LEAGUE OF NATIONS

Statistical versus Theoretical Relations in Economic Macrodynamics.

A memorandum prepared by Prof. Frisch for the Business Cycle Conference at Cambridge, England, July 18th - 20th, 1938, to discuss Professor J. Tinbergen's publications of 1938, for the League of Nations.

Introduction.

The present memorandum has been written rather hurriedly, and the text is therefore not as carefully polished as it ought to be in a manuscript ready for publication. It should, however, be clear enough to bring out my point of view.

The present memorandum does not discuss details of the various equations which Tinbergen has obtained and whose coefficients he has determined statistically. My main concern has been to discuss what equations of this type really mean, and to what extent they can be looked upon as "A Statistical Test of Business Cycle Theories". (The title of one of the volumes which Tinbergen has presented for discussion).

My conclusion is that the work which Tinbergen is now presenting is of paramount importance, perhaps the most important single step forward in Business Cycle Analysis of recent years. But I do not think that it can be looked upon as "A Test of Business Cycle Theories". The question of what connection there is between the relations we work with in theory and those we get by fitting curves to actual statistical data is a very delicate one. I think it has never been exhaustively and satisfactorily discussed. Tinbergen in his work hardly mentions it. He more or less takes it for granted that the relations he has found are in their nature the same as those of theory. See for instance his discussion in Vol.II p.109 - 123 where he constantly refers to the coefficients of his equations and takes the signs and magnitudes of these as tests of whether certain theoretical contentions are right or wrong. This is, in my opinion, unsatisfactory. In a work of this sort, the connection between statistical and theoretical relations must be thoroughly understood and the nature of the information which the statistical relations furnish - although they are not identical with the theoretical relations - should be clearly brought out.

The present memorandum is an attempt to bring some contribution to this question. It will be divided into 7 sections viz:

1. Some remarks on terminology.

2. Functional equations and their solutions.

3. The irreducibility of a functional equation with respect to a set of functions.

4. Coflux and superflux relations. The nature of passive observations.

6. Aberrations versus stimuli. Confluence analysis and shock-theory.

7. Interpretation of Professor Tinbergen's results.

1. SOME REMARKS ON TERMINOLOGY.

In any macrodynamic analysis there will be some constants or functions of time that are taken as data while others are considered as the variates to be "explained". A determinate theory is one that considers just as many independent relations as there are variates to be explained.

We shall use the expression "nature" or "constitution" of the system of phenomena studied as the whole of all those characteristics that describe the "way of functioning" of these phenomena. When we speak of the "structure" of the system, we think more specifically of those features of the "constitution" that can be quantitatively described. We speak for instance of the "structural equations" for the system. We do not intend however to draw any sharp line of demarcation between constitution and structure. The difference between them is only one of degree and one that is not very important. The precise definition of a structure is a matter of theorising although of course the leading ideas of the theoretical definitions will frequently be suggested by facts.

A disturbance is a deviation from that situation which should have existed as a consequence of the structure. In other words , it is something incompatible with the structure; something new and spontaneous introduced in addition to the structure. Such disturbances may be of two sorts: aberrations and stimuli. A stimulus is a disturbance that carries on its effects to the subsequent states of the system, - through the structural equations. In other words at any given moment it is the magnitudes of the variates including the stimuli that are taken as influencing the further evolution; that is, the stimuli act as a sort of permanently changing initial conditions. An aberration is also a departure from the value which a variate should have had according to the structure, but this departure acts only at the actual moment at which it occurs; it is a sort of instantaneous addition - unexplained by the structure - and without any consequence for the subsequent states. In other words it is the magnitudes of the variates exclusive of the aberrations that act as initial conditions for the subsequent states,

The existence of aberrations leads to the application of the methods of Confluence analysis.* The existence of stimuli leads to the shock-theory.** There may also be mixed cases but I shall not go into this question here.

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* See the publication "Confluence Analysis" of the University Institute of Economics, Oslo.

