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OVERDETERMINATENESS AND OPTIMUM EQUILIBRIUM

By *Ragnar Frisch**).

Economic models depicting how the economic equilibrium process works, may be constructed in a variety of ways. It is interesting to note that by imposing a number of the traditional conditions, each of which is so near at hand as to be nearly inevitable, such as the condition expressing the profit maximalization of the enterprises (or some other condition to replace it), the condition expressing the quantity adaption of the consumers (or some other condition to replace it) etc. one will frequently run into *overdeterminate* systems. This fact merits, in my view, more attention than is usually given to it. By studying the nature of this overdeterminateness and such changes in the models that can overcome it, one may get on the track of some very essential features of the economic mechanism at play in this complicated and mysterious world of ours. This applies in particular when we are studying in what sense the equilibrium arrived at is an *optimum* equilibrium as judged by some sort of social standard.

The object of the present note is to set forth some simple examples of this and to draw certain conclusions from them. Amongst these are some conclusions regarding specific forms of taxation and wage payments and their effects on the incentive to work. The examples are on purpose made extremely simplified so as to retain only the most basic aspects of the problem.

1. *A simplified model.*

Consider for a moment a production process where a single commodity — a consumption good — is produced and where there is used only a single factor of production, namely labour. Suppose that all the workers are equal and that the entrepreneurs do not consume. The number of workers is supposed given so that it makes no difference whether the argument is carried through in terms of magnitudes per worker or in terms of magnitudes relating to the society as a whole. We use the following notation:

*) Professor, University of Oslo.

v = input of labour.

$f(v)$ = output of the consumption good as a function of labour input.

x = consumption of the consumption good.

$W(x)$ = total utility of consumption.

$U(v)$ = total disutility of labour.

$w(x) = \frac{dW(x)}{dx}$ = marginal utility of consumption.

$u(v) = \frac{dU(v)}{dv}$ = marginal disutility of labour.

$f'(v) = \frac{df(v)}{dv}$ = marginal productivity.

p = price of the consumption good (in monetary units).

q = wage rate (in monetary units).

$\frac{q}{p}$ = real wage rate.

The social optimum in this case may be defined as follows: Use labour input v up to a point where the *difference* between the total utility obtained by consuming the amount of the consumption good that is produced though the input v *and* the total disutility involved in making the input v, is maximized. That is to say the problem is to maximize the difference

$$W(f(v)) - U(v) \tag{1.0}$$

considered as a function of the sole variable v. Since

$$\frac{d}{dv}\left[W(f(v)) - U(v)\right] = w(f(v))\left[f'(v) - \frac{u(v)}{w(f(v))}\right] \tag{1.1}$$

we see that the social optimum is determined by the condition

$$f'(v) = \frac{w(f(v))}{u(v)} \tag{1.2}$$

In other words it is determined by the intersection between the marginal productivity curve $f'(v)$ and the curve of the function

$$\frac{u(v)}{w(f(v))}. \tag{1.3}$$

This intersection point is indicated by A in fig. (1.4). In this figure the marginal productivity curve and the average productivity curve are assumed to have the usual shapes.

This shows that the shape of the curve (1.3) plays an important role in the optimum problem. It may be assumed to have a *positive* ordinate and to be *monotonically increasing* with v over the relevant v-range. Indeed we may assume $f(v)$ and $u(v)$ to be monotonically increasing functions of v, and $w(f)$ to be a monotonically decreasing function of f.

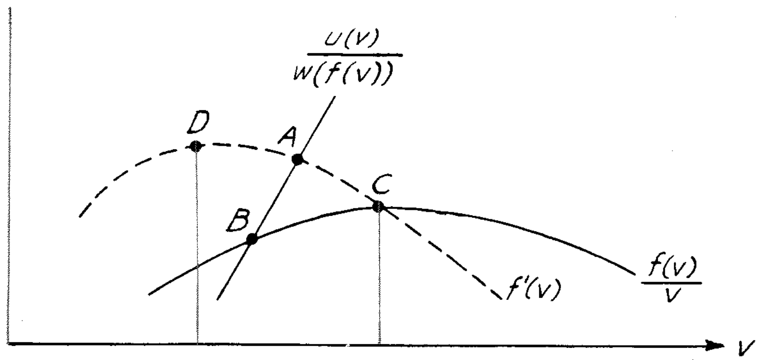

Fig. (1.4).

Now consider various actual ways of organizing production and consumption, or rather consider certain *conditions* which such ways may lead to. For the present purpose we are not interested in the concrete ways in which these conditions are brought about, only in the existence of the conditions themselves. We consider the following four conditions:

Consumption equals production i. e. $x = f(v)$. (1.5)

This may be looked upon as a condition for a *stationary* solution. In this case there is neither addition to nor subtraction from a stock of the consumption good, and no part of the production is thrown away.

