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THE MULTIPLEX METHOD FOR LINEAR PROGRAMMING

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1. The problem

In standard form the linear programming problem can be formulated as follows. Consider $(n+m)$ real variables

$$x_1, x_2, \ldots, x_{n+m} \qquad \ldots \quad (1.1)$$

satisfying m linearly independent linear equations.

The number of degrees of freedom is consequently n and it is always possible at least in one way to express all the variables in terms of a set of n basis variables, linearly independent amongst themselves. Let

$$x_u, x_v, \ldots, x_w \qquad \ldots \quad (1.2)$$

be such a basis set. The equations may then be written in the standard form

$$x_j = b_{j0} + \sum_{k-u,\, v \ldots w} b_{jk} x_k \qquad (j = 1, 2, \ldots, n+m) \qquad \ldots \quad (1.3)$$

where the b_{j0} and b_{jk} are constants. Obviously

$$b_{j0} = 0 \quad \text{and} \quad b_{jk} = 0 \begin{bmatrix} 1 & \text{if } k = j \\ 0 & \text{otherwise} \end{bmatrix} \quad \text{when } j = u,\ v \ldots w. \qquad \ldots \quad (1.4)$$

If the equations (1.3) are taken for all $j = 1, 2, \ldots, n+m$, we get a system of equations that are linearly dependent, but if we take (1.3) only for[1] $j = 1, 2 \ldots) u, v \ldots w(\ldots n+m$, we get a system of equations that are linearly independent. More precisely : If the coefficients b_{j0} and b_{jk} have any values whatsoever, m equations of the form (1.3) for $j = 1, 2 \ldots) u, v \ldots w(\ldots n+m$ are always linearly independent.

We consider a linear *preference function*

$$f = p_0 + p_u x_u + p_v x_v + \ldots + p_w x_w \qquad \ldots \quad (1.5)$$

where the p_k $(k = 0, u, v \ldots w)$ are any given constants positive, negative or zero.

It does not restrict generality if we assume that only basis variables occur in (1.5). Indeed, any linear function of all the $n+m$ variables will assume the form (1.5) when all the variables are expressed in terms of the basis variables by means of (1.3).

The linear programming problem is the problem of determining that one or those sets of values of the variables that will maximize (1.5) subject to two sets of conditions. In the first place the *equations* (1.3), and in the second place the non-negativity conditions expressed by the *inequalities*

$$x_j \geqslant 0 \qquad (j = 1, 2, \ldots, n+m). \qquad \ldots \quad (1.6)$$

[1]We use the inverted parenthesis)...(to denote "exclusion of".

A great variety of problems can be reduced to this standard form. If there should be one of the variables that is *not* subject to the non-negativity condition, this variable could be eliminated and the problem reduced to one with the same number of degrees of freedom but one equation and one variable less. A problem containing any linear inequality, that is the condition that a given linear function of the variables shall be non-negative, can be reduced to the above standard form simply by putting the linear function in question equal to a new variable that is entered in the list (1.1).

The inequalities (1.6) introduce *discontinuities* in the boundary conditions, and this makes the method of Lagrange multipliers, which serves so well in many maximum problems with side conditions, inapplicable here.

The neoclassical method for handling the linear problem is the simplex method due to George B. Dantzig. At the Oslo University Institute of Economics, considerable effort has been made to handle the problem in a different way in the hope of finding one or some methods that may be more advantageous in cases with a great number of variables, particularly in problems of the type occurring in macroeconomic planning.[1]

It is an easy form of mental exercise to imagine iteration processes of one form or another which may look nice on paper, but in most cases such methods do not converge, or if they do converge in principle, the convergency will in general be so slow as to make the method entirely useless in practice. At the Oslo Institute we have accumulated a considerable junk-pile of such methods. Some of the methods we have used, do, however, contain features that may make them useful under certain circumstances. Some aspects of methods which have actually been used successfully on medium sized examples, are described in a number of mimeographed memoranda from the Oslo Institute of which the following might be mentioned here.[2]

21 June 1954 : Methods of solving linear programming problems. Synopsis of a lecture to be given at the International Seminar on Input-Output Analysis, Varenna (Lake Como) June—July 1954.

18 October 1954 : Principles of linear programming. With particular reference to the double gradient form of the logarithmic potential method.

29 March 1955 : A labour saving method of performing freedom truncations in linear programming. Part I.

13 May 1955 : The logarithmic potential method of convex programming. With particular application to the dynamics of planning for national development. Synopsis of a communication to be presented at the international colloquium of econometrics in Paris 23—28 May 1955.

The present memorandum describes—without proofs—a method which we have recently applied quite successfully to numerical examples of medium size. It may be called the multiplex method as distinguished from the simplex method.

It should always be remembered that any mathematical method and particularly methods in linear programming must be judged with reference to the type of computing

[1] In a national planning problem of some size, one may easily run into several hundred variables and perhaps a hundred or more degrees of freedom.

[2] Subsequently a great number of other memoranda have been produced. I also discussed the problem in my lectures at the Institute for Social Studies, The Hague, in the Spring of 1957.

machinery available. In all our work we have been guided by the possibilities and desiderata in a situation where the available equipment consists of desk machines or IBM 602A (or its electronic improvement 626, or similar types of calculating punches) or electronic automatic computors with a small high speed memory, as for instance the Oslo machine NUSSE (whose name academically stands for "Norwegian Universal etc.", but is better understood when referred to the meaning of the Norwegian word nusse which can appropriately be translated as "small cute girl"). Our outlook may perhaps be changed when we get used to the super modern, high capacity electronic computor that will be available here from the middle of next year.

2. The initial point

Start from a point in the interior of the admissible region, that is such a combination of the values of the basis variables (1.2) as will make all the $(n+m)$ variables (1.1) effectively positive (not zero). If necessary, use the $S(\lambda)$ method of Section 6 in "Principles of linear programming" for finding such a point.[1]

From the initial point move by some method in a direction which is influenced by the direction of the preference vector, i.e. the vector with components p_k $(k = u, v \ldots w)$. A highly effective way of determining the direction of such a movement and at the same time taking account of the boundary conditions is the double gradient form of the logarithmic potential method, but this does involve a higher computational cost than simply to move, say, in the preference direction. At present I cannot say definitely which one of the alternatives for the initial step will be computationally most profitable when account is taken of what may happen in the subsequent steps. It is, of course, always tempting to start in the cheapest way.

The computations involved in a movement in the direction determined by the double gradient form of the logarithmic potential method, are described in detail in "Principles of linear programming" (in particular (12.29)—(12.31) and § 13). A movement in the preference direction is simply determined by putting

$$x_k = x_k^0 + \lambda p_k \qquad (k = u, v \ldots w) \qquad \ldots \quad (2.1)$$

x_k^0 being the coordinates of the initial point, and letting λ increase from 0 through positive values.

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[1] The $S(\lambda)$ method is well adapted for mechanical computation. It is in essence a method of finding solutions to a set of linear inequalities. I suspect that from the procedures involved in the $S(\lambda)$ method we can also extract a necessary and sufficient criteria for the case where an admissible region with a positive content exists, but I have not had an opportunity to follow up this idea. One of my associates in a working team at the Indian Statistical Institute in the winter of 1954-55 suggested that when the linear programming problem is transformed into a problem of solving a set of linear inequalities (which is possible by a theorem of J. von Neumann), we may solve the whole linear programming problem by the $S(\lambda)$ method. I do not know how fruitful this idea is. I suspect it will depend on how large the number of degrees of freedom in the optimum point is. The pointset that in the original formulation was the optimum point set (possibly only a single point) will appear as the admissible region in the transformed problem. In order to apply the $S(\lambda)$ method to (1.6) we do not need to know beforehand the dimensionality of the pointset that actually satisfies (1.6).

The change in the other variables in the course of this movement is determined by

$$x_j = x_j^0 + \lambda d_j \qquad (j = 1, 2, \ldots, n+m) \qquad \ldots \quad (2.2)$$

where

$$d_j = \sum_{k=u,\, v \ldots w} b_{jk}\, p_k \qquad (j = 1, 2, \ldots, n+m). \qquad \ldots \quad (2.3)$$

The largest value permissible for λ when we shall stay within the admissible region is[1]

$$\lambda_{\max} = \operatorname*{Min}_{j} \lambda_j \;\Big|\; \lambda_j \gtreqless 0 \qquad \ldots \quad (2.4)$$

where

$$\lambda_j = \frac{x_j^0}{-d_j} \;\Big|\; d_j < 0. \qquad \ldots \quad (2.5)$$

The value (2.4) of λ determines the breaking-out-point where we must stop.

We consider also the values of λ_j that are positive larger than $\lambda_{\max}$, and *come closest to* $\lambda_{\max}$. This determines the priority order of what may be called the *optimum candidates* determined by the breaking-out-point considered. If $j = \alpha$ is the value of j that furnishes the minimum (2.4), $j = \beta$ the value that gives the next to smallest positive value of λ_j etc. ...up to some value $j = \gamma$, we say that the list $\alpha, \beta \ldots \gamma$ is the list of optimum candidates corresponding to the breaking-out-point considered. The variable x_α may be called the leading canditate. Some of the candidates may give the same value of λ. For instance it may happen that $\lambda_\alpha = \lambda_\beta$ (in the double gradient form this is always so) which expresses the fact that in the breaking-out-point we hit simultaneously two of the boundary planes.[2]

3. THE NUMBER OF CANDIDATES TO CONSIDER

At each breaking-out-point which has been reached through a movement whose direction is to a large extent determined by (local) considerations on the maximization of the preference function, we pick a certain number of the optimum candidates and make provisionally the guess that there exists at least one optimum point where these candidates are zero. Much depends on a happy choice of this number. If we make it too *large*, we run a great risk of making a wrong guess and will then have to face the possibility that to correct the mistake we must make additional computations at the end of the work when it turns out that the point we have reached is not optimal (compare § 9 below). And if we make the number of variables included in the guess at each step too *small*, we shall proceed through many calculations that are in fact unnecessary. We can in principle never be completely protected against making a wrong guess. Even a leading candidate may finally turn out to be different from zero in the optimum point. But if we are willing to run *some* risk of having to make correctional computations in the end, we may very often make short cuts that can save us a good deal of work.

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[1] An indication written after a vertical bar expresses the condition imposed on the suffix over which a maximization or minimization shall take place.

[2] The idea of using the values of the λ for ranging the candiates in a priority order was suggested some years ago by Mrs. Inger Haugstad, Chief computor in the University Institute of Economics, Oslo.

Since we have an exact necessary and sufficient criterion by which to decide whether a given point is optimal or not, and since we have a method of proceeding to correctional computations at the end if the optimality criterion should turn out in the negative, it is a sound procedure to work with guesses on the optimum candidates.

Empirically we have found that it pays fairly well to include at each round additional candidates equal in number to *the square root of the number of degrees of freedom with which we arrived in the breaking-out-point in question.*

Since this rule will usually give fractional numbers, we use a standard procedure for the rounding off to integers.

We first compute by the following recurrence formulae

$$N_t = N_{t-1} - \sqrt{N_{t-1}} \qquad \ldots \quad (3.1)$$

the number of degrees of freedom N_t which should theoretically be retained in round number t, the initial value being $N_0 = n$.

These numbers N_t are then rounded off to the nearest integer

$$n_t = \text{nearest integer to } N_t \qquad \ldots \quad (3.2)$$

and $n_{t-1} - n_t$ is taken as the number of new candidates to include in our optimality guess before we start on round number t.

Or otherwise expressed : When going from round number $t-1$ to round number t, we *reduce the number of degrees of freedom* by

$$n_{t-1} - n_t. \qquad \ldots \quad (3.3)$$

For instance, if $n = 12$, we get the scheme indicated in Table (3.4).

TABLE (3.4). EXAMPLE OF COMPUTATIONS (3.1)—(3.3)

round number	solution of (3.1) N_t	number of degrees of freedom with which this round is to be performed n_t	number of variables to be put equal to zero when starting on this round $n_t - t-1$
$t =$ 0	12.0000	12	0
1	8.5359	9	3
2	5.6143	6	3
3	3.2448	3	3
4	1.4435	1	2
5	0.2421	0	1

Towards the end of the work when the rule indicates that two variables shall be put equal to zero, we only take *one* (the leading candidate), because it will involve no more work to use (4.2) twice than to use first (5.2) (to bring one variable to zero) and then (4.2).

4. A PREFERENCE INCREASING MOVEMENT

Consider a point where $\nu(\leqslant n)$ variables

$$x_\alpha, x_\beta, \ldots, x_\gamma \qquad \ldots (4.1)$$

—basis variables or dependent vriables—are equal to zero, all the other variables being effectively positive. That is to say we are in a point on the boundary of the admissible region.

Consider a movement from this point in the direction

$$d_k = p_k + B_\alpha b_{\alpha k} + B_\beta b_{\beta k} + \ldots + B_\gamma b_{\gamma k} \qquad (k = u, v \ldots w) \qquad \ldots (4.2)$$

where the $B_s (s = \alpha, \beta \ldots \gamma)$—independent of k—are the solutions of the system of linear equations

$$M_{0s} + B_\alpha M_{\alpha s} + B_\beta M_{\beta s} + \ldots + B_\gamma M_{\gamma s} = 0 \qquad (s = \alpha, \beta \ldots \gamma) \qquad \ldots (4.3)$$

where

$$M_{rs} = \sum_{k=u, v \ldots w} b_{rk} b_{sk} \qquad \begin{pmatrix} r = 0, \alpha, \beta \ldots \gamma \\ s = 0, \alpha, \beta \ldots \gamma \end{pmatrix}. \qquad \ldots (4.4)$$

We assume that the boundary vectors $b_{\alpha k}, b_{\beta k} \ldots b_{\gamma k}$ are linearly independent so that (4.3) has a unique solution. The B are called the regression coefficients.