2. FUNCTIONAL EQUATIONS AND THEIR SOLUTION.

The structure of a macrodynamic system will be described by means of a number of functional equations. We shall in particular consider linear lag-equations, and taking as our variates the deviations from certain trend values, it will be sufficient for our purpose to consider homogeneous equations. Let $\bar{\bar{x}}_1(t) \dots \bar{\bar{x}}_N(t)$ be a number of variates whose movement is to be explained; t designating time. Let $\bar{x}_1(t) \dots \bar{x}_N(t)$ be trend values determined in some way or other, and let $x_1(t) \dots x_N(t)$ be deviations from trend, i.e.

$$x_i(t) = \bar{\bar{x}}_i(t) - \bar{x}_i(t) \tag{2.1}$$

Between the variates x_i we assume a number of relations (structure equations) of the form

$$\sum_{i\theta} \alpha_{ki\theta}\, x_i(t-\theta) = 0 \qquad (k = 1,2\dots) \tag{2.2}$$

k represents different equations, while the summation $i\theta$ represents the terms of each equation; i runs through all or some of the variate numbers 1 ... N, and θ runs through a certain range of lag numbers, in general different for each variate. The $i\theta$ range in each equation determines the nature of the terms involved, we shall call it the form of the equation; the α's are the coefficients of the equation. The distinction between the form and the coefficients of the equation is essential for the discussion in Section 4. A similar distinction may of course be made for more general types of functional equations.

For the discussion of the following Sections it is necessary to summarise some of the classical facts of the theory of linear lag-equations (difference equations).

A certain number of equations of the form (2.2) - equal or unequal to the number of variates N - are said to be linearly independent if it is impossible to deduce any one of the equations from the others no matter what the time shapes of the variates are. A necessary and sufficient condition for the independence of a set of m equations of the form (2.2), is that there should not exist any set of numbers $\lambda_1\, \lambda_2 \dots \lambda_m$ not all zero, such that

$$\sum_k \lambda_k \alpha_{ki\theta} = 0 \quad \text{for any } i\,\theta \tag{2.3}$$

In terms of the coefficients α the criterion can be formulated by considering the m rowed and M columned matrix

$$\left\| \alpha_{ki\theta} \right\| \tag{2.4}$$

where all $i\theta$ combinations are written as columns, M being the number of different $i\theta$ combinations that exist in all the m equations and k representing the rows.

The equations are independent when and only when this matrix is of rank m. Or again the criterion can be formulated in terms of the moments $[\alpha_h \alpha_k]$ the summation being extended over all $i\theta$ combinations. The equations are independent when and only when the symmetric determinant of the magnitudes $[\alpha_h \alpha_k]$ is different from zero.

For each equation of the kind (2.2) that we add, we make less general the class of functions that satisfy the equations. If the number of equations becomes equal to N the system is <u>determinate.</u> This means that the nature of the solution has been restricted as much as it is possible to do so by means of functional time equations. It does not mean that the set of functions $x_1(t) \ldots x_N(t)$ is completely determined, a considerable amount of freedom is still left and will have to be determined by a set of <u>initial</u> conditions. But this determination is in point of principle different from that achieved by the functional equations. This is shown clearly by the fact that we cannot, say, replace the initial conditions by one or more <u>additional</u> equations of the form (2.2). Indeed if there are <u>more</u> than N independent equations of the form (2.2), there will in general exist no functions satisfying the system.

The <u>solution</u> of a determinate system of the form (2.2) can be achieved either <u>directly</u> or by means of expansions in series of complex exponentials. The direct method is applicable only in the simplest cases. Take as an example the system

(2.5)
$$\alpha_1 x_1(t) - \alpha_2 x_2(t) = 0$$
$$\alpha_1 x_1(t) + \alpha_2 x_2(t-\theta) = 0$$

From this follows immediately $\alpha_1 x_1 + \alpha_1 x_1(t-\theta) = 0$

(2.6) hence $$x_1(t) + x_1(t-\theta) = 0$$

A solution of (2.6) is obtained by choosing arbitrarily the shape of x, over an interval of length θ and then repeating this shape antiperiodically for each subsequent θ interval. (2.6) shows that no more general form than this can be an X solution of (2.5). If we further put $x_2(t) = \frac{\alpha_2}{\alpha_1} x_1(t)$ we get a complete solution of (2.5). Obviously this is the most general form of the solution. Any function that can be a solution, must be a special case of this. The arbitrary shape of X_1 over the original θ interval is here the initial condition. Putting this equal to a sine function with period 2θ, we get, both for X_1 and X_2, over the complete t range, sine functions with this period.

