THE PRINCIPLE OF SUBSTITUTION.

AN EXAMPLE OF ITS APPLICATION IN THE CHOCOLATE INDUSTRI [1]

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1. INTRODUCTION.

Various parts of economic theory are at present emerging into a truly quantitative stage. Many principles, which so far have been formulated quantitatively only in the abstract, are now being confronted with the facts. Attempts are being made at measuring numerically the forces at play within difinite fields, for instance, within a given market or in the technical production in a given industry or a given firm, and so on. Many of the quantitative data which are being utilised in this work are in themselves not novel. Some of them have, to a smaller or larger extent, already been utilised a long time by engineers and cost accountants. The particular way in which they are now being utilised is, however, novel. By being interpreted in the light of modern economic theory they receive a new significance and throw new light on the many problems which are of special concern to the industry or to the firm and also on problems which are of a much more general economic interest. On the whole this is a field of study in which the engineer, the cost accountant and the economist have much to learn from each other.

The following is an example of this type of analysis. It is a very modest piece of study. It was originally made only in order to serve as an illustration in connection with my lectures last spring on productivity theory. But it did arouse a certain interest both with the students and, not least, with the representatives of the firm from which the data were taken. It may perhaps, therefore, be worth while to give a brief account of it in the columns of Nordisk Tidsskrift for Teknisk Økonomi.

The data were furnished by *Freia Chocolade Fabrik,* Oslo. The Managing Director, *J. Throne Holst,* and the Chief-Engineer, *H. Throne Holst,* have kindly permitted these data to be used for the present purpose.

[1]) Presented as Publication No. 10 from the University Institute of Economics Oslo.

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I am particularly concerned with giving a clear-cut presentation of a case which shows how the *principle of substitution* works in practise in a factory of the *Freia* type. In order to make the argument as definite and simple as possible, I shall confine the analysis to one particular aspect of the production process. The whole theoretical productivity apparatus, the notion of marginal productivity, the principle of substitution, etc. may of course be applied just as well within a *part* of a given production process as to the process as a whole. This simply means that the *rest* of the factors are considered as data in the particular study concerned. As an example I will consider a certain phase of the production of ordinary nut chocolate for eating purposes.

The basic raw material in chocolate production is the *paste* (cocoa mass). Is is composed according to certain recipes. Each kind of chocolate has its characteristic recipe. The main parts of the paste are generally cocoa beans and sugar. After the paste is throughly mixed it is heated and poured into moulds, subsequently these moulds are brought into cooling cabinets (with a temperature of between + 1° and + 2° C.).

There always exists a certain risk that the paste will not fill the moulds completely. A certain number of the castings must therefore be scrapped. The essential fact from our view point is that the probability of this scrapping is all the lower the more liquid the paste, which in turn means the higher its fat content. The scrapping percentage may indeed be looked upon as a technically rather well defined function of the fat content.

The cocoa beans contain some fat so that the paste will also get a certain percentage of "natural" fat content. In the case of the nut chocolate the recipe—as determined by the taste type and other qualities of the nut chocolate which it is desired to produce—is such that the paste gets a "natural" fat content of about 30 per cent, that is to say 30 kilogrammes fat per 100 kilogrammes mass. With this fat content the moulding is exceedingly difficult and consequently the scrapping percentage very high. In order to increase the liquidity of the paste, one therefore resorts to adding some pure cocoa fat (technically called "butter") to the paste. The quantities of pure cocoa fat which it is here a question of adding are so small that they will not change the character of the produce. Since the cocoa fat is more expensive than the rest of the paste, the cocoa fat may be looked upon as an economic *factor of production,* which may be used in varying quantities.

To study the effect of this factor of production we must consider in more detail the effect of the scrapping. If a certain number of the castings are scrapped, they may, without any appreciable loss in paste, immediately be thrown back into the liquid mass and remoulded. In other words, the only loss which the scrapping entails is the work involved in the moulding and cooling. For our present purpose it is therefore convenient to consider moulding and cooling as one single process. We shall call it the moulding-cooling work. If the scrapping percentage is high, we may simply look upon this as meaning that *much* moulding-cooling work is incorporated in the finished product.

Thus we have here a rather clear-cut case where there are present two factors of production: 1) the addition of pure cocoa fat, 2) the moulding-cooling work. These may to a certain extent *substitute* each other. The problem is: How far shall we go in this substitution? In other words, how much cocoa fat will it pay to add?