With the direction numbers (4.2) we perform a movement

$$x_k = x_k^0 + \lambda d_k \qquad (k = u, v \ldots w) \qquad \ldots (4.5)$$

x_k^0 being the initial values and λ being a parameter that increases from 0 through positive values. During this movement all the variables change according to

$$x_j = x_j^0 + \lambda d_j \qquad (j = 1, 2, \ldots, n+m) \qquad \ldots (4.6)$$

where

$$d_j = \sum_{k=u, v \ldots w} b_{jk} d_k \qquad (j = 1, 2, \ldots, n+m). \qquad \ldots (4.7)$$

The preference function will also change. More precisely, for any positive value of λ it will have *increased* as compared to its initial value. We have indeed

$$f - f^0 = \lambda \sum_{k=u, v \ldots w} p_k d_k = \lambda \frac{\begin{vmatrix} M_{00} & M_{0\alpha} & M_{0\beta} & \ldots & M_{0\gamma} \\ M_{\alpha 0} & M_{\alpha\alpha} & M_{\alpha\beta} & \ldots & M_{\alpha\gamma} \\ M_{\beta 0} & M_{\beta\alpha} & M_{\beta\beta} & \ldots & M_{\beta\gamma} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ M_{\gamma 0} & M_{\gamma\alpha} & M_{\gamma\beta} & \ldots & M_{\gamma\gamma} \end{vmatrix}}{\begin{vmatrix} M_{\alpha\alpha} & M_{\alpha\beta} & \ldots & M_{\alpha\gamma} \\ M_{\beta\alpha} & M_{\beta\beta} & \ldots & M_{\beta\sim} \\ \ldots & \ldots & \ldots & \ldots \\ M_{\gamma\alpha} & M_{\gamma\beta} & \ldots & M_{\gamma\gamma} \end{vmatrix}}. \qquad \ldots (4.8)$$

By (4.4) the expression to the right in (4.8) is the ratio between two moment matrices and hence positive definite.

All the variables $x_\alpha, x_\beta, \ldots, x_\gamma$ on the contrary will remain *unchanged* during the λ variation because by (4.2) and (4.3)

$$x_s - x_s^0 = \lambda \sum_{k=u,\, v \ldots w} b_{sk} d_k$$

$$= \lambda(M_{0s} + B_\alpha M_{\alpha s} + B_\beta M_{\beta s} + \ldots + B_\gamma M_{\gamma s}) \quad (s = \alpha, \beta \ldots \gamma). \quad \ldots \quad (4.9)$$

The parenthesis to the right in (4.9) is zero by (4.3).

Since the rest of the variables are all effectively positive, we can let λ increase a certain non-zero amount before we hit any boundary plane. Its maximum value will be determined in exactly the same way as in (2.4)—(2.5) and we will consequently have a new determination of some basis candidates, their number being determined as explained in § 3.

This guess being made, we try to move to a new point where these new variables are zero as well as those that were previously brought to zero. In doing so we retain the *same basis form of the equations*, that is to say we retain the same basis variables (1.2), the reduction in the number of degrees of freedom being produced by conditions imposed on the direction of the movement. This is done as follows.

5. A VARIABLE-ANNIHILATING MOVEMENT

Suppose that we are in any point x_k^0 in the admissible region and want to move towards a *direction point* where the ν variables $x_\alpha, x_\beta, \ldots, x_\gamma$ have reached prescribed (positive, negative or zero) increments

$$\Delta x_\alpha, \Delta x_\beta, \ldots, \Delta x_\gamma \quad \ldots \quad (5.1)$$

the direction of the movement being further specified by the requirement that it shall have components of the form

$$d_k = C_\alpha b_{\alpha k} + C_\beta\, b_{\beta k} + \ldots + C_\gamma b_{\gamma k} \qquad (k = u, v \ldots w) \quad \ldots \quad (5.2)$$

where $C_\alpha, C_\beta, \ldots, C_\gamma$ are constants independent of k.

The values of these ν constants will be determined by the requirement that the ν increments (5.1) shall have prescribed values. Indeed the coefficients C must then be solutions of the system

$$C_\alpha M_{\alpha s} + C_\beta\, M \quad + \ldots + C_\gamma M_{\gamma s} = \Delta x_s \qquad (s = \alpha, \beta, \ldots, \gamma). \quad \ldots \quad (5.3)$$

We assume that the matrix in the left member of (5.3) is non-singular so that the system has a unique solution.

Inserting the C values thus obtained into (5.2), we get a set of well defined direction numbers, and can proceed to a movement of the type indicated by (4.5)—(4.7). In particular we may choose the increments (5.1) as

$$\Delta x_s = -x_s^0 \qquad (s = \alpha, \beta, \ldots, \gamma) \quad \ldots \quad (5.4)$$

in which case all the ν variables $x_\alpha, x_\beta, \ldots, x_\gamma$ will be zero in the direction point.

As λ increases from 0 to 1, we pass from the initial point to the direction point where the prescribed increments (5.1) are reached. Two cases are now to be distinguished : We may be able to proceed unhindered up to the point $\lambda = 1$ where the increments (5.1) are realized, or we may en route be stopped by the boundary plane of a variable which is *not* in the set $x_\alpha, x_\beta, \ldots, x_\gamma$.

If the set $x_\alpha, x_\beta, \ldots, x_\gamma$ has been successfully selected, we will as a rule have the first alternative. In this case all is well, and we can proceed from the point thus obtained by using (4.2)—(4.3). Computationally this is now an extremely simple matter if the system (5.3) has been solved by the Gaussian elimination-algorithm.[1] The computation of the B_s ($s = \alpha, \beta \ldots \gamma$) that satisfies (4.3) will then only be a matter of one more back solution.

In the other case we will meet (at least) one other boundary plane before reaching $\lambda = 1$. In this case we start from the point reached and use (4.2) with a set $x_\alpha, x_\beta, \ldots, x_\gamma$ that contains all the variables that are zero in this point. This is the variable which just stopped us together with those variables that remain zero during the movement up to the point where we now are. In the breaking-out-point to which this will lead us, a new selection of candidates can be made according to the standard rules of § 3 and § 4. Since the last movement was made by (4.2)—not by (5.2)—we will be justified in counting it as a round which means that a correspondingly larger number of variables will be included in the optimum group in the next round. In any case should the total net number of degrees of freedom n_t to be used in round number t be taken according to the rule of § 3.

6. THE ACCURACY OF THE COMPUTATIONS

As we proceed in the work the number of variables entering into the set $x_\alpha, x_\beta, \ldots, x_\gamma$ will increase. This means that the order of the system of linear equations—that is (4.3) or (5.3)—will increase. Since we do not need to consider an entirely new system at each round, but only have to add a certain number of rows and columns, the computational cost involved will not be prohibitive if the Gaussian algorithm is used. But some attention must be paid to the accuracy of the results. In general one will need all the more decimal-places the higher the system. One may find for instance that the variables that should in principle be maintained exactly equal to zero under the λ variation, do not arrive in the point $\lambda = 1$ with a value which is sufficiently close to zero. This is annoying for the subsequent work. The simplest way to handle this situation is to start by computing the moments (4.4) with great accuracy and in the subsequent work to make it a standard procedure at each step to improve the Gaussian algorithm values obtained for the B_s or C_s by at least one round of an *iteration process*.

Letting B_s and C_s be the values obtained through the Gaussian algorithm, the improved values B'_s and C'_s are then computed by

$$B'_s = \frac{-M_{0s} - \sum_{r<s} B'_r M_{rs} - \sum_{r>s} B_r M_{rs}}{M_{ss}} \qquad (s = \alpha, \beta \ldots \gamma) \qquad \ldots \quad (6.1)$$

$$C'_{s_1} = \frac{\Delta x_s - \sum_{r<s} C'_r M_{rs} - \sum_{r>s} C_r M_{rs}}{M_{ss}} \qquad (s = \alpha, \beta \ldots \gamma). \qquad \ldots \quad (6.2)$$

[1] And the elimination is performed in such a way as to store the triangular matrix obtained. This is always the case if the computations are done on desk machines, but it is not necessarily so when electronic automatic computors are used.

To perform one such iteration round is equivalent to testing the solutions by inserting in the original equations. The iteration is therefore also a reliable check. If judged necessary by the change produced from the uncorrected to the corrected values, one more iteration round may be performed. The computational cost involved in one iteration round is of the same order as that involved in the back solution by which B_s or C_s is obtained.

7. Sufficient criteria for optimality

In the work as here outlined each round will consist of two movements : First one of type (5.2) through which certain additional variables are brought down to zero, and next one of type (4.2) where all previous 0-variables are maintained as 0-variables and (usually) one more 0-variable is introduced, and at the same time a selection is made of certain variables which one will in the next step try to bring down to zero.

If the Gaussian elimination algorithm is used, each round will involve *adding* to the previous matrix as many rows and columns as there are variables which now are to be forced down to zero, elimination-treating these added rows and columns and then performing two back solutions of the *complete* system.

When (5.2) is used, the fact that the computation of the solution of (5.3) for the C_s ($s = \alpha, \beta \dots \gamma$) in the case where not all Δx_s are zero, proceeds regularly and does *not* by (5.2) give a set of d_k ($k = u, v \dots w$) that are all zero, assures us that the variables $x_\alpha, x_\beta, \dots, x_\gamma$ are not linearly dependent.

When we reach a stage where all the d_k defined by (4.2) turn out to be zero for $k = u, v \dots w$, we must consider the possibility that we have reached a point in the optimum region.

If all the coefficients $B_\alpha, B_\beta, \dots, B_\gamma$ entering into the d_k in question are non-negative, we can say immediately that we have actually reached an optimum point and that there exists a $(n-\nu)$ dimensional region of such points, ν being the number of variables in the set $x_\alpha, x_\beta, \dots, x_\gamma$.[1]

This is seen simply by noticing that if all the d_k defined by (4.2) are zero— a fact that only expresses a feature of the boundary vectors b_{sk} and the preference vector p_k and has nothing to do with the particular point, i.e. the particular values of $x_u, x_v, \dots, x_w$, which we may happen to consider — we get by equating (4.2) to zero

$$f = p_0 + \sum_{k=u,\, v \dots w} p_k x_k = p_0 - \sum_{s=\alpha,\, \beta \dots \gamma} \; \sum_{k=u,\, v \dots w} B_s b_{sk} x_k = p_0 - \sum_{s=\alpha\ \beta \dots \gamma} B_s (x_s - b_{so})$$

that is

$$f = \Big[p_0 + \sum_{s=\alpha,\, \beta \dots \gamma} B_s b_{so} \Big] - \sum_{s=\alpha,\, \beta \dots \gamma} B_s x_s. \qquad \dots \quad (7.1)$$

Since this formula holds good for any values of the basis variables inside or outside the admissible region or on its boundary, we can immediately conclude that if all the coefficients $B_s (s = \alpha, \beta \dots \gamma)$ are non-negative, we can nowhere in the admissible region find a point which produces a value of f larger than the value obtained by putting

$$x_\alpha = x_\beta = \dots = x_\gamma = 0. \qquad \dots \quad (7.2)$$

[1] If N of the coefficients B are actually zero, there even exists an $(n-\nu+N)$ dimensional region of such points. The corresponding N conditions in (7.2) can then be dropped.

This optimum value for the preference function is

$$f_{opt} = p_0 + \sum_{s=\alpha, \beta \ldots \gamma} B_s b_{s0}. \qquad \ldots \quad (7.3)$$

If the computation of the coefficients $B_\alpha, B_\beta, \ldots, B_\gamma$ has proceeded regularly, the boundary vectors $b_{\alpha k}, b_{\beta k}, \ldots, b_{\gamma k}$ cannot be linearly dependent. Hence the linear manifold defined by (7.2) must be of dimensionality $n-\nu$ where ν is the number of variables in (7.2). That is to say there exists (at least) one set of $n-\nu$ of the variables which are linearly independent and will, as they vary within the confines defined by the admissible region, generate a linear manifold where every point is optimal, i.e. produces the preference value (7.3).

If all the d_k defined by (4.2) are zero for $k = u, v \ldots w$, but one or more of the coefficients B are negative, we may have one or the other of the following two situations :

(I) We are actually in an optimal point, and if so, it must be possible to transform the expression (7.1) in such a way as to obtain a linear form with only non-negative coefficients B. This is the case where the optimum region is *multiply determined*, that is there exists at least one other set of ν variables such that the condition that *these* ν variables shall be zero, is equivalent to the condition (7.2).[1]

(II) We are not in an optimal point. One or more of our guesses of optimum candidates must then have been wrong, and in order to proceed towards the optimum point, we must again let loose one or more of the variables we have tentatively fixed at zero value.

Case (II) is discussed in § 9. In the present section we shall consider case (I). A few simple graphical illustrations will indicate the possiblities and what can be done in order to bring the expression (7.1) over into a form with only non-negative coefficients B so that a sufficient criterion for optimality emerges.

Fig. (7.4) illustrates a case where the admissible region is two-dimensional (being generated by the two basis variables x_u, x_v). The optimum region is the point $A(\nu = n = 2)$. The two straight lines that pass through A and *actually form the boundary* in the vicinity of A, are the two lines numbers (1) and (4), representing the conditions $x_1 = x_4 = 0$. There are, however, also two other lines passing through A, namely the lines numbers (2) and (3), representing the conditions $x_2 = 0$ and $x_3 = 0$. These two lines are situated entirely outside of the admissible region—with the exception of the point A—and the solution of the linear programming problem would have remained completely unchanged if one of or both the lines (2) and (3) had been omitted. They only add an unnecessary and undesirable complication in the picture. The optimum point can now be determined by putting *any two* of the four variables x_1, x_2, x_3, x_4 equal to zero. This might not be discovered until we get towards the end of the calculations and this is what causes the undesirable complication.

