ON WELFARE THEORY AND PARETO REGIONS*

By RAGNAR FRISCH

1. Introduction

The following considerations constitute an attempt at discussing—and questioning—certain basic propositions of welfare theory which are frequently taken more or less for granted. They are, I believe, very much in need of being scrutinized more cautiously. In particular I shall discuss the way in which *conditions* enter into the argument about optimality. I shall concentrate on these basic points without making any attempt at surveying the whole field of welfare economics.[1]

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The purpose of a macro-economic decision model[2] is to discuss which economic policy or policies might be designated as "good" or perhaps as "the best" under given circumstances. In this connection it becomes necessary to consider *criteria* which may, in point of principle, be used to define the meaning of these words as applied to an economic policy.

An essential problem in this connection is how to *reconcile* the desires of those in power (the dictator, the democratic public authority, the influential non-official personalities, etc.) with those of the mass of citizens. A conflict may arise on this point. This conflict is formally similar to the conflict between the interests of individual citizens. The discussion of the general problem may therefore well take as its starting-point a theoretical device for describing the preferences of *any* economic entity, whether a person, a family, a business enterprise, a public authority, etc. Such a device is the *preference function* of that entity.[3] In what follows we assume the existence of such a function for each such entity. For shortness such an entity will be referred to as an "individual". In other words, we assume that each individual acts *as if* there exists a function which this individual tries to maximize.

In the most general case the preference function of any individual may be envisaged as depending not only on the values assumed by all the *variables* of

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[1] An excellent survey of the opinions of different authors is given by Tibor Scitovsky in the *American Economic Review*, 1951.

[2] The concept of a decision model is studied at length in two papers in the *Revue d'Économie Politique*, 1951.

[3] The application of this concept to the "homme d'état" is considered in a very interesting article by Maurice Fréchet: "Sur les fonctions de bien-être", which I have had an opportunity to read in manuscript.

the theoretical model used, but as depending also explicitly on the nature of the various *relations* that form part of the model (*i.e.* on the parameters of these relations). This means that the value of the preference function will depend on the nature of these relations in a manner which is so complicated that it is not sufficient to indicate the *result* of the relations, that is, the values of the various variables as they *actually exist*. In what follows we shall, for simplicity, only consider the case where the preference functions *can* be expressed simply in terms of these variables.

Some of the conditions—in the form of equations or inequalities—which we impose on the variables of a model are more or less *obligatory*, unavoidable. For instance, certain definitional equations between the variables, certain technically given production functions depending on the laws of nature, certain existing limitations on natural resources, etc. Others of the conditions we impose on the variables of the model are of a more or less *facultative* sort. They may be changed if man changes the "rules of the game", *i.e.* changes the rules and regulations by which production and distribution take place. The choice of an economic policy just consists in choosing a specific form of these facultative conditions. On closer scrutiny it might in border cases be found difficult to draw the line of demarcation between obligatory and facultative conditions in a hard and fast way, but some sort of distinction between these two kinds of conditions has to be made if it shall be possible at all to discuss the question of economic policy in a systematic way. The distinction between obligatory and facultative conditions is more important than a distinction between "technical" and "behaviouristic" conditions.

A certain variable or a certain set of variables may belong to a given individual as his *parameter(s) of action* (variable(s) of action).[4] That is, the individual in question is free to choose whatever magnitude he wishes of this or these parameters; *e.g.* the quantity bought by a buyer in an atomistic market, or the price fixed by a monopolistic seller who is not under public regulation, etc. In choosing a specific magnitude of this or these parameters the individual will be guided by the nature of his preferences—as expressed by his preference function—and by the conditions that are *conjectural* to him, that is, the conditions which he *believes* will hold good *while he makes his choice*. For instance, the buyer in the atomistic market believes that by changing the quantity which he buys, he will not affect the price, the monopolistic seller believes that if he changes his price by so much, the quantity demanded will be changed by so much, etc.

The conjectural conditions are not the same thing as the *objective* (obligatory or facultative) conditions referred to above, but the conjectural conditions will, in general, depend on some of these objective conditions and on the psychology of the individual in question. The set of *all* these conditions: the objective conditions (the obligatory and facultative ones) and the conjectural conditions, we

[4] M. Lutfalla, editor of the *Revue d'Économie Politique*, has suggested the term *variable potestative*, which would refer to the meaning of the term *potestas* in Roman law.

shall call *a régime*. Thus a régime is simply the complete set of conditions which the variables of a model will satisfy under a given type of economic policy. If nothing is said to the contrary, we assume that the régime is specified in such a way as to lead to a uniquely determined point or at any rate to a pointset of measure zero.

When speaking of a purpose to be realized by an economic policy, two types of problems should be distinguished: the *selection* problems and the *realization* problems (problems of régime).

In the first type of problems we ask whether *there exist* points (a pointset of zero or higher measure) satisfying some preassigned set of conditions. Some of these conditions may be objective (obligatory or facultative) and some may be conjectural and, perhaps, some may be *additional* conditions imposed by the policy-makers in the manner discussed in detail in the paper on decision models already referred to (*e.g.* "the part of the national income that goes to the workers shall not decrease").

If such points exist, we classify them according to their desirability defined in some way or another. We may, for instance, pick out one specific *subset* of points which we consider particularly desirable. The consideration of *Pareto conditions* is an example of a principle by which to select such a subset.

In the second type of problems we ask whether there exists *a régime* that will lead to a point in the selected subset.