In the more complicated cases one must resort to the indirect method. It consists in trying to satisfy the equations by expansions of the form

(2.7)

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$$x_i(t) = \sum C_{i\gamma} e^{\gamma t} \qquad (i = 1, 2, \ldots N)$$

where $C_{i\gamma}$ are constants and the summation over γ runs over certain values to be determined. Complex numbers are admitted both as C's and γ's.

Inserting (2.7) in (2.2) we get - if the system is determinate -

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(2.8) $$\sum_{\gamma i \theta} \alpha_{ki\theta} C_{i\gamma} e^{\gamma(t-\theta)} = 0 \qquad (k = 1, 2 \ldots N)$$

Any number of exponential functions with different exponentials are linearly independent, therefore if the γ's are different, (2.8) cannot vanish identically in t unless the terms of (2.8) vanish separately for each γ. Assuming for the moment all the γ's to be different we see that we must have -

(2.9) $$\sum_{i\theta} \alpha_{ki\theta} C_{i\theta} e^{-\gamma\theta} = 0 \text{ for all } \gamma \text{ and k.}$$

For any given value of γ this is a system of linear homogeneous equations (k = 1, 2 ... N) in the set of N numbers . If this system is to have a solution apart from the trivial $C_{1\gamma} = \cdots = C_{N\gamma} = 0$ the determinant of the coefficients must vanish, i.e. we must have

(2.10) $$\left| \sum_{\theta} \alpha_{ki\theta} e^{-\gamma\theta} \right| = 0 \text{ for any } \gamma$$

k and i designate rows and columns respectively in the N rowed determinant (2.10). (2.10) is the characteristic equation whose roots $\gamma_0, \gamma_1, \ldots$ (in general infinite in number) give the exponents of the expression (2.7). Under very general conditions this expansion is valid even though some of the γ's are equal, the only difference being that in this case the multiple terms are replaced by a polynomial in t (of the order equal to $\varkappa - 1$ if $\varkappa$ is the multiplicity of the γ-root) multiplied by the exponential in question. We need not consider this case here. The characteristic equation could also have been obtained by eliminating - in a way similar to that used to obtain (2.6) - a certain number of the variates in order to get a "final equation" in one or a few variates, and then forming the characteristic equation for this. This procedure is often useful when it is wanted to give a concrete interpretation of the mechanism of the solution, but in point of principle it is just as easy to form the characteristic equation directly as in (2.10).

It will be noted that the set of exponents as determined by (2.10) is the same for all the variates $x_1 \ldots x_N$. In other words all the variates contain the same sort of components (if γ is a real number the component in question is a real exponential, if γ is a complex number its conjugate must also be a solution of the characteristic equation and these two terms together will form a real, damped, undamped or antidamped sine function). But the intensities with which the components occur in the various variates will be different. These intensities, - amplitudes - are represented by the numbers $C_{i\gamma}$. The distribution of these numbers and in particular the extent to which it is determined by the functional equation is essential for the interpretation of the relation between statistical and theoretical relations in economic macrodynamics.

For any given γ the corresponding numbers $C_{1\gamma} \dots C_{N\gamma}$ will - if (2.10) is of rank N - 1 for this value of γ - be uniquely determined apart from a common factor of proportionality c_γ. Indeed, the numbers $C_{1\gamma} \dots C_{N\gamma}$ will, when (2.10) is of rank N - 1, be proportional to the elements in a row of its adjoint (the elements of all these rows are proportional). There exists at least one row which does not consist exclusively of zeros and hence determines the proportions in question. Let $\hat{C}_{1\gamma} \dots \hat{C}_{N\gamma}$ be one such set of proportionality numbers. We may then put

(2.11) $$C_{i\gamma} = c_\gamma \hat{C}_{i\gamma}$$ where c_γ is an arbitrary number.