If the two lines numbers (2) and (3) had not been present, the situation in fig. (7.4) would have been very simple. The necessary and sufficient condition for the point A to be an optimum point, would then obviously have been that the preference vector p_k had been situated in the convex angle (the two-dimensional case of a convex polyhedral cone) between the two negative boundary vectors $(-b_{1k})$ and $(-b_{4k})$. They are denoted 1 and 4 and drawn

[1] When a region (in the special case $\nu = n$, a point) is multiply determined, it is customary to speak of "degeneracy". This term is not a happy one. It seems more appropriate and suggestive to speak of "a multiply determined" region.

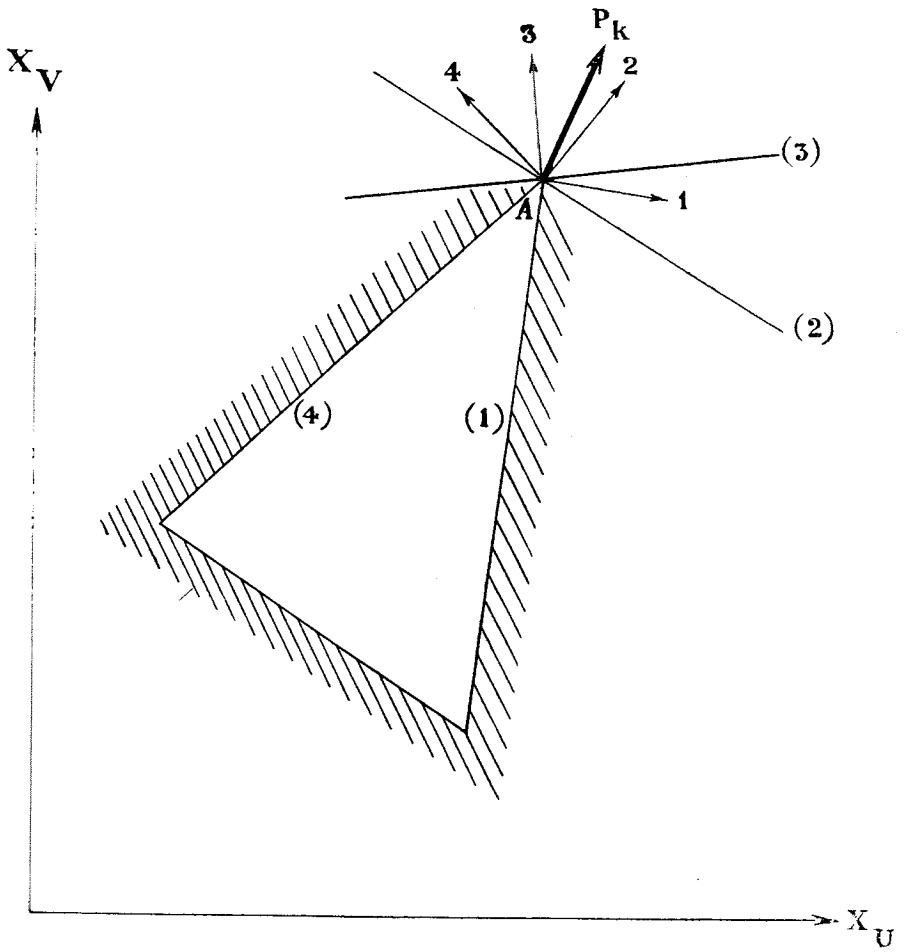

Fig. 7.4

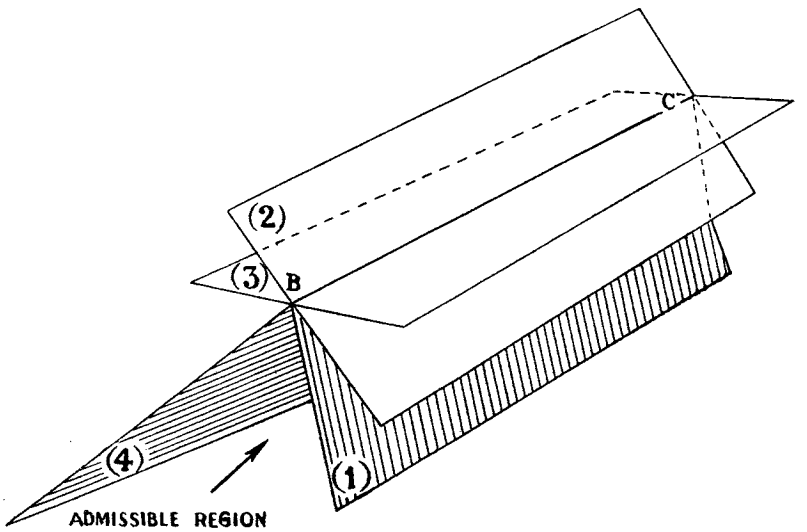

Fig. 7.5

as light arrows in fig. (7.4). This is the same as to say that the preference vector could have been expressed as a linear form in the two negative boundary vectors and with *non-negative* coefficients, i.e. as a linear form in the boundary vectors themselves, b_{1k} and b_{4k} and with *non-positive* coefficients. It is easily seen by a graphical inspection that this is the necessary and sufficient condition that there are no points that fall on the desired side of the preference plane through A and at the same time in the admissible region.

As the situation actually is in fig. (7.4) the preference vector may in $\binom{4}{2} = 6$ ways be expressed as a linear form in two boundary vectors $b_{\alpha k}$, $b_{\beta k}(\alpha \neq \beta)$, but not all these ways will give both coefficients of the boundary vectors non-positive. For instance, if we use the set 3, 4 the coefficient of b_{3k} will be negative but that of b_{4k} positive. If we use the set 2, 3 both coefficients will be negative, so that 2, 3 will actually furnish a *sufficient* criterion for optimality. Geometrically : If the preference vector p_k is situated in the angle 2, 3 it must *a fortiori* be situated in the angle 1, 4. If we use the set 1,4 we will also get a sufficient criterion. This latter set-being characterized by the fact that all the boundary vectors in A are included in the convex angle (1, 4) — has a further property not shared by any of the other sets, namely of furnishing a *necessary* criterion for optimality.

The above considerations show that if we are in an optimum point and this point is multiply determined, it will in general be possible to transform the expression for the preference *in different ways* to a linear function where all the coefficients of the variables are non-negative. Any of these transformations will give a sufficient criterion for optimality. The special set 1, 4 has the property of furnishing the *weakest possible* of the sufficient criteria for optimality. The preference vector in fig. (7.4) may indeed change to any direction between 1 and 4 without depriving the point A of its character of optimum point. The set of boundary vectors which has this property, we may call the minimum set. And the same designation may be used on the corresponding set of variables, that is x_1 and x_4.

Finally fig. (7.4) suggests that if we express the preference vector in terms of a set (θ, δ) which is not a minimum set and by so doing we get a positive coefficient of one of the vectors, say $b_{\theta k}$ and a negative coefficient for the other—in fig. (7.4) for instance $\theta = 4$, $\delta = 3$ —and we want to retain only minimum vectors, that is vectors permitting to express the optimum condition in its weakest form, it is not the variable x_θ whose vector $b_{\theta k}$ got a positive coefficient, that should be eliminated, but we should eliminate the variable which is such that the inclusion of its vector caused *some* coefficient or coefficients to become positive. For instance in fig. (7.4) if p_k is expressed in terms of b_{4k} and b_{3k} we will find that b_{4k} gets a positive and b_{3k} a negative coefficient. In this case it is x_3 that should be eliminated, not x_4, if we want to work towards a minimum set.

The situation is analogous if we have an optimum region of higher dimensionality. Fig. (7.5) illustrates the case $\nu = 2$, $n = 3$. There are four planes, any two of which define the optimum region, which is now one-dimensional, namely the segment of straight line BC.

The two planes (1) and (4) (shaded in fig. (7.5)) form the minimum set, while (2) and (3) only introduce unnecessary and undesired complications. Any two of the four planes will define the straight line that carries the segment BC which forms the optimal region. The vectors b_{1k},b_{2k}, b_{3k}, b_{4k} that are normal to these four planes, will—when translated to a common point on BC—be in a plane that is perpendicular to BC, and in this plane the situation is similar to the one exhibited around the point A in fig. (7.4).

This suggests the following heuristic rules for handling the case where we have reached a point which we suspect of being optimal and multiply determined.

Suppose that a movement is made by (4.2) and that the breaking out point to which this leads is determined by the fact that a certain number ω of the variables, namely

$$x_\varphi \ \dots \ x_\psi \qquad (\phi \dots \psi \text{ being } \omega \text{ affixes}) \qquad \dots \quad (7.6)$$

become zero simultaneously.

Also suppose that a further movement of the form (4.2) but now with a subset

$$x_\theta \ \dots \ x_\delta \qquad (\theta \dots \delta \text{ being } \mu \text{ affixes amongst those in (7.6))} \qquad \dots \quad (7.7)$$

included, i.e.,

$$d_k = p_k + B_\alpha b_{\alpha k} + \dots + B_\gamma b_{\gamma k} + B_\theta b_{\theta k} + \dots + B_\delta b_{\delta k} \qquad (k = u, v \dots w) \qquad \dots \quad (7.8)$$

makes all the d_k for $k = u, v \dots w$ equal to zero, but at least one of the B negative.

Regardless of the signs of the B, we know that if all the d_k are zero, the preference function can be written in the form

$$f = \left[p_o + \sum_{s=\alpha, \beta \dots \gamma, \theta \dots \delta} B_s b_{so} \right] - \sum_{s=\alpha, \beta \dots \gamma, \theta \dots \delta} B_s x_s. \qquad \dots \quad (7.9)$$

In the case now considered at least one B is negative.

We want to replace one or more of the variables numbers $\alpha, \beta \dots \gamma, \theta \dots \delta$ in (7.9) by an equal number of variables chosen among $\alpha, \beta \dots \gamma, \phi \dots \psi$ (compare (7.6)) in such a way as to obtain an expression for the preference function with only non-negative coefficients B.

In principle this can be done by testing all possible $\binom{\nu+\omega}{\omega-\mu}$ combinations of $(\nu+\mu)$ variables picked in the set consisting of the ν variables numbers $\alpha, \beta \dots \gamma$ and the variables (7.6). Since this may involve a tedious or even prohibitive work if done in a random order, we try to test the most promising combinations first.

For instance if one of the ν variables number $\alpha, \beta \dots \gamma$ is such that in the course of the previous work its inclusion produced one or more negative coefficients B, we may try to take this variable out. And then in a systematic way test one by one the $\binom{\omega}{\omega-\mu-1}$ alternatives obtained by adding $(\mu+1)$ of the variables (7.6).

In so doing we need to express some of the $(\nu+\omega)$ variables in terms of $(\nu+\mu)$ of them. In the multiply determined optimum region now considered, we can assume that any $(\nu+\mu+1)$ boundary vectors from the set of $(\nu+\omega)$ boundary vectors numbers $\alpha, \beta \dots \gamma, \phi \dots \psi$, are linearly dependent. For instance let numbers $\alpha, \beta \dots \gamma, \theta \dots \delta, \kappa$ be linearly dependent. That is we have

$$\sum_{s=\alpha, \beta \dots \gamma, \theta \dots \delta, \kappa} D_s b_{sk} = 0 \qquad \text{for all } k = u, v \dots w \qquad \dots \quad (7.10)$$

where the coefficients D are independent of k and not all zero.

They can be determined from the non-homogeneous system of order $(\nu+\mu)$

$$\sum_{s=\alpha, \beta \dots \gamma, \theta \dots \delta, \kappa} D_r M_{rs} = 0 \qquad (s = \alpha, \beta \dots \gamma, \theta \dots \delta) \qquad \dots \quad (7.11)$$

with the convention

$$D_\kappa = 1. \qquad \dots \quad (7.12)$$

If the matrix M_{rs} of rows and columns $\begin{pmatrix} r = \alpha, \beta \ldots \gamma, \theta \ldots \delta \\ s = \alpha, \beta \ldots \gamma, \theta \ldots \delta \end{pmatrix}$ is non-singular, (7.11) in conjunction with the convention (7.12) will determine the coefficients D uniquely. This assumption amounts to the same as to say that the $(\nu+\mu+1)$ set $(\alpha, \beta \ldots \gamma, \theta \ldots \delta, \kappa)$ is linearly dependent but the $(\nu+\mu)$ set $(\alpha, \beta \ldots \gamma, \theta \ldots \delta)$ is not.

If (7.10) holds, we have

$$\sum_{s=\alpha, \beta \ldots \gamma, \theta \ldots \delta, \kappa} D_s(x_s - b_{s0}) = 0 \qquad \ldots \quad (7.13)$$

Indeed, the left member of (7.13) is equal to

$$\sum_{k=u, v \ldots w} x_k \sum_{s=\alpha, \beta \ldots \gamma, \theta \ldots \delta, \kappa} D_s b_{sk}$$

and the last sum in this expression is zero for all k by (7.10).

Through the regression equation (7.13)—which holds good for any point inside or outside of the admissible region or on its boundary—any of the variables that appear with a non-zero coefficient in (7.13) can be expressed in terms of the others. Choosing the affixes $\theta \ldots \delta, \kappa$ differently, we obtain the necessary formulae for testing the various combinations and see if any of them can produce an expression of the form (7.9) with all the coefficients B non-negative. The first combination that yields this result, is sufficient to prove that the point considered is an optimum point.

A numerical example where the above method rapidly yields a sufficient criterion for optimality, is given in § 10.

If there is no combination that makes the sufficient optimality criterion verified, the region considered cannot be optimal.

If at any stage we find that by using (4.2) no boundary plane intervenes to stop the increase of λ to infinity, the linear programming problem is such that the preference function can be rendered arbitrarily great. In practice this is a trivial case.