The distinction between these two types of problems is absolutely fundamental if it shall be possible at all to attach any meaning to a statement about whether a régime is "good" or not. First we must define what we mean by "good". This is the selection problem. Secondly, we must define the régime and find out whether it leads to a point that has the property which was defined as "good". The distinction between these two types of problems brings out very forcibly the necessity of being fully aware of the significance of *conditions*. Indeed in the selection problem, *i.e.* in the definition of what is "good" and in the definition of a régime, two entirely different sets of conditions enter and fundamental fallacies may be introduced if we are not aware of the difference in the meaning of an "optimality" produced under these two sets of conditions.

This classification of the problems in the two categories: selection problems and realization problems, is discussed fully in Section 9 of the article in the *Revue d'Économie Politique*. In that article a number of examples of formulations are also given. In what follows the rôle of conditions will be considered.[5]

2. The Rôle of Conditions in Problems of Choice

In the discussion of welfare problems, the reasoning about Pareto conditions is not always carried through as carefully as one could wish. A brief systematic

[5] *Note added in March* 1959. It should be stressed that the main purpose of the paper is to elucidate the basic rôle of conditions, and to warn against the errors which may creep into the reasoning if this rôle is not clearly recognized.

discussion of some essential points—starting from a rigorous statement of definitions—might therefore be useful.

Consider a model containing the m variables $X_i (i=1, 2 \ldots m)$ and the n individuals Nos. $j=1, 2 \ldots n$, each of whom has a preference function $\Omega_j(X_1, X_2 \ldots X_m)$. A point in m-dimensional space with cartesian co-ordinates $(X_1, X_2 \ldots X_m)$ we denote for shortness by X. Sometimes we may speak of "the space X".

Let C be any given set of compatible conditions (constraints) imposed on the variables $X_1, X_2 \ldots X_m$. The set of points X satisfying these conditions we denote by

$$X[C] \tag{2.1}$$

In particular the total space we denote $X[O]$. $X[C]$ may be called the admissible region under the conditions C.

If the conditions consist of s independent *equations*, the normal case will be the one where the locus of points satisfying C is of dimensionality $m-s$. On the other hand, if the condition consists of any number of independent *inequalities*, the locus of points satisfying C will, in the normal case, still be of dimensionality m. That is, a set of inequalities will, in general, not restrict the dimensionality of the admissible space (but it will restrict its volume). If the conditions are in the form of s equations and any number of inequalities, the dimensionality of the admissible space will in the normal case be $m-s$.

Definition (2.2). Let X and X' be any two points. If X' is such that for *any* of the individuals whose preference functions are $\Omega_1 \ldots \Omega_n$, X' is either preferred to X or indifferent to X, and for *at least one* of the individuals X' is actually preferred to X, we shall say that X' is *Pareto-preferred* to X under the preference function $\Omega_1 \ldots \Omega_n$, *i.e.* by the individuals Nos. $1 \ldots n$.

The property of Pareto-preference is *transitive* provided the individual comparisons are completely transitive. This means that for *any* of the individuals the comparison of points is transitive with respect to preference and also

From the criteria discussed there follow also certain conclusions which have a practical bearing on the question of régime. It is true that the régime needed to produce *exactly* a Pareto-optimal solution may be so complicated that we do not find it feasible to organize it. This simply means that in practice we should choose a solution that does not deviate *too grossly* from one that has the criteria in question. This can often be done *ad hoc*. Or, if a more refined analysis is wanted, we can formulate the problem of choosing amongst a set of feasible régimes that (or those) coming *closest* to Pareto optimality.

Such extensions of the theory may, in particular, be needed in cases where it is found necessary to let the utility of one individual depend not only on the quantities received by him, but also on those received by other individuals. Similar complications may be considered in the technical field. (The subsequent general parts of the argument do take account of such complications, but the examples which are discussed in detail do not.)

It is, of course, perfectly proper to try to improve the analysis by proceeding to a closer scrutiny of more general cases, provided one can do it in such a way as actually to reach precise conclusions and to show that the simplified treatment *eliminated something that is quantitatively important*. This cannot be done by general philosophical discussions, but demands a precise mathematical analysis. One should not be led to the negative, perfectionist attitude of saying that, since the principle of Pareto optimality does not solve all intricacies, the principle should be abandoned altogether. If we did abandon it, we would lose a lot of relevant—although approximate—insight.

transitive with respect to indifference and transitive with respect to comparisons with one preferent and one indifferent term, preference being in this case dominant (if X'' is preferred to X' and X' indifferent to X, X'' is preferred to X, and similarly if X'' is indifferent to X' and X' preferred to X). Then the property defined in Definition (2.2) is transitive. Indeed, suppose that X'' is Pareto-preferred to X' and X' Pareto-preferred to X. Then there can be no individual for whom X is preferred to X'' (because for any individual X'' is preferred or indifferent to X', and X' preferred or indifferent to X, hence X'' preferred or indifferent to X). Furthermore, there must be at least one individual who prefers X'' to X'. For this particular individual X' is either preferred or indifferent to X, hence this individual will actually prefer X'' to X. Consequently X'' is Pareto-preferred to X.

The property of Pareto-preference being transitive, if X' is Pareto-preferred to X, and if we determine the set of points which are Pareto-preferred to X and the set of points that are Pareto-preferred to X', we will find that the latter set is included in the first.

