on a *voluntary* adherence from the various persons or groups. This being the case, the request-matrix, as determined by questioning the parties, will never, of course, show an exact demand-supply balance, that is, it will not have row-column equality. The sum in the first row will not be exactly equal to the sum in the first column, the sum in the second row not equal to that in the second column, etc. The problem thus arises: In what way can modifications be introduced that will bring about such a balance? From the purely formal point of view this may be done in an infinity of ways, indeed, there are n^2 elements that may be corrected, and n conditions imposed, of which only $n-1$ are independent. In this general setting the solutions has therefore, a great number of degrees of freedom.

But this purely formal aspect of the problem has little interest from the economic viewpoint. The crux of the matter, so far as applications are concerned, is that the correcting of the request-matrix must be done in such a way as to leave the originally requested quantities *as far as possible unchanged.* This should be done in order to satisfy the expressed desires of the parties to the largest possible extent. This ought to be the fundamental optimum consideration in the planning of the circulation.

Which features of the matrix should be changed, and to what extent the corrections should be carried through, will depend on a *weighing* of the various desires towards each other. Of course, it is hardly possible to find any absolutely *objective* and *exact* scale for making such comparisons. A considerable amount of estimate will always be involved, the scale of comparison must be constructed more or less by plausibility considerations. But nevertheless some sort of plausible weighing will always be possible. We have here one of those cases—so frequent in economic practice—where it can be "proved" by abstract reasoning that a solution is not possible, but where life itself compels us nevertheless to find a way out. I have been told that during the war a slogan was created in the U. S. Departments: "*It can't be done—but here it is.*" There is much to be learned from this slogan, particularly for the theoretical economist who is facing the problem of constructing a scale of comparison for the desires of various groups.

The correcting of the request-matrix ought to be done by means of certain well defined *principles* that can be applied in a rather mechanical way to any given request-matrix. Whatever estimate and plausibility considerations are necessary should be incorporated in the definition of the principles themselves. From the practical viewpoint such an arrangement seems imperative. If some committee or board of directors should decide each case upon its own merits, confusion would undoutedly arise. There would be arbitrariness and lack of co-ordination in the handling of the many figures in the request-matrix.

In this and the following sections I shall discuss certain principles for the correction of the request-matrix that may be applied either separately or in conjunction.

Many—if not most—of the persons or groups who are likely to take part in the planned exchange will probably consider it a fair and straight arrangement if the necessary adjustment is made on a strict percentage basis for each person, so that the *relative* distribution of the purchases for any given person is maintained unchanged while his total volume of purchase is adjusted. This means that there is prescribed for each person a certain *percentage* by which he has to reduce all his figures. This we shall call *the principle of partakers' percentages.*

This principle seems not only fair and straight, but, for many partakers, it will be the only natural solution. We only have to think of an enterprise that requests certain factors of production. To the extent that the coefficients of production are constant for variations in output of the order of magnitudes here considered, the principle of partakers' percentages would be the only correct solution.

Does there exist any set of partakers' percentages that will, when applied to the original request-matrix, furnish a matrix with complete row-column equality? Such a set of percentages always exists, and it is in general uniquely determined, apart from an arbitrary factor of proportionality that is common to all the percentages. This is easily proved by the lemmas of Section 21.

Indeed, let z_i be the reduction percentage applied for partaker No. 1. The elements of the corrected matrix will then be

$$(1 + z_i)a_{ij} \tag{4.1}$$

where a_{ij} are the elements of the original request-matrix.

If such a set of percentages z_i is given, the total purchase c_i of the partaker No. 1 is also given; it is simply equal to

$$c_i = (1 + z_i)a_{i0} \tag{4.2}$$

$a_{i0} = \sum_k a_{ik}$ being the total request made from partaker No. 1. The magnitudes c_i will be called the *volume* numbers. Vice versa, if the quantities c_i are given, the percentages z_i are also determined. The problem may therefore just as well be formulated in terms of the c's as in terms of the z's; this transformation is convenient and will be used in the following. In terms of the c's the elements of the corrected matrix are

$$c_i\alpha_{ij} \tag{4.3}$$

where

$$\alpha_{ij} = \frac{a_{ij}}{a_{i0}} \tag{4.4}$$

are the *percentages* with which partaker No. 1 originally proposed to distribute his purchase over the various groups. Of course we have

$$\alpha_{i0} = \sum_k \alpha_{ik} = 1. \tag{4.5}$$

The sum of the i-th column in the corrected matrix is $\sum_k c_k \alpha_{ki}$ and the condition that the corrected matrix shall have complete row-column equality is, therefore, expressed by

$$\sum_k c_k \alpha_{ki} = c_i \tag{4.6}$$

that is,

$$\sum_k c_k(\alpha_{ki} - e_{ki}) = 0 \qquad (i = 1, 2 \cdots) \tag{4.7}$$

where e_{ki} are the unit numbers

$$e_{ki} = \begin{cases} 1 \text{ if } k = i \\ 0 \text{ if } k \neq i \end{cases} \tag{4.8}$$

(4.7) form a system of n homogeneous equations in the n quantities c_k. The matrix of the coefficients has the property that the sum in any row is zero. By (4.5) we get indeed $\sum_i(\alpha_{ki} - e_{ki}) = 0$. The determinant of the system is consequently zero, so that there certainly exists a solution in the c's. The determinant will not (except by coincidence) be of lower rank; we may therefore conclude that in general there exists a solution in the c's that is uniquely determined apart from an arbitrary factor of proportionality. From the theory of linear equations we know that the c's are proportional to the elements in any row of the adjoint of the matrix $(\alpha_{ki} - e_{ki})$. Since $(\alpha_{ki} - e_{ki})$ is a singular matrix it does not matter which row we pick out, since all the rows in the adjoint of a singular matrix are proportional. Incidentally, it follows from the concluding remark under Lemma II in Section 21 that the rows of the adjoint of $(\alpha_{ki} - e_{ki})$ are not only proportional but *equal*.

Let

$$P_1, P_2 \cdots P_n \tag{4.9}$$

be the numbers obtained from the elements in any row of the adjoint of $(\alpha_{ki} - e_{ki})$ by reducing these elements with a common factor so that their sum becomes equal to unity. Such a reduction is possible since by Lemma III of Section 21 the sum in question is not zero. For the numbers P_i thus defined we have

$$\sum_k P_k = 1. \tag{4.10}$$

The numbers c_i that form the solution of (4.7) are then proportional to the P_i, and by virtue of (4.10) the proportionality factor is nothing but the *sum* of all the c_i. Denoting this sum by

$$C = \sum_k c_k \tag{4.11}$$

we consequently have

$$c_i = CP_i \tag{4.12}$$

Thus there can always be determined a set of numbers c_i that will satisfy (4.7) and therefore produce complete demand-supply equality in the exchange matrix. But that is not yet the answer to our problem. From the practical point of view the essential question is of course if these numbers will be *non-negative*. Only in this case will they have a concrete meaning as *volume numbers* for the partakers and thus furnish a real solution of the problem before us.

The non-negativity of all the magnitudes c_i determined by (4.12) follows from Lemma III in Section 21. Indeed, the matrix $(\alpha_{ki}-e_{ki})$ is equal to the matrix (21.4) apart from the factors $a_{10}, a_{20} \cdots a_{n0}$. All the elements of the adjoint of $(\alpha_{ki}-e_{ki})$ thus have the same sign, the reduced numbers P_i are consequently all non-negative, and hence all the c_i non-negative, provided of course that the arbitrary parameter C (which is equal to the sum of the c's) is selected non-negative. The numbers P_i may be looked upon as positive percentages expressing how the total volume of operation C is distributed over the various partakers; the P_i may therefore be called the *relative volume* members.

The total volume C which so far appears as an arbitrary parameter must be determined by certain supplementary conditions. During the tentative organization period of the exchange system, C may, for instance, be determined as a rather small number, thus making it possible to experiment on a small scale, until it is found safe to let the operations assume larger proportions. Later, when experience is gained, the main concern in determining C will be to assure the fullest utilization of the available capacity, without, however, impelling any of the parties to participate with a larger volume than they originally requested. In certain circumstances this latter condition may impose rather severe restrictions on the total volume of operations. We shall later discuss certain modifications that will in part overcome this difficulty. At present let us study the nature of the limitation that follows from a strict application of the condition considered.

If we require that the *relative* distribution of each person's purchases shall be exactly maintained, and also require that for none of the persons will the total volume of operations after correction be larger than the volume originally requested by him, then it is clear that all the adaptation must be an adaptation *downwards.* In this process some of the commodities (or services) may act as a troublesome *minimum factor.* Such a situation arises if the supply of a given commodity is small, which means that one of the parties only requests to partake for a small total sum, and if the other parties manifest an effective demand for this commodity. It is clear that if the supply of this commodity is not to be increased, a strict maintenance of the same *relative* distribution as originally claimed by the various parties, is only compatible with a heavy curtailment of the total volume of transactions for the community.

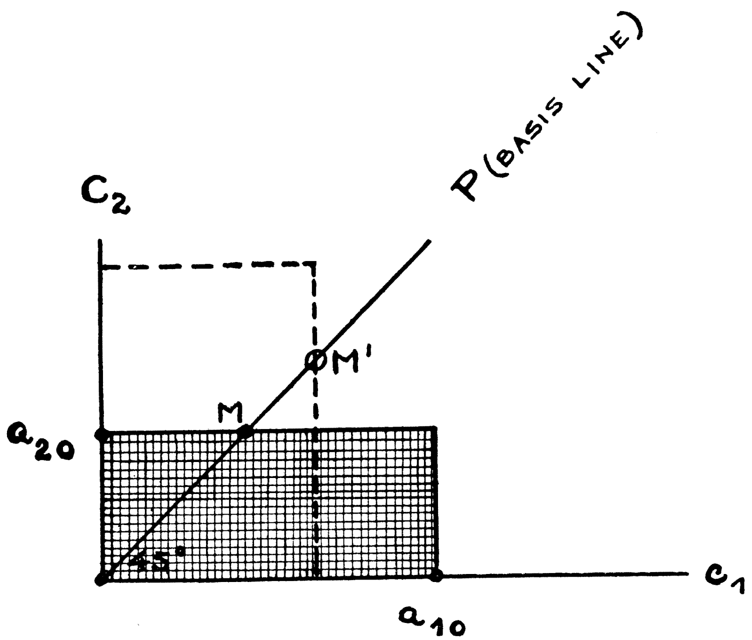


FIGURE 1

The exact point to which the transactions must be brought down may be illustrated graphically as follows. Consider first the case of only two persons (or groups). Let c_1 and c_2 be their total volume of operation after correction of the request matrix. Let c_1 and c_2 be measured along the two axes in a rectangular system of coordinates. (See Figure 1.) Of the various points in this diagram only those lying on the straight line P under 45° are possible. Indeed if the two persons can trade only with each other, the amount which one buys (and sells) must be equal to the amount which the other buys (and sells). The line P we shall call the *basis line.* Trying to make the total volume of operation for the community as large as possible will mean that we attempt to move on the basis line outwards as far as possible. How far can we go? This is limited by the exchange requests originally made by the person. If No. 1 originally requested a total of a_{10} and No. 2 a total of a_{20}, and c_1 and c_2 shall not surpass these limits, the point (c_1, c_2) is bound to lie somewhere within the shaded rectangle in Figure 1, or on its borders. This means that the point (c_1, c_2) to be finally selected is the point M where the basis line leaves the shaded rectangle. If the point of exit is on the upper (horizontal) side of the rectangle, the request of No. 2 is used to full capacity but not that of No. 1. If the rectangle had been the dotted one, where the point of exit is M'. we would have had the opposite situation.

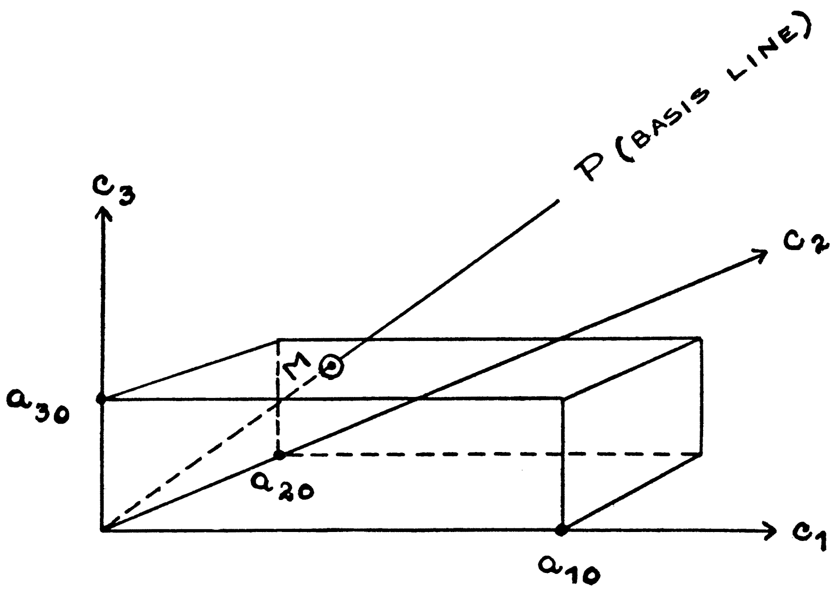


FIGURE 2

By virtue of (4.12) we would expect the direction numbers of the line P to be the numbers P_1 and P_2 determined by the elements of the adjoint of $(\alpha_{ki}-e_{ki})$. In the present case this matrix is

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \text{ and its adjoint consequently} \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$$

so that $P_1 = P_2 = \frac{1}{2}$ which is in accordance with Figure 1.

A similar representation can be used in three dimensions. See Figure 2.

In the case represented in Figure 2 it is only the request of No. 3 that will be fully utilized.

In the general case of n persons the point of exit M may be determined algebraically as follows. Along the basis line we have $c_i = CP_i$ where C is an arbitrary positive parameter. The boundary condition is that for each i we shall have

$$c_i \leqq a_{i0}$$

hence

$$C \leqq \frac{a_{i0}}{P_i} \text{ for } i = 1, 2 \cdots n. \tag{4.13}$$

If we want to make C as large as possible under the conditions (4.13) we consequently have to put

$$C = \text{the smallest of the numbers } \frac{a_{i0}}{P_i} \qquad (i = 1, 2 \cdots n). \tag{4.14}$$

The numbers $\frac{a_{i0}}{P_i}$ we shall call the *absolute limitation numbers;* they indicate a boundary which the total volume of transactions of the community cannot surpass as long as none of the partakers shall be impelled to participate with a larger amount than he himself originally proposed, and the adaptation is made only by the principle of partakers' percentages.

Since a_{i0} and P_i are uniquely determined by the elements of the original request-matrix a_{ij}, C as defined by (4.14) is also determined by the elements of the request-matrix and consequently by (4.12) all the c_i. The partakers' percentages z_i may finally be determined by (4.2).

As an example let us consider the request-matrix given in (4.15).

(4.15)	Wants to buy from			Total
	Shoemaker	Tailor	Farmer	
Shoemaker	0	75	150	225
Tailor	80	0	360	440
Farmer	330	510	0	840
Total	410	585	510	1505

The matrix $(\alpha_{ki} - e_{ki})$ is in this case

(4.16)	$i=1$	2	3	Sum check
$k=1$	-1.00000	0.33333	0.66667	0
2	0.18182	-1.00000	0.81818	0
3	0.39286	0.60714	-1.00000	0

The adjoint of this is

(4.17)	$i=1$	2	3	
$k=1$	0.50325	0.73809	0.93939	(Check: all
2	0.50325	0.73809	0.93939	rows *equal*)
3	0.50325	0.73809	0.93939	

and consequently the relative volume numbers

$$\begin{aligned} P_1 &= 0.23077 \\ P_2 &= 0.33846 \\ P_3 &= 0.43077. \\ \hline \text{Total } P_0 &= 1.00000. \end{aligned} \tag{4.18}$$

The limitation numbers are consequently

$$\frac{a_{10}}{P_1} = 975 \qquad \frac{a_{20}}{P_2} = 1300 \qquad \frac{a_{30}}{P_3} = 1950. \tag{4.19}$$

Hence partaker No. 1 (the shoemaker) is the minimum factor, and the net total volume of the operations for the whole community will be

$$C = 975. \tag{4.20}$$

The corrected sales of the individual partakers are thus by (4.12)

$$\begin{aligned} c_1 &= 225 \\ c_2 &= 330 \\ c_3 &= 420 \\ \hline \text{Total } &975. \end{aligned} \tag{4.21}$$

which by (4.2) corresponds to the following partakers' percentages

$$\begin{aligned} 1 + z_1 &= 1.000 \\ 1 + z_2 &= 0.750 \\ 1 + z_3 &= 0.500. \end{aligned} \tag{4.22}$$

Using these percentages we finally get the corrected exchange-matrix expressed in (4.23).