8. CONTINUOUS WORK WITH NON-NEGATIVE REGRESSION COEFFICIENTS

In some cases the concrete setting of the problem may be such that it is not absolutely necessary to verify exactly that an optimum point has been reached. It may be sufficient to note that a *high* value of the preference function has been reached. If so, one does not need to worry about the trouble that may be involved in finding a sufficient criterion for optimality in the case where the optimal region is multiply determined. But if it is necessary to make definitely sure that no better solution exists, it may pay *already from the start* to be prepared for a very difficult case of a multiply determined optimal region, and to proceed accordingly. One way to do this is to try to work throughout only with non-negative regression coefficients.[1]

Suppose that one move of the kind (4.2) has been made and that all the coefficients B in (4.2) have—through the solution of (4.3)—turned out to be non-negative. And further suppose that when the move (4.2) with these B coefficient is performed, we reach a breaking out point indicating that the additional set $\theta \ldots \delta$ shall be included, so that the next move that contains the preference function, will be of the form (7.8). Finally, suppose that one

[1]Intuitively one would think that this is always possible because the admissible region is convex, but I have not gone through an exact proof.

or more of the coefficients B in (7.8) turn out to be negative. We may then decide *not* to make the move according to (7.8), but instead drop one or perhaps more of the variables numbers $\theta \ldots \delta$, and see if this should produce a determination of the direction numbers d_k which involve only non-negative coefficients B. We may first try to drop the variable that was the last candidate in the priority order for the variables numbers $\theta \ldots \delta$, and next—if necessary—try to drop that variable which was next in priority order, and so on. Finally —if necessary—we may try to drop more than one of the variables numbers $\theta \ldots \delta$. As soon as a set is found that gives only non-negative B, we proceed.

In the new breaking out point thus determined, the number of new candidates to include in the optimum set may be determined *by starting a new count* on the basis of the square root rule formulated in the beginning of § 3.

By following the above rule in all the rounds of the work it will usually be easier to handle the optimality criterion at the end. This is illustrated in the example in § 10.

It is interesting to note how the value of the preference function is gradually increased. Towards the end the rate of increase gradually decreases from round to round in such a way that *most* of the increase can be realized without carrying the work through to the exact optimum. In many practical cases this may be of considerable value.

9. Correctional computations in the case where the optimality criterion turns out in the negative

If we have reached a *corner* on the boundary, that is with $\nu = n$ linearly independent variables $x_\alpha, x_\beta, \ldots, x_\gamma$ equal to zero, or we have reached a stage where $(n-\nu)$ is equal to the dimensionality of the optimum point, as we happen to know it beforehand, and if in either case the optimality criterion turns out in the negative, we must at one stage or another have made at least one wrong guess. At least one of the vriables must therefore be set free from the condition of being zero. The same applies in the case (II) of Section 7.

Sometimes the general aspect of the numerical results will give an indication of which one or which ones of the variables in the set $x_\alpha, x_\beta, \ldots, x_\gamma$ should be set free from the condition of being 0.

One way to set a variable free is, of course, to all make the computations over again with this variable kept out of the set $x_\alpha, x_\beta, \ldots, x_\gamma$. This may easily be done if the variable in question was one of the *latest* to be included in the set $x_\alpha, x_\beta, \ldots, x_\gamma$. Only a slight recomputation will then be needed.[1] But if the variable in question occurs somewhere in the first

[1] If the computations have been made in such a way as to store the triangular matrix which is obtained in the course of the work *or* the computations proceed by means of the inverse. This will often be the case when one works with an automatic electronic computor. If N is the order of the inverse which the machine can handle directly and the order of the matrix n is larger than N, perhaps very much larger, the best procedure is probably to increase stepwise the order of the matrix by means of formulae similar to those of § 4 in "Principles of linear programming". One will then begin by computing the inverse of the upper left $N \times N$ corner directly, end then compute the inverse of the $2N \times 2N$ upper left corner by means of one new $N \times N$ inversion and some extra work. Further the inverse of the upper left $3N \times 3N$ corner is computed by means of one new $N \times N$ inversion and some extra work. And so on. As one proceeds, the extra work needed becomes larger and larger in proportion to the $N \times N$ inversion, but even so this procedure is probably the most effective when the capacity of the machine is not sufficient to handle the whole inversion directly. The procedure is particularly adapted to the method of the present paper which proceeds in any case by adding rows and columns round by round. Inversion formulae are given in section 13.

part of the list, the recomputation will virtually mean to start afresh. The problem therefore arises if it should be possible to perform the operation of setting one or more variables free by a computational procedure that permits us to use most of the previous results. This is done as follows.

Suppose that by an application of (5.2)—with not all the Δx_s in (5.3) equal to 0—we have reached a point where all the variables $x_\alpha, x_\beta, \ldots, x_\gamma$ have actually been reduced to zero, and suppose that we subsequently decide that we should have wanted to make this movement *without making any assumptions* about a certain *subset* $x_\theta \ldots x_\delta$ of the variables in the set $x_\alpha, x_\beta, \ldots, x_\gamma$.

To do this we insert in the ν dimensional system (5.3) *not* the values $\Delta x_\theta \ldots \Delta x_\delta$ that were prescribed in the movement as first defined, but such values $\Delta x_\theta \ldots \Delta x_\delta$ as will make $C_\theta \ldots C_\delta$ vanish in (5.2). These values of $\Delta x_\theta \ldots \Delta x_\delta$ are determined by the system of equations

$$C_r + C_r^\theta \Delta x_\theta + \ldots + C_r^\delta \Delta x_\delta = 0 \qquad (r = \theta \ldots \delta) \qquad \ldots \quad (9.1)$$

where C_s' $(s = \alpha, \beta \ldots \gamma)$ is the solution of (5.3) for $\Delta x_\theta = \ldots = \Delta x_\delta = 0$ and all the other Δx_s inserted with the values they have in the movement now wanted made, C_s^θ $(s = \alpha, \beta \ldots \gamma)$ the solution of (5.3) for $\Delta x_\theta = 1$ and all the other Δx_s equal to zero, and similarly for C_s^δ etc.

Having determined the solution $\Delta x_\theta \ldots \Delta x_\delta$ of (9.1) the direction (5.2) with

$$C_s = C_s' + C_s^\theta \Delta x_\theta + \ldots + C_s^\delta \Delta x_\delta \qquad (s = \alpha, \beta \ldots \gamma) \qquad \ldots \quad (9.2)$$

is the direction that would have been obtained by setting $x_\theta \ldots x_\delta$ free.

Obviously if the C_s are determined by (9.2) the direction thus obtained will by (9.1) not contain the terms $C_\theta b_{\theta k} + \ldots + C_\gamma b_{\gamma k}$.

Similarly, from a point where the variables $x_\alpha, x_\beta \ldots x_\gamma$ are all zero, we may have made a movement by (4.2) but afterwards decide that we should have wanted to do it *without* including certain variables $x_\theta \ldots x_\delta$ that form a subset within the set $x_\alpha, x_\beta, \ldots, x_\gamma$.

To handle this case we first consider the problem of moving from a given point in a direction

$$d_k = A_0 p_k + A_\alpha b_{\alpha k} + A_\beta b_{\beta k} + \ldots + A_\gamma b_{\gamma k} \qquad (k = 1, 2 \ldots n) \qquad \ldots \quad (9.3)$$

where the constants A are determined in such a way that the preference function f as well as the variables $x_\alpha, x_\beta, \ldots, x_\gamma$ assume given positive, negative or zero increments

$$\Delta f, \ \Delta x_\alpha, \ \Delta x_\beta \ldots \Delta x_\gamma. \qquad \ldots \quad (9.4)$$

In order to achieve this it is obviously necessary and sufficient that $A_0, A_\alpha, A_\beta, \ldots, A_\gamma$ form the solution of the $(\nu+1)$ order system

$$A_0 M_{0s} + A_\alpha M_{\alpha s} + A_\beta M_{\beta s} + \ldots + A_\gamma M_{\gamma s} = \Delta x_s \quad (s = 0, \alpha, \beta, \ldots, \gamma) \qquad \ldots \quad (9.5)$$

where for simplicity we have denoted the preference function f by x_0.

If we are only interested in the *direction* of the movement, not in the absolute values of the components (9.3), it makes no difference whether all the coefficients A are multiplied by a common positive factor. We may for instance normalize the solution by putting Δf, that is Δx_0, equal to such a number as will make $A_0 = 1$. If we do this, and put

$$\Delta x_\alpha = 0, \qquad \Delta x_\beta = 0 \ldots \Delta x_\gamma = 0, \qquad \ldots \quad (9.6)$$

we get the coefficients B_α, $B_\beta \ldots B_\gamma$ that satisfy (4.3).

This being so, consider the case where we want to determine the direction that would have been obtained if we had *not* imposed the 0-condition on the particular subset $x_\theta \ldots x_\delta$. We do it simply by *not* putting $\Delta x_\theta \ldots \Delta x_\delta$ equal to zero but to *such* values as will make $A_\theta \ldots A_\delta$ vanish. In other words we consider the system (9.5) for

$$\begin{aligned} &\Delta x_0 = \text{magnitude to be disposed of so as to make } A_0 = 1 \\ &\Delta x_s = 0 \text{ for } s = \alpha, \beta \ldots)\theta \ldots \delta(\ldots\gamma \qquad \ldots \quad (9.7) \\ &\Delta x_\theta \ldots \Delta x_\delta \text{ magnitudes to be disposed of so as to assure } A_\theta = \ldots = A_\delta = 0. \end{aligned}$$

For a moment we may disregard the equation $s = 0$ in (9.5) and only consider the equations $s = \alpha, \beta \ldots \gamma$. These equations are completely specified by the data written in two last lines of (9.7). We may (if we take account of the condition $A_0 = 1$) write these ν equations

$$M_{0s} + A_\alpha M_{\alpha s} + A_\beta M_{\beta s} + \ldots + A_\gamma M_{\gamma s} = \Delta x_s \quad (s = \alpha, \beta \ldots \gamma) \qquad \ldots \quad (9.8)$$

where the Δx_s are given by (9.7).

Let, as before, $B_s (s = \alpha, \beta \ldots \gamma)$ be the solution obtained by putting *all* the Δx_s in (9.8) equal to zero—compare the previously considered system (4.3) and let A_s^θ $(s = \alpha, \beta \ldots \gamma)$ be the solution obtained by putting $\Delta x_\theta = 1$, but all the other $\Delta x_s = 0$, A_s^δ the solution obtained by putting $\Delta x_\delta = 1$ but all the other $\Delta x_s = 0 \ldots$ etc. The general solution of (9.8) can then be written

$$A_s = B_s + A_s^\theta \, \Delta x_\theta + \ldots + A_s^\delta \, \Delta x_\delta \qquad (s = \alpha, \beta \ldots \gamma). \qquad \ldots \quad (9.9)$$

The conditions $A_\theta = \ldots = A_\delta = 0$ are consequently expressed by the system

$$B_s + A_s^\theta \, \Delta x_\theta + \ldots + A_s^\delta \, \Delta x_\delta = 0 \qquad (s = \theta \ldots \delta). \qquad \ldots \quad (9.10)$$

The order of this system is not ν but equal to the number of variables we want to set free.

When $\Delta x_\theta \ldots \Delta x_\delta$ are determined from (9.10) the direction numbers to use will be

$$d_k = p_k + A_\alpha b_{\alpha k} + \ldots) \, A_\theta b_{\theta k} + \ldots + A_\delta b_{\delta k} (\ldots + A_\gamma b_{\gamma k} \quad (k = u, v \ldots w) \qquad \ldots \quad (9.11)$$

where the A_s $(s = \alpha, \beta \ldots)\theta \ldots \delta(\ldots \gamma)$ may be determined, either by inserting $\Delta x_\theta \ldots \Delta x_\delta$ into (9.9) or by performing one back solution of (9.8) with the values for $\Delta x_\theta \ldots \Delta x_\delta$ in the right member.

The computational cost involved in this procedure consists in : 1) As many back solutions of (9.8) as there are variables to be set free, and 2) one back solution to obtain the $A_s\,(s = \alpha, \beta\ldots)\,\theta\,\ldots\,\delta(\ldots\gamma)$, or the corresponding work involved in using (9.9). The B_s can be assumed known from the previous work.

Once the back solutions needed to compute the $A_s\,(s = \alpha, \beta\,\ldots)\theta\,\ldots\,\delta(\ldots\gamma)$ are available, we may work with the number of degrees of freedom thus established in the same way as we proceeded originally to find our way towards a set of optimum candidates.

If no specific variable or variables in the set $x_\alpha, x_\beta, \ldots, x_\gamma$ distinguish themselves as the ones that need to be set free, one may make a small move into the interior of the admissible region—even at the cost of some decrease in the preference function — and from the point thus obtained make a fresh start. If the move into the interior of the admissible region is made in reasonable way, one will get a new starting point from where it is possible to make a much more rapid progress than from the original starting point.

If $x_\alpha, x_\beta, \ldots, x_\gamma$ are the zero variables, all of which one now wants to set free, one may decide on a set of effectively positive increments (5.1) which are chosen in the light of the values that were encountered in the previous work. An application of (5.2) with the coefficients C determined by (5.3) will then lead to this direction point, or possibly to some new boundary plane that prevents us from reaching the chosen direction point. If so, we may decide to stop, say, half way before this new boundary plane is reached.

No matter how the new initial point is determined, we start from it in one of the ways described in § 2. When deciding upon optimum candidates, we will now most probably notice such a conspicuous distribution of the values of the λ_j defined by (2.5) as to make it possible to jump more or less directly to an optimum point. Perhaps the distribution of the λ_j will point out fairly clearly that particular or those few particular variables that should be let loose from their zero values and also point out some other that should be equal to zero. If so, we may revert to the point where we were before the movement into the interior of the admissible region took place, and from this point make a new movement utilizing (9.1)—(9.2), or a generalization of these formulae to the case where additional conditions are imposed in the form of prescribed increments for some further variables.