Definition (2.3). Let X and X' be any two points. If X is *not* Pareto-preferred to X' and X' *not* Pareto-preferred to X (with respect to a given set of preference functions), we shall say that X and X' are non-distinguishable in the Pareto-sense (with respect to these preference functions). Otherwise we shall say that X and X' are distinguishable in the Pareto-sense.

The property of being non-distinguishable in the Pareto-sense is not transitive. Indeed, even if X'' and X' are non-distinguishable and also X' and X non-distinguishable, it might well be that X'' and X are distinguishable, for instance X'' Pareto-preferred to X. It is therefore not possible to build up a theory of "indifference curves for Pareto-preference" in the same way as we do for individual preference. Some useful concepts may, however, be derived from the above definitions, as, for instance, the concept of a positive Pareto-pressure indicated in Section 5.

Definition (2.4). A point X will be called *locally Pareto-optimal under the condition* C and with respect to the preference functions $\Omega_1 \dots \Omega_n$—or shorter with respect to the individuals Nos. $1 \dots n$—if there exists *no other point* in the vicinity of X and satisfying C, which is Pareto-preferred to X with respect to the preference functions $\Omega_1 \dots \Omega_n$—*i.e.* for the individuals Nos. $1 \dots n$.

Definition (2.5). A point X will be called *globally Pareto-optimal under the condition* C if (2.4) applies when the words "in the vicinity of X" are replaced by "in the whole permissible space".

The locus of points that are globally Pareto-optimal under the conditions C will be said to form the Pareto region or the Pareto-optimal region (or space) under the conditions C and with respect to the preference function $\Omega_1 \dots \Omega_n$—*i.e.* for the individuals Nos. $1 \dots n$.

This region will be denoted

(2.6) $Par[C]$ or more explicitly $Par[C; \Omega_1 \dots \Omega_n]$

If the variables $X_1 \dots X_m$ are independent under the search for points that are—locally or globally—Pareto-optimal, that is, if *no* conditions are imposed on $X_1 \dots X_m$ during this search (apart from trivial conditions), we shall say that the points obtained are *unconditionally* Pareto-optimal—locally or globally as the case may be. The region of points having this property will be denoted

(2.7) $Par\ [O]$ or, more explicitly, $Par\ [O; \Omega_1 \dots \Omega_n]$

The above definition includes the limiting cases where there exist one or more other points, different from X, which are *for each individual* indifferent to the original point X. If necessary, we may characterize this formulation of the definition as Pareto-optimality in the *general* sense.

A slightly more restrictive definition is obtained if we call a point Pareto-optimal only in the case where all other points are such as to make at least one of these individuals worse off than in the original point X. This point may be called *strong* Pareto-optimality—local or global as the case may be.

The distinction between general and strong optimality will not play an important rôle in what follows. If nothing is said to the contrary, the optimality is to be understood in the general sense.

I shall not at this point take up a full discussion of how "correct" it is to require that a point must be—locally or globally—Pareto-optimal in order that it shall be considered a "good" or "efficient" point. I only want to take exception to one particular type of argument which does not, I think, render justice to the principle under discussion. The principle has been criticized on the ground that it makes "the initial position" a sacred one from which we are not allowed to depart if such a departure should make any individual worse off. As I see it, this way of arguing is misinterpreting the meaning of the principle. The principle does not single out any point as an "initial position". The Pareto-principle is a principle of *negation*, not one of *affirmation*. It states that if a point is *not* Pareto-optimal, then it cannot be said to be a "good" or "efficient" point. And this must be our conclusion *regardless* of how in detail we have formulated our desiderata for a "good" or "efficient" point. In other words, the principle gives a necessary condition, it segregates a class of points to which our "good" or "efficient" point must belong, if any such points shall be fixed at all. Taken in this sense it would seem that it is next to impossible not to accept the principle—unless by saying that the individuals do not know what is best for them. Indeed, if they do know this, and we want to respect it, how could we maintain a point as a "good" or "efficient" point if there exist one or more other points that are Pareto-preferred to it? Only the dictator could maintain it: "I know better than you what is best for you".

To adopt this principle is, of course, by no means the same as to say that any "initial" point which we have happened to run across and which we find on scrutiny to be Pareto-optimal, is a point from which we should not depart. Many other points may, of course, also have the property of being Pareto-optimal.

Since the Pareto-principle only forms a necessary, not a sufficient, criterion—a fact which will be clearly illustrated by some of the subsequent examples—the principle does not lead to a final fixation of what should be considered the "best" or "most efficient" point, it leaves considerable *leeway in the fixation of economic policies.* An analysis of the consequences produced by adopting some specific way of handling this leeway is just the purpose of the macro-economic decision model.

Obviously the property of a point of being globally Pareto-optimal under a given set of conditions is a special case of the property of being locally Pareto-optimal under these same conditions. In other words, any point which is globally Pareto-optimal must necessarily be locally Pareto-optimal, but the inverse does not hold. If it is wanted to construct the Pareto-optimal region in an actual case, we may therefore—if this is found convenient—begin by constructing the locus of points that are *locally* Pareto-optimal under the conditions considered. Out of this pointset we may then pick out the points that have the global property. However, in any case where it is possible to use a graphical representation it will, as a rule, be safer and more illuminating to proceed directly to an analysis from the global viewpoint. This is exemplified in the figures of Sections 4 and 5. Whenever we speak of a point or a region as being Pareto-optimal without further specification, we have the global property in mind.