This applies to any root γ that makes the rank of (2.10) N - 1. Suppose that only such roots exist (the other cases do not alter those features of the amplitude distribution in which we are here interested).

Inserting (2.11) in (2.7) we see that we can draw the following conclusions:

Each variate of the set of functions that is a solution of (2.2) can be expanded as a sum of trigonometric components. The <u>frequencies</u> and <u>damping</u> exponents of the components are determined by the equations (2.2), and so are the <u>relative</u> amplitudes, that is the ratio of the amplitude of a given component in one of the variates to that of the same component in another variate. But the <u>absolute</u> amplitudes are not determined by the equations (2.2). If these equations only are given, we may choose the absolute strengths of the various components in <u>one</u> of the variates arbitrarily (the choice of the numbers c_γ), but then the absolute strengths of these components in the other variates follow since their relations to the amplitudes of the components in the one variate we selected are determined by the equations (2.2). Briefly, the <u>relative</u> amplitude distribution is determined by the equations (2.2), but the <u>absolute</u> amplitude distribution has to be fixed by the initial conditions.

A similar situation exists for the <u>phase</u> distributions. Indeed the <u>timing</u> of a given component in one variate as compared with that of the same component in another variate is determined by the equations (2.2), but the timing of the various components in one selected variate must be fixed by the initial conditions.

By elimination processes similar to that used in obtaining a final equation, many new systems of equations may be deduced from (2.2). If the correspondence between the two systems is unique in the sense that the new system may be derived from the old and vice versa, with identical variates involved, the solutions of the two systems must be identical. In particular it is of interest to consider linear-elimination processes, that is processes where the form of the equations (the specification of the functions and lag-numbers that occur in the equation) is the same but the coefficients are changed. Any transformation of the form

$$\alpha^{*}_{ki\theta} = \sum_{h} \varphi_{kh} \alpha_{hi\theta} \qquad (2.12)$$

where φ_{kh} is a non-singular matrix, independent of $i\theta$, will furnish a new system of equations, that are independent if the old system is, and has exactly the same set of functions as its solution, - and vice versa.

We shall now discuss in somewhat greater detail these various equations that have the same solutions, and introduce a classification of them which is important for our purpose. In particular we shall consider equations which have the same form but different coefficients.

3. THE IRREDUCIBILITY OF A FUNCTIONAL EQUATION WITH RESPECT TO A SET OF FUNCTIONS.

When we compare a functional equation involving one or several functions with particular set of functions, there are two questions to be asked: does the set of functions satisfy the equation and does it satisfy this equation only?

Obviously, the set will satisfy all equations which - involving identical functions - can be derived from the first equation, so we are only interested in knowing whether the set satisfies some other equation which is independent of the first. Furthermore we shall not consider all other conceivable equations but only those which are of the same form as the first but have different coefficients. In the case of homogeneous equations of the form (2.2) this means that for any given one of these equations (any given k) we are interested in knowing whether a particular set of functions considered satisfies not only this equation but also another with the same $i\theta$ range but with coefficients that are non-proportional to those of the first equation. If this is so, we shall say that the first equation is reducible with respect to this set of functions; if not it is irreducible. Thus an irreducible equation of the form (2.2) is one whose coefficients are uniquely determined and allow of no degree of freedom if the equation is to be satisfied by this set of functions (apart from the arbitrary factor of proportionality which is always present in the case of a homogeneous equation). It is clear that the property of irreducibility must be important when we are studying the nature of those equations that can be determined from the knowledge of the time shapes of the functions that are to satisfy the equations.

Obviously the first equation in the above definition is reducible, the second is also reducible. The set of functions involved in the definition may be specified in great detail or only very broadly as a general class of functions.

A similar definition may be established for a system of equations but we shall only need it for a single equation.

If an equation is given, we may consider the class of all those sets of functions (satisfying the equation) which have the property that the equation is irreducible with respect to those sets of functions. This class we may call the irreducibility class of the equation.

Let us consider some simple propositions and some examples that will help us to visualise the nature of this irreducibility definition. In the first place it is easy to see that there cannot exist two or more equations of the same form which are both irreducible with respect to the same set of functions. But if the two equations are of different forms (e.g. with different lag-numbers) each of them may be irreducible for the same set of functions.