10. A NUMERICAL EXAMPLE

As an example we consider a problem in optimal blending of aviation gasolines. The problem is a small one, containing only 22 variables and 10 linearly independent equations, i.e. 12 degrees of freedom. The example is not given here in order to give a fair illustration of the amount of work involved in the multiplex method as compared to that involved in other methods. Such a comparison can only be made on really large examples. The purpose of the present example is only to illustrate the principles underlying the multiplex method. The computations are given in details so that anyone who is interested may check for himself how the computations proceed. The form of the intermediate results given, is that which appears when the work is done on desk machines. The tables can simply be looked upon as working sheets to be used for this kind of machines. When the work is done on more or less automatic computors, intermediate results are either stored on cards or tapes or not stored at all.

The concrete nature of the problem is described in an article in Econometrica[1] and will not be considered here. We simply take as our starting point the equations expressed in basis form. They are given in Table (10.1) on page number 20. The cells of that table give the coefficients b_{jk} of (1.3). All the 22 variables are subject to the condition of being non-negative. The problem is to maximize the linear preference function given in the bottom row of Table (10.1). The cells of this bottom row give the coefficients p_k of (1.5).

As a starting point in the interior of the admissible region we use in this example the *centre* of the admissible region, i.e. the point where the logarithmic potential is (approximately) maximum. This was done simply because the example has also been handled by the logarithmic potential method. We could just as well have determined an initial point by the $S(\lambda)$ method or simply by guesswork. The way in which the initial point is chosen is irrelevant for the application of the multiplex method and will not be further discussed here. The initial point from which we start in the present example is given in column 0 in Table (10.2).

We shall describe how the work is done in the case where we decide to work throughout with non-negative regression coefficients and also in the case where we work without any sign condition on the regression coefficients. We shall describe these two alternatives independently. This will lead to some repetitions in the tables, but the explanations can then be given in a clearer and simpler form.

Working throughout with non-negative regression coefficients : From the initial point given under 0 in Table (10.2) we move in the cheapest way, i.e. by putting $d_k = p_k$ and compute the other d_j by (2.3). These numbers are given in the first of the columns in the column sector 1. The magnitudes λ_j defined by (2.5) are listed in the next columns with the priority order indicated in the small column to the right of the figures for λ_j. The value of the various λ_j (for negative d_j) need only be computed with an accuracy sufficient to determine their ranking order, but $\lambda_{\max}$ as defined by (2.4) is computed with great accuracy and listed at the bottom of the λ_j column. By means of this value and (2.2) the values of the variables in the breaking out point are computed. They are listed in the last column under 1. By the rule illustrated in (3.4) we shall pick the first three candidates. They are—as will be seen from the λ_j column in the column § 1—the variables numbers 21, 20, 22, taken in this order.

We are therefore led to using (5.2) with the set $\alpha, \beta \ldots \gamma$ made up of numbers 21, 20, 22. This means that we must solve the system (5.3)— which now consists of three equations. The matrix of the coefficients are given in the upper left 3×3 corner of Table (10·3). Since the matrix is symmetric only a triangle is filled in. Usual sum checks for this 3×3 matrix are run and listed in a column in the right part of the table.

The solution of this 3×3 matrix by the Gaussian algorithm is given in the triangle above the principal diagonal in the upper left 3×3 matrix in Table (10.4). The numbers listed below the diagonal in this 3×3 matrix are the multipliers used in working out the solution.[2] The right member as given by (5.4) is listed in its original form as the first three elements in the column marked 2 in the right hand part of Table (10.3), and the corresponding elimination treated numbers are listed in the same place in Table (10.4). The result of the back solution in the form of C_{20}, C_{21}, C_{22} are listed on the line 2 in the lower left part of Table (10.4).

[1] W. W. Cooper and B. Mellon : Blending aviation Gasolines, *Econometrica*, April 1952.

[2] A detailed description of the work sheet for the Gaussian algorithm in the symmetric case is given in Section 17 of Principles of Linear Programming.

TABLE (10.1) BASIS EQUATIONS IN THE LINEAR PROGRAMMING PROBLEM FOR THE BLENDING OF AVIATION GASOLINE.

Variable No.	Constant term	Basis variable No.: 1	2	3	4	5	6	7	8	9	10	11	12	Sum check
1 =		1.0												1.0
2 =			1.0											1.0
3 =				1.0										1.0
4 =					1.0									1.0
5 =						1.0								1.0
6 =							1.0							1.0
7 =								1.0						1.0
8 =									1.0					1.0
9 =										1.0				1.0
10 =											1.0			1.0
11 =												1.0		1.0
12 =													1.0	1.0
13 =	3800.00	- 1.0	- 1.0	- 1.0										- 3.0
14 =	2652.00				- 1.0	- 1.0	- 1.0							- 3.0
15 =	4081.00							- 1.0	- 1.0	- 1.0				- 3.0
16 =	1300.00										- 1.0	- 1.0	- 1.0	- 3.0
17 =		14.0			3.0			- 6.0			15.0			26.0
18 =			16.5			2.0			- 4.0			17.0		31.5
19 =				7.5			- 7.0			-13.0			8.0	- 4.5
20 =		1.9			- 1.1			2.9			-13.6			- 9.9
21 =			1.9			- 1.1			2.9			-13.6		- 9.9
22 =				1.9			- 1.1			2.9			-13.6	- 9.9
Sum check	11833.00	15.9	18.4	9.4	1.9	0.9	- 8.1	- 3.1	- 1.1	-10.1	1.4	3.4	- 5.6	23.3
Preference function f=		0.360092	0.888446	1.493446	0.360092	0.888446	1.493446	0.360092	0.888446	1.493446	0.360092	0.888446	1.493446	

TABLE (10.2). STEP BY STEP TABLE WHEN WORKING CONTINUOUSLY WITH NON-NEGATIVE REGRESSION COEFFICIENTS.

	0	1			2			3			
	x_j	d_j	λ_j	x_j	d_j	λ_j	x_j	d_j	λ_j		x_j
j = 1	112.8129	0.360092		123.2803	- 0.3457	356.6	7	122.9346	0.3942679		293.1977
2	140.3695	0.888446		166.1955	0			166.1955	0.9727674		586.2815
3	1072.2389	1.493446		1115.6516	- 8.0606	138.4	5	1107.5910	1.6351873		1813.7406
4	116.2668	0.360092		126.7342	0.2001			126.9343	0.3403059		273.8942
5	140.3808	0.888446		166.2068	0			166.2068	0.8396283		528.7972
6	170.6608	1.493446		214.0735	4.6667			218.7402	1.4113852		826.2417
7	114.5645	0.360092		125.0319	- 0.5277	236.9	6	124.5042	0.4122553		302.5381
8	149.4271	0.888446		169.2531	0			169.2531	1.0171472		608.5044
9	53.6579	1.493446		97.0706	- 12.3030	7.9	3	84.7676	1.7097881		823.1333
10	20.5147	0.360092		30.9821	2.4745			33.4566	0.1154642		83.3194
11	20.0400	0.888446		45.8660	0			45.8660	0.2848819		168.8712
12	54.0105	1.493446		97.4232	57.6970			155.1202	0.4788763		361.9212
13	2474.5787	- 2.741984	902.5	6 2394.8726	8.4063			2403.2789	- 3.0022226	300.5	2 1106.7801
14	2224.6916	- 2.741984	811.3	5 2144.9855	- 4.8668	440.7	8	2140.1187	- 2.5913194	825.9	3 1021.0669
15	3769.3505	- 2.741984	2374.7	8 3689.6444	12.8307			3702.4751	- 3.1391906	1179.4	4 2346.8271
16	1205.4348	- 2.741984	439.6	4 1125.7287	- 60.1715	18.7	4	1065.5572	- 0.8792224	1212.0	5 685.8682
17	1548.5145	9.362732		1820.6675	36.0442			1856.7117	5.7990995		4361.0316
18	2363.8300	27.986219		3177.3503	0			3177.3503	18.5043222		11168.3740
19	6581.6975	- 6.720017	979.4	7 6386.3406	528.3944			6914.7350	-16.0120266	431.8	1 0
20	139.6882	- 3.561211	39.2	2 36.0606	- 36.0606	1.0	1	0	0		0
21	255.6778	- 8.795715	29.1	1 0	0			0	0		0
22	1270.5921	-14.785115	85.9	3 840.8068	-840.8068	1.0	2	0	0		0
Sum check	23993.0001	3.482293		24094.2258	-312.4288			23781.7970	8.2913951		27362.4054
Preference function f=	2548.7944			2908.9894				2972.3629			7739.9824
Notes:		In the initial step: $d_j = p_j$	λ_{max} = 29.068780			λ_{max} = 1.0000			λ_{max} = 431.846334		

TABLE (10.2). (Continued).

	4			5			6			
	d_j	λ_j	x_j	d_j	λ_j	x_j	d_j	λ_j		x_j
j = 1	377.4841		670.6818	- 0.852842	786.4 2	62.7978	- 62.7978	1.0	1	0
2	377.4841		963.7656	- 0.274343	3513.0 7	768.2212	- 73.2700	10.5	8	694.9512
3	351.8120		2165.5526	1.127185		2968.9810	136.0678			3105.0488
4	332.7853		606.6795	- 0.305514	1985.8 5	388.9170	361.1075			750.0245
5	332.7853		861.5825	0.193809		999.7245	- 87.9571	11.4	9	911.7674
6	355.4963		1183.7380	0.111705		1263.3585	- 273.1504	4.6	5	990.2081
7	- 5.2663	57.4 4	297.2688	0.437237		608.8488	- 130.4064	4.7	6	478.3624
8	- 5.2663	115.5 5	603.2381	1.042029		1345.9696	557.9044			1903.8740
9	27.7391		850.8724	0.784432		1409.9955	288.7681			1698.7636
10	24.6973		108.0167	- 0.001223		107.1448	- 65.8048	1.6	3	41.3400
11	24.6973	0	193.5885	0.168194		313.4731	115.0429			429.3160
12	26.3118		388.2330	0.315708		613.2615	102.6781			715.9396
13	-1106.7801	1.0 1	0	0		0	0			0
14	-1021.0669	1.0 2	0	0		0	0			0
15	- 17.2065	136.4 6	2329.6206	- 2.263598	1029.2 3	716.1861	- 716.1861	1.0	2	0
16	- 75.7064	9.1 3	610.1618	- 0.482679	1264.1 4	266.1206	- 152.7162	1.7	4	113.4044
17	6685.1914		11046.2230	-15.497504	712.8 1	0	0			0
18	7334.9783		18503.3523	- 5.447848	3396.5 6	14620.2661	-1647.1579	8.9	7	12973.1082
19	0		0	0		0	0			0
20	0		0	0		0	0			0
21	0		0	0		0	0			0
22	0		0	0		0	0			0
Sum check	14020.1698		41382.5752	-20.945352		26453.2661	-1647.1579			24806.1082
Preference function			9788.0899			12807.9275				13679.8885
Notes:		λ_{max} = 1.0000			λ_{max} = 712.774328			λ_{max} = 1.0000		

TABLE (10.2). (Continued).

	7			8			9			10
	d_j	λ_j	x_j	d_j	λ_j	x_j	d_j	λ_j	x_j	d_j
j = 1	0		0	0		0	0		0	0
2	- 0.57222052	1214.5 4	334.5088	- 0.53965451	619.9 2	4.8002	- 0.00526710		1.7821	0
3	0.57222062		3465.4912	0.53965460		3795.1998	0.00526725		3798.2179	0
4	- 0.37308526	2010.3 5	515.0177	- 0.50119147	1027.6 3	208.0086	- 0.36441190	573.0 1	0	0
5	0.11387714		983.4987	0.14912033		1074.6056	0.27564854		1232.5527	0
6	0.25920804		1153.4036	0.35207108		1368.5858	0.08876325		1419.4473	0
7	- 0.44929401	1064.7 2	195.3517	- 0.19010748	1027.6 4	79.2032	- 0.13822554	573.0 2	0	0
8	0.18818327		2022.4108	0.01672140		2032.6269	0.19430674		2143.9647	0
9	0.26111073		1863.2375	0.17338023		1969.1699	- 0.05608117		1937.0353	0
10	- 0.06562935	629.9 1	0	0		0	0		0	0
11	- 0.04902594	8756.9 6	398.4345	- 0.08388858	4740.5 6	347.1017	0.01840212		357.7261	0
12	0.11465529		788.1611	0.08388866		839.4139	- 0.01840195		828.8695	0
13	0		0	0		0	0		0	0
14	0		0	0		0	0		0	0
15	0		0	0		0	0		0	0
16	0		113.4044	0		113.4044	0		113.4044	0
17	0.59206803		372.9443	- 0.36292953	1027.6 5	151.2080	- 0.26388246	573.0 3	0	0
18	-10.80005836	1201.2 3	6170.1395	-10.09905022	611.0 1	0	0		0	0
19	0		0	0		0	0		0	0
20	0		0	0		0	0		0	0
21	0		0	0		0	0		0	0
22	0		0	0		0	0		0	0
Sum check	-10.20799032		18376.0838	-10.46197949		11984.2080	- 0.26388222		11833.0000	0
Preference function			14435.4771			15083.3391			15242.2877	
Notes:		λ_{max} = 629.901104			λ_{max} = 610.962354			λ_{max} = 573.0010		

Using these three C-values the d_k are computed by (5.2) and the other d_j by (4.7). The result is given in the first column under column section 2 in Table (10.2). Using (4.5) we find that we can unhindered reach the direction point where the three variables Nos. 20, 21, 22 are zero. The rest of the variables in this point ar obtained by puting $\lambda = 1$ in (4.6). The result is listed in the last column under column section 2.