Obviously the answer to this question depends in the first place on the technical curve that expresses how the scrapping percentage varies with the fat addition, and in the second place it depends on certain economic data, primarily on the price ratio between ordinary paste and pure cocoa fat and the special costs involved in the moulding-cooling apparatus.

2. THE DATA.

There are of course certain difficulties in getting an absolutely exact determination of these data. This would involve prolonged and systematic studies of the individual processes concerned and would necessitate a more elaborate cost accounting system than that which is at present used in the *Freia* factory. A first approximation to the necessary data may, however, be obtained. An estimate of the technical curve indicating the relation between scrapping percentage and fat addition as determined by *Freia*'s experience is presented in Fig. 1. In order to get an idea of the cost data involved we must make a short survey of the various cost elements entering into the *Freia* accounting system. The various cost items pertaining to production proper are classified under the following five groups:—

I. *The cost of raw materials* is in general calculated as an *average* cost per kilogramme of the paste used for the particular chocolate in question. This average raw material cost depends on the weight composition, according to the recipe, and on the price which the factory has paid or will probably pay for the various component parts. A new calculation is effected every new year and also each time an essential price change takes place. During the war, for instance, the paste price calculations had to be made several times each year. At present the average cost of the paste which enters into the nut chocolate is 105 øre per kilogramme when no extra cocoa fat is added. The

cocoa fat itself is calculated at 155 øre per kilogramme. Of course neither these figures nor any of those used in the following must be detached and separately compared with the sale's price of the product. There are a number of cost elements that are not computed in the present analysis.

II. *Labour Costs.* This includes in the first place the direct wages paid to the workers who are immediately occupied with the product.

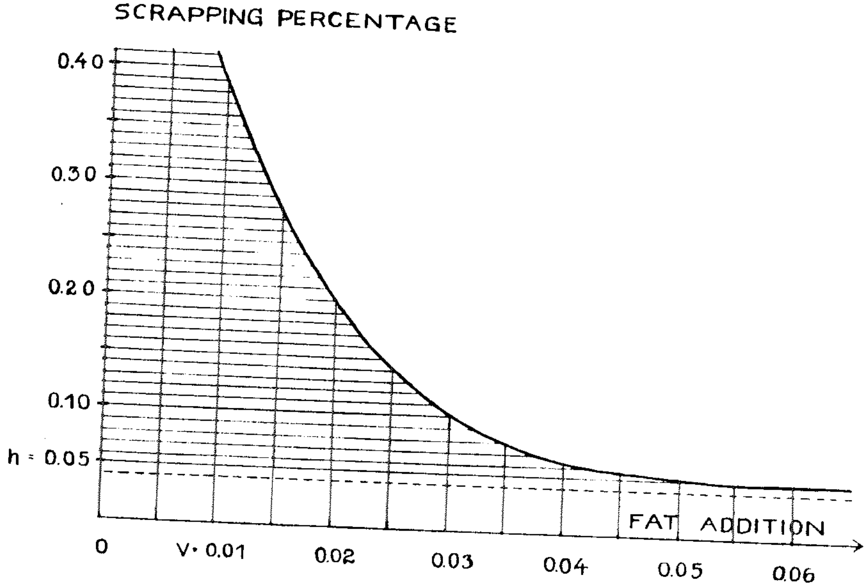

Fig. 1.

On the basis of experiences over a long period of years the labour costs are calculated as a certain cost per kilogramme paste for the various types of chocolate. These labour costs are available separately for the various processes that a product must go through. For moulding and cooling in the case of the nut chocolate the labour cost is 12 øre per kilogramme. In addition to the direct labour cost must be reckoned an *indirect* labour cost designated to cover expenses for wages to foremen, charwomen, and liftmen, holiday money, health insurance, and contributions to the Employers' Associations and to *Freia*'s own institutions for the benefit of the workers. This indirect labour cost is calculated at 25 per cent of the direct labour cost. For nut chocolate the direct and indirect labour costs for moulding-cooling will therefore be 15 øre per kilogramme paste. For all operations taken together the labour cost is 42 øre per kilogramme paste.

III. *Running costs*. These are expenses for power, coal, water, light, gas, watchmen, depreciation of machines and buildings, and necessaries for production (brushes, spattles, etc.). These costs, too, are calculated as a certain sum per kilogramme paste. No distinction is made between the various sorts of chocolate, but a distinction is of course made between the different processes. For moulding the figure is 4.6 øre and for cooling 3.1 øre, giving a total of 7.7 øre per kilogramme paste.