It should be emphasized that it is absolutely essential to indicate explicitly the *conditions* under which the search for Pareto-optimal points is to take place. If these conditions are not indicated, the definition of a Pareto-optimal region has no sense. It is as if one would speak of "the derivative" of a function of several variables without indicating the variable *with respect to which* the derivation is to be performed.

This necessity of specifying conditions when speaking of Pareto-optimal points is only a special manifestation of a basic principle underlying the whole theory of choice. The absurdities which may be produced by carelessness on this point may, perhaps, be illustrated by the following "theoretical analysis" of the "régime" which consists in forcing people to do abominable things under the threat of being shot. Firstly: this régime has the important property that any person subject to it is *perfectly free to choose* himself the alternative which he likes. Secondly: this being so, everybody will, of course, choose the alternative which gives him *the highest possible satisfaction.* Thirdly: any régime which allows everybody subject to it to reach the highest possible satisfaction must be *a very desirable régime* for these persons. Therefore: the régime considered must be a very desirable régime for those concerned. *Quod erat demonstrandum.*

I am not suggesting that all attempts at "proving theoretically" the superiority of the régime of free competition proceed on logical lines similar to the above, but I think it is fair to say that some of these attempts come dangerously close to this form of logic. Translated into economic terms: "the régime of free competition is the best of all régimes within the class of régimes which consists of the régime of free competition". It is even possible that Pareto himself has at one time been thinking more or less along such lines, but has at a later stage recognized the fallacy. My suspicion in this direction has been confirmed by prolonged conversations with such an eminent *connaisseur* of Pareto as Professor Gustavo Del Vecchio. The essence of our conversations on this point is given in Section 7.

3. The Unconditional Pareto Region

To simplify we only consider points where all the preference functions have continuous first order partial derivatives and none of the preference functions has all its first order partial derivatives equal to zero. Any point where the conditions are not fulfilled would need a special investigation which is not necessary for the present purpose.

Proposition (3.1). If all the preference functions have continuous first order partial derivatives and none of the preference functions has all its first order partial derivates equal to zero, a necessary and sufficient condition for a point X' to be unconditionally locally Pareto-optimal is that there exists at least one set of n numbers $\Theta_1 \ldots \Theta_n$, functions of the point X, all of them effectively positive (*i.e.* none of the Θ equal to zero), such that

$$\Theta_1 d\Omega_1 + \ldots + \Theta_n d\Omega_n = 0 \quad (\Theta_j > 0) \tag{3.2}$$

for all variations $d\Omega_1 \ldots d\Omega_n$ around the given point X, *i.e.* for all values of $dX_1 \ldots dX_m$ around the point X.

The proposition holds no matter whether $m > n$, $m = n$ or $m < n$.

Another way to express the criterion (when the indifference surfaces have the usual convexity property) is to say that it is necessary and sufficient that there exists at least one set of n numbers $\Theta_1 \ldots \Theta_n$ with the specified properties, such that

$$\Theta_1\omega_{1i} + \Theta_2\omega_{2i} + \ldots + \Theta_n\omega_{ni} = 0 \quad \text{for } i = 1, 2 \ldots m \tag{3.3}$$

where

$$\omega_{ji} = \frac{\partial \Omega_j}{\partial X_i} \tag{3.4}$$

If we consider $\omega_1 \ldots \omega_n$ as n *vectors*—preference vectors—in m-dimensional space, with components $(\omega_{11} \ldots \omega_{1m}) \ldots (\omega_{n1} \ldots \omega_{nm})$, we can interpret (3.3) by saying that it is necessary and sufficient that *any* of the n preference vectors $\omega_1 \ldots \omega_n$ is expressible as a linear form in the *other* $(n-1)$ preference vectors, with coefficients all of which are effectively negative.

Another symbolic way to express (3.3) is to say that for a *given* point X, the n "variables" $\omega_1 \ldots \omega_n$ (so far without specification of the secondary subscript) shall be perfectly correlated (around their natural origin) over the "field of variation" generated by maintaining X fixed, but putting successively $i=1, 2\dots m$, all the "perfect regression coefficients" in the homogeneous form of the equation, *i.e.* all the $\Theta_1 \dots \Theta_n$ (constants under the variation over i) being effectively positive. In the terminology of confluence analysis we may say that the variables $\omega_1 \dots \omega_n$ form a *closed* set of variables.

Finally, we may interpret the situation by saying that if there exists a convex corner on X such that all the n preference vectors $\omega_1 \dots \omega_n$ from X lie in this corner or on its boundary, and at least one of them lies in the interior of it (as a limiting case the convex corner may be a half-space), then, and only then, will there exist points in the vicinity of X that are Pareto-preferred to X. Consequently the necessary and sufficient condition that no such points shall exist in the vicinity of X is that the n preference vectors $\omega_1 \dots \omega_n$ do *not* have the property in question, a condition that is precisely expressed by the existence of the n effectively positive numbers $\Theta_1 \dots \Theta_n$.

The *sufficiency* of the criterion in (3.1) follows immediately when the proposition is considered in the form (3.2), that is, when the variations around the given point X are expressed in terms of the increments $d\Omega_1 \dots d\Omega_n$. Indeed, if $\Theta_1 \dots \Theta_n$ is any set of n numbers, all of which are positive, non-zero, it is impossible to find a set of non-negative numbers $d\Omega_1 \dots d\Omega_n$, at least one of them different from zero, such that the linear form written in the left member of (3.2) vanishes. Hence, if there actually exist n effectively positive numbers $\Theta_1 \dots \Theta_n$ such that (3.2) holds for *any* variations around the given point X, this point must be unconditionally locally Pareto-optimal.