Profit maximalization of the enterprises. (1.6)

For simplicity we assume this to take place under a regime where the enterprises act *as if* the price of the product and the wage rate are not affected by their adaptation. In this case the condition is expressed by $pf'(v) = q$. For our present purpose it is not necessary to study other and more complicated conditions that may replace the one put down.

The workers consume all they earn, i. e. $px = qv$. (1.7)

The workers are allowed to act as quantity adapters. (1.8)

This is expressed by the *Gossenian* balance

$$\frac{w(x)}{p} = \frac{u(v)}{q}$$

The equations (1.7) and (1.8) determine together the supply of labour and the demand for the consumption good in terms of the real wage rate.

It is easy to see that all the four conditions (1.5)—(1.8) cannot be satisfied at the same time if the function shapes $f(v)$, $w(x)$ and $u(v)$ are arbitrarily given. We have indeed only three variables to dispose of, namely x, v and $\frac{q}{p}$.

One must not fall into the error of thinking that the overdeterminateness is only apparent because we have forgotten to introduce the profit of the entrepreneurs as a variable.[1]) We shall subsequently revert to the question of the profit, but may already at this stage note that the profit, r, is definitionally related to the variables already introduced, namely by the equation

$$r = pf(v) - qv. \qquad (1.9)$$

Adding the variable r together with the equation (1.9) to the system we get a total of five equations between the four variables x, v, $\frac{q}{p}$ and $\frac{r}{p}$, and are thus, from the viewpoint of determinateness in the fixation of x and v, exactly where we were.

One way of exhibiting the nature of the overdeterminateness is for instance to note that (1.7) in conjunction with (1.5) and (1.9) requires that the profit shall be zero, while (1.6) requires that it shall be maximized under a certain type of variation. In order to achieve the fulfilment of both these conditions the real wage rate $\frac{q}{p}$ must have a certain magnitude. But it would only be by coincidence that this magnitude of the real wage rate becomes equal to that determined by (1.5) and (1.8). There are of course also many other ways of exhibiting the

[1]) I do not understand why, but my experience during several discussions seems to indicate that there exist some mysterious force that pushes the reasoning into this particular fallacy.

situation. From whatever angle one chooses to look upon the matter, the essential fact to be retained is that the system (1.5)—(1.9) contains a basic overdeterminateness, when the functions $f(v)$, $w(x)$ and $u(v)$ have arbitrarily given shapes.

This being so we are led to consider the properties of the systems obtained by dropping one of the four conditions (1.5)—(1.8). I do not propose to study the system obtained by dropping the last equation (1.8) since this would more or less be to accept the necessity of using forced labour, or — as the case may be — to accept a state of involuntary unemployment. But the three systems obtained by dropping (1.5), (1.6) and (1.7) respectively, will be considered.

2. Dropping the condition (1.5).

In this case the three variables x, v and $\frac{q}{p}$ are determined by the three equations (1.6)—(1.8). The nature of the solution in this case is best brought out by eliminating $\frac{q}{p}$ so as to get the following two equations in x and v,

$$f'(v) = \frac{u(v)}{w(x)} = \frac{x}{v}. \tag{2,1}$$

These two equations determine consumption x and labour input v. Through the determination of v the total production $f(v)$ is also determined. If it so happens that total production in the equilibrium point is larger than consumption, part of the production may be looked upon as being added to a stock of the consumption good, and if it so happens that total production in the equilibrium point is less than consumption, the difference may be looked upon as being drawn from stock.

It is easy to formulate a criterion for these two cases by introducing the *passus-coefficient* ϵ. It is defined by

$$\epsilon = \frac{d \log f(v)}{d \log v} \tag{2.2}$$

From this follows

$$v \cdot f'(v) = \epsilon \cdot f(v). \tag{2.3}$$

Hence by (2.1)

$$\frac{x - f(v)}{f(v)} = \epsilon - 1. \tag{2.4}$$

That is to say: The percentage with which consumption exceeds (or in the negative case: falls short of) production is simply equal to $\epsilon - 1$. Since ϵ is larger than unity in the technically pre-optimal stage (to the left of the vertical through C in fig. (1.4)) and ϵ is less than unity in the technically post-optimal stage (to the right of the vertical through C in fig. (1.4)), we can formulate this conclusion:

Production falls short of or exceeds consumption accordingly as the shapes of the datum functions $f(v)$, $w(x)$, $u(v)$ are such as to place the equilibrium point in the technically pre-optimal or in the technically post-optimal stage. (2.5)

Thus, the equilibrium in this case can coincide with the social optimum only if by chance the curve (1.3) passes through the technical optimum point (C in fig. (1.4)). More precisely: This is the necessary and sufficient condition that the two equilibria shall coincide.