In the second place we notice that <u>any</u> functional equation is irreducible with respect to the most general set of functions that satisfy this equation. Indeed, if the set of functions should also satisfy <u>another</u> equation - independent of the first - this would represent a <u>restriction</u> to the set, so that it could not actually be the most general set that satisfied the first equation. But if we consider a set of functions that satisfy two independent equations, neither of the equations need be irreducible with respect to this set. This is indeed a more special set of functions and the requirement that this set shall be a solution is less rigorous and therefore places less restrictions on the coefficients of the equation.

As a more particular example let us consider the equation (2.6). Pure sine functions with period 2θ is a solution, and for functions of this sort the equation is irreducible, because there do not exist any values of p and q which will make the equation

$$(3.1) \qquad p x_1(t) + q x_1(t-\theta) = 0$$

an equation satisfied by pure sine functions of period 2θ, except the values $p = q$. And in this case the equation is the same as (2.6).

On the other hand take the equation

$$(3.2) \qquad 0.6 x_1(t) + x_1(t-\theta) + 0.4 x_1(t-2\theta) = 0$$

This equation is also satisfied by a sine function of period 2θ (which is easily seen by insertion), but it is not irreducible with respect to this function. The equation would also be satisfied by this function if we let the first coefficient be 0.9 and the last 0.1, or quite generally if the sum of the first and last coefficients are equal to the middle coefficient. In this case the coefficients of the equation have a one dimensional degree of arbitrariness (even apart from the arbitrary factor of proportionality which is always present in the homogeneous equations).

The following is a general rule about the reducibility of equations of the form (2.2).

(3.3) Rule about reducibility: If the functions with respect to which reducibility is defined are made up of n exponential components (two complex exponentials correspond to one damped, undamped or antidamped sine function), the equation is certainly reducible - and hence its coefficients are affected in a more or less arbitrary manner - if it contains more than n + 1 terms. And it may be reducible even if it contains n + 1 terms or less.

Let us first consider as an example the following three term equation in one function

(3.4) $$\alpha_1 \cdot x(t-\theta_1) + \alpha_2 \cdot x(t-\theta_2) + \alpha_3 (t-\theta_3) = 0$$

If x is simply an exponential $x(t) = Ce^{\gamma t}$, the left-hand side of (3.4) becomes $Ce^{\gamma t}[\alpha_1 e^{-\gamma\theta_1} + \alpha_2 e^{-\gamma\theta_2} + \alpha_3 e^{\gamma\theta_3}]$.
In order that this expression should vanish identically in t it is necessary and sufficient that the bracket should disappear. This leaves a one dimensional arbitrariness in the α's even apart from their arbitrary common factor of proportionality.

If the function considered is of the form

(3.5) $$x(t) = Ae^{\beta t}\sin(a+\alpha t)$$

it is equivalent to two exponential components (i.e), and since the number of terms in the equation is only 3, the equation may be irreducible. But it may also be reducible if the lag-numbers satisfy certain special conditions. Inserting from (3.5) into the left-hand side of (3.4) we get

(3.6) $$Ae^{\beta t}\sin(a+\alpha t)[\alpha_1 c_1 + \alpha_2 c_2 + \alpha_3 c_3] + Ae^{\beta t}\cos(a+\alpha t)[\alpha_1 s_1 + \alpha_2 s_2 + \alpha_3 s_3]$$

where

(3.7) $$c_i = e^{-\beta\theta_i}\cos\alpha\theta_i \qquad s_i = e^{-\beta\theta_i}\sin\alpha\theta_i$$

In order that (3.6) shall vanish identically in t it is necessary and sufficient that the two brackets should disappear separately (because the two time functions in front of the brackets are linearly independent). I.e. we must have

(3.8) $$\alpha_1 c_1 + \alpha_2 c_2 + \alpha_3 c_3 = 0$$
$$\alpha_1 s_1 + \alpha_2 s_2 + \alpha_3 s_3 = 0$$

These are two equations in the α's If the coefficients of the two equations are proportional, i.e. if

(3.9) $$\frac{c_1}{s_1} = \frac{c_2}{s_2} = \frac{c_3}{s_3}$$

a set of α's that satisfies one of the equations would automatically satisfy the other. Hence there would again be only one condition for the three α's consequently the α's would have a one dimensional arbitrariness, even apart from the usual factor of proportionality. The condition (3.9) is equivalent to the condition that all the three two-rowed determinants in the matrix of coefficients in (3.8) should vanish (if any two of these determinants vanish, the third vanishes automatically).