IV. *General costs* include office expenses at the central administration, salaries to engineers, expenses of the central sales and buying office (not including the expenses mentioned under V.), maintenance of machines, etc. At the *Freia* factory interest on capital is for special reasons not included, otherwise interest charges ought to have been included, partly here and partly under Running Costs. The General Costs are also calculated with a certain average figure per kilogramme paste, namely, as follows:—

Factory administration	3	øre per kg.
Maintenance of machines	2.7	- - -
Office expenses and other general expenses..	17.3	- - -
Total of general costs	23	øre per kg.

V. *Sales' costs* include expenses for warehousing and handling of finished products, and outgoing transport (truck hauling, railroad charges, etc.). Further, the sales costs proper:— agents, publicity, samples and loss on customers.

3. THE COEFFICIENT OF SUBSTITUTION AND THE POINT OF PROFITABLENESS.

These figures will determine how far it pays to go in the direction of adding cocoa fat. To make the connection clear we shall introduce the following two cost-coefficients ("coefficients of fabrication" reckoned in money):

w_1 = fat cost per kilogramme finished paste.
w_2 = moulding-cooling cost per kilogramme finished (not scrapped) paste.

The fat cost has to be reckoned as follows:— If to one kilogramme of the original paste (where no extra fat is added) we add v kg. cocoa fat, we get $1 + v$ kg. paste. And the raw material cost for this portion will obviously, according to the above data, be $105 + 155v$ øre. The average cost of the new paste will consequently be

$$\frac{105 + 155v}{1 + v} \text{ øre per kg.}$$

The *additional* price per kg. paste which is caused by the fat addition will consequently be:

$$(1) \qquad w_1 = \frac{105 + 155v}{1 + v} - 105 = \frac{50v}{1 + v}$$

This is the fat cost coefficient.

In order to determine w_2 we first have to compute the moulding-cooling costs on the assumption that there is no scrapping. These will be: First 15 øre for direct and indirect labour costs, then 7.7 øre running costs, and finally a part of the general costs. In order to determine what part of the general costs should be allocated to moulding-cooling we notice that the factory administration and the depreciation on the machines will increase approximately proportionally to the volume of the moulding-cooling operations, while the office expenses at the central administration and the other general costs will be approximately unchanged. The additional moulding-cooling ought therefore to be allocated a certain part of the first two items of the general costs specified above, viz. 3 + 2.7 øre per kg. What part of this 5.7 øre shall be taken into consideration? Obviously, not the whole 5.7 øre because this has reference to all the various processes. The accounting system in *Freia* is not organised in such a way that one can see directly what part of the 5.7 øre is attributable to moulding-cooling, but we can get a sufficiently accurate estimate by a pro rata allocation based on the labour costs. From the above data will be seen that the labour costs involved in moulding-cooling constitutes 15/42 of the total labour costs. We ought therefore to allocate moulding-cooling with a general cost of $15/42 \times 5.7 = 2$ øre per kg. paste. Of the cost mentioned under I and V nothing will be attributable to the moulding-cooling operation. The total costs involved in this operation will consequently be $15 + 7.7 + 2 = 24.7$ øre per kg. paste.

If the scrapping percentage is equal to h, this means that for every kilogramme paste which is put into the process, and whose moulding-cooling cost is 24.7 øre, only a finished (not scrapped) quantity of $1-h$ kg. comes out. The moulding-cooling cost per kg. finished paste will consequently be:—

$$(2) \qquad w_2 = \frac{24.7}{1 - h(v)}$$

where now $h(v)$ designates the scrapping percentage h as a function of the fat addition v. This function is given by Fig. 1. To every magnitude of v one can by graphic reading in Fig. 1 determine the corresponding magnitude of h. This means that by (1) and (2) both w_1 and w_2 will depend on v. For instance, if the fat addition is $v = 0.02$, the fat cost will be $w_1 = 1/1.02 = 0.98$ and the scrapping percentage will by Fig. 1 be $h = 0.20$, and consequently the moulding-cooling cost $w_2 = 24.7/0.80 = 30.875$ øre per kg.