By taking account of the second order conditions we see that if (1.6) shall give a true maximum (not a minimum) the equilibrium point must fall to the right of D in fig. (1.4) (the highest ordinate on the marginal productivity curve). But nothing prevents the equilibrium point from falling either to the left or to the right of C (the highest ordinate on the average productivity curve).

3. *Dropping the condition (1.6).*

Eliminating in this case $\frac{p}{q}$ by (1.7) and (1.8), and x by (1.5) we get the following equation in v alone

$$\frac{f(v)}{v} = \frac{u(v)}{w(f(v))}. \tag{3.1}$$

Hence, in this case the equilibrium is determined by the intersection of the curve (1.3) with the *average* productivity curve, i. e. by the point B in fig. (1.4). Since the social optimum was determined by the intersection of the curve (1.3) with the *marginal* productivity curve, i. e. by the point A in fig. (1.4) we see that also in the present case the equilibrium point will coincide with the social optimum when, and only when, the curve (1.3) happens to pass through the technical optimum point (C in fig. (1.4)).

4. Dropping the condition (1.7).

This is a very interesting case. Eliminating in this case $\frac{p}{q}$ between (1.6) and (1.8) and using (1.5) we get

$$f'(v) = \frac{u(v)}{w(f(v))} \tag{4.1}$$

which is identical with (1.2). In other words: *By dropping the budget condition (1.7) for the workers we get the social optimum.*

This being so let us consider the difference between the value of what the workers earn and the value of what they consume. This difference, namely

$$T = qv - px \tag{4.2}$$

may be looked upon as a tax or, if T happens to be negative, as a *subsidy* paid to the workers. The difference $qv - T$ is the disposable income of the workers, the assumption being that they consume all their disposable income. This assumption may be looked upon as one that now takes the place of (1.7).

By a similar argument as in Section 2 we now get

$$\frac{T}{px} = \epsilon - 1. \tag{4.3}$$

In other words: The income transfer needed in order to maintain the social optimum under the regime considered, is positive or negative accordingly as the social optimum (as defined by A in fig. (1.4)) happens to fall in the technical pre-optimal or in the technical postoptimal stage.

Comparing (4.2) with (1.9) it may appear as if we are now considering the same situation as in (1.9), the only difference being the sign of a new parameter introduced (T being equal to $(-r)$). There is however a fundamental difference. By (1.9) we introduced the parameter r in a situation where *all four* equations (1.5)—(1.8) were retained. Now we introduce (4.2) *in replacement* of one of the equations (1.5)—(1.8). The parameter T now introduced is, as we have seen, a tax which is put on the workers, the sum collected being given as a subsidy to the enterprises, or if T turns out to be negative, it represents a surplus taken from the enterprises and distributed as a subsidy to the workers in order to enable them to consume the total product, and thus to maintain the equilibrium. Certainly such an income transfer is not "profit" in the traditional sense of the equilibrium theory.

If we want to introduce profit in the traditional sense, that is in the sense of something *that may be retained by the entrepreneurs* it seems more natural to do it in the form of case 2, namely by dropping the assumption production equals consumption. This leads to a constantly increasing or a constantly decreasing stock (which would be the form in which profit is invested). Whether the stock shall increase or decrease is uniquely determined by the shapes of the three functions $f(v)$, $w(x)$, $u(v)$. This case throws, in my opinion, an interesting light on a much discussed structural tension inherent in the capitalistic system: If profits are allowed to accumulate (positively or negatively) we do not get a *stationary situation*. But the situation may be made stationary by introducing, as above, the income transfer. And by this device the stationary situation even becomes identical with the socially defined optimum situation[1]).

It should be noted that in the case where income transfers are not used, but the stock is allowed to change, the direction and extent of this change is *determined* by the conditions put up. It is not possible to add at this stage a condition regarding, say, the entrepreneurs' propensity to save. Adding such a condition would again make the system overdeterminate. The increase (or decrease) in stock that is required in order to satisfy the other conditions may not be the same as that which would be required in order to match with the propensity to save. This is only another way of pointing out the structural tension that may emerge in a capitalistic system.

5. *The case with consuming entrepreneurs.*

Prima facie one might think that the overdeterminateness considered in previous sections would be lifted by introducing the *consumption* of the entrepreneurs. This, however. is not so. From the viewpoint of determinateness nothing essentially new is brought in by this device. This is easily seen as follows. We denote by x_1 and x_2 the consumption of the workers and the entrepreneurs respectively, so that $x = x_1 + x_2$. The conditions (1.5) to (1.8) remain as they stand with the exception that x_1 is inserted instead of x in (1.7) and (1.8) and $x_1 + x_2$ instead of x in (1.5). We now have four variables x_1, x_2, v and $\frac{q}{p}$ and consequently exact determinateness. It is however only typographically that this case gives something new. In reality it is identical with the case discussed in

[1]) It should be remembered that the social optimum here considered is defined under a technically given production function $f(v)$. If by any means whatsoever it should be possible to change this function, a still higher optimum may be reached.