(4.23)	Is allowed to buy from			Total
	Shoemaker	Tailor	Farmer	
Shoemaker	0	75	150	225
Tailor	60	0	270	330
Farmer	165	255	0	420
Total	225	330	420	975

5. Generalized Principles of Correcting the Request-Matrix

The analysis of the last Section shows that if each person's *relative* distribution of purchases shall be strictly maintained, it may be necessary to reduce the *total* volume of transactions for the community considerably in order to find a complete demand-supply balance. We shall now discuss certain generalizations of the correction principles that may allow a better utilization of the productive capacity of the partakers.

Instead of using a set of reduction percentages z_i to be applied to each person, the idea may present itself to use a set of percentages x_j for each commodity: these percentages would then be a sort of *rationing-percentages*, it being understood that the rationing may be either positive or negative.

If we do not require the demand-supply balance to be fulfilled, we can, of course, always, by using an appropriate system of rationing-coefficients, assure full utilization of the capacity of the parties. We can simply let everybody produce to capacity and then determine the distribution by means of the rationing-percentages. If the demand-supply balance shall be fulfilled for each person, such an arrangement is not, however, possible. In this case no real advantage would be gained by using rationing-percentages instead of partakers' percentages. Indeed, in such a closed system everybody is at the same time a supplier and a consumer, so that the rationing of the consumption would at the same time be a rationing of the supply. Working the problem through mathematically, we would be led to exactly the same kind of solution as we discussed in Section 4, the only difference being that we would now have interchanged rows and columns of the request-matrix.

If a real advantage is to be gained we must use *both* partakers' percentages and rationing-percentages. This will introduce new degrees of freedom in the system, and open up the possibility of a better adaptation.

It may be a question whether one should combine the two percentages additively or multiplicatively. This is largely a question of ease in computation; indeed, if none of the percentages is very large, the additive composition can simply be looked upon as an approximation to the multiplicative. In the following we will use the additive composition, so that the elements of the corrected matrix become

$$(1 + z_i + x_j)a_{ij}. \tag{5.1}$$

In general we will not *a priori* make any assumption *about the sign* of the percentages z_i or x_j. The only thing we need to assume is that for any i, j $1+z_i+x_j$ is non-negative. For certain types of partakers an adjustment such as the one here considered will be more desirable than an adjustment strictly by partakers' percentages. For an ordinary consumer, for instance, it will in general be more advantageous to keep up as far as possible the total volume of his purchases than to maintain exactly the *relative* distribution over the various items in his budget.

If there is a great surplus of one of the commodities, that is, if there is in the request-matrix a big supply and a small demand, it would be necessary to make the corresponding rationing percentage x_j *very large* in order to assure even a reasonable utilization of the productive capacity for this commodity. But, on the other hand, it may be that the commodity in question takes only a small part of the total production capacity of the community, so that it would hardly be perceptible if the surplus of this commodity—or at least a part of it—were distributed over the *whole* group of partakers. This leads to introducing a third type of coefficiently y_j, to be applied to the *total* request of each person. In other words the elements of the corrected matrix would be

$$(1 + z_i + x_j)a_{ij} + a_{i0}y_j. \tag{5.2}$$

The percentages y_j must be assumed non-negative, otherwise it may happen that some of the elements in the corrected matrix become negative. For instance, if person No. 1 did not request the commodity j at all, but did express a desire to take some part in the exchange activity, the corrected element (5.2) would become negative if y_j was negative. The percentages y_j are thus essentially suited only to dispose of an existing *surplus*. They may, therefore, be called the *surplus coefficients*.

The percentages y_j have the effect of adjusting the surplus (or part of it) according to the *total* purchases proposed by each person. No regard is taken to the relative composition of the person's request. For instance, if a person has made a total request for \$2000, and it is decided to dispose of a butter surplus by putting the surplus coefficient for butter equal to $0.01=1$ per cent, then the partaker in question would receive a warrant on butter for \$20 (in addition to what he may himself have requested of butter). If the partaker is, say, a shoe factory whose original request only involved certain factors of production like hide, labor hours of a certain category, etc., it would have no immediate use for the butter received. Of course, there would always be possibilities of utilizing it in a further indirect exchange, but that is a different question. The organized exchange should of course aim at producing the best possible harmony right from the beginning. It may, therefore, be found expedient to discriminate between various types of partakers, and let these types be affected differently by the surplus percentages. The simplest way in which this can be obtained seems to be to make the discrimination according to the relative composition of the purchases as it is displayed in the original requests from the various partakers. In other words, instead of the term $a_{i0}y_j$ in (5.2) one would introduce the weighted sum

$$\sum_k a_{ik}y_{kj} \tag{5.3}$$

where y_{kj} is a set of percentages. If these percentages are independent of k, i.e., if we have $y_{kj}=y_j$, we get back to the case expressed by the last term in (5.2).

If the last term in (5.2) is replaced by an expression like (5.3), it will from the formal point of view be unnecessary to retain the term $x_j\ a_{ij}$. Indeed, the effect of this term is represented by the term $k=j$ in (5.3). If this interpretation is adopted, the matrix y_{kj} must be assumed to consist exclusively of non-negative elements, except possibly the diagonal elements that may be negative. For convenience in handling the formulae it will, however, be better to retain also the term with x_j. If this is done, it does not restrict generality to assume that the matrix y_{kj} has all its diagonal elements equal to any set of given numbers, for instance zeros. The term with z_i in (5.2) must under any cirsumstances be retained, as its effect is not in general taken account of by (5.3).

We are thus led to considering the following form of the elements of the corrected matrix.

$$(1 + z_i + x_j)a_{ij} + \sum_k a_{ik}y_{kj}. \tag{5.4}$$

Finally it may be desirable to introduce a generalization with regard to the demand-supply balance. To explain the reason for this, let us imagine that one of the parties involved would attempt to make an extra profit by raising the price of its product and at the same time trying to force its sale through the exchange organization. Such a situation is quite conceivable if the correcting of the request matrix is made with a view to utilizing as far as possible the supply from the partakers as expressed by the request-matrix. To meet this situation it is desirable to introduce into the planned circulation mechanism some element that may be manipulated so as to produce an effect similar to the one which would in a free market be produced by the adjustment of relative prices. This can be obtained simply by deciding that in the correcting of the request-matrix there shall *not* be arranged for an exact equality between the value of the goods delivered and those received by each partaker, but that the former value should *exceed*, or possibly be short of, the latter by a certain amount. The difference would—if it is positive—appear as a *service tax* levied by the exchange organization. In the general theoretical set-up it will be desirable to introduce such a parameter for each partaker. This means that the demand-supply balance would now be written in the form

$$(5.5)\quad \begin{cases} \quad c_i = \text{sum of row No. 1 in the corrected matrix} \\ c_i + \rho_i = \text{sum of column No. 1 in the corrected matrix} \end{cases}$$

where ρ_i is the service tax levied on partaker No. 1.

Since the sum of the rows must be equal to the sum of the columns, we must have

$$(5.6)\qquad \rho_0 = \sum_k \rho_k = 0.$$

This means that some of these taxes must be positive and some negative. To avoid this limitation and make the scheme perfectly general, we only have now to let one of the partakers be the organization itself. This would establish a close accounting system where the condition (5.6) is fulfilled, but where no assumption is made on the service parameters ρ_i for the real partakers.

The parameters ρ_i may be used for various purposes. Besides the one already mentioned, they may be used in an attempt to counteract the disastrous effect on the whole exchange activity of the community that is occasionally produced by price changes that deprive certain classes of producers of the greater part of their purchasing power. The service parameter for this group of partakers would then have to be made negative.

The service parameters may also be utilized to allocate to the partakers the administration costs of the system in case it is found desirable that the costs should be covered by the system itself and not, for instance, by federal or state funds. In this respect the service parameters would play exactly the same rôle as the surcharge made to the net actuarial premium in the life insurance companies.

If the exchange system becomes widely used, it would also, through the technique of the service parameters, be possible to realize various forms of communication between the exchange system and the state or federal budgets. Conceivably the service parameters may even be used for regular fiscal purposes.

6. The Limitational Condition for the Total Volume of Operations

How shall these various parameters be fixed in order to assure the "best possible" adaptation of the system to the desires of the partakers? At a first glance the great number of parameters and their very different nature seems to make it nearly impossible to keep track of the various effects and to recognize clearly the actions and counteractions. By a suitable grouping of the questions that arise, and by making a few—as it seems, rather plausible—assumptions, it is, however, possible to bring order into the matter and work out a technique amenable to numerical computation.

We shall assume that the application of the partakers' percentages z_i does not give rise to any optimum consideration. The general line of analysis will therefore be to start by investigating what the total volume of operation of the community C will be if all the *other* parameters are *given* and the z_i are simply adapted so as to realize the largest possible volume compatible with the condition that none of the originally proposed capacities a_{i0} shall be surpassed. The total volume C as determined by this procedure will be a function of the other parameters, that is of the x, y, and ρ. The next step will then be to make this function C as large as possible. This must be looked upon as involving a certain "cost" which is the *inconvenience* created by forcing any of the variables in C, namely the x, y, and ρ, out of their "natural" position, which is zero. These two considerations, that of increasing the total volume C and that of keeping down the inconvenience must now be weighted against each other.

Logically this problem is very much like the general problem we encounter in productivity theory: C may be considered as the "product" and the x, y, and ρ, the "factors of production." As in productivity theory, it will also here be found expedient to separate the problem of adaptation in two parts: First, the "substitution" by which the minimum cost is obtained which corresponds to a given volume, then the "volume adaptation," which involves a study of how total volume varies as a function of total cost along the optimum curve. This optimum curve (or the "substitumal" as I call it in my lectures on productivity) is defined as the locus of points in factor space where to any given magnitude of the product the substitution is completely realized. Much of the technique of this productivity analysis can be applied to the present case, although some aspects of the problem now are more complicated than it is customary to consider it in productivity theory.

The first step in the analysis of the adaptation process, as here outlined, is to eliminate the partakers' percentages. Taking the sum over j and the sum over i respectively in (5.4) and using the balancing principle expressed in (5.5) we get the following two fundamental equations.

$$(6.1) \qquad (1 + z_i)a_{i0} + \sum_k a_{ik}x_k + \sum_k a_{ik}y_{k0} = c_i$$

$$(6.2) \qquad a_{0j} + \sum_k z_k a_{kj} + a_{0j}x_j + \sum_k a_{0k}y_{kj} = c_j + \rho_j.$$

In the second of these equations we change j to i, and insert the expression for the z's taken from the first equation. This gives

$$a_{0i} + \sum_k \left\{ c_k - a_{k0} - \sum_h a_{kh}(x_h + y_{h0}) \right\} \alpha_{ki}$$
$$+ a_{0i}x_i + \sum_h a_{0h}y_{hi} = c_i + \rho_i$$

which reduces to

$$(6.3) \qquad \sum_k c_k(\alpha_{ki} - e_{ki}) = r_i$$

where

$$(6.4) \qquad r_i = \rho_i + \sum_k (\gamma_{ik} - e_{ik})a_{0k}x_k + \sum_k (\gamma_{ik}y_{k0} - y_{ki})a_0$$

and

$$(6.5) \qquad \gamma_{ij} = \sum_k \alpha_{ki}\beta_{kj}$$

$$(6.6) \qquad \beta_{ij} = \frac{a_{ij}}{a_{0j}}.$$

The matrices β and γ are determined by the data in the original request-matrix.

If it is desired to avoid the row sums y_{k0} in (6.4) and express the formula only by means of the elements of the matrix y_{ki}, the last term of formula (6.4) becomes a double sum, we get

$$(6.7) \qquad r_i = \rho_i + \sum_k (\gamma_{ik} - e_{ik})a_{0k}x_k + \sum_{hk} (\gamma_{ik} - e_{ih})a_{0k}y_{kh}.$$

The matrices β and γ have obviously the properties

$$(6.8) \qquad \beta_{0j} = \sum_k \beta_{kj} = 1$$

$$(6.9) \qquad \gamma_{0j} = \sum_k \gamma_{kj} = 1$$

which, since $\rho_0=0$, shows that, quite regardless of what values are selected for the parameters x and y, we must always have

$$r_0 = \sum_i r_i = 0. \tag{6.10}$$

From (6.10) it follows that the system (6.3) always has a solution. Indeed the matrix of the coefficients has by (4.5) the property that the sum in any row is zero. The determinant of the coefficients is therefore zero, and it will not (except by coincidence) be of lower rank. But any matrix obtained from $(\alpha_{n_i}-e_{n_i})$ by replacing one of the rows by the numbers $r_1 \cdot \cdot \cdot r_n$ will by (6.10) have exactly the same property, which shows that (6.3) is a singular system of compatible equations.

The coefficients in the left member of (6.3) depend only on the elements of the originally given request-matrix. *The effect of the parameters x, y, and ρ, is thus completely incorporated in the quantities r_i that form the right member of* of (6.3). Furthermore, we note that it is only in this second member that the present equation distinguishes it-

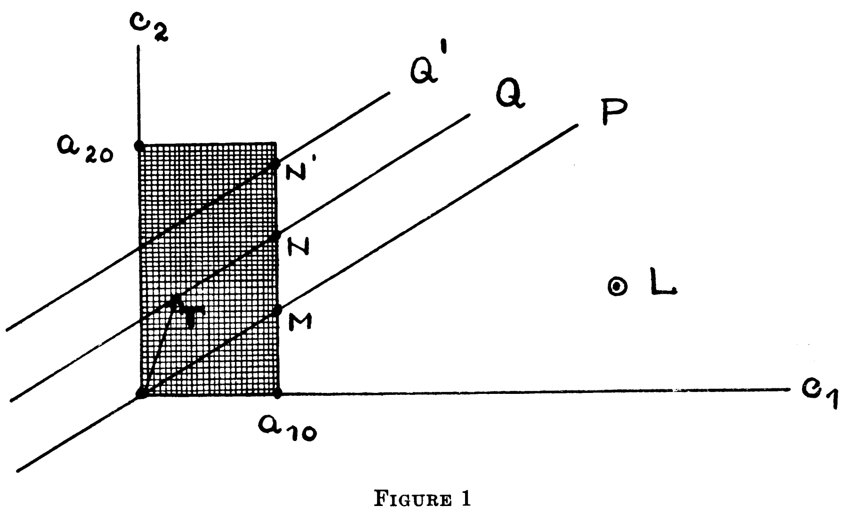


FIGURE 1

self from (4.7), (6.3) is simply the inhomogeneous equation corresponding to the homogeneous equation (4.7). It is, therefore, to be expected that some of the essential features of the simpler case discussed in Section 4 will be retrieved in the present case.

The fact that the effect of all the parameters x, y, and ρ, is incorporated in the r's makes the further line of analysis clear: We must determine what the maximum volume of transactions will be if the r's are considered as a single set of *given* parameters.

We shall first illustrate the matter graphically.