Since

(3.10) $$\begin{vmatrix} c_i & c_j \\ d_i & d_j \end{vmatrix} = e^{-\beta(\theta_i+\theta_j)} \sin\alpha(\theta_j - \theta_i)$$

we see that in terms of the lag-numbers the condition (3.9) is reduced to

(3.11) $$\theta_2 - \theta_1 = \frac{h\pi}{\alpha} \qquad \theta_3 - \theta_1 = \frac{k\pi}{\alpha}$$

where h and k are integers (h = k or h = o or k = o represent trivial cases). Thus, if (3.11) is fulfilled, (3.4) is reducible with respect to (3.5).

A general criterion for the case when even the (n + 1) terms equation is reducible with respect to a series consisting of n exponential components, is provided by the n rowed and (n + 1) columned matrix

(3.12) $$\left\| C_{i\gamma} e^{-\gamma\theta_{i1}}, C_{i\gamma} e^{-\gamma\theta_{i2}} \dots C_{j\gamma} e^{-\gamma\theta_{j1}}, C_{j\gamma} e^{-\gamma\theta_{j2}} \right\|$$

where i, j... are the affixes of the variables x_i, x_j ... which occur in the equation considered, and $\theta_{i1}, \theta_{i2} \dots \theta_{j1}, \theta_{j2} \dots$ are the lag-numbers. The rows of (3.12) are produced by letting γ run through the characteristic numbers of which we here suppose that there exist n. For the (n + 1) term equation in question to be reducible with respect to the set of functions considered, it is necessary and sufficient that the matrix (3.12) should be less than n. More precisely: if it is of rank r $\gtreqless$ n (which is a criterion that depends only on the nature of the functions in question and the distribution of the lag-numbers) the equation will have an (n - r) dimensional reducibility, i.e. its coefficients will have an (n - r) dimensional degree of arbitrariness, in addition to the arbitrary factor of proportionality associated with the homogeneity of the equation.

4. COFLUX AND SUPERFLUX EQUATIONS. THE NATURE OF PASSIVE OBSERVATIONS.

If a determinate system of the form (2.2) is given, it is of particular interest to consider the reducibility of the various equations with respect to that class of functions which is a solution of the complete system. This, of course, is a much more special class of functions than that which satisfies each equation taken separately, and the reducibility of the equation is correspondingly higher. The specialisation of the functions is still further increased by the initial conditions. We have indeed seen that even though the solution of the equations themselves may contain a large - perhaps infinite - number of components, the equations do not say anything about the absolute amplitude distribution. It may indeed happen that in the actual solution all components will disappear except, say, one which is a pure sine curve. In this case all the original equations that consisted of more than three terms would certainly be reducible, and even some of the three-term equations might be reducible. An equation which is irreducible with respect to the set of functions that forms the actual solution of the complete system (including those determined by the initial conditions) we shall call a coflux equation. The others - those that are reducible with respect to their set of functions - will be called superflux equations. These latter equations are of course in a particular sense irreducible, but with respect to more general classes of functions. If any one of them is not reducible for any more special class of function it is at least irreducible for that class which consists of its own most general solution. The word "flux" in this connection suggests that the reducibility is here defined with respect to the time shape - the "flux" - actually possessed by the phenomena.

The notion of coflux relations is fundamental when we ask what sorts of equations it is possible to determine from the knowledge of the time shapes that are actually produced. The answer is obviously that all coflux equations and no other equations are discoverable from the knowledge of the time shapes of the functions that form the actual solution.