Section 2. Indeed the consumption of the entrepreneurs in the present scheme *is not left to be decided by the entrepreneurs themselves.* The magnitude x_2 is "forced down the throat of the entrepreneurs" or on the contrary (if x_2 in the equilibrium point happens to be a negative magnitude) it is "extracted from the entrepreneurs". In any case x_2 is determined simply as an amount necessary to make it possible to fulfil the specified conditions of the regime, and these conditions do not contain any allusion to the consumption preferences of the entrepreneurs. Whether we think of the discrepancy between production and the workers' consumption as being taken care of by such a forced consumption or possibly "deconsumption" on the part of the entrepreneurs or we think of it as a change in stock, is irrelevant from the determinateness point of view. From the viewpoint of concrete interpretation the idea of the forced consumption on the part of the entrepreneurs is, of course, so unrealistic that it cannot be taken seriously.

On the other hand if we did *not* want to consider the consumption of the entrepreneurs as a forced one, but wanted to let it be determined by some sort of utility or demand relation we would again run into an overdeterminate system.

6. *A more general analysis: The wage function.*

Let us reconsider the whole problem. Suppose that the wage rate q is not a number but some *function* of v, denoted $q(v)$. That is to say the money wage paid to each worker per unit of labour input is not a constant to him but some function $q(v)$ of his total labour input v. Instead of the wage rate $q(v)$ we may consider the wage sum $v \cdot q(v)$. And we may further deflate this sum by dividing it by p, p being as before a number, not a function. That is, the real value of the total wage, namely

$$c(v) = \frac{v \cdot q(v)}{p} \tag{6.1}$$

is considered as some function of v. We will study the problem in terms of the properties of this function $c(v)$.

We impose simultaneously all the conditions as formulated in words under (1.5)—(1.8). These may now be summarized in the form of the following three conditions on the single variable v,

$c(v) = f(v)$	(The budget equation of the workers combined with the assumption that consumption equals production).

$c'(v) = f'(v)$ (Profit maximalization). (6.2)

$f'(v) = \frac{u(v)}{w(f(v))}$ (The Gossenian balance for the workers transformed by means of the two conditions above).

where

$$c'(v) = \frac{dc(v)}{dv}$$

The second order condition in the equilibrium point is

$$f''(v) < c''(v) < \frac{d}{dv}\left[\frac{u(v)}{w(f(v))}\right] \tag{6.4}$$

where f'' and c'' stand for second order derivatives.

(6.2) are three conditions on the single variable v. In order to satisfy them, the function $c(v)$ must consequently satisfy *two point conditions.* One of these correspondends to the fixation of the real wage rate in the case where this wage rate was a number. The other point condition which the function $c(v)$ must satisfy corresponds to what was the over-determinateness when the problem was formulated with the real wage rate as a number. Since we now have more degrees of freedom at disposal (even a "number" of degrees of freedom having a higher power than that of the continuum) all the conditions may be fulfilled.

As an example of a case where the function $c(v)$ satisfies the point conditions in question may be mentioned the case discussed in Section 4. This obviously may be formulated as a very special case of the function $c(v)$.

The general conditions on the nature of the function $c(v)$ which follow from the regime now considered, may be formulated in the following two propositions:

In order that the wage function $c(v)$ shall satisfy simultaneously the regime conditions (1.5)—(1.8) it is necessary and sufficient that it has first order contact with the product function in the special point that represents the social optimum. (6.5)

If the wage function $c(v)$ satisfies the conditions mentioned in (6.5) as well as the second order condition (6.4) the equilibrium point reached by using this wage function under the regime (1.5)—(1.8) will automatically lead to the social optimum point. (6.6)

The latter proposition suggests an interesting possibility of influencing the incentive to work and indeed the whole configuration of the economy in a desirable direction. In principle the procedure would be, first to determine the location of the social optimum point from a knowledge of the shapes of the functions $f(v)$, $w(x)$ and $u(v)$, and then to apply some wage function $c(v)$ that has the contact properties mentioned.

Whether the application of a given wage function $c(v)$ is looked upon as a method of "wage fixing" or as method of "taxation" is immaterial. Indeed if both the wage and the tax are considered from the general viewpoint of "functions of v", there does not remain any possibility of distinguishing between the concept of a wage and that of a tax.

Since there is a great class of wage (or tax) functions that may in principle achieve the social optimum, one may *add more conditions,* for instance conditions that make the equilibrium point as stable as possible according to some criterion, or that has certain other properties which may be desirable from the viewpoint of applications.