A system of two equations in the two unknowns c_1, c_2 is completely determinate when the matrix of the coefficients is non-singular. In this case the solution is represented by a *point* in (c_1, c_2) space, say the point L in Figure 1. If the matrix of the coefficients is singular (but not of a rank lower than 1) and the system is homogeneous, i.e., if the right member is equal to zero, there also always exists a solution, but this solution has one degree of freedom; more precisely it is represented by *a straight line* through the origin, say the line P in Figure 1. Finally, if the system is singular (but not of a rank lower than 1) and inhomogeneous with a right member that satisfies the compatibility conditions, the solution is represented by a straight line that does *not* go through origin, say the line Q in Figure 1. The *direction* of the line Q is just the same as the direction of the line P that represents the solution of the corresponding homogeneous equation. All possible systems with the

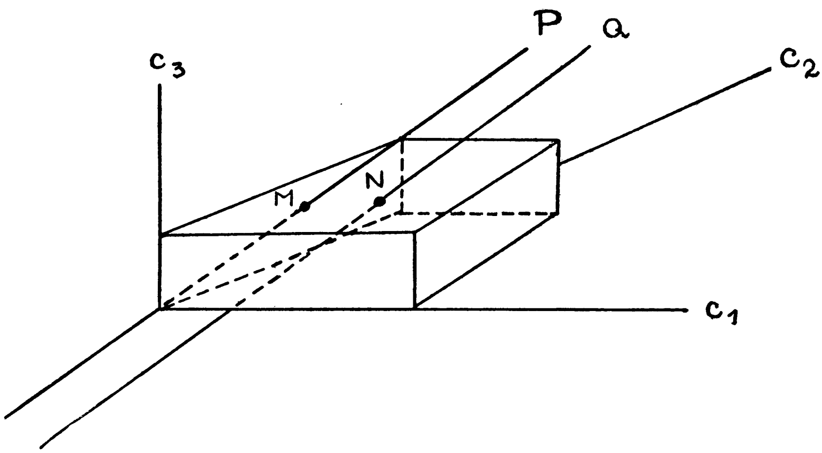


FIGURE 2

same matrix of coefficients in the left member will thus have the same basis direction, and only differ with regard to a parallel displacement, a *translation* of the basis line, this translation depending on the right member of the system of equations. It seems natural, therefore, in this case to look upon the solution as being made up of two components: first a *translation* T from origin to some point on the line Q (see Figure 1) and then a movement *along* Q.

A similar interpretation applies in the case of three equations with three unknowns. If the three-rowed matrix of the coefficients is non-singular, the solution is represented by a *point* in (c_1, c_2, c_3) space. If the matrix is of rank 2, and the system homogeneous, i.e., without second member, the solution is a straight line through origin (see P in Figure 2). Finally, if the matrix is of rank 2 and the system inhomogeneous with a right member satisfying the compatibility conditions, the solution is a straight line that does *not* go through origin (see Q in Figure 2).

(6.3) is an example of a system whose solution will be represented by a straight line as indicated in Figure 1 and Figure 2. If the r's are all zero, the straight line in question will go through origin, otherwise the solution contains a translation from origin in addition to a movement along the line.

These diagrams illustrate in a striking fashion what is obtained by introducing the parameters x, y, and ρ, in addition to the partakers' percentages. If only the latter are used, we have to move on the line P, which is independent of the x, y, and ρ, and uniquely determined by the original request-matrix. In Figure 1 this gives the point of exit M, where the total capacity of No. 1 is used, but only a small fraction of the capacity of partaker No. 2. The introduction of the parameters x, y, and ρ, *defines the translation* T, and by going on the new line Q we reach a point N where a much larger portion of the capacity of No. 2 is used. A still larger translation would give the line Q' with the point of exit N', where nearly the full capacity of No. 2 is used. The *direction* of the line Q is given by the elements of the request-matrix and cannot be altered, but a *translation* may be produced by using a set of values of the adaptation parameters x, y, and ρ.

A similar interpretation applies in the case of three variables. Figure 2 exhibits, for instance, a situation where the points M and N are such that only No. 3 is utilized to capacity, while both No. 1 and No. 2 have unused capacity.

Now let us express this algebraically. The equation (6.3) is of the form (22.5). Consequently an application of the formula (22.21) for $A_{ki}=\alpha_{ki}$ and $s_i=r_i$ would in point of principle furnish a solution of our system. In practice, however, it is convenient to make a slight modification which will greatly increase the rapidity with which the iteration process of Section 22 converges. To see the nature of this procedure, let us put $A_{ki}=\alpha_{ki}$ and $s_i=0$ in (22.6), and let us introduce a tentative set of numbers for the c's in the left member. By so doing we shall have determined a *new* set of c's, and this new set may be taken as a second approximation to the correct c's, that satisfy the homogeneous equation obtained from (22.5) by equating the s's to zero. Incidentally, the equation (22.11) shows that the *sum* of the c's in the second approximation is the same as in the first, which furnishes a convenient check on the computation. From the second approximations we may determine a third, and so on.

By using this process on a matrix A_{ki} that has the usual properties of a relative request-matrix α_{ki}, one will find that the approximations for, say, c_i fall alternately on the upper and lower side of the true value; both series approaching about equally rapidly to the true value. This suggests taking always two steps at a time and using the average value of c_i obtained at the two steps. Instead of doing this numerically we may determine quite generally the formula of the new iteration process that thus arises. Doing this we find that the new process corresponds to putting

$$A_{ki} = \frac{\alpha_{ki} + e_{ki}}{2} \tag{6.11}$$

$$s_i = \tfrac{1}{2} r_i \tag{6.12}$$

and then using this new matrix A_{ki} in the formulae (22.9), (22.18) and (22.17) that serve to determine the matrix Q_{ki}.

That this method will lead to a determination of the c's also in case of the inhomogeneous equation can be verified directly. From (6.3) we get indeed immediately by adding and subtracting e_{ki} in the parentheses under the summation sign

$$\sum_k c_k(\alpha_{ki} + e_{ki} - 2e_{ki}) = r_i$$

that is

$$\sum_k c_k\left(\frac{\alpha_{ki} + e_{ki}}{2} - e_{ki}\right) = \frac{1}{2} r_i .$$

This is nothing but equation (22.5) if we put A_{ki} and s_i equal to the expressions (6.11) and (6.12) respectively.

The solution c_i of (6.3) can consequently be written in the form

$$c_i = CP_i + \tfrac{1}{2}\sum_k r_k Q_{ki} \tag{6.13}$$

where Q_{ki} is the matrix (22.20), A_{ki} being defined by (6.11). The numbers P_i are determined by (22.14).

Incidentally, the numbers P_i in (6.13) are identical with the numbers P_i defined by (4.9). Indeed, by putting all the r's equal to zero in (6.13) we see that the P_i that occurs in (6.13) must be proportional to the elements in a row of the adjoint of $(A_{ki} - e_{ki})$. But we have

$$A_{ki} - e_{ki} = \tfrac{1}{2}(\alpha_{ki} - e_{ki}) \tag{6.14}$$

so that the elements of a row in the adjoint of $(A_{ki} - e_{ki})$ are proportional to elements in a row of the adjoint of $(\alpha_{ki} - e_{ki})$, i.e., proportional to the numbers P_i defined by (4.9). Since the sum of the P_i in (6.13) is equal to the sum of those defined by (4.9), namely 1, the two sets must be identical.

For brevity let us put

$$(6.15) \qquad u_i = -\frac{1}{2P_i}\sum_k r_k Q_{ki}.$$

The numbers u_i are going to play a fundamental rôle in the following. They satisfy the identity

$$(6.16) \qquad \sum_i u_i P_i = 0.$$

This follows by multiplying (6.15) by P_i and performing a summation over i, using (22.19).

In terms of the u_i formula (6.13) may be written

$$(6.17) \qquad c_i = (C - u_i)P_i.$$

Performing here a summation over i we see by (4.10) and (6.16) that equation (4.11) is satisfied also in the present general case.

If a set of parameters x, y, and ρ, are given, the numbers u_i are determined. This means by (6.17) that we have determined the straight line on which the point $(c_1 \cdots c_n)$ is bound to lie. The parameter C that represents the movement along the line is, however, still undetermined. It is finally fixed by requiring that none of the c_i shall be larger than the corresponding a_{i0}. That is to say, we must have

$$(C - u_i)P_i \leqq a_{i0} \text{ for all } i$$

which is equivalent to

$$(6.18) \qquad C \leqq \frac{a_{i0}}{P_i} + u_i \text{ for all } i.$$

The largest C that satisfies the condition (6.18) is obviously

$$(6.19) \qquad C = \text{smallest of the numbers } C_1\, C_2 \cdots C_n$$

where

$$(6.20) \qquad C_i = \frac{a_{i0}}{P_i} + u_i.$$

The first term in the right member of (6.20) is independent of the parameters x, y, and ρ, and depends only on the elements of the original request matrix.

Since also in the present general case, C designates the total volume of operations for the community, the preceding formulae may be given the following interesting interpretation:

If the total volume of operation for the community shall be brought up *to a given size C, it is necessary and sufficient that there be fixed a set of values of the parameters $u_1, u_2 \cdots u_n$ such that the smallest of the numbers* (6.20) *becomes not less than C. This we shall call the limitational condition.*

Suppose for instance that we have four commodities (or services) and that the magnitudes a_{i0}/P_i, as determined from the elements of the original request-matrix, turn out to be

i	$\dfrac{a_{i0}}{P_i}$
1	1300
2	975
3	625
4	800.

Then, if only partakers' percentages are used, the total volume of transactions for the community would be 625. If a larger total volume will be possible, say a volume of 650, it is necessary to have $u_3 \geqq 25$. If we would arrange for a total volume of 825, it would be necessary to have $u_3 \geqq 200$ and $u_4 \geqq 25$, and so on.

The parameters u_i may thus be looked upon as a set of limitational factors that concur in determining the size of C. For any given set of values of the u's one or more of them will be *minimum factors*. The situation is exactly as for limitational factors in productivity theory. A lower limit for each u_i is determined as a function of C. The expression "limitational factors" or simply "factors" will frequently be used in the following to designate the u's.

The fixation of the factors $u_1, u_2 \cdots u_n$ involves a cost element, namely the inconvenience attached to giving x, y, and ρ, such magnitudes as will produce the set of u's considered. How can the parameters x, y, and ρ, be determined so as to produce the necessary set of u's with the least possible inconvenience? This will be the topic of the next Section.

7. *A General Differential Method of Adaptation for the Minimum Factors*

A given set of the u's can be produced by many different choices of the parameters x, y, and ρ. This selection may also be described in productivity terms, but now the u's have to be considered as the things produced and the x, y, and ρ, the factors of production; we may perhaps call them the *elementary* factors as distinct from the u's that are factors in the production of C.

In considering the relations between the u's and the elementary factors, we have essentially a problem in *joint* production: There are n "products" $u_1 \cdots u_n$ being produced simultaneously by the use of a number of common factors. The principle of substitution leading to the minimum cost combination is in this case much more complex than the corresponding principle for a single commodity production, of the type usually considered in productivity theory. But, on the other hand, the form of the functional relationship between products and elementary factors is simpler in the present case. It is indeed a linear relationship. By (6.15) the u's are expressed linearly in terms of the r's, and by (6.7) the r's are expressed linearly in terms of the x, y, and ρ; it would not be difficult, therefore, to write out the general form of the relations connecting the u's with the elementary factors.

In the following we shall not, however, use this general formula. For simplicity we shall carry the rest of the discussion through on the assumption that $\rho_i = 0$, and that $y_{kh} = y_h =$ independent of k. This means that we consider the case where the corrected matrix is of the form (5.2). In this case the expression (6.7) reduces to

$$(7.1) \qquad r_i = \sum_k (\gamma_{ik} - e_{ik}) a_{0k} x_k + y_0 a_{0i} - y_i a_{00}.$$

Inserting this into (6.15) we get

$$(7.2) \qquad u_i = \sum_k f_{ik}' x_k + \sum_k f_{ik}'' y_k$$

where

$$(7.3) \qquad f_{ik}' = -\frac{a_{0k}}{2P_i} \sum_h (\gamma_{hk} - e_{hk}) Q_{hi}$$

$$(7.4) \qquad f_{ik}'' = -\frac{1}{2P_i} \sum_h (a_{0h} - e_{hk} a_{00}) Q_{hi} = \frac{1}{2P_i} \left\{ a_{00} Q_{ki} - \sum_h a_{0h} Q_{hi} \right\}.$$

The matrices (7.3) and (7.4) satisfy the identities

$$(7.5) \qquad \sum_i P_i f_{ik}' = 0 \qquad \sum_i P_i f_{ik}'' = 0 \text{ for all } k.$$

By (7.2) u_i is defined as a linear form in the $2n$ variables $x_1, x_2 \cdots x_n$, $y_1, y_2 \cdots y_n$. For convenience in handling the formulae, it is better to denote these variables consecutively

$$(7.6) \qquad x_1, x_2 \cdots x_N$$

where $N = 2n$ and $x_K = y_{K-n}$ when $K > n$. With this notation we may write

$$(7.7) \qquad u_i = \sum_K f_{iK} x_K$$

where K runs from 1 to N, and f_{iK}, is the n rowed and N columned matrix obtained by joining f_{ik}'' on to the right side of f_{ik}'. The matrix f_{ik} obviously satisfies the relation

$$\sum_i P_i f_{iK} = 0 \text{ for all } K. \tag{7.8}$$

All the elements of the matrix f_{ik} are by the preceding formulae determined in terms of the elements of the original request-matrix a_{ij}.

At this stage of the analysis we have to introduce an assumption about the nature of the inconvenience connected with a given deviation of the percentages $x_1 \cdots x_N$ from zero.

Suppose for a moment that we had succeeded in defining numerically an index function $\Omega(x_1 \cdots x_N)$ that could roughly be taken as indicating the total inconvenience caused by letting the parameters x_K assume given magnitudes. The nature of the variation of the function Ω could, for instance, be indicated by means of tables or graphs, or perhaps in part by analytical expressions with given numerical parameters. The function Ω being thus defined, we could also by a numerical differentiation process indicate roughly the value of the marginal inconveniences

$$\Omega_K = \frac{\partial \Omega}{\partial x_K} \tag{7.9}$$

in the various positions. Possibly we should be obliged in certain points to distinguish between forward and backward derivatives. This would in particular be the case for $\Omega_{n+1} \cdots \Omega_N$ in points where any of the $x_{n+1} \cdots x_N$ is zero. Indeed, we assume that all the $x_{n+1} \cdots x_N$ are non-negative, which can be interpreted by saying that an infinite inconvenience would be caused if any of the $x_{n+1} \cdots x_N$ passed from a non-negative to a negative value.

Instead of assuming Ω to be numerically given we could also take the Ω_K as the primary data; that is to say, by tables, graphs, or otherwise, we could attempt to indicate the intensity of *increase* in total convenience that would follow if, from a given point x_K we would increase one of the parameters x_K. From the theoretical viewpoint this would be a more general definition of the inconvenience, and *in practice it would probably be easier to formulate it this way.*

I believe that it would not be beyond practical possibilities actually to carry through a numerical definition of Ω or the Ω_K in a given exchange system. Suppose, for instance, that it is decided to work only with the rationing coefficients $x_1 \cdots x_n$. These may be either positive or negative. As a first approximation it seems fair to assume that a small positive value of any of them would be about as undesirable as a small negative value (the total purchase of each partaker being in any case adjusted to his total sales). As a first approximation we may further assume the marginal inconveniences to be independent. This suggests representing the part of the total inconvenience that is due to a given value of x_K by a parabola around origin, in other words by a function of the form

$$\frac{x_K^2}{2\epsilon_K} \tag{7.10}$$

where ϵ_K is a constant. The shape of such a curve is indicated in Figure 1.