Indeed all other equations will have coefficients with at least a one-dimensional degree of arbitrariness. If an attempt were made to fit such an equation to the data, the coefficients would be of the $\frac{0}{0}$ form when no errors (aberrations) were present, and otherwise they would have a fictitious determinateness, their magnitudes being determined solely by the errors, and not by the structure.

This is the nature of passive observations, where the investigator is restricted to observing what happens when all equations in a large determinate system are actually fulfilled simultaneously. The very fact that these equations are fulfilled prevents the observer from being able to discover them; unless they happen to be coflux equations, that is, irreducible with respect to the functions that form the actual solution.

But why bother about these other equations that are not discoverable through passive observations?

The answer is that some of these other equations frequently have a higher degree of "autonomy" than the coflux equations, and are therefore very well worth knowing. The "autonomy" of an equation is not, like the irreducibility a mathematical property of a closed system like (2.2), but is built on some sort of knowledge outside this system. I shall now proceed to a discussion of this point.

5. THE AUTONOMY OF A FUNCTIONAL EQUATION - NATURE OF "EXPLANATIONS", EXPERIMENTATION AND REFORM.

Suppose that, from a knowledge of the time shapes of the two functions $x_1(t)$ and $x_2(t)$, I have determined a relation of the form

$$x_1(t) = a x_1(t-\theta_1) + b x_2(t) + c x_2(t-\theta_2) \tag{5.1}$$

What does this equation mean? It means that so long as x_1 and x_2 continue to move with the same time shapes as they have had in the past I can compute the value of x_1 at any point of time t from the knowledge of x_2 at this same point and x_1 and x_2 at certain earlier moments as indicated in the formula. In other words the equation is simply a description of the "routine of change" which x_1 and x_2 follow. The equation determined in this empirical way does not state that if a situation occurs where $x_1(t-\theta_1)$, $x_2(t)$ and $x_2(t-\theta_2)$ have some arbitrary values. I can again compute $x_1(t)$ by (5.1). To assume that (5.1) should hold good for any values whatsoever inserted for the variables on the right-hand side of the equation would indeed imply that I conceived of the possibility of another structure than the one which prevailed when the equation (5.1) was determined. For instance, if the original structure was taken as defined by two equations of the form (2.2), I could not conceive of a free variation of the variates on the right-hand side of (5.1) without giving up at least one of the two structural equations that determine the course of x_1 and x_2. But that would mean giving up the very assumption on which (5.1) was determined.

This situation can also be interpreted in terms of irreducibility. If I conceive of the possibility that the constants, a, b and c in (5.1) may have definite values, I must also conceive of the existence of some time shapes of x_1 and x_2 for which (5.1) becomes an irreducible equation, that is, has determinate coefficients without any arbitrariness. And the same applies to any other structural equation.

In a big system of structural equations it would be quite exceptional if all the equations should be irreducible with respect to that particular solution which turns out to be the final one. We only have to think of a case where the initial conditions are such that only one single component is left with an amplitude different from zero, while many of the structual equations contain a large number of terms. The fact that I reckon with such a system of equations, must mean that I conceive of the possibility that the structure may have been different from what it actually is, thus giving a chance of producing a time shapes complicated enough to make the big structural equations irreducible with respect to these time shapes.

But when we start speaking of the possibility of a structure different from what it actually is, we have introduced a fundamentally new idea. The big question will now be: in what directions should we conceive of a possibility of changing the structure. There is nothing in the nature of the equations that describe the actual structure, which can suggest an answer. It is true that if a system of equations is given, it would be natural to imagine in turn all equations omitted except one; this remaining equation would then certainly be irreducible with respect to the general class of functions which now satisfy the equation (see the second example in Section 2). But this solution is only apparent, because there exist an infinity of ways of writing the system of structural equations. (Compare for instance the transformation (2.12)).

To get a real answer we must introduce some fundamentally new information. We do this by investigating what features of our structure are in fact the most autonomous in the sense that they could be maintained unaltered while other features of the structure were changed. This investigation must use not only empirical but also abstract methods. So we are led to constructing a sort of super-structure, which helps us to pick out those particular equations in the main structure to sense. The higher this degree of autonomy, the more fundamental is the equation, the deeper is the insight which it gives us into the way in which the system functions; in short, the nearer it comes to being a real explanation. Such relations form the essence of "theory".