TABLE (10.3). MATRIX DATUM TABLE (MOMENT TABLE) WHEN WORKING CONTINUOUSLY WITH NON-NEGATIVE REGRESSION COEFFICIENTS.

	Left members:											Sum checks for left members in move No.:						Right members:			
	s=20	21	22	19	13	14	1	15	(17)	10	18	4	2	4	6	8	9	10	0	Δ 2	Δ 4	Δ 6
s=20	198.19	0	0	0	-1.90	1.10	1.90	-2.90	-198.10	-13.60	0	-1.10	198.19	197.39	- 1.71	182.79	182.79	181.69	3.5649108	- 36.0606	0	0
21		198.19	0	0	-1.90	1.10	0	-2.90	0	0	-213.65	0	198.19	197.39	194.49	194.49	-19.16	-19.16	8.7956154	0	0	0
22			198.19	-124.55	-1.90	1.10	0	-2.90	0	0	0	0	198.19	72.84	69.94	69.94	69.94	69.94	14.7851154	-840.8068	0	0
19				338.25	-7.50	7.00	0	13.00	0	0	0	0		213.20	226.20	226.20	226.20	226.20	6.7205070		0	0
13					3.00	0	-1.00	0	- 14.00	0	- 16.50	0		-10.20	-25.20	-11.20	-27.70	-27.70	2.7419840		-1106.7801	0
14						3.00	0	0	- 3.00	0	- 2.00	-1.00		13.30	10.30	13.30	11.30	10.30	2.7419840		-1021.0669	0
1							1.00	0	14.00	0	0	0			15.90	1.90	1.90	1.90	- 0.3600920			- 62.7978
15								3.00	6.00	0	4.00	0			13.30	7.30	11.30	11.30	2.7419840			-716.1861
(17)									466.00						270.90				(-9.3623920)			0
10										1.00	0	0				-12.60	-12.60	-12.60	- 0.3600920			
18											581.25	0					353.10	353.10	-27.9860490			
4												1.00						- 1.10	- 0.3600920			
Sum check													594.57	683.92	774.12	672.12	797.07	793.87	3.6433836			

TABLE (10.4). MATRIX SOLUTION TABLE WHEN WORKING CONTINUOUSLY WITH NON-NEGATIVE REGRESSION COEFFICIENTS.

		Left members:												
		s=20	21	22	19	13	14	1	15	(17)	10	18	4
	s=20	198.19	0	0	0	-1.90	1.10	1.90	-2.90	-198.10	-13.60	0	-1.10
	21	0	198.19	0	0	-1.90	1.10	0	-2.90	0	0	-213.65	0
	22	0	0	198.19	-124.55	-1.90	1.10	0	-2.90	0	0	0	0
	19	0	0	0.628493736	252.97812681	-8.69403098	7.69128110	0	11.17753166	0	0	0	0
	13	0.00958676	0.00958676	0.00958676	0.03344139	2.65461499	0.28884344	-0.98178516	0.29038738	-15.89913716	- 0.13037994	-18.54821127	-0.01054544
	14	-0.00555023	-0.00555023	-0.00555023	- 0.02958434	-0.10880803	2.72271428	0.09628067	-0.31398937	- 0.17054564	0.08966951	1.20400097	-0.99274732
	1	-0.00958676	0	0	0	0.36984088	-0.03536202	0.61527619	0.14630203	10.02501712	0.07898921	- 6.90246268	0.04175085
	15	0.01463242	0.01463242	0.01463242	- 0.04299412	-0.10938964	0.11532219	-0.23778269	2.28936641	2.43707525	- 0.19318008	14.68289979	-0.13935552
	(17)	0.99954589	0	0	0	5.98924408	0.06263809	-16.29352359	-1.06451953	6.81829629			
	10	0.06862102	0	0	0	0.04911444	-0.03293387	-0.12838009	-0.08438146		0.03095597	0.33065128	-0.06042504
	18	0	1.07800595	0	0	6.98715683	-0.44220614	11.21847845	-2.04550035		-10.68134127	130.25659995	1.70416957
	4	0.00555023	0	0	0	0.00397289	-0.36461678	-0.06786709	0.06087078		1.95196726	- 0.01354381	0.47872348
	3	0.01798734	0.04437971	0.07460071									
B	5	0.02656730	0.05295968	0.13660615	0.08501321	1.26941199	0.63638197						
in	7a	-0.06734733	0.07043042	0.13359925	0.05242820	1.58137988	0.65471006	2.56494227	0.95260273	- 0.08689532			
move	7b	0.03130304	0.06893176	0.13399507	0.05544278	1.59163685	0.69874992	1.17206906	0.90016883				
No.:	8	0.18236592	0.07249519	0.13185051	0.04795120	1.56394137	0.66068096	0.85735411	1.07906065		2.12008443		
	9	0.12147286	0.16088879	0.16495892	0.09042917	2.47867947	0.59088375	1.68778900	0.85058883		1.29193889	0.07753198	
	10	0.29882509	0.14935469	0.14935469	0.06722244	2.28138660	0.85860020	1.46752691	1.05260523		2.83792918	0.06722222	0.76121572
0 in	2	-0.181950	0	-4.242428									
move	4	-1.815979	-1.815979	-3.671656	-2.952842	-380.934473	-390.787773						
No.:	6	291.108111	-8.517857	6.613869	24.078337	57.086056	97.326728	892.531659	-582.606293	259.551037			

TABLE (10.4) (continued).

	sum checks for left members in move no.:						right members :			
	2	4	6	8	9	10	0	Δ		
								2	4	6
$s=20$	198.19	197.39	− 1.71	182.79	182.79	181.69	3.56491080	− 36.0606	0	0
21	198.19	197.39	194.49	194.49	−19.16	−19.16	8.79561549	0	0	0
22	198.19	72.84	69.94	69.94	69.94	69.94	14.78511540	−840.8068	0	0
19		258.97537730	270.15290896	270.15290896	270.15290896	270.15290896	16.01202589		0	0
13		2.94345730	−13.64707762	2.12167960	−16.42659168	−16.43707711	3.53768715		−1106.7801	0
14		2.72271404	2.33445969	2.59467485	3.79867582	2.80592850	1.73268546		− 900.6405	0
1			10.78659495	0.84056705	− 6.06189564	− 6.02014479	0.85284213			− 62.797800
15			4.72644114	2.09618581	6.77908560	6.63973008	2.06080713			−701.253870
(17)			6.81829506				(−0.59206893)			1769.69587
10				0.03095581	0.36160709	0.30118205	0.06562927			
18					130.25653890	132.02070854	10.09904824			
4						0.47872294	0.36441184			

From the point thus obtained we perform a preference increasing movement, holding constant the three variables which have now reached zero, namely numbers. 20, 21, 22. To do this we have to use the *same* 3×3 matrix which we just used, but now with the right member given as the first three elements in the column marked 0 in the right hand part of Table (10.3) and Table (10.4).[1] The result is listed as the three coefficients B on line 3 in the lower left part of Table (10.4). Since all these three regression coefficients B turn out to be non-negative, we proceed. The coefficients d_j now to be used are listed in the first column in column section 3 in Table (10.2) and the λ_j listed in the subsequent column. The breaking out point defines a certain priority order of the remaining candidates. By (3.4) we shall also now pick three such candidates. They turn out to be numbers 19, 13, 14, taken in this order. We are thus led to consider the 6×6 system in the upper left corner of Table (10.3) and Table (10.4) with a right member which have the first four items equal to zero and the following two equal to the values listed on the fifth and sixth place in the right member column marked 4 of Table (10.3). The corresponding elimination treated elements are listed in the same places in Table (10.4). The back solution yields the six coefficients C listed on line 4 in the lower left part of Table (10.4). Computing the d_j — listed in the first column under column section 4 in Table (10.2)—we find that we can unhindered reach the direction point where all six variables numbers 20, 21, 22, 19, 13, 14 are zero. The values of all the variables in this point are listed in the last column in column section 4 in Table (10.2).

From the point thus obtained we want to perform a preference increasing movement holding constant all the six variables that have so far reached zero, namely numbers 20, 21, 22, 19, 13, 14. To do this we have to use the *same* 6×6 matrix as we just used, but now with the right member given as the first six elements in the column marked 0 in the right hand part of Tables (10.3) and (10.4). The result is listed as the six coefficients B on line 5 in the lower left part of Table (10.4). Since all these six regression coefficients B turn out to be non-negative, we proceed. The breaking out point is determined through the d_j built on these six B and through the corresponding λ_j. The values of the x_j in this breaking out point are listed in the last column in the column section 5 of Table (10.2). We select againt hree new candidates, namely numbers 17, 1, 15, of which the last two have to be forced down to zero. This is done by a 9×9 system with the right member indicated in column 6 in the right hand part of Tables (10.3) and (10.4). The resulting nine C coefficients are given on line 6 in the lower left part of Table (10.4). Using (4.5) we find that we can unhindered reach the direction point where all the nine variables numbers 20, 21, 22, 19, 13, 14, 17, 1, 15 are zero. The values of all the variables in this point are listed in the last column in column section 6 in Tab. (10.2).

From the point thus obtained we attempt again a preference increasing movement. This time, however, we find that *not* all the corresponding regression coefficients — namely the nine coefficients B listed on line 7a in the lower left part of Table (10.4)—are non-negative. We therefore decide that the preference increasing movement shall *not* be carried through with all the above nine variables equal to zero. The first candidate which came into the picture and produced some negative numbers in the list of regression coefficients was the leading candidate in the last breaking out point, namely number 17. This variable was also the only

[1] If the work is done on desk machines and the tables actually used as work sheets, the paper is folded so that the right member column to be used always comes in convenient proximity to the other columns used.

one of the newly added variables that got itself a negative regression coefficient. We therefore decide to throw number 17 out of the set of nine zero variables and perform the preferenc increasing movement by keeping only the remaining eight variables equal to zero, namely numbers 20, 21, 22, 19, 13, 14, 1, 15.

To do this we must use an 8×8 system. It so happens that number 17 was added as the last row and column in the Tables (10.3) and (10.4). To carry the computations through without number 17 is therefore an easy matter. The result is the eight regression coefficients listed on line 7b in the lower left part of Table (10.4). Since all these eight coefficients turn out to be non-negative we proceed.

To decide on the number of degrees of freedom to retain in the next round we note that in the last breaking out point — the one built on the B coefficients on line 7b—we arrived with 12—8 = 4 degrees of freedom. According to the square root rule we should consequently in the next move put 2 variables equal to zero. As it involves just as much work first to force one variable to zero (the leading candidate is already zero) and next to make a preference increasing move — as to make two separate preference increasing moves, the latter alternative is chosen. This leads through the moves whose column sections in Table (10.2) are designated 7, 8 and 9 to the final attempt number 10 which produces a situation where all the d_k are zero, see the column number 10 in Table (10.2). Since the eleven coefficients B that lead to these zero d_k are all non-negative —as will be seen from line 10 in the lower left part of Table (10.4)—we can conclude that an optimal point has been reached.

Since exactly eleven B coefficients in this last round are effectively positive, we know that the optimal region is 12—11 = 1 dimensional. This one dimensional region may be generated in different ways. In Table (10.5) we have used the variable No. 6 to generate the

TABLE 10.5. OPTIMAL SOLUTION OF THE PROBLEM OF BLENDING AVIATION GASOLINES. ONE DEGREE OF FREEDOM GENERATED BY x_6, WHERE x_6 IS LYING IN THE INTERVAL (10.6)

		constant term	x_6	values of the variables in the lower and upper endpoints of the interval (10.6)	
				(1)	(2)
variable no.	$j=1$	0		0	0
	2	−65.3310	0.047281	0	60.0582
	3	3865.3310	−0.047281	3800.0000	3739.9418
	4	0	0	0	0
	5	2652.0000	−1.000000	1270.2399	0
	6	0	1.000000	1381.7601	2652.0000
	7	0	0	0	0
	8	1131.6719	0.713160	2117.0879	3022.9722
	9	2949.3281	−0.713160	1963.9121	1058.0278
	10	0	0	0	0
	11	17.6853	0.239559	348.6984	652.9958
	12	1168.9103	−0.239559	837.8972	533.5998
	13	0	0	0	0
	14	0	0	0	0
	15	0	0	0	0
	16	113.4044	0	113.4044	113.4044
	17	0	0	0	0
	18	0	0	0	0
	19	0	0	0	0
	20	0	0	0	0
	21	0	0	0	0
	22	0	0	0	0

optimum region. It is easy to see that when the eleven variables specified in the last move shall be zero, the point will be situated in the admissible region when and only when x_6 lies between the limits

$$1381.76 \leqslant x_6 \leqslant 2652.00 \qquad \ldots \quad (10.6)$$

In other words we can attribute to x_6 any value between the limits (10.6) and then by Table (10.5) compute the corresponding values of all the variables. Any such point will be situated in the admissible region and will produce the same value of the preference function, namely $f = 15242.29$ and in all other admissible points than those generated in this way, will the preference function be actually less than the value indicated.

In the present example we did not need to use the procedure suggested in § 9.