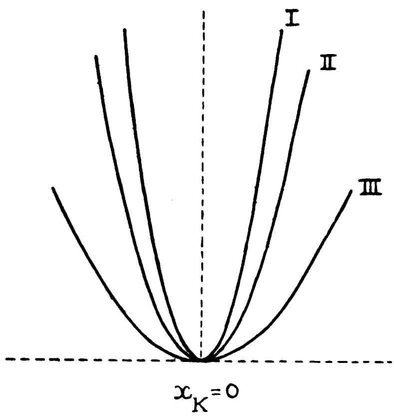


FIGURE 1

A small value of the constant ϵ_K means that the inconvenience increases very rapidly if x_K deviates from zero, (see curve I in Figure 1) while a small ϵ_k means a small increase in inconvenience. Thus the size of ϵ_K will be characteristic for the commodity (or service) in question. It does not seem to be beyond practical possibilities to fix roughly the value of ϵ_K for some of the main commodities that would be likely to enter into the exchange system. For the parameters $y_1 \cdots y_n$ the marginal inconvenience curves would be asymmetric, shooting up to infinity on the negative side.

Assuming the functions Ω_K to be numerically given, let us now consider the substitution problem, that is to say, the problem of how the parameters x_K are to be adapted so as to realize the smallest cost or inconvenience that is possible when it is given that the total volume of operation C shall have a given size.

If C shall have a given magnitude, it follows from the limitational condition of Section 6 that there are certain of the factors u_i that must necessarily have at least certain given magnitudes. This limitational effect of the factors u we shall now consider a little more closely, and particularly in relation to a given point in x_K space.

If a point in x_K space is given, then all the factors $u_1 \cdots u_n$ are given by (7.7) and consequently also the total volume of transaction which it is possible to realize in this point. Those factors for which a_{i0}/P_i+u_i is just equal to this total volume are the minimum factors in the point x_K considered; we may also say that these factors are *minimal* in this point.

Now suppose that in addition to a given point x_K we consider some *given* value C of the total volume, which may be equal to or different from the maximum volume belonging to the point x_K. Each factor u_i which in the point x_K is less than what is needed to produce C we shall call a *deficiency factor* for C in the point x_K, or, shorter, we may say that this factor is *deficient* for C in the point x_K. Those factors for which $a_{i0}/P_i+u_i=C$ may be called *barely sufficient* for C in the point x_K, and those for which $a_{i0}/P_i+u_i>C$ *plentiful* for C in the point x_K.

If it is desired to realize a certain given value C of the total volume, each point in x_K space may thus be classified according to its properties with regard to the given value of C. This means that the space x_K is divided into an extremely complicated system of compartments, partly overlapping and partly containing each other, and such that in each compartment a definite set of factors (perhaps all) is deficient for C, some other set is barely sufficient, and others plentiful, for C.

To seek the minimum of the total inconvenience function Ω in these circumstances will, therefore, involve such a mixture of problems in continuous variation and in multidimensional boundary conditions that, *prima facie*, it seems impossible to keep track of all the various alternatives.

There exists, however, a way out which I believe will be applicable even in the case of a large number of variables. It is quite impossible to consider the x_K space in *its totality* and by *a priori* considerations to pick out that particular point where the minimum of Ω is realized. It is impossible here to proceed by the same methods which are—alas, so superficially—used in general economic equilibrium theory, namely to determine the minimum (or maximum) of an index function of several variables just by equating its partial derivatives to zero.

In the present case it will not, indeed, be possible to study the x_K space *at large*, we must be satisfied by studying it *differentially*, but this will furnish a solution that satisfies most of the practical requirement. *It means that we have to seek the solution of the adaptation process by small tentative steps.*

Following strictly the theoretical distinction between the substitution process and the volume adaptation, we should then have to consider first a tentative movement by small steps during which we keep all the time the total volume constant, while constantly reducing the total inconvenience. Having achieved this we may start to change the volume. In practice it will be found better to have both objects in view during each step. In other words, if we start tentatively in a given point x_K (for instance, where all the x_K are zero), we may decide to make a movement such that total volume changes *a certain amount* which we fix tentatively. This is the form of movement we shall consider in the following. It contains of course as a special case the movement where total volume is kept constant.

Suppose then that we start in some given point x_K where the total volume turns out to be C. We decide that we want to move to some neighboring point

$$x_K' = x_K + \delta x_K \tag{7.11}$$

where total volume is

$$C' = C + \delta C \tag{7.12}$$

δC we look upon as given, and the quantities δx_K we consider as unknowns to be determined in such a way that the total inconvenience Ω is decreased as much as possible (or increased as little as possible) by the passage from x_K to x_K'.

If we make a small step from the starting point x_K to some point x_K' not imposing any other condition on the movement than the minimalization of Ω, it may conceivably happen that, in the new point, the total volume actually turns out to have the prescribed magnitude (7.12). But in most cases that will not, of course, be so. With the shape of the function Ω that is appropriate to the optimum problem in circulation planning, the cost minimalization will—if not checked by side relations—*have a tendency to lower the u's;* and this may have the effect that some or more of the factors $u_1 \cdots u_n$ will in the new point turn out to be deficiency factors for the prescribed magnitude $C+\delta C$.

It will therefore be necessary to pick out at least a certain group of factors, say the m factors

$$u_p, u_q \cdots u_t \tag{7.13}$$

and impose on the movement from x_K to x_K' the side condition that these factors shall *not* be deficiency factors in the new point.

Conceivably the manner in which the total inconvenience depends on the various parameters may be such that, for one or more of the factors u, it will be easier to arrange for a higher value than for a smaller; it might for instance be easier to make u_3 equal to 30 than to 25. In practice such cases will, however, be very exceptional. It therefore seems plausible to replace the condition that a given factor shall not be deficient for $C+\delta C$, in the new point, with the condition that it shall be barely sufficient. In other words the side condition for the factors (7.13) will be that

$$\sum_K f_{iK} x_K' = u_i' \qquad (i = p, q \cdots t) \tag{7.14}$$

where u_p', $u_q' \cdots u_t'$ are the magnitudes of the factors considered that will barely be sufficient for the prescribed volume $C'=C+\delta C$. By (6.19) these magnitudes are

$$u_i' = C' - \frac{a_{i0}}{P_i} \,. \tag{7.15}$$

Introducing the increments instead of the total values we can write (7.14) in the form

$$\sum_{K=1,2\cdots N} f_{iK} \cdot \delta x_K = \delta u_i \qquad (i = p, q \cdots t) \tag{7.16}$$

where

$$\delta u_i = u_i' - u_i \tag{7.17}$$

u_i being the value of the factors in the starting point x_K.

The selection of the group of factors, for which the non-deficiency condition (7.16) is imposed, must in practice to some extent be based on a rough estimate of the kind of variations in the u_i that are likely to accompany a given change in the x_K. It is always desirable, of course to make the group as small as possible, because the smaller the number of side conditions, the larger the possibility of obtaining a large net reduction in Ω. It will therefore be desirable to make a first attempt with a rather small group of conditional factors, and then little by little to increase the groups, if it is found that some of the free factors actually turn out to become deficiency factors in the new point.

Since the number of variables is finite, there is a finite number of possibilities to consider, a complete solution of the problem is therefore always possible in principle, but in practice there can never, of course, be any question of trying more than a very few of those combinations that seem really important. For instance, if the step considered involves a *non-decrease* in total volume, i.e., if δC in (7.12) is non-negative, it will probably be practical first to make a step where one includes in the conditional factor group *those factors that are deficient or barely sufficient for* $C+\delta C$ *in the starting point* x_K, and possibly some others that are very close to having this property, i.e., for which the difference $C+\delta C-(a_{i0}/P_i+u_i)$ is very small in the starting point.

So much for the side conditions. Now let us consider the minimalization process itself.

If the marginal inconveniences Ω_K change in a continuous way in the vicinity of the starting point x_K, the change in total inconvenience by going from x_K to x_K' will be approximately

$$\sum_K \Omega_K \cdot \delta x_K \tag{7.18}$$

provided the δx_K are small.

If some of the Ω_K should change discontinuously in the vicinity of the point considered, for instance, become infinite for a backward movement, we may simply consider the corresponding δx_K as a constant equal to zero, and not as a variable. We may, therefore, always—by a proper interpretation of the δx_K—take the minimalization of (7.18) as the object of the adaptation.

This minimalization is to take place under the side condition (7.16). Since we have formulated the minimalization problem in the differential form (7.18), there is, however, also another kind of side condition to take into account; viz., that all the δx_K shall be "small." This condition will in practice have to be interpreted *cum grano salis*, but it cannot simply be neglected.

The condition can be formulated by defining a closed surface around x_K, say a sphere, an ellipsoid, or a little "box," within which the point $x_K + \delta x_K$ shall lie. The form of the surface involves a definition of the *metric* of the space x_K, i.e., of the notions of direction and length. In point of principle, the x_K cannot be considered as comparable in the same sense as the coordinates in a physical space, so that the metric of the space x_K is not defined by any obvious *a priori* principle; but for practical purposes we may simply make use of the facts that all the x_K are here pure numbers (percentages) and that they will, furthermore, in our problem, be of the same order of magnitude. It seems therefore, plausible to use the notions of direction and length in the ordinary way and to define the surface as a small N dimensional sphere.

This being so, if *no* other side condition than that of the "smallness" of the δx_K were imposed on the movement, the cost minimalization would be obtained by going from x_K *along the gradient of* Ω, that is along the vector whose components are $(\Omega_1 \cdots \Omega_N)$, and in the negative direction. The length of the movement would be determined by the radius of the limiting sphere.

If some other side condition is imposed, for instance, that the movement shall lie in a given *plane*, then the cost minimalization would be realized by going along the *projection* of the gradient on to this plane. Quite generally, if the movement is bound to lie in a certain $N-m$

dimensional linear manifold, as the one defined by (7.16) (where the f_{iK} and the δu_i now must be looked upon as given), the cost minimalization would be realized by going a certain "small" distance in the negative direction of the projection of $(\Omega_1 \cdots \Omega_N)$ on to this $N-m$ dimensional manifold.

We therefore have to determine the projection of the vector

$$-\lambda\Omega_K \tag{7.19}$$

on the $N-m$ dimensional manifold defined by (7.16), λ being a "small" positive quantity.

The solution of this problem is given by the formulae of Section 23. Let us first assume that all the derivatives Ω_K can be looked upon as continuous in the vicinity of the point considered. In this case all the δx_K are free variables (apart from the conditions (7.16)). The solution of the problem is, therefore, obtained by putting $-\lambda\Omega_K$ for x_K and δu_i for u_i in (23.35). This gives

$$\delta x_H = \lambda \sum_{K=1,2\cdots N} (F_{HK} - e_{HK})\Omega_K + \sum_{k=p,q\cdots t} \delta u_k \cdot \overset{*}{f}_{kH} \tag{7.20}$$

where F_{HK} is the projection matrix for the m-rowed and N-columned matrix consisting of the rows $p, q \cdots t$ from the matrix f_{iK} in (7.7). Choosing tentatively some small value of λ in (7.20) means that we decide upon the "length" of the step to take from the initial point x_K. The length of this step has a certain connection with the selection of the set of variables to be included in the conditional group (7.13). The smaller the step, the smaller the probability that any of the free factors shall break through and become deficiency factors in the new point.

If, in the starting point, some of the $x_{n+1} \cdots x_N$ are zero, we must take care not to make a step that will decrease any of these parameters, but, on the other hand, we must leave the possibility open of *increasing* any of them if that should fit in with the object pursued. In this case the derivatives $\Omega_1 \cdots \Omega_n$ are continuous, while the $\Omega_{n+1} \cdots \Omega_N$ are not all continuous; as a matter of fact if $x_K=0$, $K>n$, Ω_K should be looked upon as being defined only for a forward movement.

A practical way of handling this situation seems to be the following: First the formula (7.20) is used *as if* all the variables had been free (apart from (7.16)). In other words (7.20) is used as it stands. If this gives a negative value for some of the $\delta x_{n+1} \cdots \delta x_N$ for which the corresponding $x_{n+1} \cdots x_N$ is zero, the computation should be repeated, this time considering these variables as *constants* so that the equation (7.16) now take on the form

$$\sum_{K=P,Q\cdots T} f_{iK}\cdot\delta x_K = \delta u_i \qquad (i = p, q \cdots t) \tag{7.21}$$

where δx_P, $\delta x_Q \cdots \delta x_T$ are all the $\delta x_1 \cdots \delta x_N$ except the increments for those of the magnitudes $x_{n+1} \cdots x_N$ that are now considered as constants. The problem is thus now to determine the projection of the vector $-\lambda\Omega_K(K=P, Q \cdots T)$ on to the manifold defined by (7.21). By (23.35) the solution of this problem is

$$\delta x_H = \lambda \sum_{K=P,Q\cdots T} (F_{HK} - e_{HK})\Omega_K + \sum_{k=p,q\cdots} \delta u_k \cdot \overset{*}{f}_{kH} \tag{7.22}$$

where F_{HK} now is the projection matrix for the m-rowed and M-columned matrix formed by the rows $p, q \cdots t$ and the columns P, $Q \cdots T$ of the matrix f_{iK} of (7.7).

By the formulae (7.20) and (7.22) a series of tentative steps may be made, realizing more and more perfectly the minimalization of the total inconvenience, and increasing at the same time the total volume C little by little.

In each point that is successively passed during the computation, the magnitude Ω (or, more generally, the integral of the scalar product $\Sigma_K\Omega_K\cdot\delta x_K$ along the path followed) should be computed. The size of this quantity will be the guide by which to judge the feasibility of a further increase in C. If a point is reached where Ω (or the integral) has become so large that a further increase is deemed undesirable, the process should be stopped. The point x_K thus determined will then define the final corrected request-matrix.

It would perhaps be possible to combine the study of the increasing cost function Ω with the study of an increasing total utility function ϕ representing the "desirability" of an increase in the quantities consumed by the various parties. The difference $\phi-\Omega$ would then be an index function to be maximized. In this case the adaptation process itself may lead to a definite final maximum, without that amount of arbitrariness which in the preceding method was involved in judging when Ω had become so large as to make it undesirable to continue. I shall not, however, discuss this possibility any further here.

8. An Analytic Method of Adaptation

The method of adaptation described in the preceding Section is perfectly general in the sense that no assumption is made about the analytical form of the functions that define the total inconvenience Ω. The method of the preceding Section may indeed be applied even if Ω or the marginal inconveniences Ω_K are given only in the form of tables or graphs.

If certain assumptions are made about the analytical form of the functions Ω or Ω_K, the work may in many respects be simplified, particularly because it may then be possible to indicate in explicit form the complete solution of the minimalization problem for each given set of values of the limitational factors $u_p, u_q \cdots u_t$, so that at least this part of the work need not be done by a series of tentative steps.

We shall in particular consider the case where only rationing coefficients $x_1 \cdots x_n$ are introduced (besides the partakers' percentages $z_1 \cdots z_n$). In this case it seems to be a reasonable working hypothesis to assume that Ω varies as indicated by the curves in Fig. 1 of Section 7. This means that we put

$$2\Omega = \sum_{k=1,2\cdots n} \frac{x_k{}^2}{\epsilon_k} \cdot \tag{8.1}$$

In this case, the increment in total inconvenience is defined, not only for infinitesimal steps, but for finite movements. In the present adaptation process, we need not, therefore, confine the discussion to a series of small displacements in x_k space, but we may study what can be obtained by going directly to some *a priori* given point in this space. Also for the limitational factors $u_1 \cdots u_n$ we may now, if we wish, consider large jumps, but in practice this will not always be found convenient, at least in parts of the adaptation process it will be found useful to work with small tentative variations of the $u_1 \cdots u_n$. The calculations will take the following form.

First, let a certain magnitude C of the total volume of transactions be given. By the principles of the preceding Sections we associate with this magnitude C a certain set of conditional parameters $u_p, u_q \cdots u_t$ whose values are determined when C is given. The problem will then be to minimize (8.1) under the side conditions

$$\sum_{k=1,2\cdots n} f_{ik}x_k = u_i \qquad (i = p, q \cdots t) \tag{8.2}$$

where the u_i are given.