These figures do not yet give any information as to whether it will pay to add the fat or not; this we get only by comparing these figures with the corresponding figures that prevail when no fat is added. Since with a fat addition of zero there is practically nothing but scrapping, we can see without any computation that the above fat addition of 0.02 will be more profitable than no fat addition. But when it comes to asking whether it will pay to add *more* than 0.02, or perhaps *less*, we must be guided by the run of the figures. For instance, will it pay to add $v = 0.04$? This new alternative gives $h = 0.06$, $w_1 = \frac{2}{1.04} =$ 1.923 øre per kg., $w_2 = \frac{24.7}{0.94} =$ 26.277 øre per kg. In other words, we see that w_2 has declined much sharper than w_1 has gone up. To increase the fat addition from 2 kg. to 4 kg. per 100 kg. is therefore decidedly profitable. We express it by saying that it pays to *substitute* the factor moulding-cooling with the factor fat.

In order to measure the strength of this profitableness of substitution we can simply compare the increase in w_1 with the decrease (that is to say with the negative increase) in w_2. Let d be an increment symbol so that dw_1 designates the new magnitude of w_1 minus the old, and similarly for dw_2. In the present case we have:—

$$(3)\qquad \begin{aligned} dw_1 &= 1.923 - 0.98 = 0.943 \text{ öre per kg.} \\ dw_2 &= 26.277 - 30.875 = -4.598 \text{ öre pr. kg.} \end{aligned}$$

Here dw_1 is positive and dw_2 negative, which expresses the fact that w_1 increased and w_2 decreased. We now take the ratio between dw_1 and dw_2 and change signs in order to get a positive coefficient, in other words we form the ratio

$$(4)\qquad s_{12} = -\frac{dw_1}{dw_2}$$

In the present case we have $s_{12} = \frac{0.943}{4.598} = 0.205$. This ratio we call the *substitution coeffcient* beween factors No. 1 and No. 2, that is to say between fat and moulding-cooling. *This coefficient expresses how much cheaper it is to use factor No. 1 (fat) than factor No. 2 (moulding-cooling).* More precisely the coefficient s_{12} expresses how much we must increase the expenses for factor No. 1 (fat) in order to save Kr. 1.—. of the expenses to factor No. 2 (and in such a way that we obtain the same production result as before). For instance, when $s_{12} = 0.205$ as above, an increase in the fat cost of 20.5 øre can save a moulding-cooling cost of Kr. 1.—. The above argument shows immediately that it pays to increase the use of fat as long as the substitution coefficient is less than unity. Now from the above data it follows that the substitution coefficient increases as the fat addition increases. There will consequently come a point where the substitution coefficient passes unity, and this is just the point where it will be profitable to stop.

If it is desired to determine the point of profitableness exactly, the above computation should be sligtly refined, viz. as follows. The above value of the substitution coefficient 0.205 refers to the *interval* between $v = 0.02$ and $v = 0.04$. In other words the value $s_{12} = 0.205$ refers to the interval whose mid-point is $v = 0.03$. To use the mid-point of the interval is particularly desirable when we want to study the continuous variation of the substitution coefficient and we want to determine the point of profitableness exactly. Suppose, for instance, that the substitution coefficient for a certain interval is less than unity. This means that the upper limit of the interval is more profitable than the lower limit (for instance in the above numerical example the point $v = 0.04$ is more profitable than $v = 0.02$), but this does not of course exclude the possibility that there is some point in the interior of the interval that is still more profitable. It may indeed be that the profitableness has increased in the first part of the interval and decreased in the latter part, the decrease being only a little less than the increase. If the increase in the first part is practically equal to the decrease in the latter part, the substitution coefficient for the whole interval would turn out to be virtually equal to unity, and in this case it would be a fair approximation to say that the exact point of profitableness is the *mid-point* of the interval. We can, therefore, express the above rule more precisely by saying that it pays to stop in a point that is such that the substitution coefficient computed in an interval *around* this point (with this point as a mid-point) is equal to unity. If the substitution coefficient thus computed is less than unity, it pays to increase the use of fat. In the opposite case it pays to decrease it.