In parameter form the solution (10.5)—(10.6) can be expressed by considering the two *endpoints* of the interval (10.6). In the lower endpoint, denoted (1), the variable x_2 is zero and in the upper endpoint, denoted (2), the variable x_5 is zero. The complete list of the values of all the variables in these two points are given in columns (1) and (2) in Table (10.5). Any point on the straight line segment generated by the variation of x_6 between the two limits considered can be expressed by introducing two non-negative parameters t_1 and t_2 subject to the condition

$$t_1+t_2 = 1. \qquad \ldots \quad (10.7)$$

If $x_j^{(1)}$ and $x_j^{(2)}$ are the values of the variable x_j $(j = 1, 2, \ldots n+m)$ in the two endpoints of the segment, i.e. the values read off in columns (1) and (2) respectively in Table (10.5), any point on the optimal segment can be expressed in the form

$$x_j = t_1 x_j^{(1)} + t_2 x_j^{(2)} \qquad (j = 1, 2 \ldots n+m). \qquad \ldots \quad (10.8)$$

That is to say if we choose any two non-negative numbers t_1 and t_2 satisfying (10.7) and insert them in (10.8) we get a point on the optimal segment. Conversely any point on the optimal segment can be expressed in the form (10.8) where t_1 and t_2 are two non-negative numbers satisfying (10.7).

This parameter form has certain advantages for some practical purposes, for instance if one wants to add certain *secondary* preference considerations for comparing alternative points within the optimum region. If the optimum region is of dimensionality δ a similar expression with $(\delta+1)$ non-negative parameters adding up to unity, applies.

TABLE (10.9). STEP BY STEP TABLE WHEN WORKING WITHOUT SIGN CONDITIONS ON THE REGRESSION COEFFICIENTS.

	6	11			12				13
	x_j	d_j	λ_j		x_j	d_j	λ_j		d_j
j = 1	0	0			0	0			0
2	694.9512	- 0.559116	1248.9	2	3.6957	-0.00419965	880.0	4	1.7822	0
3	3105.0488	0.559116			3796.3043	0.00419976			3798.2178	0
4	750.0245	- 0.481042	1559.2	3	155.2949	-0.34083860	455.6	1	0	0
5	911.7674	0.156263			1104.9604	0.28003680			1232.5525	0
6	990.2081	0.324779			1391.7447	0.06080173			1419.4475	0
7	478.3624	- 0.306806	1559.2	4	99.0469	-0.21738513	455.6	2	0	0
8	1903.8740	0.100091			2027.6207	0.25534842			2143.9647	0
9	1698.7636	0.206715			1954.3324	-0.03796329			1937.0353	0
10	41.3400	- 0.026514	1559.2	5	8.5596	-0.01876633	455.6	3	0	0
11	429.3160	- 0.069408	6185.4	6	343.5047	0.03121257			357.7260	0
12	715.9396	0.095922			834.5313	-0.01242615			828.8696	0
13	0	0			0	0			0	0
14	0	0			0	0			0	0
15	0	0			0	0			0	0
16	113.4044	0			113.4044	0			113.4044	0
17	0	0			0	0			0	0
18	12973.1082	-10.493187	1236.3	1	0	0			0	0
19	0	0			0	0			0	0
20	0	0			0	0			0	0
21	0	0			0	0			0	0
22	0	0			0	0			0	0
Sum check	24806.1082	-10.493187			11833.0000	0			11833.0000	0
Preference function =	13679.8885				15099.3550				15242.2878	
Notes:			λ_{max} = 1236.33632				λ_{max} = 455.6259			

TABLE (10.10). MATRIX DATUM TABLE (MOMENT TABLE) WHEN WORKING WITHOUT SIGN CONDITIONS ON THE REGRESSION COEFFICIENTS.

	Left members:												Sum checks for left members in move No.:						Right members:			
	s=20	21	22	19	13	14	1	15	17	18	4	7	2	4	6	12	13	14	0	△ 2	△ 4	△ 6
s=20	198.19	0	0	0	-1.90	1.10	1.90	-2.90	-198.10	0	-1.10	2.90	198.19	197.39	- 1.71	- 1.71	- 2.81	0.09	3.5649108	- 36.0606	0	0
21		198.19	0	0	-1.90	1.10	0	-2.90	0	-213.65	0	0	198.19	197.39	194.49	-19.16	-19.16	-19.16	8.7956154	0	0	0
22			198.19	-124.55	-1.90	1.10	0	-2.90	0	0	0	0	198.19	72.84	69.94	69.94	69.94	69.94	14.7851154	-840.8068	0	0
19				338.25	-7.50	7.00	0	13.00	0	0	0	0		213.20	226.20	226.20	226.20	226.20	6.7205070		0	0
13					3.00	0	-1.00	0	- 14.00	- 16.50	0	0		-10.20	-25.20	-41.70	-41.70	-41.70	2.7419840		-1106.7801	0
14						3.00	0	0	- 3.00	- 2.00	-1.00	0		13.30	10.30	8.30	7.30	7.30	2.7419840		-1021.0669	0
1							1.00	0	14.00	0	0	0			15.90	15.90	15.90	15.90	- 0.3600920			- 62.7978
15								3.00	6.00	4.00	0	-1.00			13.30	17.30	17.30	16.30	2.7419840			-716.1861
17									466.00	0	3.00	-6.00			270.90	270.90	273.90	267.90	- 9.3623920			0
18										581.25	0	0				353.10	353.10	353.10	-27.9860490			
4											1.00	0					1.90	1.90	- 0.3600920			
Sum check													594.57	683.92	774.12	899.07	901.87	897.77	4.0234756			

TABLE (10.11). MATRIX SOLUTION TABLE WHEN WORKING WITHOUT SIGN CONDITIONS ON THE REGRESSION COEFFICIENTS.

		Left members:											
		s=20	21	22	19	13	14	1	15	17	18	4	7
	s=20	198.19	0	0	0	- 1.90	1.10	1.90	- 2.90	-198.10	0	-1.10	2.90
	21	0	198.19	0	0	- 1.90	1.10	0	- 2.90	0	-213.65	0	0
	22	0	0	198.19	-124.55	- 1.90	1.10	0	- 2.90	0	0	0	0
	19	0	0	0.62843736	259.97812681	-8.69403098	7.69128110	0	11.17753166	0	0	0	0
	13	0.00958676	0.00958676	0.00958676	0.03344139	2.65461499	0.28884344	-0.98178516	0.29038738	-15.89913716	-18.54821127	-0.01054544	0.02780160
	14	-0.00555023	-0.00555023	-0.00555023	-0.02956434	-0.10880803	2.72271428	0.09628067	-0.31398937	- 0.17054564	1.20400097	-0.99274732	-0.01912070
	1	-0.00958676	0	0	0	0.36984088	-0.03536202	0.61527619	0.14630203	10.02501712	- 6.30246268	0.04175685	-0.01684329
	15	0.01463242	0.01463242	0.01463242	-0.04299412	-0.10938964	0.11532219	-0.23778269	2.28936641	2.43707525	4.68289979	-0.13935552	-0.95860719
	17	0,99954659	0	0	0	5.90924408	0.06263809	-6.29352359	-1.04451953	6.81829629	- 3.53394800	1.24323472	-1.64089851
	18	0	1.07800595	0	0	6.98715683	-0.44220614	11.21847845	-2.04450035	0.51830367	131.95669090	1.76312221	1.12451007
	4	0.00555023	0	0	0	0.00397249	0.36461678	-0.06786709	0.06087078	- 0.18233803	- 0.01336137	0.37031806	0.23618718
B	3	0.01798734	0.04437971	0.07460071									
In	5	0.02656730	0.05295968	0.13660615	0.08501321	1.26341199	0.63638197						
move	11	-0.06734733	0.07043042	0.13359925	0.05242620	1.58137988	0.65471006	2.56894227	0.99260273	- 0.08683532			
No.:	12	-0.02245731	0.16243177	0.16626987	0.09442476	2.51334469	0.58877409	2.83459944	0.78607003	- 0.04561985	0.07951932		
	13	-0.21596752	0.14935465	0.14935466	0.06722218	2.28138607	0.85860046	5.40906583	1.05268608	- 0.21981668	0.06722219	0.92039421	
C In	2	-0.181950	0	-4.242428									
move	4	-1.815979	-1.815979	-3.671656	-2.952842	-380.934473	-330.787773						
No.:	6	291.108111	-8.517857	6.613869	24.078337	97.086056	97.326728	-432.531659	-582.606293	259.551037			
D	14	0.39370085	0	0	0	0	0	-5.74540777	0	0.35695544	0	-0.63779547	1.00000000
B	15	0.02647735	0.14935465	0.14935466	0.06722218	2.28138607	0.85860046	1.07098707	1.05268608		0.06722219	0.52763347	0.61580986

TABLE (10.11). Continued.

	Sum checks for left members in move No.:						Right members:			
	2	4	6	12	13	14	0	△ 2	△ 4	△ 6
s=20	198.19	197.39	- 1.71	- 1.71	- 2.81	0.09	3.56491080	- 36.0606	0	0
21	198.19	197.39	194.49	-19.16	-19.16	-19.16	8.79561540	0	0	0
22	198.19	72.84	69.94	69.94	69.94	69.94	14.78511540	-840.8068	0	0
19		258.97537730	270.15290896	270.15290896	270.15290896	270.15290896	16.01202589		0	0
13		2.94345730	-13.64707762	-32.19528890	-32.20583493	-32.17803273	3.53768715		-1106.7801	0
14		2.72271404	2.33445969	3.53846066	2.54571334	2.52659264	1.73268546		- 900.6405	0
1			10.78559495	3.88413226	3.92588312	3.90903983	0.85284213			- 62.797800
15			4.72644114	9.40934094	9.26998541	8.31117822	2.06080713			-701.253870
17			6.81829560	3.28434770	4.52758233	2.88668382	-0.59206893			1769.695876
18				131.95667757	133.71979994	134.84431001	10.49318537			
4					0.37031837	0.60650555	0.34083860			

Working without any sign conditions on the regression coefficients: We will then show how the solution of the problem (10.1) can be obtained by working without any sign condition on the regression coefficients until we reach the last stage.

From the point listed in the last column in the column section 6 in Table (10.2) we now proceed, regardless of the fact that not all the nine regression coefficients on line 7a in the lower left part of Table (10.4) are non-negaive. That is, we make a preference increasing movement with the nine variables 20,21, 22, 19, 13, 14, 17, 1, 15 equal to zero. The corresponding computations are given in the column section 11 of Table (10.9). This leads to a breaking out point where the x_j have the values listed in the last column in the column section 11. According to the standard rule—see line $t = 4$ of Tab. (3.4)—we shall now pick two optimum candidates. As before we pick instead only one since the total work involved will not be any

larger than by picking two. This leads through the move number 12—see Table (10.9), (10.10) and (10.11)—to the final attempt No. 13 where the eleven variables

$$\text{numbers } 20, 21, 22, 19, 13, 14, 17, 1, 15, 18, 4 \qquad \ldots \quad (10.12)$$

are zero and all the d_k are zero.

From the solution obtained by the first method we know that the point we have now reached—and listed in the last column in the column section 12 in Table (10.9)— is actually an optimum point, but by the method now followed we have not yet got a sufficient criterion for the optimality of the point reached because not all the eleven coefficients on the line 13 of Table (10.11) are non-negative. These coefficients are now the B of (7.9).

Looking back we see by the same reasoning as in the first method that number 17 is a black sheep. In the expression for the preference function now obtained —compare (7.9) —we therefore wish to express number 17 in term of eleven other variables, namely the ten variables that are left in the set (10.12) when No. 17 is taken out and another variable added so as to retain the total number eleven.

As such a variable to be added we pick the *next* candidate in the priority order in the last breaking out point, namely the variable number 7. In other words we wish to express number 17 in terms of the 11 variables

$$\text{numbers } 20, 21, 22, 19, 13, 14, 1, 15, 18, 4, 7. \qquad \ldots \quad (10.13)$$

To do this we must consider the equation between the corresponding twelve boundary vectors, namely the boundary vector for No. 17 and those in (10.13) or — which is the same — the boundary vector for No. 7 and those in (10.12). Compare (7.10) where now $\kappa = 7$. This is done by the previously used 11×11 system and with a right member consisting of the moments defined through the newly added boundary vector No. 7. See (7.11) and (7.12) with $\kappa = 7$. Doing this we find the set of twelve D coefficients listed on line 14 in the lower left part of Table (10.11). Using these coefficients to express the variable No. 17 in terms of those indicated in (10.13) — compare (7.13)—and inserting this expression for No. 17 in the expression for the preference function—compare (7.9)—we get the set of eleven B coefficients listed on line 15 in the lower part of Table (10.11). Since all these are non-negative we know that an optimum point has been reached. We also see that the optimum region is one dimensional and can be expressed in any of the two forms indicated in (10.5)—(10.8).

This example shows that we will as a rule actually reach an optimum point quicker by going ahead without paying any attention to the sign of the regression coefficients, but in return we have to make a transformation at the end if we are absolutely set on getting a sure optimality criterion. When a successful transformation is found so immediately as in this example, it will probably pay to go ahead regardless of the signs of the B-coefficients, but in more complicated cases where a high degree of freedom in the optimum region is to be expected and we are set on getting an optimality criterion, it may be safest to work with non-negative regression coefficients throughout.

11. ILLUSTRATION OF THE REVERSAL TECHNIQUE

In the example described in § 10 of the present paper we did not need to use the technique developed in § 9. The numerical material in the example can nevertheless be used to illustrate how the formulae of § 9 work.

Consider for instance the point listed in the last column of column § 6 of Table (10.2) Here the following nine variables

$$20, 21, 22, 19, 13, 14, 1, 15, 17 \qquad \ldots \quad (11.1)$$

are zero.

The moment matrix for the boundary vectors of these variables are given in the first nine rows and columns of Table (10.3), and the corresponding matrix solution table is given by the first nine rows and columns of Table (10.4).