If we let $\lambda_p, \lambda_q \cdots \lambda_t$ be a set of m arbitrary parameters and consider the derivatives

$$\frac{\partial\left(\Omega - \sum_{i=p,q\cdots t} \lambda_i \sum_{h=1,2\cdots n} f_{ih}x_h\right)}{\partial x_k} = \frac{x_k}{\epsilon_k} - \sum_{i=p,q\cdots t} \lambda_i f_{ik} \tag{8.3}$$

we see that the necessary condition for a minimum is

$$x_k = \sum_{i=p,q\cdots t} \lambda_i \bar{f}_{ik} \qquad (k = 1, 2 \cdots n) \tag{8.4}$$

where

$$\bar{f}_{ik} = f_{ik}\epsilon_k. \tag{8.5}$$

The values of the λ_i that correspond to the extremum point are determined by inserting into (8.2) the expression for x_k taken from (8.4). This gives

$$\sum_{j=p,q\cdots t} \phi_{ij}\lambda_j = u_i \qquad (i = p, q \cdots t) \tag{8.6}$$

where

$$\phi_{ij} = \sum_{k=1,2\cdots n} f_{ik}\bar{f}_{jk} = \sum_{k=1,2\cdots n} \epsilon_k f_{ik} f_{jk}. \tag{8.7}$$

So long as the number m of affixes in the set $p, q \cdots t$ is less than n, the m-rowed and n-columned matrix f_{ik} will not be singular (except by coincidence). Since we do not consider the case where *all* the $u_1 \cdots u_n$ are included in the conditional set, we may consequently assume that f_{ik} is of rank m, and hence the linear system (8.6) nonsingular

Let

(8.8) $\hat{\phi}_{ij(p,q\cdots t)}$ = the adjoint of the m-rowed matrix ϕ_{ij},

the adjunction being made within the set $p, q \cdots t$. Further let

(8.9) $|\phi_{(p,q\cdots t)}|$ = determinant value of the coefficients in the left member of (8.6).

With this notation we have

$$\lambda_i = \sum_{j=p,q\cdots t} \hat{\phi}_{ij(p,q\cdots t)} u_j / |\phi_{(p,q\cdots t)}| \qquad (i = p, q \cdots t). \tag{8.10}$$

When $\lambda_p, \lambda_q \cdots \lambda_t$ are computed by this formula, all the $x_1 \cdots x_n$ are determined by (8.4), hence by (7.7) all the $u_1 \cdots u_n$ are determined (not only those corresponding to $i = p, q \cdots t$). It is easily seen that these values of $u_1 \cdots u_n$ are obtained simply by using (8.6) also for all the other values of i in the set $1, 2 \cdots n$. The definition (8.7) applies, of course, for any values of $i, j = 1, 2 \cdots n$.

This computation of the complete set of u's tells us whether a deficiency factor has come in by the minimalization process. If no such factor appears, the selection of the conditional set $u_p, u_q \cdots u_t$ has been successful, the point x_k thus determined will then actually represent a minimum of Ω under the given magnitude of C.

If a deficiency factor has come in, the computation will in any case yield information about a situation in x_k space that is not very far from a minimum combination, but it will not give exactly such a point;

if a more exact result is wanted, another selection of the conditional set $u_p, u_q \cdots u_t$ must therefore be tried, possibly in connection with a small modification in the magnitude of C for which the minimalization is carried through.

The value which Ω assumes in the point x_k that is fixed by the above procedure, may be easily expressed in terms of the corresponding values of $\lambda_p, \lambda_q \cdots \lambda_t$. Inserting into (8.1) the expression for x_k taken from (8.4) we get

$$2\Omega = \sum_k \frac{1}{\epsilon_k}\left(\sum_i \lambda_i \bar{f}_{ik}\right)^2 = \sum_k \sum_{ij} \frac{1}{\epsilon_k}\lambda_i\lambda_j\bar{f}_{ik}\bar{f}_{jk}$$

where k runs through $1, 2 \cdots n$, and i, j, independently of each other, through $p, q \cdots t$; hence

$$2\Omega = \sum_{ij}\phi_{ij}\lambda_i\lambda_j \tag{8.11}$$

where i, j runs through $p, q \cdots t$. By (8.6) the expression (8.11) can also be written

$$2\Omega = \sum_{i=p,q\cdots t} \lambda_i u_i. \tag{8.12}$$

The last formula is very convenient for computation purposes, since both the λ_i and u_i are already known when we get to the stage where it is desired to compute Ω.

The above computations may be made for a series of different values of C. To each chosen magnitude of C there will correspond a value of Ω. The curve exhibiting this relation between Ω and C is the optimum curve. *It shows the inevitable total inconvenience of adjustment which must be incurred if a given total volume of transaction for the community shall be assured.*

Under the assumption here made, the optimum curve is continuous, and it has even a continuously varying tangent, except at the points where one or more new minimum factor comes in. It follows from the above argument that as C increases, a larger and larger number of factors $u_1 \cdots u_n$ must be included in the conditional set; for each value C where a new factor is thus included, the tangent of the optimum curve changes discontinuously.

Under our assumptions the optimum curve will everywhere be sloping upwards, that is to say Ω will be constantly increasing with C, and the discontinuous changes in the slope of the tangent will always be an *increase* in the steepness. In most practical cases the element of discontinuity introduced by the inclusion of new minimum factors will not be very prominent; it will indeed hardly be perceptible unless the computations are carried through with a great number of decimal places and the graph is drawn on a very large scale; Figure 1 in Section 9 is an example of this.

The feature which will dominate the picture is the *rapid increase in* Ω *that takes place after a certain level of* C *is reached.* This is explained by the fact that for higher C a rapidly increasing number of minimum factors must be taken account of, and this imposes *new side conditions* that become more and more severe, thus making it more and more difficult to bring Ω down to a reasonable magnitude. This is clearly shown in the numerical example of the next Section.

9. *A Numerical Example Showing Details of the Computation*

In order to illustrate the use of the technique developed in the preceding sections we shall now work through in detail a simplified numerical example.

Suppose that there is given the three-rowed request-matrix (4.15). If only partakers' percentages z_1, z_2, z_3 are used, in other words, if it is required that the *relative* distribution of each person's purchase shall be maintained unaltered, then total sales will, as we have seen, only be 975 and the individual sales will be given by (4.21). The problem before us is how we can, by modifying somewhat the relative distribution for each partaker, under due consideration of his desires, increase total sales. For simplicity we shall assume that only rationing coefficients x_1, x_2, x_3 are employed, (besides the partakers' percentages z_1, z_2, z_3) and that the total inconvenience in applying such a set of coefficients may be represented by a function of the form (8.1). As an illustration we put

$$\begin{aligned} \epsilon_1 &= 0.1 \\ \epsilon_2 &= 0.2 \\ \epsilon_3 &= 0.7. \end{aligned} \tag{9.1}$$

The first thing to do is to form the row-relative request-matrix, namely

(9.2)

α_{ij}	$j=1$	2	3	Row check:
$i=1$	.00000	.33333	.66667	1.00000
2	.18182	.00000	.81818	1.00000
3	.39286	.60714	.00000	1.00000

Similarly we form the column-relative request matrix, namely

(9.3)

β_{ij}	$j=1$	2	3
i=1	.00000	.12821	.29412
2	.19512	.00000	.70588
3	.80488	.87180	.00000
Column check:	1.00000	1.00000	1.00000

On the basis of these two matrices we form $\gamma_{ij}=\sum_k \alpha_{ki}\beta_{kj}$. The result is given in (9.4)

(9.4)

γ_{ij}	$j=1$	2	3
$i=1$	.35168	.34250	.12834
2	.48867	.57204	.09804
3	.15964	.08547	.77362
Column check by formula (6.9)	.99999	1.00001	1.00000

The small deviations from unity in the column sums in (9.4) are due to the dropping of decimal places.

Subtracting e_{ij} from (9.4) we get

(9.5)

$\gamma_{ij}-e_{ij}$	$j=1$	2	3
$i=1$	−.64832	.34250	.12834
2	.48867	−.42796	.09804
3	.15964	.08547	−.22638
Column check by formula (6.9)	−.00001	.00001	.00000

The above fundamental matrices are derived by straightforward product-summing, a work that may be easily performed even in the case of a large number of variables.

Starting from (9.2) we now form the matrix $A_{ij}=A_{ij}^{(1)}=\alpha_{ij}+e_{ij}/2$ and its successive powers $A_{ij}^{(\nu)}$ defined by (22.9). In all these matrices each row shall add up to unity. If a listing adding machine is used and there is only a small number of variables, it is quicker to verify that the sum of the three column-sums is equal to the number of variables, in the example considered. This is done in Table 1.

TABLE 1.—ITERATION PROCESS $A_{ij}.^{(\nu)}$

	.50000	.16667	.33333
$\nu=1$	.09091	.50000	.40909
	.19643	.30357	.50000
3.00000	.78734	.97024	1.24242
	.33063	.26786	.40151
$\nu=2$	.17127	.38934	.43939
	.22403	.33631	.43966
3.00000	.72593	.99351	1.28056
	.26854	.31092	.42054
$\nu=3$	.20734	.35660	.43606
	.22895	.33896	.43209
3.00000	.70483	1.00648	1.28869
	.24514	.32788	.42698
$\nu=4$	.22175	.34523	.43302
	.23017	.33881	.43102
3.00000	.69706	1.01192	1.29102
	.23625	.33442	.42933
$\nu=5$	.22732	.34103	.43165
	.23055	.33861	.43084
3.00000	.69412	1.01406	1.29182
	.23286	.33692	.43022
$\nu=6$	.22945	.33944	.43111
	.23069	.33852	.43079
3.00000	.69300	1.01488	1.29212
	.23157	.33787	.43056
$\nu=7$	.23027	.33883	.43090
	.23074	.33848	.43078
3.00000	.69258	1.01518	1.29224
	.23108	.33824	.43068
$\nu=8$	.23058	.33860	.43082
	.23076	.33847	.43077
3.00000	.69242	1.01531	1.29227
	.23089	.33838	.43074
$\nu=9$	.23070	.33852	.43077
	.23077	.33846	.43077
3.00000	.69236	1.01536	1.29228

	.23082	.33843	.43076
$\nu=10$	.23074	.33848	.43077
	.23077	.33846	.43077
3.00000	.69233	1.01537	1.29230
	.23078	.33845	.43077
$\nu=11$	.23076	.33847	.43077
	.23077	.33846	.43077
3.00000	.69231	1.01538	1.29231
	.23077	.33846	.43077
$\nu=12$	.23077	.33846	.43077
	.23077	.33846	.43077
3.00000	.69231	1.01538	1.29231

Table 1 exhibits clearly how the iteration process converges. The elements of the final matrix $A_{ij}^{(\infty)}$ obtained must by (22.14) be proportional to those of the adjoint of $(\alpha_{ij}-e_{ij})$. The rows of this adjoint are not only proportional (which follows simply from the fact that $(\alpha_{ij}-e_{ij})$ is singular), but they are (by Lemma II of Section 21) even *equal*. This is clearly seen by considering the last part ($\nu=12$) in Table 1. Furthermore, the elements in one of these equal rows are just the numbers (4.18) that were determined directly as the proportionality numbers for the adjoint of $(\alpha_{ij}-e_{ij})$ in the present example.

In the present simple case these proportionality numbers are easily determined either directly as in (4.17), or by the iteration process in the above Table 1; the direct method is here the simplest, but in the case of a great number of variables it is the iteration process that must be relied upon.

The next step is to construct the matrix Q_{ij}. For this purpose we first form the differences $D_{ij}^{(\nu)}$ defined by (22.18); they are given in Table 2. The check in this case is similar to the one used in Table 1, only the sum in each row should now be zero, and hence the sum of the column-sums should also be zero; it will be seen that this checks everywhere in the table.

TABLE 2.—SERIES $D_{ij}^{(\nu)}$.

	−.50000	.16667	.33333
$\nu=1$	.09091	−.50000	.40909
	.19643	.30357	−.50000
.00000	−.21266	−.02976	.24242
	−.16937	.10119	.06818
$\nu=2$	.08036	−.11066	.03030
	.02760	.03274	−.06034
.00000	−.06141	.02327	.03814

	−.06209	.04306	.01903
$\nu=3$	.03607	−.03274	−.00333
	.00492	.00265	−.00757
.00000	−.02110	.01297	.00813
	−.02340	.01696	.00644
$\nu=4$	.01441	−.01137	−.00304
	.00122	−.00015	−.00107
.00000	−.00777	.00544	.00233
	−.00889	.00654	.00235
$\nu=5$	.00557	−.00420	−.00137
	.00038	−.00020	−.00018
.00000	−.00294	.00214	.00080
	−.00339	.00250	.00089
$\nu=6$	.00213	−.00159	−.00054
	.00014	−.00009	−.00005
.00000	−.00112	.00082	.00030
	−.00129	.00095	.00034
$\nu=7$	.00082	−.00061	−.00021
	.00005	−.00004	−.00001
.00000	−.00042	.00030	.00012
	−.00068	.00051	.00017
$\nu=8$	.00043	−.00031	−.00012
	.00003	−.00002	−.00001
.00000	−.00022	.00018	.00004
	−.00019	.00014	.00006
$\nu=9$	.00012	−.00008	−.00005
	.00001	−.00001	.00000
.00000	−.00006	.00005	.00001
	−.00007	.00005	.00002
$\nu=10$	.00004	−.00004	.00000
	.00000	.00000	.00000
.00000	−.00003	.00001	.00002
	−.00004	.00002	.00001
$\nu=11$	.00002	−.00001	.00000
	.00000	.00000	.00000
.00000	−.00002	.00001	.00001
	−.00001	.00001	.00000
$\nu=12$	.00001	−.00001	.00000
	.00000	.00000	.00000
.00000	.00000	.00000	.00000

$$D_{ij}^{(13)} = .00000$$

From the data in Table 2 the successive approximations to a given element in the matrix Q_{ij} are formed by continuous summation, using (22.17). When a listing adding machine is used the result will come in the form of a column of figures (the successive approximations) for each given (i, j) combination. It is therefore now more convenient to tabulate the result in the form used in Table 3.

TABLE 3.—SUCCESSIVE APPROXIMATIONS TO Q_{ij}.

	$j=-1$	2	3
$i=1$	−.50000	.16667	.33333
	−.83874	.36905	.46969
	−1.02501	.49823	.52678
	−1.11861	.56607	.55254
	−1.16306	.59877	.56429
	−1.18340	.61377	.56963
	−1.19243	.62042	.57201
	−1.19787	.62450	.57337
	−1.19958	.62576	.57391
	−1.20028	.62626	.57411
	−1.20072	.62648	.57422
	−1.20084	.62660	.57422
2	.09091	−.50000	.40909
	.25163	−.72132	.46969
	.35984	−.81954	.45970
	.41748	−.86502	.44754
	.44533	−.88602	.44069
	.45811	−.89556	.43745
	.46385	−.89983	.43598
	.46729	−.90231	.43502
	.46837	−.90303	.43457
	.46877	−.90343	.43457
	.46899	−.90354	.43457
	.46911	−.90366	.43457
3	.19643	.30357	−.50000
	.25163	.36905	−.62068
	.26639	.37700	−.64339
	.27127	.37640	−.64767
	.27317	.37540	−.64857
	.27401	.37486	−.64887
	.27436	.37458	−.64894
	.27460	.37442	−.64902
	.27469	.37433	−.64902
	.27469	.37433	−.64902
	.27469	.37433	−.64902
	.27469	.37433	−.64902

Each of the 9 cells in Table 3 represents one of the elements of Q_{ij}, the first figure in the cell being $(A_{ij}-e_{ij})$, and the last the 12th approximation to Q_{ij}. On each level of approximation we have the same check as before, namely that the sum of the elements in a given row is zero.

The next step is to compute the matrix whose (i, j)th element is $\sum_k(\gamma_{kj}-e_{kj})Q_{ki}$. It is convenient to write this matrix in such a way that the second affix on Q designates the row number and the second affix on γ the column number. The result is given in (9.6).