To determine the exact point of profitableness it will therefore be well to compute the substitution coefficient for a number of other alternatives. For the interval $v = 0.04$ to $v = 0.06$, that is to say for the interval *around* $v = 0.05$ we find the following figures. On the upper limit of the interval we have $h = 0.044$, $w_1 = \frac{3}{1.06} = 2.83$ øre per kg., $w_2 = \frac{24.7}{0.956} = 25.837$ øre per kg., consequently $s_{12} = \frac{0.907}{0.440} = 2.06$. In other words, already for the interval around $v = 0.05$ the substitution coefficient is greater than unity. Similarly we may compute the substitution coefficient for the intervals around $v = 0.035$, 0.040, 0.045. In doing this we get the points shown in Fig. 2. By interpolating a curve through the points computed and making a graphical reading we find that the substitution coefficient passes unity in the point $v = 0.0435$. A fat addition of 4.35 kg per 100 kg. paste is therefore the most profitable.

Obviously the above definition of the point of profitableness amounts to the same thing as to determine a point where the *sum* of the two cost coefficients w_1 and w_2 is the smallest possible.

In table 1 is given a resumé of some of the figures. This table also expresses how much is lost by not using the most profitable addition of fat. Expressed per kg. the saving is not very large, but on a large production it certainly amounts to a sum that is not negligible. For a production of 1.000 tons, for instance, the difference between an addition of 3 kg. and the most profitable addition is nearly Kr. 8.000.-

Table 1. Resumé.

Fat addition per 100 kg. of the original paste	Scrapping percentage	Fat cost per kg. of the finished (not scrapped) product	Moulding-cooling cost per kg of the finished (not scrapped) product	In all	Loss by not using the most profitable addition
$100\,v$ kg	$100\,h(v)$	w_1 öre per kg	w_2 öre per kg	$w_1 + w_2$ öre per kg	öre per kg
2	20 %	0.980	30.875	31.855	3.661
3	10.2 »	1.156	27.506	28.962	0.768
4	6 »	1.923	26.277	28.200	0.006
4.35	5.4 »	2.084	26.110	28.194	0.000
4.5	5.3 »	2.153	26.082	28.235	0.041
6	4.4 »	2.830	25.837	28.667	0.473

If some simple rules of the differential calculus are used, the determination of the point of profitableness may also be carried through in another way, which has the advantage of describing the variations

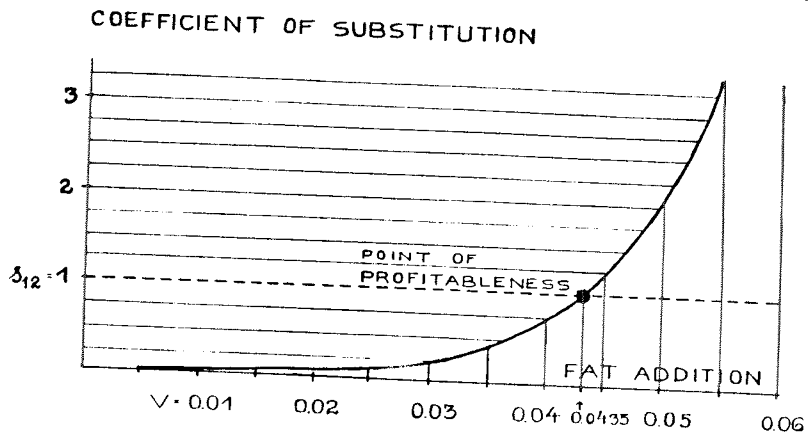

Fig. 2.

in a more continuous way, and avoiding the approximation involved in using the *mid-point* of the interval of definition for s_{12}. The differential method is particularly applicable to cases where the original data are determined with great accuracy. We first determine the differential coefficient of w_1 and w_2 with respect to v. This gives:

$$\frac{dw_1}{dv} = \frac{50}{(1+v)^2} \tag{5}$$

$$\frac{dw_2}{dv} = \frac{24.7\,h'(v)}{(1-h(v))^2} \tag{6}$$

where $h'(v)$ is the differential coefficient of $h(v)$ with respect to v. In other words, it simply expresses the slope of the tangent to the curve in Fig. 1. The slope curve $h'(v)$ may simply be determined by letting a ruler touch successive points on the curve in Fig. 1, and for each point read off the steepness of the ruler. The results of these readings are given in Fig. 3.