The *necessity* of the criterion in (3.1) is most easily discussed in terms of (3.3), which we now prefer to look upon as a vector equation in the n preference vectors $\omega_1 \dots \omega_n$. We may first note that in any point which is unconditionally locally Pareto-optimal the preference vectors must be *linearly dependent*. This is trivial in the case $m<n$ because any set of more than m vectors are necessarily linearly dependent in m-dimensional space. In the case $m \gtreqless n$, suppose that the n preference vectors $\omega_1 \dots \omega_n$ in the given point X are *not* linearly dependent. Hence the matrix $[\omega_{ji}]$ $(m \gtreqless n)$ is of rank n and the equations

(3.5) $$\omega_{j1}dX_1 + \dots + \omega_{jm}dX_m = \delta_j \quad j=1, 2\dots n \; (\lesseqgtr m)$$

has solutions $dX_1 \dots dX_m$ for arbitrarily given $\delta_1 \dots \delta_n$. In particular we may choose all the δ_j non-negative and at least one of them positive. Any set of values $dX_1 \dots dX_m$ satisfying the equations (3.5) with this choice of the δ's, will produce increments $d\Omega_j = \omega_{j1}dX_1 + \dots + \omega_{jm}dX_m$, $j=1, 2\dots n$, which are non-negative and at least one of them positive.

Thus, in any point X that is unconditionally locally Pareto-optimal, there must exist at least one set of n numbers $\Theta_1 \dots \Theta_n$, independent of i and at least one of them different from zero, such that (3.3) holds for all i (and consequently (3.2) holds for all variations of $d\Omega_1 \dots d\Omega_n$ around X). This applies no matter whether $m>n$, $m=n$ or $m<n$.

To complete the proof of the necessity of the criterion in (3.1) it therefore suffices to consider the *sign condition* for the Θ's and dispense with all cases where at least one of the Θ's is zero or of different sign from the other Θ's. The necessity of the sign condition for the Θ's in (3.1) follows from a general principle of the signs of the coefficients in linear vector aggregates. However, rather than to approach the problem from this general and abstract viewpoint, we prefer to discuss the meaning of the sign condition in terms of some simple examples, maintaining all the time the concrete interpretation of the vectors as preference vectors. From these examples the necessity of the sign conditions in (3.1) in the general case will follow more or less intuitively.

To simplify the wording we shall say that a given point X' is *deferred* to X (by a given individual) if it is such that X is preferred to X'. We shall say that the point is *non-deferred* to X if it is either indifferent to X or preferred to X (by a given individual).

Example 1. In the case m arbitrary ($\geqq 1$), $n=2$, we have an immediate geometric interpretation of the necessity and sufficiency of (3.3). Indeed, in this case a point X, where none of the preference functions has all its partial derivatives equal to zero, is obviously unconditionally locally Pareto-optimal when, and only when, the preference vector of one of the two individuals points exactly in the *opposite* direction of the preference vector of the other individual, that is, when, and only when, there exist two effectively positive numbers Θ_1 and Θ_2 such that $\Theta_1\omega_{1i}+\Theta_2\omega_{2i}=0$ for all i.

Example 2. Next consider the case $m=2$, $n=3$. Consider the two preference vectors ω_1 and ω_2 going out from a given point X (none of these preference vectors having both its preference components equal to zero). First suppose that they form an angle different from 0 and from 180° as indicated in Figure 1 (3.6). Extend these two vectors forwards and backwards (towards A and B). We have the proposition:

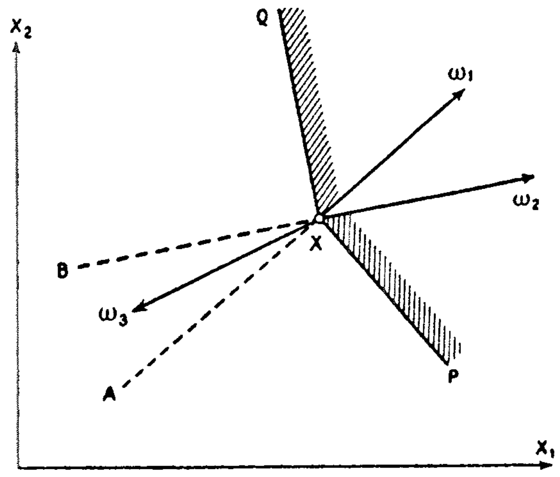

FIG. 1 (3.6)

Proposition (3.7). In order that the point X shall be unconditionally locally Pareto-optimal for the three individuals Nos. 1, 2, 3—when the two preference vectors ω_1 and ω_2 are given (and all the indifference surfaces have the usual convexity property)—it is necessary and sufficient that the third preference vector ω_3 is directed to a point in the *interior* of the opposite of the (infinite) convex corner formed by the vectors ω_1 and ω_2, *i.e.* it is necessary and sufficient that ω_3 is directed to a point in the interior of the (infinite) convex corner AXB in Figure 1 (3.6), not to a point on the boundary lines AX or BX of this corner, and not to a point in any other sector of the plane.