(9.6)

$\sum_k(\gamma_{kj}-e_{kj})Q_{ki}$	$j=1$	2	3
$i=1$	1.05162	−.58857	−.17031
2	−.78807	.63333	−.09292
3	−.26353	−.04478	.26323
Column check by (22.19)	.00002	−.00002	.00000

From the matrix (9.6) we compute easily f_{ij} by (7.3). The result is given in (9.7). Each element in (9.7) is obtained from the corresponding element in (9.6) by multiplication with the row and column factors indicated by (7.3).

(9.7)

f_{ij}	$j=1$	2	3
$i=1$	−934.18623	746.01018	188.19197
2	477.32202	−547.32936	70.00712
3	125.41158	30.40631	−155.82218
Column check by (7.5)	−.00420	.00580	.00015

From (9.7) we get $\bar{f}_{ij}$ by (8.5), the result is given in (9.8).

(9.8)

$\bar{f}_{ij}$	$j=1$	2	3
$i=1$	−93.41862	149.20204	131.73438
2	47.73220	−109.46587	49.00498
3	12.54116	6.08126	−109.07553
Column check: $\bar{f}_{0j}=\epsilon_j f_{0j}$	−33.14526	45.81743	71.66383

Finally the symmetric matrix ϕ_{ij} is computed by (8.7), the result is given in (9.9)

(9.9)

ϕ_{ij}	$j=1$	2	3
$i=1$	223368	-117031	-27706
2	-117031	86128	-4978
3	-27706	-4978	18754
Column check: $\sum_i P_i\phi_{ij}=0$	1.40	$-.73$	.09

In the three subsets (12), (13) and (23) the adjoints of ϕ_{ij} are

(9.10)

$\hat{\phi}_{ij(12)}$ (symmetric)	$j = 1$	2
$i = 1$	86128	
2	117031	223368

(9.11)

$\hat{\phi}_{ij(13)}$ (symmetric)	$j = 1$	3
$i = 1$	18754	
3	27706	223368

(9.12)

$\hat{\phi}_{ij(23)}$ (symmetric)	$j = 2$	3
$i = 2$	18754	
3	4978	86128

The determinant values are

$$|\phi_{(12)}| = 5542.10^6 \tag{9.13}$$

$$|\phi_{(13)}| = 3421.10^6 \tag{9.14}$$

$$|\phi_{(23)}| = 1590.10^6. \tag{9.15}$$

The check on the computation of the adjoint (9.10) is that the same value (9.13) is obtained by taking the product sum of the first row of (9.10) with corresponding elements of (9.9) as by taking the similar product sum in the second row. The adjoints (9.11) and (9.12) are checked in a similar way.

We now have the necessary tools to construct the optimum curve. The calculations are best arranged in a scheme as indicated in Table 4.

TABLE 4.—CONSTRUCTION OF THE OPTIMUM CURVE

	Starting point	1. Computed Point	2. Computed Point
u_1	0.0000	50.0000	100.0000
u_2	0.0000	−26.1969	−52.3938
u_3	0.0000	−6.2019	−12.4037
λ_1	0.0000	.0002238459	.0004476917
λ_2	0.0000		
λ_3	0.0000		
Check by (9.16)	0.0000	0.0000	0.0000
Check $\sum_i u_i P_i = 0$	0.0000	0.0000	0.0000
C_1	975.0000	1025.0000	1075.0000
C_2	1300.0000	1273.8031	1247.6062
C_3	1950.0000	1943.7981	1937.5963
Check $\sum_i C_i P_i - a_{00}0$	0.0000	0.0006	0.0009
Abscissa $C = \min [C_i]$	975.0000	1025.0000	1075.0000
Ordinate $2\Omega = \sum_i u_i \lambda_i$	0.0000	0.0112	0.0448

As the starting point on the optimum curve we take the one corresponding to $\Omega = 0$. In this point all the limitational factors u are zero; the quantities C_i are therefore by (6.20) simply the numbers (4.19), and C, the smallest of the numbers C_i, consequently 975. The situation in the starting point is thus nothing but the situation obtained when only partakers' percentages are used. The corresponding figures are inserted in the first column of Table 4.

From the magnitudes C_i in the first column of 4 is seen that it is here No. 1 that is the minimum factor; consequently if a move along the optimum curve is to be made, we must start by making u_1 a positive quantity.

If we did not need to take account of the fact that C_2 and C_3 will be lowered by the minimalization process (i.e., that u_2 and u_3 will become negative) we could immediately put u_1 so large as to bring the factor No. 1 on the same level as the next factor, which means that we would put $u_1 = 325$. But such a choice would certainly introduce at least No. 2 as a deficiency factor in the new point computed, thus keeping the total volume below the level 1300 which corresponds to the choice $u_1 = 325$. We shall therefore move by shorter steps, say, first increasing C_1 by 50, i.e., putting $u_1 = 50$. Moving by so small a step has also the advantage that we get information about the shape of the optimum curve in more frequent points.

The computations for the new point are carried out thus: first the figure $u_1 = 50$ is entered on the first line in the column headed "1. computed point" in Table 4. We now consider u_1 as the only conditional factor, and determine the corresponding λ_1 by (8.10) which in the present case simply means putting $\lambda_1 = u_1/\phi_{11} = 0.0002238459$. This figure is entered on the corresponding line in the column in question. An application of the sum check

$$\sum_{j=p,q\cdots t} \phi_{0j(p,q\cdots t)}\lambda_j - (u_p + u_q + \cdots + u_t) = 0 \tag{9.16}$$

where

$$\phi_{0j(p,q\cdots t)} = \sum_{i=p,q\cdots t} \phi_{ij} \tag{9.17}$$

may, for the sake of principle, be used even in this simple case where there is only one conditional factor.

The parameter λ_1 being determined, the two remaining factors u_2 and u_3 are computed (by applying (8.6) also for $i=2, 3$), and inserted in their place, whereafter the check $\sum_i u_i P_i = 0$, i.e., $u_1P_1 + u_2P_2 + u_3P_3 = 0$ is applied and entered in its proper place. Adding the quantities u_1, u_2, u_3, to a_{10}/P_1, a_{20}/P_2, a_{30}/P_3 respectively we get the new quantities $C_1 = 1025$, $C_2 = 1273.8031$, and $C_3 = 1943.7981$ on which the check sum $\sum_i C_i P_i - a_{00} = 0$ is run.

It thus turns out that the smallest of the C_i is *still* No. 1, which means that no deficiency factor has come in by the movement from the starting point to the first computed point. The first computed point is therefore the true minimum point which will be obtained when it is required that the total volume C shall be kept constant at 1025. The magnitude $C = 1025$ is the abscissa of the point considered on the optimum curve. Its ordinate (affected with the conventional factor 2) is obtained by taking the product sum $\sum_i u_i \lambda_i$ which gives 0.0112.

It is seen that the reaction in C_2 and C_3 corresponding to the chosen increase of 50 in the minimum factor C_1 is only small; indeed, C_2 has only dropped from 1300 to about 1274 and C_3 only from 1950 to about 1944; there is thus still a long way to go until any of the factors Nos. 2 or 3 become minimum factors. If only a rapid orientation is needed, it therefore now appears that there would be no risk in taking a bigger step during which factor No. 1 is still kept as the only conditional factor. We may put for instance $u_1 = 175$, which means putting $C_1 = 1150$. We shall, however, continue by small steps in order to determine a tighter set of points on the optimum curve. Putting in the next point $u_1 = 100$, that is $C_1 = 1075$, we can repeat the same computations as in the previous column. The result is that we determine a new point where $C = 1075$, and $2\Omega = 0.0448$. This point has also, as we expected, turned out to be a true minimum point, because the factor that was kept conditional (No. 1) remains as minimum factor in the point reached by the computation.

In this way we can continue. It turns out that when u_1 is equal to about 213 the second factor comes very closely down to being equal to the first, in other words, we are now very near a point where there are two minimum factors. The situation is indicated by the first column of Table 5.

TABLE 5.—CONSTRUCTION OF THE OPTIMUM CURVE CONTINUED

	6. point	7. point	8. point
u_1 u_2 u_3	213.2000 −111.7036 −26.4448	213.2632 −111.7367 −26.4526	313.2632 −11.7367 −158.5931
λ_1 λ_2 λ_3	0.0009544787	0.0009547616	0.0046205657 0.0061421693
Check by (9.16)	0.0000002	0.00001	0.0009
Check $\sum_i u_i P_i = 0$	0.0013	0.0014	0.0022
C_1 C_2 C_3	1188.2000 1188.2964 1923.5552	1188.2632 1188.2633 1923.5474	1288.2633 1288.2633 1791.4069
Check $\sum_i C_i P_i = a_{00}$	0.0016	0.0016	0.0025
Abscissa $C = \min [C_i]$	1188.2	1188.2632	1288.2633
Ordinate $2\Omega = \sum_i u_i \lambda_i$	.2035	.2036	1.3754

Making the very small step from point 6 to point 7 in table 5—still with only No. 1 as conditional factor—we reach a point where the two factors 1 and 2 coincide exactly in being minimum factors (within the accuracy carried in the computation). The point 7 where $C = 1188.2632$ may thus be taken as the exact point where factor No. 2 comes in.

Now let us decide to make a larger step forward, keeping this time *both No. 1 and No. 2 as conditional factors.* Let us, for instance, increase both these minimum factors by 100 as compared with the values they had in point 7. This means that we put $u_1 = 313.2633$ and $u_2 = -11.7367$. These figures are entered on the two first lines in the column for the 8. point. We have now to reckon with two λ parameters, namely λ_1 and λ_2. They are determined by (8.10), and from these values of the λ's are the other magnitudes in the 8. point determined. The result is given in the last column of Table 5.

It is seen that the factor that was now considered unconditional, namely the third factor, is still very far from being minimal, point 8 is therefore a true point on the optimum curve.

Continuing in this way we get a series of points which give the shape of the optimum curve exhibited in Figure 1. In all 15 points were computed on the curve.

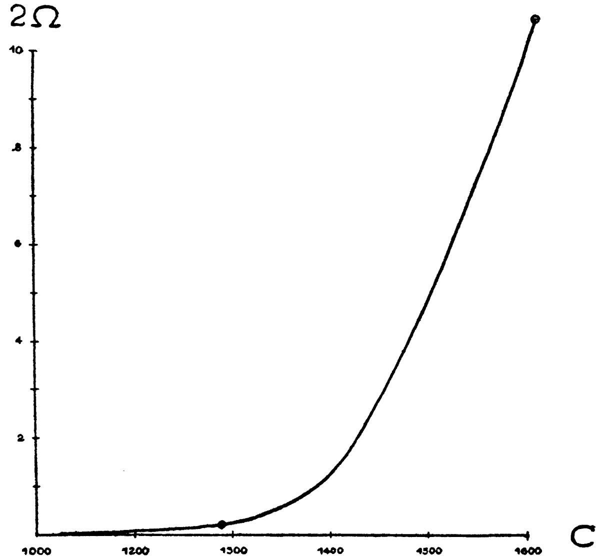


FIGURE 1

From the shape of the curve in Figure 1 is seen that the total inconvenience increases rapidly after C has passed 1200. It therefore seems indicated to fix the final adaptation in the vicinity of this point. Let us, for instance, choose the point 7 in table 5, where $C = 1188.2632$ (this happened to be just the point where factor No. 2 becomes minimal).

In the point thus selected as the final optimum point we first have to compute the rationing percentages x_1, x_2, x_3 which in the present case reduce to $x_i = \bar{f}_{i_1}\lambda_1$. The result is

$$
(9.18) \qquad \begin{array}{rrcr}
 & x_1 & = & -0.089193 \\
 & x_2 & = & 0.142452 \\
 & x_3 & = & 0.125775 \\
\hline
\text{Total} & x_0 & = & 0.179035.
\end{array}
$$

As a check on the computation we have

$$
(9.19) \qquad \sum_{j=p,q\cdots t} f_{0j}\lambda_j = x_0
$$

where, as before, the affix zero denotes the result obtained by performing a summation to the affix in question.

The values in (9.18) tell us that in the final adaptation point there is an *actual rationing* of the shoemaker's product (No. 1) while there is distributed an extra quota of the products of the tailor (No. 2) and the farmer (No. 3).

The rationing percentages being computed, we form the *rationing matrix* $a_{ij}\,x_j$; the result of this computation is given in (9.20).

THE RATIONING MATRIX

(9.20)

$a_{ij}x_j$	$j=1$	2	3	v_i
$i=1$	0.00000	10.68393	18.86624	29.55017
2	− 7.13540	0.00000	45.27897	38.14357
3	−29.43353	72.65071	0.00000	43.21718
Check: $a_{0j}x_j$	−36.56893	83.33464	64.14521	110.91092$(=v_0)$

The column sums in (9.20) ought to be equal to $a_{0j}x_j$.

The row sums in (9.20) we denote

$$v_i = \sum_j a_{ij}x_j. \tag{9.21}$$

The computation of the sums (9.21) is checked by verifying that the sum of row sums and the sum of column sums in (9.20) are equal.

On the basis of C and the u_i as determined in the optimum point (point 7 in Table 5) and the values (4.18) we compute the total purchases c_i of the partakers by the formula (6.17). The result is

$$\begin{aligned} c_1 &= 225.00 \\ c_2 &= 440.00 \\ c_3 &= 523.26 \\ \hline \text{Total } c_0 = C &= 1188.26. \end{aligned} \tag{9.22}$$

As a check on this computation we verify that the sum of the c_i is equal to the value of C in the optimum point. It is seen that the sum checks for the decimal places used.

Further it is seen from (9.22) that both the partakers No. 1 and No. 2 get a total purchase *exactly equal to the one they originally asked for* (compare (4.15)). This is only another expression for the fact that *we have now gone along the optimum curve far enough to reach a point where* *both Nos. 1 and 2 become minimum factors* (which means that both u_1 and u_2 must be considered as conditional under a *further* outward movement on the optimum curve).

By means of the values (9.22) and the row sums (9.21) it is now easy to compute the partakers' percentages

$$1 + z_i = \frac{c_i - v_i}{a_{i0}} \cdot \tag{9.23}$$

The result is given in (9.24)

$$\begin{aligned} 1 + z_1 &= .868662 \\ 1 + z_2 &= .913305 \\ 1 + z_3 &= .571483. \end{aligned} \tag{9.24}$$

As a check on the computation of (9.24) we have

$$\sum_i (1 + z_i)a_{i0} - C + v_0 = 0 \tag{9.25}$$

where $v_0 = \sum_i v_i$ is the grand total of the elements in (9.20).

Carrying out the check (9.25) we get 0.00135 which is zero within the limits of accuracy used in the computations. Indeed, the order of magnitude of the terms involved in (9.25) is between 200 and 500.

The fact that both $(1+z_1)$ and $(1+z_2)$ fall below unity must, of course, not be interpreted as indicating that neither partaker No. 1 nor No. 2 utilize their full capacity, indeed, in the present case where also rationing coefficients are used, the percentages $(1+z_i)$ do not—as in the simple case considered in Section 4—indicate the capacities used. As we have seen both Nos. 1 and 2 are actually in the present case used to full capacity.

By means of the partakers' percentages $(1+z_i)$ we compute the partakers' matrix $(1+z_i)\, a_{ij}$, which is given in (9.26).

THE PARTAKERS' MATRIX

(9.26)

$(1+z_i)a_{ij}$	$j=1$	2	3	Check: Sum of row no. i equal to $(1+z_i)a_{i0}$
$i=1$	0.00000	65.15019	130.30039	195.45058
2	73.06444	0.00000	328.78996	401.85440
3	188.58948	291.45646	0.00000	408.04294

Finally by taking the sum of the matrices (9.20) and (9.26) we get the completely corrected matrix (9.27).