The set (11.1) we now consider as the set $a = \alpha, \beta \ldots \gamma$ within which the formulae (9.1) and 9.2) are to be applied. We let the set $\theta \ldots \delta$ be the set of the three variables

$$1, 15, 17. \qquad \ldots \quad (11.2)$$

To determine the nine numbers C'_s $(s = \alpha, \beta \ldots \gamma)$, we use (5.3) with all the Δx_s equal to zero except Δx_s for $s = 13$ and $s = 14$, these two Δx_s being put equal to minus the values which the variables Nos. 13 and 14 had in the point from which we started when we wanted to force Nos. 13 and 14 down to zero. This is the point listed in the last column of column section 3 in Table (10.2). In other words we put

$$\Delta x_{13} = -1106.7801$$
$$\Delta x_{14} = -1021.0669. \qquad \ldots \quad (11.3)$$

To compute the numbers C'_s under these conditions is an easy matter because the first nine rows and columns of Table (10.4) have been computed. The result is given on the first line in Table (11.4).

TABLE (11.4). ILLUSTRATION OF THE USE OF THE TECHNIQUE OF SECTION 9.

	s=20	21	22	19	13	14	1	15	17
C'_s	-105.643766	-1.935616	-12.664797	-17.072793	-631.283421	-351.543053	901.027409	147.972876	-95.113408
C^1_s	-2.68356216	0.04550475	0.01151729	-0.05408250	0.35136164	-1.26329507	39.38586322	2.40047400	-2.35255240
C^{15}_s	-0.17116544	0.00790331	-0.01024472	-0.02887007	-0.11051708	-0.02512556	2.40047400	0.60300222	-0.15612691
C^{17}_s	0.16661973	-0.00253122	0.00065853	0.00509160	0.01732396	0.07437287	-2.35255233	-0.15612690	0.14566420
C_s	-1.815995	-1.815971	-3.671672	-2.952867	-380.934230	-330.787728	0	0	0
B_s	0.99561562	0.04144721	0.17209648	0.25980634	2.17313214	0.96500944	-11.86010261	-1.25814957	0.86725465
A_s	0.02656738	0.05295965	0.13660623	0.08501332	1.26341130	0.63638162	0	0	0

Similarly to determine C^1_s we use (5.3) but now with all the $\Delta x_s = 0$ except $\Delta x_1 = 1$. Further C^{15}_s is determined by (5.3) with all the $\Delta x_s = 0$ except $\Delta x_{15} = 1$, and similarly for C^{17}_s. The result is given on the lines Nos. 2, 3, 4 in Table (11.4).

Next the system (9.1)—which is now of order 3 —is solved, giving

$$\Delta x_1 = 377.484406, \qquad \Delta x_{15} = -77.206269, \qquad \Delta x_{17} = 6685.195685. \qquad \ldots \quad (11.5)$$

By (9.2) this gives the values C_s listed on line 5 in Table (11.4). These values are—apart from rounding errors in the last decimal places—identical with the values C listed on the next to last line in Table (10.4).

We can also illustrate the method (9.7)—(9.11). The solutions A_s^1, A_s^{15}, A_s^{17} of (9.8) are identical with the solutions C_s^1, C_s^{15}, C_s^{17} listed on lines Nos. 2, 3, 4 of Table (11.4). Hence the matrix of the system (9.10)—now of order 3—is the same as the one we just used to get (11.5), but B_s in (9.10) is not the same as C_s' in (9.2). B_s is obtained by putting all the Δx_s in (9.8) equal to zero except $\Delta x_r = M_{or}$ for $r = \theta \ldots \delta$. The result is given on the next to last line in Table (11.4). By (9.10) this gives

$$\Delta x_1 = -1.21293526 \qquad \Delta x_{15} = 0.47838524 \qquad \Delta x_{17} = -24.85991535. \tag{11.6}$$

Finally by (9.9) we get the values of A listed on the last line of Table (11.4). These values are—apart from rounding errors in the last decimal places—identical with the values B on the line marked 5 in Table (10.4).

12. The regressionally transformed problem

Once the equations have been brought over into *any* basis form, the number of equations does not have a very great influence on the amount of work involved in the multiplex method. It is primarily a large number of degrees of freedom that will entail a heavy computational cost. In many practical problems, not least in macroeconomic planning, the number of degree of freedom, although great from the viewpoint of desk machine computation, will be moderate from the viewpoint of automatic computors, while the number of equations may be extremely large. Furthermore the situation will often be such that the way in which the equations can be brought over into *some* basis form will follow without too much difficulties from the already established theory and practice of the concrete problem at hand. For instance, there may be a certain number of *parameters of action* or *steering parameters*, which will determine the constellation of the system and the way these steering parameters—to a linear approximation—determine a multitude of variables describing the system, may be known. To such cases the multiplex method is well adapted. Even if some basis form of the original equations is not already available, it may be reached through a hierarchic order of *partioning* of the equations or similar devices. Very seldom will all the variables occur in all the equations.

In the opposite case, i.e., when the number of degrees of freedom is very large and the number of equations moderate, we can transform the problem into a linear programming problem where the number *which was originally the number of degrees of freedom now appears as the number of equations and vice versa.*

In essence this is the same as to pass to what is generally called the dual problem. When looked upon from the viewpoint of (4.2) and (7.1) where the B_s are regression coefficients, we may speak of the regressionally transformed problem.

To formulate this problem we first note that in the original problem we can, if we want to, define the preference function in the form

$$f = P_0 + \sum_{j=1}^{n+m} P_j x_j \qquad \ldots \quad (12.1)$$

where the P are any set of $(n+m+1)$ given weights, positive, negative or zero. The passage from this form to (1.5) is obvious, when the equations are given in a basis form.

This being so, consider the following problem. Let $B\,(j = 1, 2, \ldots, n+m)$ be a set of $n+m$ magnitudes subject to the conditions of : (1) being non-negative, and (2) satisfying the n equations

$$p_k + \sum_{j=1}^{n+m} B_j b_{jk} = 0 \quad (k = u, v \ldots w) \qquad \ldots \quad (12.2)$$

where the p_k and b_{jk} are the coefficients in the original problem. Compare (1.3) and (1.5). To determine that or those sets of values of the magnitudes B_j which satisfy these two sets of conditions and maximize the linear function

$$F = \sum_{j=1}^{n+m} B_j b_{j0} \qquad \ldots \quad (12.3)$$

where the $b_{j0}\,(j = 1, 2, \ldots, n+m)$ are the constant terms—positive, negative or zero—in the basis form of the equations, i.e. in (1.3).

This obviously is a well defined linear programming problem where the number of variables is the same as in the original problem, namely $(n+m)$, but the number of *equations* is now n and the *number of degrees of freedom* is now m.

We know that the problem now formulated has at least one solution where at least m of the B_j are zero, hence at most n of them different from zero (and then necessarily positive). Inserting these in (4.2) we find that all the n d_k are zero, hence the original preference function f can be written in the form (7.1) with all the B_s non-negative. We therefore have a solution of the original problem.

Thus, a large number of equations need not worry us, provided we can in *some* way without prohibitive work bring the equations over into *any* basis form, nor need a large number of degrees of freedom do so, if in the regressionally transformed problem we can in *some* way bring the equations over into *any* basic form.

13. SOLVING LINEAR EQUATIONS AND INVERTING MATRICES WHEN HIGH SPEED MEMORY IS LIMITED

The following are explicit formulae that can be used in the case where the capacity of the automatic machinery is not large enough to handle the whole problem in one stroke. The essence of this method is that one proceeds step by step as one would with a desk machine and a work sheet, eliminating, however, not *one single* variable at each step, but eliminating *as many* variables at a time as is determined by the order of an inverse which the machine can handle directly in one stroke. While each step in the ordinary Gaussian elimination algorithm involves *one division*, i.e., the formation of one reciprocal number (and several multiplications), each step in the extended algorithm involves the formation of one inverse matrix (and some matrix multiplications). This procedure may, for instance, be used when we want to work the equations of a linear programming problem over into a basis form and there is available automatic machinery with a capacity large enough to handle a good sized inversion but not large enough to make the whole of the desired inversion in one stroke.

Let

$$a_{i0}+\sum_{j=1}^{n+m} a_{ij}x_j = 0 \qquad (i = 1, 2, \ldots, m) \qquad \ldots \quad (13.1)$$

be m linearly independent equations connecting the $n+m$ variables $x_1, x_2, \ldots, x_{n+m}$, the a being given constants.

We pick a set of ν variables ($\nu < m$) , let it be Nos. $\alpha \ldots \gamma$, and also pick a set of ν equations, let it be Nos. $\alpha' \ldots \gamma'$, and use these equations to express the ν variables in terms of the other $n+m-\nu$ variables. To do this we consider the matrix

$$\begin{pmatrix} a_{\alpha'\alpha} & \ldots & a_{\alpha'\gamma} \\ \ldots\ldots & \ldots & \ldots\ldots \\ a_{\gamma'\alpha} & \ldots & a_{\gamma'\gamma} \end{pmatrix} \text{ and its inverse } \begin{pmatrix} a^{-1}_{\alpha\alpha'} & \ldots & a^{-1}_{\alpha\gamma'} \\ \ldots\ldots & \ldots & \ldots\ldots \\ a^{-1}_{\gamma\alpha'} & \ldots & a^{-1}_{\gamma\gamma'} \end{pmatrix} . \qquad \ldots \quad (13.2)$$

In terms of this inverse, we have

$$x_r = b^{(1)}_{r0}+\sum_{j=k,\,2..)\alpha..\gamma(..n+m} b^{(1)}_{rj}x_j \quad (r = \alpha \ldots \gamma) \qquad \ldots \quad (13.3)$$

where

$$b^{(1)}_{rj} = -\sum_{r'=\alpha'..\gamma'} a^{-1}_{rr'}\, a_{r'j} \qquad \begin{pmatrix} r = \alpha \ldots \gamma \\ j = 0, 1\ldots)\alpha \ldots \gamma(\ldots n+m \end{pmatrix} . \quad \ldots \quad (13.4)$$

Inserting (12.6) in all the equations (12.4) except the equations Nos. $\alpha' \ldots \gamma'$, we get

$$a^{(1)}_{i0} + \sum_{j=1,2..)\alpha..\gamma(..n+m} a^{(1)}_{ij}\, x_j = 0 \quad (i = 1, 2\ldots)\alpha' \ldots \gamma'(\ldots m) \qquad \ldots \quad (13.5)$$

where

$$a^{(1)}_{ij} = a_{ij}+\sum_{r=\alpha..\gamma} a_{ir}\, b^{(1)}_{rj} \qquad \begin{pmatrix} i = 1, 2\ldots)\alpha' \ldots \gamma'(\ldots m \\ j = 0, 1\ldots)\alpha \ldots \gamma(\ldots m+n \end{pmatrix} . \quad \ldots \quad (13.6)$$

We next pick a new set of variables, now μ in number, ($\mu \leqslant m-\nu$), let it be Nos. $\theta \ldots \delta$, and also pick μ equations in (13.5), let it be Nos. $\theta' \ldots \delta'$. None of the variables Nos. $\theta \ldots \delta$ occur in the set $\alpha \ldots \gamma$, and none of the equations Nos. $\theta' \ldots \delta'$ occur in the set $\alpha' \ldots \gamma'$. Through the equations Nos. $\theta' \ldots \delta$ in (13.5) we express the variables Nos. $\theta \ldots \delta$ in terms of the other $n+m-\nu-\mu$ variables. To do this we consider the matrix

$$\begin{pmatrix} a^{(1)}_{\theta'\theta} & \ldots & a^{(1)}_{\theta'\delta} \\ \ldots\ldots & \ldots & \ldots\ldots \\ a^{(1)}_{\gamma'\theta} & \ldots & a_{\gamma'\delta} \end{pmatrix} \text{ and its inverse } \begin{pmatrix} a^{-1(1)}_{\theta\theta'} & \ldots & a^{-1(1)}_{\theta\delta'} \\ \ldots\ldots & \ldots & \ldots\ldots \\ a^{-1(1)}_{\delta\theta'} & \ldots & a^{-1(1)}_{\delta\delta'} \end{pmatrix} . \qquad \ldots \quad (13.7)$$

In terms of this inverse we have

$$x_r = b^{(2)}_{r0} + \sum_{j=1,\,2..)\alpha..\gamma,\,\theta..\delta(..n+m} b^{(2)}_{r}\, x_j \quad (r = \theta \ldots \delta) \qquad \ldots \quad (13.8)$$

where

$$b^{(2)}_{rj} = -\sum_{r'=\theta'..\delta'} a^{-1(1)}_{rr'}\, a^{(1)}_{r'j} \qquad \begin{pmatrix} r = \theta \ldots \delta \\ j = 0, 1\ldots)\alpha \ldots \gamma, \theta \ldots \delta\ (\ldots n+m) \end{pmatrix} \quad \ldots \quad (13.9)$$

Inserting (13.8) in the right member of (12.6) we get

$$x_r = b_{r0}^{(2)} + \sum_{j=1,\,2..)\alpha..\gamma,\theta..\delta(..n+m} b_{rj}^{(2)}\, x_j \quad (r = \alpha \ldots \gamma) \qquad \ldots \quad (13.10)$$

where

$$b_{rj}^{(2)} = b_{rj}^{(1)} + \sum_{s=\theta..\delta} b_{rs}^{(1)}\, b_{sj}^{(2)} \qquad \begin{pmatrix} r = \alpha \ldots \gamma \\ j = 0,1\ldots)\alpha\ldots\gamma,\theta\ldots\delta(\ldots n+m \end{pmatrix} \ldots \quad (13.11)$$

Through (13.10) and (13.8) the $\nu+\mu$ variables Nos. $\alpha \ldots \gamma, \theta \ldots \delta$ are expressed in terms of the remaining $n+m-\nu-\mu$ variables.

In this way we can continue until we are left with only n variables in the right member. The equations are then brought over into a basis form.

A possibility that might be considered is to express all the variables linearly in *some other* set of n parameters $t_1, t_2 \ldots t_n$. The whole programming theory could easily be expressed in this way. Practical experience will decide if any such transformation will pay.

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