Indeed, draw the straight half-line XP perpendicular to ω_1 and the straight half-line XQ perpendicular to ω_2. The local region around X which is non-deferred by both individuals 1 and 2 and actually preferred by at least one of them is the convex corner PXQ (North-East of X), the boundary lines XP and XQ included. No point (except X itself) has the property of being indifferent to X for both individuals as long as we confine our attention to the vicinity of X. Hence, in order that the point X shall be unconditionally locally Pareto-optimal with respect to all three individuals—when the two preference vectors ω_1 and ω_2 are given and form an angle different from 0 and from 180°—it is necessary that the preference vector ω_3 is such that all points in the convex corner PXQ, boundary lines included, become actually deferred to X for individual No. 3. This happens when, and only when, the preference vector ω_3 is directed to some point in the interior of the (infinite) convex corner AXB, boundary lines not included. This happens when, and only when, the vector $(-\omega_3)$ is directed to some point in the *interior* of the (infinite) convex corner formed by the two preference vectors ω_1 and ω_2, boundary lines *not* included. According to a familiar proposition in vector algebra, this happens when, and only when, the vector $(-\omega_3)$ can be written as a linear form in the two vectors ω_1 and ω_2 with coefficients, each of which is positive, not zero. In other words, if the point X shall be unconditionally locally Pareto-optimal when the two vectors ω_1 and ω_2 are given and form an angle different from 0 and from 180°, it is *necessary* that there exist at least one equation of the form

$$\Theta_1\omega_1 + \Theta_2\omega_2 + \Theta_3\omega_3 = 0 \qquad (3.8)$$

where all the coefficients Θ are positive, not zero. In the case now considered these coefficients are obviously uniquely determined apart from a common factor which is positive, but otherwise arbitrary.

If the two vectors ω_1 and ω_2 point in the *same* direction (*i.e.* form an angle of 0 degrees), the point X will obviously be unconditionally locally Pareto-optimal when, and only when, the vector ω_3 points in the opposite direction of ω_1 and ω_2, *i.e.* when, and only when, $\omega_3 = -\alpha\omega_1$ where α is positive. If ω_1 and ω_2 point in the *same* direction, the condition on ω_3 can also be expressed by saying that we must have $\omega_3 = -\beta\omega_2$ where β is positive. Multiplying these two equations by arbitrary positive constants and adding, we see that in order that the point X shall be unconditionally locally Paretian it is also now necessary that there exist at least one equation of the form (3.8) with all the Θ's positive, not zero. In the present case those Θ's have a two-dimensional arbitrariness.

If the two vectors ω_1 and ω_2 point in *opposite* directions, *i.e.* form an angle of 180°, the point X will be unconditionally locally Pareto-optimal for the two individuals 1 and 2. This applies no matter whether we define the local structure of the preferences as a linear structure expressed through the single datum consisting of the preference vectors, or we define it more exactly in its curved form depending also on the higher order derivatives (with the usual convexity assumption). The difference between the two viewpoints is only of avail if we attribute importance to a refined distinction between limiting cases. In the formulation (3.2) no assumption about first order approximation—*i.e.* linearity—is implied, but in (3.3)—where only the first order derivatives are involved—such an approximation is involved. In a point X, where the preference vectors ω_1 and ω_2 of two individuals point in opposite directions, the situation is as follows: when due account is taken of the curvature of the indifference lines (with the usual convexity assumption), any point in the vicinity of X will make at least one of the individuals worse off than he is in X. When first order derivatives only are considered, any point in the vicinity will make at least one of the individuals worse off, *except* points on the straight line through X which is normal to the direction of ω_1 and ω_2. Along this line any point will—in terms of the linear approximation—be indifferent to both individuals. Whichever of the two ways of looking at the situation we use, the point X will in the general sense (compare the definitions (2.4)-(2.5)) be locally Pareto-optimal for the two individuals. If we introduce a third individual, we must take account of the difference between the curvilinear and the linear way of looking at the situation. From the curvilinear point of view the point X will be strongly Pareto-optimal for the two individuals 1 and 2, and consequently also be strongly Pareto-optimal for the three individuals 1, 2, 3 (compare Proposition (3.13) below). From the linear point of view the point X is only Pareto-optimal in the general sense (not in the strong sense) for the two individuals 1 and 2. If in this case the preference vector ω_3 does not fall in the direction line of ω_1 and ω_2, the point X is *not* Pareto-optimal (neither in the general nor in the strong sense) for the three individuals 1, 2, 3. Indeed, if the vectors ω_1 and ω_2 point in opposite directions and the vector ω_3 does not fall in the line defined by ω_1 and ω_2, then some points along the line through X normal to ω_1 and ω_2 will be Pareto-preferred by the three individuals, namely the points on that half of the perpendicular considered which is situated to the same side of X as ω_3. In this case it is impossible to satisfy (3.3) which is the appropriate formula to consider when the linear viewpoint is adopted. Only if ω_3 falls in the same line as ω_1 and ω_2 (no matter whether it is to the side of ω_1 or to that of ω_2), will the point X be unconditionally locally Pareto-optimal for the three individuals. When this happens, we put $\omega_2 = -\alpha\omega_1$ and $\omega_3 = \beta\omega_1$, where α is effectively positive and β either effectively positive or effectively negative. Multiplying these two equations by effectively positive numbers Θ_2 and Θ_3

and adding, we get (3.8) with $\Theta_3 = \alpha\Theta_2 - \beta\Theta_3$. If β is negative, Θ_3 will be effectively positive for any choice of the two positive numbers Θ_2 and Θ_3. If β is positive, we only have to impose the condition $\frac{\Theta_2}{\Theta_3} > \frac{\beta}{\alpha}$. Thus we reach the conclusion that there must exist at least one equation of the form (3.8) where all the coefficients are positive, not zero. This checks with (3.3).