THE COMPLETELY CORRECTED MATRIX

	$(1+z_i)a_{ij}+a_{ij}x_j$	Is allowed to buy from			
		Shoemaker	Tailor	Farmer	Total
(9.27)	Shoemaker	0.00	75.83	149.17	225.00
	Tailor	65.93	0.00	347.07	440.00
	Farmer	159.16	364.11	0.00	523.26
	Total	225.08	439.94	523.24	1188.26

The check on (9.27) consists in verifying that the sum of the first row is equal to the sum of the first column, and similarly for the second row and column and for the third row and column. It is seen that this checks within the accuracy carried. Finally, it should be checked that the grand total of (9.27) is equal to the abscissa C of the optimum curve in the point that was selected as the equilibrium point (point 7 in Table 5).

10. Estimate of the Amount of Work Involved when the Number of Variables is Very Great

The computations of the preceding sections can be easily carried through when the number of commodities (or services) involved is small. What will be the situation if this number is very large? Will the amount of work be prohibitive? I believe it will not. Let us try to make a rough estimate of the work.

The bulk of the work involved will consist of taking large product sums. For instance, nearly all the preceding formulae expressing one variable in terms of others are *linear forms*. And the inversion processes for the matrices considered are also built on a repeated use of product sums; this is, for instance, the case with the series leading to the determination of the matrix Q_{ij}. In practice some similar process, for instance the Neumann-series, will have to be used in order to determine the adjoint matrices of the ϕ's in the construction of the optimum curve. If only a rough estimate is desired, it therefore seems plausible to measure the amount of work by the number of multiplications involved.

The number of multiplications will for some of the computations be proportional to the number of variables n, for others it will be proportional to n^2 and for still others proportional to n^3. If we follow the method of Section 9 the handling of the basic matrices require the following number of multiplications:

	n^3	n^2
		matrix α
		matrix β
	matrix γ	
(10.1)	each step in the series for matrix Q	
	matrix f	
		matrix $\bar{f}$
	matrix ϕ	

If n is a large number, say 50, we need not take account of those parts of the work where the number of multiplications is proportional either to n or n^2; we get a sufficiently accurate estimate by considering only those phases of the work that are listed in the first column of (10.1). If ν steps are required in the iteration process performed to get matrix Q, the total number of multiplications involved in the construction of the basic matrices will therefore be

$$(\nu + 3)n^3. \tag{10.2}$$

For the determination of one point on the optimum curve we need a Neumann-series, or a similar series, by which to invert the m-rowed submatrix ϕ, m being the number of conditional factors. This involves a number of multiplications equal to m^3; if μ steps are needed before the process converges, this gives μm^3 multiplications. We may therefore conclude that the total number of multiplications needed to construct the optimum curve by means of the basic matrices is

$$\omega \mu m^3 \tag{10.3}$$

where ω is the number of points on the curve, m the number of conditional factors needed in the determination of one such point and μ the average number of steps needed in the convergency process.

The computations necessary to finish the work and finally to arrive at the corrected request-matrix, only involve a number of multiplications proportional to n^2 and may therefore here be neglected.

As an example, suppose that $n=50$, and that $\nu=7$ steps are needed in the iteration process performed on Q. This gives by (10.2) a total number of multiplications equal to 1250000. From the time records in my statistical laboratory at the University Institute of Economics in Oslo, it appears that one should strike a safe average by saying that when everything is included: preparatory steps (arrangement of tables, etc.), checkings, search for errors, occasional rests, etc., 100 products of 6 digit numbers take on the average 1 hour. Of course, for any particular small job, the time used may be considerably less, but a safe long time average would, I think, be somewhere near the one mentioned. On the basis of this the handling of the basic matrix would require about 300 weeks' work.

The construction of the optimum curve would in all probability take somewhat less; indeed, the number of variables which on the average is included in the conditional set would as a rule hardly be larger than one-third or one-half of n; and it would seldom be necessary to determine more than, say, 6 or 7 points on the curve. At any rate it seem that this item, as well as the other smaller items would be largely covered by doubling the 300 weeks previously found, thus giving 600 weeks. If the work is to be completed in 2 weeks time, a staff of some 300 computors would thus be needed.

I have here not made any allowance for the use of special labor saving devices that may be developed. If the work was organized on a permanent basis, it is quite obvious that such devices would be worked out to great perfection. One only has to think of the possibility of punching two factors separately on two loose halves of Hollerith cards; in the present case where repeated product sums are used *with one fixed factor*, this would be a great help. Various kinds of special short cut arrangements of the computations would also undoubtedly be invented as experience was gained in the actual work.

But even assuming that a staff of 300 people is needed, would this be a big item to reckon with? I think it is right to say that it would—even in a small country—be nothing but a *quantité négligeable* if the system actually was able to secure some of the results aimed at.

PART II. APPLICATION

11. Introduction: General Remarks on the Practical Organization of the Planned Exchange Service

The administration of a planned exchange service built on the principles expounded in the preceding sections will of course raise a series of practical problems. Will it be possible to solve them in such a way that the system actually becomes practicable? I believe it will.

It is well known that in various parts of the world (particularly in the United States and Germany) isolated attempts have been made to establish exchange organizations or introduce new types of money-substitutes which have had the object of pressing forward that exchange of goods and services which under the old system had stagnated. According to the reports which are available these measures seem to have done much good. They have, however, all been of a more or less local and sporadic character. None of them have been organized on a national basis. There is reason to believe that if such measures were organized on a national basis, and supported by federal or state authority, they would be capable of yielding great results.

If the economic activity *within* each of the big economic groups in the country, for instance, within the shoe industry, the textile industry, agriculture, etc., had been perfectly planned and controlled, the problem of the national exchange services would only have been to organize the circulation *between* the groups. This could have been done almost exactly according to the method of the preceding sections.

The activity within each such group is not, however, at present organized as a planned unity. The problem is thus how the *individual* persons or concerns within a group can be fitted into the exchange service.

It is clear that it will not be possible to use an accounting system which is so complete that each individual person, who can be expected to become a participant in the exchange transaction, receives an individual account, which is balanced against all the other accounts by the preceding methods.

It is only the *large groups* which can be treated in this manner. But at the same time something must be done so that there is established an equity between services and counterservices also for the individual persons or concerns within a group. It is the arrangement of this question which is the actual practical problem in the carrying out of the exchange system. The principle according to which a solution must be sought, must, in my opinion, be the following: The warrants are drawn on one of the large groups, not on a particular person. The distribution, of the warrants, on the other hand, must be made to particular persons, perhaps, through the medium of certain centrals (see below). A complete account must be kept of what the individual persons or concerns are allocated of goods warrants. This being done, efforts must be made to arrange matters in such a way that the warrants *by themselves*, more or less automatically flow through the usual trade channels and finally find their way to the proper suppliers who participate in the exchange service. The following is a suggestion of how this may be carried out in practice.

12. Membership in the National Exchange Organization

To a certain extent the technique in the organization of such an exchange service will resemble that which was employed in the rationing during the war. There is, however, the difference that the exchange service should be based on voluntary participation, and must embrace many more kinds of goods and services. It will first obtain its full effectiveness when there is a comparatively large choice of goods and services.

Everybody who desires to participate in the exchange service, individual persons or concerns, ought to subscribe as *members* of a particular branch (textile, agriculture, etc.). The nomenclature of this division is an important question which would have to be looked into with great care. Transversally of this division there must be a geographical division. The most expedient would presumably be that, in each city or country district, there be established an exchange office. To commence with, this could assume a very modest form. It may perhaps be arranged simply by some taxation officer undertaking to keep the necessary lists. If a member had a considerable interest in more than one group of suppliers, it would probably be more practical to let him enter with a separate membership into each of the groups in question, treating him as an independent person in each of the groups.

The wage earners form a class of suppliers which it is especially important to include in the exchange service. The increase in the exchange of their "goods" means nothing less than a reduction in unemployment. It is of course only the unemployed who could form part of the exchange organization. A fully occupied workman has indeed no "goods" which he can deliver. It would not be practical that each unemployed person should be a personal member of the exchange organization. It would be more efficient if the various *trade unions* were members. When the total exchange table for the country had been balanced and the warrants issued, the trade union in question would then have to decide *which* of its members should start activity and thus get the warrants. To start with, it is of course very improbable that the balancing of the whole exchange matrix for the country could take place in such a manner that all the unemployed in all the trade groups received occupation.

13. The Exchange Requests and Their Balancing

The exchange could best be organized in definite periods, for instance by *months*. For the sake of simplicity I will in the following speak of the exchange period as a month, but the reasoning is quite general and applies irrespective of the length of time one finds it advisable to choose for the exchange period.

The first step towards planning the exchange in a definite month is the collecting of exchange requests from the members. In these requests the members would have to specify the amounts for which they desire warrants on the various groups of suppliers, and furthermore give a description of the commodities or services they propose to deliver. These requests would first have to be criticized by the district office or, if the case arises, by a district committee appointed especially for this purpose. It would have to be verified that the total amount which the member in question had requested was in a reasonable proportion to what he could be expected to be able to supply.

In the beginning, this preparatory criticism of the request would have to be made very strict. There could be fixed certain comparatively low maximum limits for the various categories of members. For an ordinary productive concern one should perhaps in the beginning maintain the total sum of goods warrants within say 20 per cent of the production capacity of the concern in question. For other categories the maximum limit could be fixed higher, for instance for unemployed workers virtually at 100 per cent. After the system had started to operate and experience had been gained, it would naturally be an easy matter to increase some of the limits.

The question of preliminary criticism of the exchange requests is connected with the question as to the manner in which one would formulate the legal claims which the National Exchange Organization got on the supplies by the fact that the Organization gives them warrants at the beginning of the month. More precise priority rules for these claims would have to be worked out.

The criticized and, if it should be the case, reduced exchange requests would have to be added up by the district offices, and the totals for for each individual group of suppliers would have to be sent to the national headquarters of the Organization. Here the figures for the entire country would have to be added up in order to get the total requests matrix for the country. The figures in this matrix would then have to be dealt with on some such principles as explained in the preceding Sections. On the basis of the balancing method which may have been adapted the necessary percentages, partakers' percentages, rationing coefficients, etc., would have to be computed and the final exchange figures for each group fixed. The total corrected exchange matrix thus obtained would then show complete demand—supply balance apart from the service taxes that may have been introduced.

Irrespective of the principles of balancing that had been chosen, it would be practical not to carry out the calculation of the necessary percentages to many decimal places, but only use the adopted balancing principle to compute a table that showed *approximate* agreement between demand and supply, and then make the final balancing of the exact amounts more or less by a rule of thumb.

This balancing of the exchange matrix should be done on a *national* basis. No attempt should, for instance, be made to make the exchange service a closed cycle within certain geographical spheres. It must be sufficient that demand and supply for each group balance when taken as a whole for the entire country. To attempt a balance for any special geographical sphere would be in direct conflict with the fundamental idea underlying the exchange service. The aim should just be to arrange for a cooperation between the various parts of the country, and between the special branches of industry within the national border.

14. The Issuance and Utilization of the Exchange Warrants

When the national headquarters had balanced the exchange accounts for the various groups, the necessary number of exchange warrants would have to be printed. These should be sent to the district offices which must then see to the distribution to the individual members in the district. The national office would have to keep account of the amount of warrants issued to the various suppliers' groups in each district, and the district offices would have to keep the further detailed account of the amounts issued to the individual members in the groups. This latter accounting would presumably be done most easily in the form of loose leaf, the amounts could simply be entered in a special column on the original request blanks received from the members.

It would probably be advantageous only to issue warrants in a few denominations, e.g., only use warrants for $10 and $100. It is desirable to simplify the technical printing apparatus as much as possible because the printing of the warrants is something which—if the system acts according to intention—will take place each month.

Safe methods of auditing the accounts would naturally have to be arranged. In addition to the local auditing, which could be left to the district authorities to arrange, there would have to be organized an audit by travelling auditors directly under the National Headquarters. The danger of embezzling will in all probability scarcely be great in such a system where all service is to be *entirely balanced at short intervals.*

It must be a fundamental principle that one endeavors to make the use of the warrants as *simple* and *convenient* as possible. In various ways efforts must be made to make the warrants easily exchangeable so that there is created the least possible friction during the process whereby the warrants flow back to those who at the end of the month are again to deliver them up to the Exchange Organization.

In order to obtain this, the warrants ought to be *made out to bearer,* so that anyone who obtains possession of a warrant for the month in which it applies can use it. It means, for example, that townspeople may simply hand over to their grocer those of the warrants which apply to farm products. The grocer for his part would then use the warrants for his purchases. A certain grocer will perhaps be used to dealing with a particular farmer at a particular place. There is naturally nothing to prevent him from using the warrants as payment just *vis à vis* of this farmer, on the assumption of course that the farmer is a participant in the exchange service during the month in question. Before putting through his order the grocer would probably ascertain that the farmer is a member. In this manner it would be possible to use without any alteration the whole technical exchange apparatus that already exists.

Another condition for the warrants being easily used in the manner indicated, is that they are not *drawn on* specified suppliers, but only on large groups of suppliers. Even if the warrants are drawn in this general manner there is every prospect that the exchange process will lead to such a result that each single individual supplier at the end of the month comes into possession of the quantity of warrants which he is to account for. The guarantee that this result will be attained lies in the fact that none of the participants has the right to acquire (through sales of his products) warrants for an amount *greater* than that which he was allocated at the commencement of the month. The most practical course would probably be simply to lay down the rule that any amount which a supplier may have acquired of warrants during the month over and above the amount which he was granted at the beginning of the month *shall be valueless*. That is to say, the National Organization undertakes no obligation to pay him in cash the countervalue of the surplus of warrants which he may thus hold.

This method of assuring the right distribution of the back-flowing warrants can be illustrated by the arrangement used when there are issued *non-numbered* tickets to a theatre. The method will not break down just because the tickets are not numbered. It only means that one leaves it to the individuals themselves to find their places. Of course this may result in some having slightly better places than others. Those who obtain the best places are simply those who take the trouble to arrive in good time before the performance commences. Every participant is, however, in the same position in this respect so that the arrangement contains no injustice. And, furthermore, it must be admitted that it is of subordinate importance whether some obtain a little better places than others, provided only that all the places are fairly good.

This naturally does not mean that one shall not do what one can in order that the process of adjustment can take place as easily as possible and give as great advantage as possible for those who use the warrants. In this regard the theatre illustration might be supplemented somewhat. If there is a big rush one would naturally take auxiliary measures in order to conduct the public to their places. One would establish queue formations, give out notices at short intervals as to where seats are still available, etc. The main point is that when everybody knows that there is sufficient seating for everybody provided with a ticket, the question of getting people to take their places quietly will always be a problem which can be surmounted.

One of the things which will contribute towards a proper distribution of the back-flow of the warrants to the suppliers is that an easy opportunity be given for *voluntary clearing agreements* within the various branches. For instance, the shoeshops in a given city may establish a collegiate agreement to the effect that if the shop X in the course of the month has been able to acquire a greater amount in warrants than it is to account for, while on the other hand the shop Y has not been able to acquire a sufficient amount, Y shall have opportunity to buy a number of X's excess warrants *at a reduced price*. By such an arrangement X and Y would share the extra profit which X has obtained, because its exchange has increased by a greater amount than that which was originally the intention.

Also with regard to the distribution of the warrants which the participants receive at the beginning of the month, there should be open full access to voluntary clearings. Two workmen who have both received warrants but not exactly with the desired distribution may, for instance, exchange between themselves shoe warrants and warrants for farm products.

If the exchange service gets going on a really large scale, there would certainly develop such arrangements and many other forms of clearing activity. All this should not only be permitted but encouraged. It would do much to increase the effectivity of the entire system.

15. Information Service. The "Redeemability" of the Warrants

The National Exchange Organization should arrange a comprehensive information service regarding the system and its functioning. There should, at frequent intervals, be printed up-to-date lists of the suppliers within the various branches and within the various parts of the country. Alphabetical and systematical lists of the kinds of goods involved would also be useful. If the system once gets going well, advertizing by private firms would undoubtedly also help in this information activity.