Taking the ratio between (5) and (6) we get

$$s_{12} = -\frac{dw_1}{dw_2} = \frac{2.02}{-h'(v)} \cdot \left(\frac{1-h(v)}{1+v}\right)^2 \tag{7}$$

This formula expresses the substitution coefficient directly in terms of the slope curve in Fig. 3. To every magnitude of v one can read $h(v)$ from Fig. 1 and $h'(v)$ from Fig. 3, and inserting these values into the right member of (7) s_{12} as a function of v can be determined. The curve in Fig. 2 was actually determined this way. The computations exemplified in (3) gives an approximation to the substitution coefficient, while those based on (7) give the exact value (provided the curves in Fig. 1 and Fig. 3 are exactly drawn). Comparing the closeness of the approximation we notice for instance that the figures given in (3) led to $s_{12} = 0.205$, while from Fig. 2 a reading gives about 0.20.

The expressions (5) and (6) may also be used to show that the point of profitableness defined, as the point where s_{12} passes zero, is at the same time the point where the sum of the two cost coefficients w_1 and

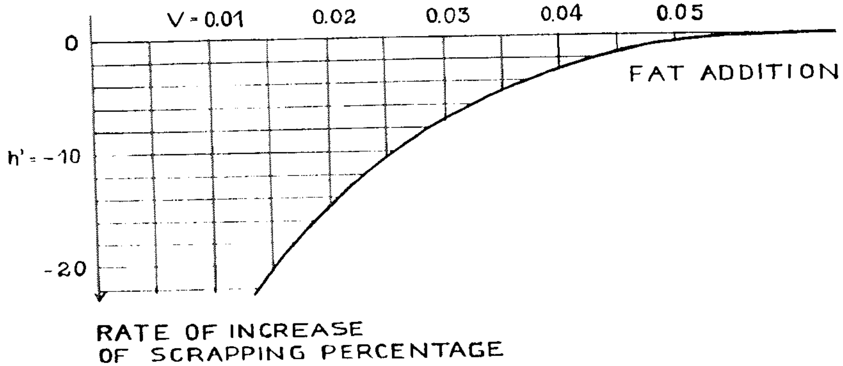

Fig. 3.

w_2 is the smallest possible. Indeed, taking the differential coefficient of $(w_1 + w_2)$ with regard to v we get by (4)

$$\frac{d(w_1 + w_2)}{dv} = \frac{dw_2}{dv}(1 - s_{12}) \tag{8}$$

Since $\frac{dw_2}{dv}$ is different from zero in the range considered, the condition that (8) shall be zero is the same as the condition that s_{12} is equal to unity.

4. THE ADAPTATION PROCESS FORMULATED IN GENERAL PRODUCTIVITY TERMS.

The method developed above is particularly adapted to the case where one studies the substitution possibilities between *two* factors. If several factors are considered and it is wanted to consider *simultaneously* their effects on the products, it is more convenient to formulate the analysis by means of some other and more general notions: Product functions, marginal productivity, etc. As an example I shall show how

the *same data* as those which we have treated above may be analysed by this more general method.

Above we considered the cost coefficients w_1 and w_2, both being reckoned as costs *per kilogramme of the product*. This means that we considered all the time a product quantity equal to 1 kg. and investigated how the costs for this 1 kg. changed. Now we want to adopt the point of view that we have given quantities of the two factors, fat and moulding-cooling, and we want to investigate how large a product these two factór quantities make it possible to produce. Of course, concretely speaking, these two factors alone cannot produce the product; a number of other factors are also needed. However, we assume that these other factors are present in those quantities that are needed. In other words, we only consider the limits that are placed on the product by the two *special* factors here considered. To express these explicitly we may say that the two factors considered create a certain *potential* quantity of the product. What we want to investigate is how this potential quantity of the product changes as a function of the two factors considered.

The potential product quantity we designate u and the factor quantities $v_1 = u \cdot w_1$ and $v_2 = u \cdot w_2$. In the present case u is measured in kilogrammes while v_1 and v_2 are measured in ore (not in ore per kg as w_1 and w_2). When we want to investigate how u depends on v_1 and v_2 in the present case we have to assume that the law of productivity in these factors is a *pari-passu* law. By this I mean that the law is such that, if both factors increase in the same proportion (for instance are doubled or trebled, etc.), then the potential product quantity will also increase in this same proportion (that is to say will be doubled or trebled, etc.). This assumption is already implicitly contained in the assumption that the costs can be expressed as certain fixed amounts *per kilogramme*; this means indeed that we assume that these cost figures do not vary appreciably if the product quantity gets larger or smaller. If this condition is not fulfilled, for instance, if we find that the running costs (amongst others depreciation on the machines) are different, according to when the production takes place in larger or smaller portions, then we will say that we have an *ultra-passum* law.