The above difference in interpretation of the point X according to whether we adopt the exact, curvilinear, or the approximate, linear, point of view, only applies to the point X itself, not to some *near-by* point such as, for instance, X' in Figure 2 (3.9) with preference vectors ω_1', ω_2' and ω_3'. This near-by point X' will, as a rule, appear as non-Pareto-optimal for the three individuals 1, 2, 3, no matter whether we adopt the exact, curvilinear, or the approximate, linear, point of view. This illustrates the fact that the difference between the two viewpoints is only of avail when we want to classify in detail limiting cases, and in

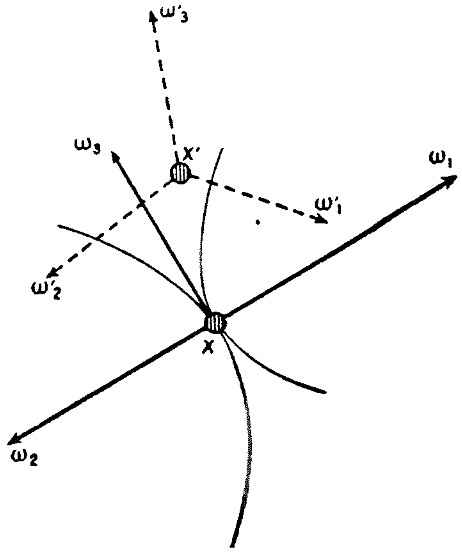

Fig. 2 (3.9)

particular when we want to be careful in stating exactly whether some boundary points belong to a certain region or not. In any case which is simple enough to admit of a graphical illustration with curvilinear indifference lines—such as the examples below—it is easy to check whether boundary points should, or should not, be considered as belonging to the region.

Since (3.8) is symmetric in the three vectors ω_1, ω_2 and ω_3, and the criterion only specifies the *sign* of the Θ's, we will reach the same criterion no matter which set of two vectors we start with.

From the above analysis we conclude that in all cases where the point X in two-dimensional space is unconditionally locally Pareto-optimal with respect to three individuals, there must exist at least one equation of the form (3.8) where each of the three coefficients Θ is positive, not zero.

We can also express the conclusions of Example 2 in the following proposition:

Proposition (3.10). If the point X is unconditionally locally Pareto-optimal, the preference vectors must be distributed around X in such a way that *any* vector around the point X which can be written as a linear form in the preference vectors can be written as such a form with all the coefficients *positive*, not zero.

If we had wanted to, we could have aimed at first proving (3.10) and from (3.10) deduce (3.8). It is easily seen that if (3.10) holds, the existence of an equation of the form (3.8) is assured. Indeed, if ω is any vector which can be written as a linear combination of the preference vectors, the vector $(-\omega)$ has the same property. Write each of these two vectors as a linear combination of the preference vectors with coefficients that are *positive*, not zero. The sum of these two vector equations is a vector equation of the form (3.8) with all the Θ's positive, not zero.

Example 3. $m=2$, n arbitrary>2. If all the n preference vectors are situated in the same line, the point X is unconditionally locally Pareto-optimal when, and only when, there is at least one vector pointing in one direction along this line, and at least one vector pointing in the other direction. In this case obviously any vector that can be written as a linear combination of the preference vectors (*i.e.* any vector directed to a point on the line considered) can be written as such a combination with coefficients that are positive, not zero, hence by the same argument as in (3.10) there must exist an equation of the form (3.3).

If not all the preference vectors are situated in the same line, draw a circle around X and follow the circumference. If the point X shall be locally Pareto-optimal, we must nowhere pass as much as 180° between one preference vector and the next. Indeed, suppose that somewhere we pass exactly 180°. There would then be at least two preference vectors pointing in opposite directions and at least one preference vector pointing in the sector covered by the 180° which we have *not* passed. Any points in this sector and situated on the line which is perpendicular to the line of the two preference vectors that point in opposite directions would be Pareto-preferred by the group consisting of all the individuals. If we passed *more* than 180°, there would even be a whole convex corner of Pareto-preferred points.

Thus we only have to consider the case where we nowhere pass as much as 180° when going from one preference vector to the next. In this case there must exist a set of three preference vectors ω_1, ω_2, ω_3, such that the counterclockwise angle between any two consecutive vectors is less than 180°; as indicated in Figure 3 (3.11). Such a triple of vectors is, for instance, constructed by starting with any preference vector ω_1 and moving counterclockwise until we reach the *last* (one of the last) preference vectors that form an angle of less than 180° with ω_1. Let it be ω_2. From ω_2 we continue the counterclockwise movement until we reach the first (one of the first) vectors ω_3 such that the counterclockwise angle from ω_1 to ω_3 is more than 180°. Since there is no empty angle of as much as 180°, ω_3 must be situated somewhere in the interior of the counterclockwise angle between A_1 (the negative of ω_1) and ω_1. Hence the counterclockwise angle from ω_3 to ω_1 is less than 180°. Any vector around X can be expressed