All this would help the majority of people, who have warrants, to find out comparatively easily where and how they could use their warrants. The National Organization must, however, be prepared for all eventualities and therefore take special measures whereby everyone who has a valid goods warrant can be quite certain *under all circumstances* to obtain that for it for which it is made out. This arrangement is for the warrants what gold redemption is for the ordinary paper notes. The National Exchange Organization must regard it as its duty always *to help the public* with the redemption. It must never permit itself to do what many central banks of issue frequently have done, namely to maintain redemption in name and carry on an actual sabotage *vis à vis* those who desire to avail themselves of it. If the National Exchange Organization lives up to this duty of always helping the public, the warrants will obtain a "redeemableness" which makes them immeasureably more secure than any of the usual kinds of money which are only based on gold.

Let us look at a few practical measures which the National Exchange Organization could adopt to ease the public's use of the warrants. Let us suppose that a person has a warrant but is late in utilizing it. The end of the month is perhaps drawing near so that the majority of the shops at the place have already used their quota. The person in question is of course certain that there actually exist suppliers who are prepared to deliver goods for his warrant. Indeed he knows that to each warrant issued there corresponds an obligation to deliver. The only problem for him is to *find* the right supplier.

For such cases the county offices and, if the case arises, the National office itself, ought to organize a special form of information service. The ideal would be that the National office keep a complete current survey of the progress of the activity within the various branches and in the different parts of the country based on current reports from the suppliers telling how much of their quota they had delivered up to that time. A speedy publication of these summaries, for instance through the daily papers, would greatly assist in directing the movement of the still floating warrants. Such an arrangement, however, would cause rather a large apparatus, which in the beginning should perhaps be avoided. To start with, one could perhaps content oneself with an arrangement whereby the suppliers who had not yet delivered their quotas, say, after the end of the first three weeks of the month, reported this. Or one could arrange reports on the basis of *estimates*.

16. Special Measures for Those who have not Succeeded in Placing their Orders

Even if the information service is fairly well organized, it is conceivable that some participants at the end of the month have still not found suppliers for their warrants. The simplest arrangement for the National Exchange Organization would then be to declare these warrants valueless. I do not believe, however, that this would be a wise policy. Even if it should cause a certain extra apparatus, one should do what one can in order to secure these "laggards" also. This will be another feature of the National Exchange Office's endeavor to assist the public.

The following arrangement might be possible for the laggards. Those who at the end of the month have not yet placed their warrants should within a certain period, e.g., ten days, send in the warrants to the National Exchange Organization with a detailed *order* for the goods they desire. At the same time every district office should send a report of the amount of warrants which the participants had properly accounted for by the end of the month, and also a survey of the amounts which still remained open within the various branches.

Against these latter amounts the National Exchange Organization would now be able to draw *supplementary warrants*. These supplementary warrants should (in contradistinction to the original main warrants for the month) be drawn, not only on a group of suppliers but more specifically on such a group in a particular district. It is possible that they might even be drawn on a particular supplier; in this latter case the name of the supplier should be added by the district office inasmuch as it will scarcely be practical to send names of suppliers to the National office. The laggards would receive such supplementary warrants corresponding exactly in kinds of goods and amounts to the orders they had placed with the National Exchange Organization. And these supplementary warrants would now contain such an exact indication of supplier that no difficulty would be caused in finally obtaining delivery.

It is conceivable that, even after the issue of the supplementary warrants, there would be some participants who remain with a balance for which they had not handed in a settlement. Such balances would be of two kinds. In the first place it might be a balance with a participant who was capable of delivering but where customers had simply not appeared. Such cases will only arise if a number of warrants for one reason or another are *lost* on their way through the exchange machinery. The number of such warrants will scarcely be greater than the number of unfetched prizes in a money lottery. Where such cases arise the National Exchange Organization could simply close the account and dis regard the balance or, if it is found practical, make a certain deduction in the amount of warrants which the participant in question receives in the next month.

Another kind of unsettled account will arise in cases where one of the participants has failed and been put under receivership in the course of the month. There must then be present a direct case of fraud, as the person concerned must already at the beginning of the month have known his position. In such cases there will at the end of the month arise a corresponding number of *unredeemed* warrants (they will presumably be presented to the National Exchange Organization in the manner mentioned in the preceding Section). The National Exchange Organization must honor these warrants, if the case arises, by a direct payment in cash. And the National Exchange Organization itself must then prosecute its claim *vis à vis* the supplier and obtain what is possible of the claim. If the preliminary criticism of the requests is properly carried out, the loss to the National Exchange Organization on this score would be very small in comparison with the whole month's volume of transactions. Such losses must be covered in the same way as the ordinary administration expenses.

It should be a general principle that all effects of a transaction which has taken place in a particular exchange period shall be completely deleted *shortly after the end of that period.* Old accounts, old uncovered items, and the like, should not exist. Even if a number of small losses must be written off, it is better to make a clean sweep so that the Organization and its members are always concerned with the present services and counterservices. As part of this policy it should be laid down that the supplementary warrants issued by the National Office should only have validity for a certain shorter period, for example 20 days, and that thereafter all the items which must still remain open should be regarded as proscribed.

17. Connections between the Exchange Economy and the regular Money Economy

The activity which is set on foot by the exchange operations will in several ways affect the ordinary money economy. Even if the circle of participants in the exchange transactions is made very large, it is not likely that *all* activity will be carried on within this circle.

When a participant sets up his exchange request there is, of course, nothing to prevent him from reckoning that the carrying out of the exchange proposed may involve also certain *monetary* transactions for him. Some of these monetary transactions will perhaps take place between the participant in question and certain National activities which are outside the exchange circle, while others will take place between him and certain activities which affect importation or exportation. In both cases the exchange transaction will probably act to stimulate also that economic activity which takes place outside the exchange circle.

This will always be a benefit in so far as the national activities are concerned. With regard to the activities connected with the international exchange, it is conceivable that situations will arise where the stimulation of the activity will increase importation more than is desirable. This stimulation of importation is, however, a question which it is scarcely correct to judge on the basis of the mechanism of the planned exchange service only. Importation is a problem which must be taken up for special consideration. If it should prove that it had in a given case been too great—either because of the ordinary money economy activity, or on account of the exchange service—an effort must be made to regulate the circumstance by the means which are known in this sphere: rise in customs duties, rationing of foreign exchange, import prohibitions, and the like.

Some simplified examples will illustrate the way in which a connection can be made between the exchange service activity and the ordinary money economy. Let us first consider a boot and shoe factory in Norway which uses Norwegian hides and employs Norwegian laborers only, and where no accessories are imported from abroad. Let us assume that the owner of the factory is granted Kr.1200 warrants for farm products and Kr.1000 warrants for Norwegian boot and shoe workers. Let us further suppose that of the Kr.1200 he uses Kr.700 to purchase hides which are used as raw material for the manufacture of boots and shoes, and that he further uses warrants for Kr.500 for the purchase of butter and eggs for his own housekeeping. The Kr.1000 warrants for Norwegian boot and shoe laborers he uses as wages. The sales value of the shoes we assume to be equal to the total of the warrants received, namely Kr.2200. Here the circle is entirely closed. The net result for the owner is that he has received Kr. 500 worth of butter and eggs. Apparently he has obtained it gratis. In reality it is, of course, not gratis. From an economic point of view it is an allowance for his contribution towards the result of the production process.

Next, let us modify the example to the effect that the above production costs of Kr.1000 does not in its entirety consist of wages, but that, for example, Kr.700 is wages and Kr.300 is depreciation on machinery. This machinery may have been imported or made in the country. This is of secondary importance in this connection, the main thing is that the machines represent a service from someone who does not participate in the exchange circle. In this case the Kr.300 amortization must appear in the manufacturers' ordinary money accounts. But this naturally does not prevent him from participating in the exchange transaction.

He can do it in two ways. We assume that he will produce the same quantities of shoes as assumed in the first example, that is, shoes valued at Kr.2200. He can then either reckon that of the Kr.2200 value of products only Kr.1900 is to be disposed of through the exchange service, while Kr.300 of shoes is to be sold in the usual manner for cash. In this case, in his request warrant he will demand: Kr.1200 warrants for farm products and Kr.700 warrants for shoemakers' work. Of this he will now, as previously, use Kr.700 for the purchase of hides and

Kr.500 for butter and eggs, which together amounts to Kr.1200 warrants for farm products. In addition he will pay out as wages Kr.700 warrants for shoemakers work.

He can, however, also reckon that the whole of the product value, namely Kr.2200, shall be disposed of through the exchange service. In this case he must add a new item in his request warrant. It must be an item which is not in connection with the footwear production but covers certain other expenditure which otherwise would have gone *via* his ordinary money economy. He may for example, want to have his house painted. Suppose that it would have cost Kr.300. By adding in his request for Kr.300 warrants for painting, he will obtain an exchange budget through which the whole of the product value, Kr.2200, can be disposed of. He will now ask for Kr.1200 warrants for farm products, Kr.700 warrants for shoemakers' work and Kr.300 warrants for painters' work. This exchange budget has the same effect for him as if the whole of the activity in the manufacture of boots and shoes—inclusive of amortization—was settled in and with the exchange transaction, while he had had Kr.300 cash outlay for painters' work. His profit on the transaction is naturally the same as in the first case, namely Kr.500.

18. Cooperative Construction Companies in Connection with the National Exchange Organization

The exchange service as it has been sketched in the foregoing does not apply only to articles of consumption. The system opens a certain opportunity for private concerns to use the exchange service as a link in the construction of permanent means of production and plant. When, for example, a building is to be erected, the builder can pay part of the wages in the form of warrants. This applies even if the constructural work is of such a kind that its result does not enter into the exchange system but must continue to be owned by the person who has erected it. Only in this case the person in question must carry on some *other* activity also, the products of which can enter into the exchange service. He must have certain products by the sales of which he can acquire warrants, (since he will scarcely have any interest in purchasing warrants for cash).

This naturally limits the importance of the exchange service for the actual construction activity. This fact is connected with one of the principles which formed the basis for the balancing of the exchange matrix, namely, that the participants should not do any *saving* by the exchange system. This would complicate the acounting considerably. This question can however be solved by the organization of a certain kind of concern: *cooperative construction companies* in connection with the National Exchange Organization.

By means of these companies the exchange service can be extended so as to embrace also such activities as those where the participants give work or goods, without immediately receiving in exchange other goods or services, but where they become participants in a saved-up value. It is probable that such an arrangement would be of considerable interest both to large and small savers, not least because the saving would be invested in *real values*, that would not suffer from monetary depreciation.

The cooperative construction companies could be organized in approximately the following manner: They must be under public control. Their financing should be based mainly on goods and services obtained through the National Exchange Service. The technique for the accounting in connection therewith would simply be that the construction companies, like other members of the National Exchange Organization, should send in their requests each month. As a counter-service, however, these companies would not offer to deliver other goods and services, but *property shares* in the company. The property shares could either be issued in the form of participation certificates or the participants could simply be credited on a current account with the company.

It would be a sound principle if these shares or accounts established through the National Exchange Organization were, practically speaking, the only financing basis for the construction companies. One should not perhaps definitely prohibit the construction companies from taking up loans in the usual forms (bank credits, bond loans, etc.) but there should in any case be fixed very strict limits as to how large a part of the total capital could be obtained in this manner. If capital should be obtained over and above that which is saved through the National Exchange Organization, it would be more reasonable that the State should enter as a share owner.

With regard to the economic exploitation of such enterprises distinction may be made between the following three cases:

a. The plants erected are of such a nature that they can be kept going on a regular profit basis. In this case they should be managed and exploited as private enterprises.

b. They are only partly rentable in a private economic sense (e.g., harbor constructions). In this case it would be reasonable that the State pay a certain *annual fee* to the construction company *for the social service* which the plant in question renders.

c. The plants are wholly unrentable in a private economic sense, that is to say, it is found impractical to institute such rules for their exploitation as would make them privately economically rentable (e.g., tolls on roads). In this case the State must pay an annual fee large enough to keep the plant on a paying basis. The principle here is the same as under (b), only now the *whole* of that which is necessary for upkeep and interest would be regarded as a fee payable by the community for the use of the plant.

The main advantage of the arrangement here outlined is that the financing of the constructional works does not come over the State Budget, so that no new taxes or Government loans are involved.

19. Some Further Possibilities

Many city and county expenses are of such a kind that they can be paid in warrants. This applies particularly to small communities in country districts. Here the use of warrants will in many cases be a clear advantage. And even in large city municipalities undoubtedly certain kinds of payment could, without inconvenience, be made in the form of warrants.

A portion of the *municipal taxes* could then also be paid in warrants. In reality this means that a system is effected where the citizens, to a certain extent, pay their taxes *in the kinds of goods and services which they themselves produce.* In many cases this would undoubtedly contribute towards making taxation less burdensome. Such taxes paid through the Exchange Organization would resemble the old taxes *in natura* but the whole system would now be more elastic and more efficient.

If first certain municipal expenses were settled in warrants, it is also conceivable that the *banks*, to a certain extent, would let some of their payments take this form. In any case this could be done where the municipalities take up bank loans. To the extent to which the banks find they are able to use warrants in their payments they would also be able to accept payments of *debt* instalments in the form of warrants. This would be a help towards liquidating frozen credits.

The methods which are developed in the foregoing for balancing a closed national exchange circle can in principle also be applied to exchange between countries. International Trade has at present developed more and more in the direction of contingents and quota agreements. Such agreements between *two* countries have, however, only limited possibilities. In the international as well as in the national sphere it is the *indirect* exchange that brings the big advantages. Sooner or later the question of three-cornered or many-cornered contigent contracts will therefore probably arise. The technique of the request matrix seems to be the logical technique to apply to the situation.

20. Conclusion

1. The organization of a National Exchange service for a country seems, both on theoretical and practical grounds, to be possible. There can scarcely be any doubt that such an arrangement would help greatly to *break through* the obstacles of a purely circulation kind which the

depression has created. Pessimism and lack of confidence would not be able to stop the exchange within such a system inasmuch as the participants would be placed face to face with the *fait accompli* that they *have* those things which will enable them to buy the goods they want. They would not first have to obtain the means by increased sale of their own products. The vicious phase connection between sales and purchases would be broken.

2. The arrangement involves practically no *financial responsibility for the State*. The State's expenses will only be for administration and to cover losses caused by direct frauds.

3. The arrangement is enormously *elastic*. According to the desires of the participants, the volume of the exchange may be increased or decreased, or, if the case arises, may be diverted in new directions, without necessitating any great alteration in the administration and the technical apparatus.

4. If the system gradually works itself in, one will thereby obtain a real *survey* of the forces that exist both on the demand and supply side in economic life.

To be concluded in an early issue

AVIS DE LA RÉUNION DE LA SOCIÉTÉ D'ÉCONOMÉTRIE À STRESA, LAGO MAGGIORE, ITALIE, SEPTEMBRE 1934.

LA 4° réunion européenne de la Société internationale d'Économétrie se tiendra à Stresa, Lago Maggiore (Italie), vers le 25 septembre 1934.

Le programme des travaux sera arrêté, au cours de l'été, dans le même esprit que les prédécents, par le Comité permanent présidé par Monsieur Bowley.

Les membres qui envisagent de participer à la réunion sont priés de le faire connaître dès que possible à Monsieur G. DEL VECCHIO, 8–10 via delle Lame, BOLOGNA (Italie), qui a bien voulu se charger de recueillir les adhésions et leur fera parvenir les circulaires ultérieures.

Ils sont priés, en outre, de faire connaître s'ils désirent participer à un arrangement collectif pour le logement et les repas, en vue de réduire les frais de séjour.

Les adhésions devront être accompagnées du versement de vingt lires pour la couverture des frais d'organisation.

Les membres désireux de proposer une communication devront en adresser le texte ou tout au moins une analyse assez détailée à Monsieur R. G. D. ALLEN, London School of Economics, Houghton Street, Aldwych, LONDON, WC 2, le plus tôt possible.

Le Vice-Président
F. DIVISIA

5 Mai 1934