Supposing now that the production follows a pari-passu law, we see that if we have at our disposal Kr. 14.56 to be used for fat and Kr. 275.06 to be used for moulding-cooling, we may produce a potential product of 1,000 kg. This follows from the figures in the second line, under columns Nos. 3 and 4 of Table 1. But from this table also follows that a product quantity of 1,000 kg. can be created if Kr. 19.23 is used instead for fat and Kr. 262.77 for moulding-cooling. This is expressed in the third line of the table. Similarly for the other lines. This may be exhibited graphically in a factor diagram by means of *isoquants*. A factor diagram is simply a diagram, as Fig. 4, where one factor (the

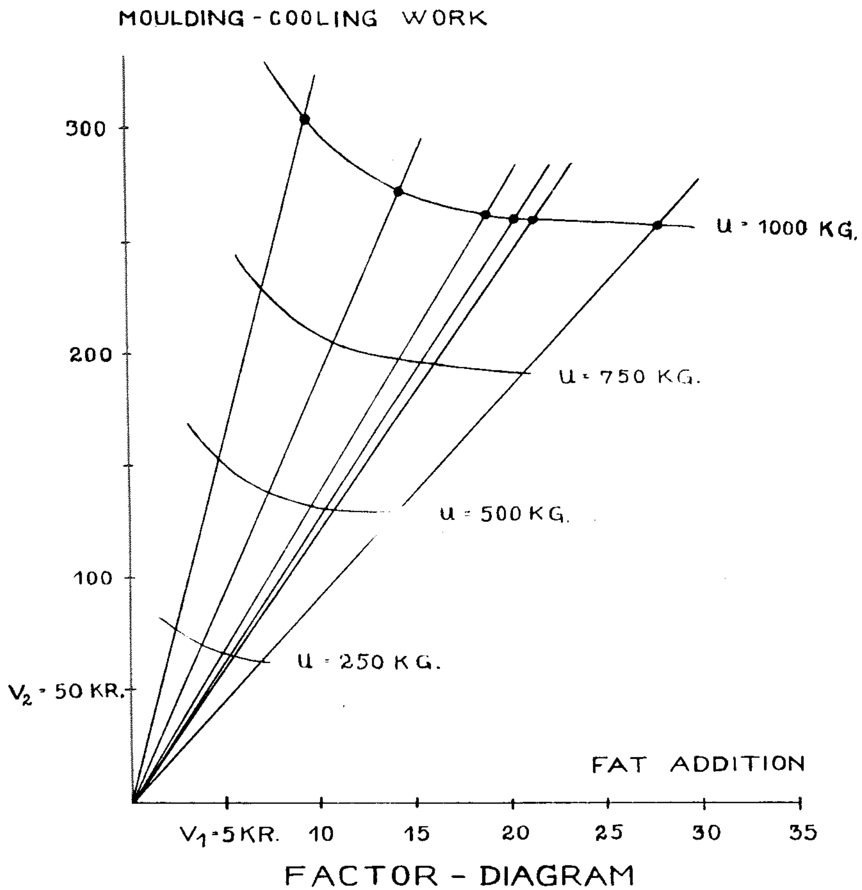

Fig. 4.

fat addition) is measured along the horizontal axis, and the other factor (the moulding-cooling) is measured along the other axis. A point in this diagram represents a given quantitative combination of these two factors. An isoquant in this diagram is a line connecting points where the potential product quantity is constant. In Fig. 4 are indicated four such isoquants. The above argument shows that we get the isoquant

corresponding to $u = 1000$ kg. by multiplying the figures in the columns w_1 and w_2 in Table 1 by 1000. The two columns thus obtained will simply give the abscisse and the ordinate respectively along the isoquant in question.