Once such a basic system of structural equations to which we can attach the label "autonomous" has been selected, it is easy to derive others that have a greater or lesser degree of autonomy. Equations that are obtained by long elimination processes, based on several autonomous equations will have a low degree of autonomy, they will in fact depend on the preservation of a great many features of the total system.

The coflux relations that can be determined by observation of the actual time shapes may or may not come near to resembling an autonomous relation; that depends on the general constitution of the phenomena studied. To give two extreme examples: the demand function for a consumers commodity as depending on price and income and perhaps on some secondary variables will, if the coefficients can be determined with any degree of accuracy, come fairly near to being an autonomous relation. It will not be much changed by a change in monetary policy, in the organisation of production etc. But the time relation between the Harvard A, B and C curves is a pure coflux relation, with only a small degree of autonomy.

Such I believe is in essence the relation between the equations of pure theory and those that can be determined by passive observations.

If the situation is such that the coflux relations are far from giving information about the autonomous structural relations, recourse must be had to experimentation, that is one must try to change the conditions so that one or more of the structural equations is modified. In economics the interview method is a substitute - sometimes bad, sometimes good - for experimentation.

If the results of the investigation are to be applied for economic political purposes - for reforming the existing economic organisation - it is obviously the autonomous structural relations we are interested in.

6. ABERRATIONS VERSUS STIMULI. CONFLUENCE ANALYSIS AND SHOCK-THEORY.

The existence of aberrations does not necessarily involve any important consequences for the theoretical analysis, it only concerns the statistical technique, but in this respect it is important. The existence of stimuli entails much more far-reaching consequences. The total time shape will now be more or less transformed, for instance damped cycles will become undamped in the long run, but will have a disturbing effect over shorter intervals. The timing between the cycles may be changed from what it is in the stimulus-free system, and entirely new cycles, pure cumulation cycles will emerge. These consequences cannot be discussed in detail here.

7. INTERPRETATION OF PROFESSOR TINBERGEN'S RESULTS.

All the way through his work Tinbergen uses approximations by which the time equations are reduced to linear forms. This is certainly admissible in a first approximation but the consequences should be clearly recognised. If the linear approximations are used for as many equations as are needed to make the system determinate (which is what Tinbergen aims at doing: "...we must continue this procedure until the number of relations obtained equals the number of phenomena..." "Business Cycles ...." p.7) - only those features of the time series are taken account of that can be approximated to by fitting to the data that type of solution which a linear system of equations admitsofnamely a number of trigonometric components (exponentials or damped, undamped or antidamped sine functions or as exceptional cases such functions multiplied by polynomials) of the time series over the interval considered. In itself there is nothing objectionable in this but it means that the significance of the results must be interpreted in the light of the various algebraic facts of the preceding sections. These become relevant with the same approximation as that involved in Tinbergen's calculations.

This being so it is clear that it is only coflux relations that are determined by Tinbergen; and the lack of agreement between these equations and those of pure theory cannot be taken as a refutation of the latter. Any number of examples could be given of statements that are in need of very much qualification on this ground. A case in point is that discussed on page 111 in "Business Cycles" or perhaps even better the attempt on page 26 to get an equation for consumers outlay. The only result of the various attempts made here is to shift from one to another amongst an infinite number of coflux equations. By a suitable choice of the variate and lag-numbers introduced one can produce practically any coefficients one likes. A comfutation from series made up of a small number of trigonometric components shows this immediately. The reasons for discarding some of the equations (p.26) are quite unsatisfactory. No other reasons seem to be given than the fact that the coefficients do not work out as the author likes. In my opinion all these equations are acceptable when interpreted as what they really are: a number of coflux equations. But none of them can, I believe, be taken as an expression of the autonomous structural equation that will characterize demand.

In concluding this memorandum, I want to stress again what I mentioned in the introduction, namely, the importance of the results obtained by Tinbergen. They will have to be taken as starting point for any further investigation aiming at obtaining limits or other sorts of information concerning the structural coefficients.

17.7.38.

(signed) RAGNAR FRISCH.