But since the production follows a pari-passu law the knowledge of any of the isoquants is sufficient to determine all the others. For instance, the isoquant for $u = 500$ kg. may be constructed by drawing from each point on the isoquant for $u = 1000$ kg. a beam through origin and simply dividing this beam in the middle. Similarly, the isoquants for $u = 250$ and $u = 750$ kg. can be obtained by dividing these beams in the rations respectively 1 : 4 and 3 : 4. Similarly, any other isoquants can be constructed. The knowledge of these isoquants and of the product quantity attached to each of them means that the product function $u\ (v_1, v_2)$ is graphically given. From this may then be derived the marginal productivities

$$u_1 = \frac{du}{dv_1} \text{ and } u_2 = \frac{du}{dv_2}$$

The determination of the product function can also be made algebraically from the formulae (1) and (2). The connection between the cost coefficients w_1 and w_2 and the corresponding sums used v_1 and v_2 are

$$w_1 = \frac{v_1}{u} \qquad w_2 = \frac{v_2}{u} \tag{9}$$

Therefore (1) and (2) can be written

$$v_1 = u\,\frac{50\,v}{1+v} \tag{10}$$

$$v_2 = u\,\frac{21.7}{1-h(v)} \tag{11}$$

The last formulae may be looked upon as two equations that define the isoquant u on parameter form, v being the variable parameter that generates the isoquant.

From (10) and (11) we find that there exist between the relative factor combination $\frac{v_1}{v_2}$ and the parameter v the unique relation

$$\frac{v_1}{v_2} = g(v) \text{ where } g(v) = 2.02\,v\,\frac{1-h(v)}{1+v} \tag{12}$$

If there are given two arbitrary values of v_1 and v_2 and the corresponding quantity u cannot be determined with a sufficient degree of accuracy by a graphic reading in Fig. 4, the determination can be made as follows. First one determines the value of the parameter v which corresponds to the given relative factor combination $\frac{v_1}{v_2}$. This implies a graphical reading of the curve in (g,v) coordinates defined by (12) or a numerical solution of the equation (12) with respect to v. When v is determined the product quantity is determined by

$$u = v_1 \frac{1+v}{50\,v} \tag{13}$$

This last equation follows directly from (10).

From the theory of productivity we know that the optimal relative factor combination is determined by $u_1 = u_2$. (If the factors had been reckoned in technical units and not in money, we would have had to divide them by the factor prices q_1 and q_2 i.e. the relation in question would have been $u_1/q_1 = u_2/q_2$). Since the condition $u_1 = u_2$ expresses that the substitution is completed, this condition must be identical with the previously considered condition $s_{12} = 1$. This can also be seen directly from the formulae. Indeed, let us in (9) consider v_1 and v_2 two independent variables, w_1 and w_2 as functions of v defined by (1) and (2) where v in turn is a function of v_1 and v_2 defined implicitly by (12), further w as the product function of v_1 and v_2. In order to determine u_1 we use the second equation in (9) and in order to determine u_2 we use the first equation. This gives

$$u_1 = -\frac{v_2}{w_2^2}\cdot\frac{dw_2}{dv}\cdot\frac{\partial v}{\partial v_1}, \quad u_2 = -\frac{v_1}{w_1^2}\cdot\frac{dw_1}{dv}\cdot\frac{\partial v}{\partial v_2} \tag{14}$$

But from (12) we get by differentiating with respect to v_1 and v_2 respectively

$$\frac{1}{v_2} = g'(v)\cdot\frac{\partial v}{\partial v_1}, \quad -\frac{v_1}{v_2^2} = g'(v)\cdot\frac{\partial v}{\partial v_2} \quad \text{where } g'(v) = \frac{dg(v)}{dv} \tag{15}$$

When this is inserted in (14) we obtain

$$u_1 = -\frac{1}{g'(v)}\cdot\frac{1}{w_2^2}\cdot\frac{dw_2}{dv} \quad u_2 = +\left(\frac{v_1}{v_2}\right)^2\cdot\frac{1}{g'(v)}\cdot\frac{1}{w_1^2}\cdot\frac{dw_1}{dv} \tag{16}$$

That is to say

$$\frac{u_2}{u_1} = -\left(\frac{w_2}{w_1}\right)^2\cdot\left(\frac{v_1}{v_2}\right)^2\cdot\frac{dw_1}{dw_2}$$

But by (9) we have $v_1/v_2 = w_1/w_2$, which on account of the definition (4) entails

$$\frac{u_2}{u_1} = s_{12} \tag{17}$$

This gives an interesting interpretation of the substitution coefficient: It is nothing else than the ratio between the marginal productivities. This shows at once that the condition $u_1 = u_2$ is the same as the condition $s_{12} = 1$.

In n variables the condition for a completed substitution can therefore be formulated

$$u_1 = u_2 = \dots = u_n \tag{18}$$

instead of by the more complicated (and redundant) set of conditions

$$s_{ij} = 1 \quad \text{for all } i, j. \tag{19}$$

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