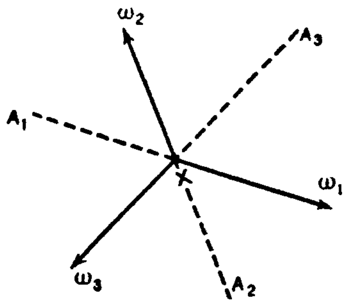

FIG. 3 (3.11)

as a linear form in the three vectors ω_1, ω_2, ω_3, with coefficients all of which are *effectively negative*. Indeed, consider first a vector directed along A_1. It can be expressed in the form $-\alpha\omega_1$, with α effectively positive, but it can also be expressed in the form $\beta_2\omega_2+\beta_3\omega_3$ with β_2 and β_3 effectively positive. Hence it can be expressed in the form $p(-\alpha)\omega_1+q(\beta_2\omega_2+\beta_3\omega_3)$ with $p+q=1$. Choosing q negative (and consequently p positive) the vector in question will be expressed in terms of ω_1, ω_2, ω_3 with coefficients all of which are effectively negative. Similarly for any vector along A_2 or along A_3 or for any vector along ω_1 or ω_2 or ω_3. Next consider a vector in the interior of the convex corner $(A_1\omega_3)$. It can be expressed in the form $\alpha_2\omega_2+\alpha_3\omega_3$ with α_2 and α_3 effectively positive, but it can also be expressed in the form $-(\beta_1\omega_1+\beta_2\omega_2)$ with the β_1 and β_2 positive; hence it can be expressed in the form $p(\alpha_2\omega_2+\alpha_3\omega_3)-q(\beta_1\omega_1+\beta_2\omega_2)$ with $p+q=1$. Choosing p negative (and consequently q positive) the vector will be expressed as a linear form in ω_1, ω_2, ω_3, with coefficients all of which are effectively negative. Similarly for vectors situated in the other sectors of Figure 3 (3.11). Thus, any vector around X can be expressed as a linear form in ω_1, ω_2, ω_3 with coefficients, each of which is effectively negative. Applying this in particular to all the *other* preference vectors ω_4, $\omega_5 \ldots \omega_n$ around X, we get an equation of the form (3.3). Hence (3.1) holds for $m=2$, $n>2$.

These examples will be enough to exhibit the meaning of the general proposition (3.1).

The conditions on the signs of the Θ's are not important from the viewpoint of manipulation of formulae in the *first stages* of the work when we wish to construct the unconditional local Pareto region in an actual case. Indeed, in this case we might begin by *eliminating* the Θ's and thus deriving a set of conditions on the point X which will be well defined when the shape of the indicators Ω_j are given. Since the Θ's enter in a homogeneous and linear fashion, the conditions after elimination of the Θ's will appear as $(m-n+1)$ equations between the derivatives of the indicator, when $m \geqq n$. The dimensionality of the subspace of the points that are Pareto-optimal will consequently—apart from degenerate cases—be

$$(3.12) \qquad m-(m-n+1)=n-1 \quad (\text{when } m \geqq n)$$

In the *further discussion* of the properties of the pointset X which is Paretian, it becomes essential to add the sign conditions. These sign conditions can then be formulated by saying that the expressions obtained for the Θ's—expressions only involving the partial derivatives of the indicators—shall be effectively positive.

* * *

If $m<n$ the unconditional Pareto region will, in general, form a closed subspace whose dimensionality is independent of n, namely equal to m, but whose volume will, in general, be all the *larger* the larger is n. In degenerate cases special situations may emerge, but the general rule is that the volume of the Pareto-optimal region increases as the number of individuals increases. One general proposition that illustrates this is the following.

Proposition (3.13). The region that is *strongly* Pareto-optimal for a group of individuals consists at least of all points that have the property of belonging to any region that is *strongly* Pareto-optimal for a subgroup of these individuals.

This proposition follows immediately from the fact that a point X, which belongs to a region that is strongly Pareto-optimal for a subgroup of the individuals, is by definition such that in any *other* point at least one of the individuals in the subgroup must be worse off. This is saying that at least one of the individuals of the total group must be worse off. In other words, the point X must belong to the region that is strongly Pareto-optimal for the total group.

A graphical illustration of a simple case will suggest the nature of the possibilities that exist. See Figure 4 (3.14). By a global analysis of Pareto conditions, such as given in Figure 4 (3.14), we get a much more elucidating picture of the situation than by a formal application of Lagrange multipliers.

The case illustrated in Figure 4 (3.14) is $m=2$, $n=3$. The two independent variables X_1 and X_2 are measured along the horizontal and vertical axis, respectively. Each individual is represented by a system of contour-lines for his preference function. The maximum points are at (1), (2), (3), respectively.

We begin by considering the individuals 1 and 2 only. Take the point A. The shaded region between A and B (enclosed by the indifference lines for individual 1 and that for individual 2, that pass through A), represents points that the individuals 1 and 2 Pareto-prefer to the initial point A. The endpoint B is not Pareto-preferred, but Pareto-indifferent to A. This, however, is unessential in this connection. What is essential is that there exist points that are Pareto-preferred to A by the individuals 1 and 2. In other words, A is not Pareto-optimal for the two individuals 1 and 2. Take any point C *in the interior* of the shaded area between A and B. A similar argument applies here. That is, the doubly-shaded area *CD*—situated entirely inside of the area *AB*—represents (with the exception of the endpoint *D*) points that the individuals 1 and 2 Pareto-prefer to *C*. Hence *C* is not Pareto-optimal for the two individuals 1 and 2. Continuing in this way, we end up by reaching a point where there is

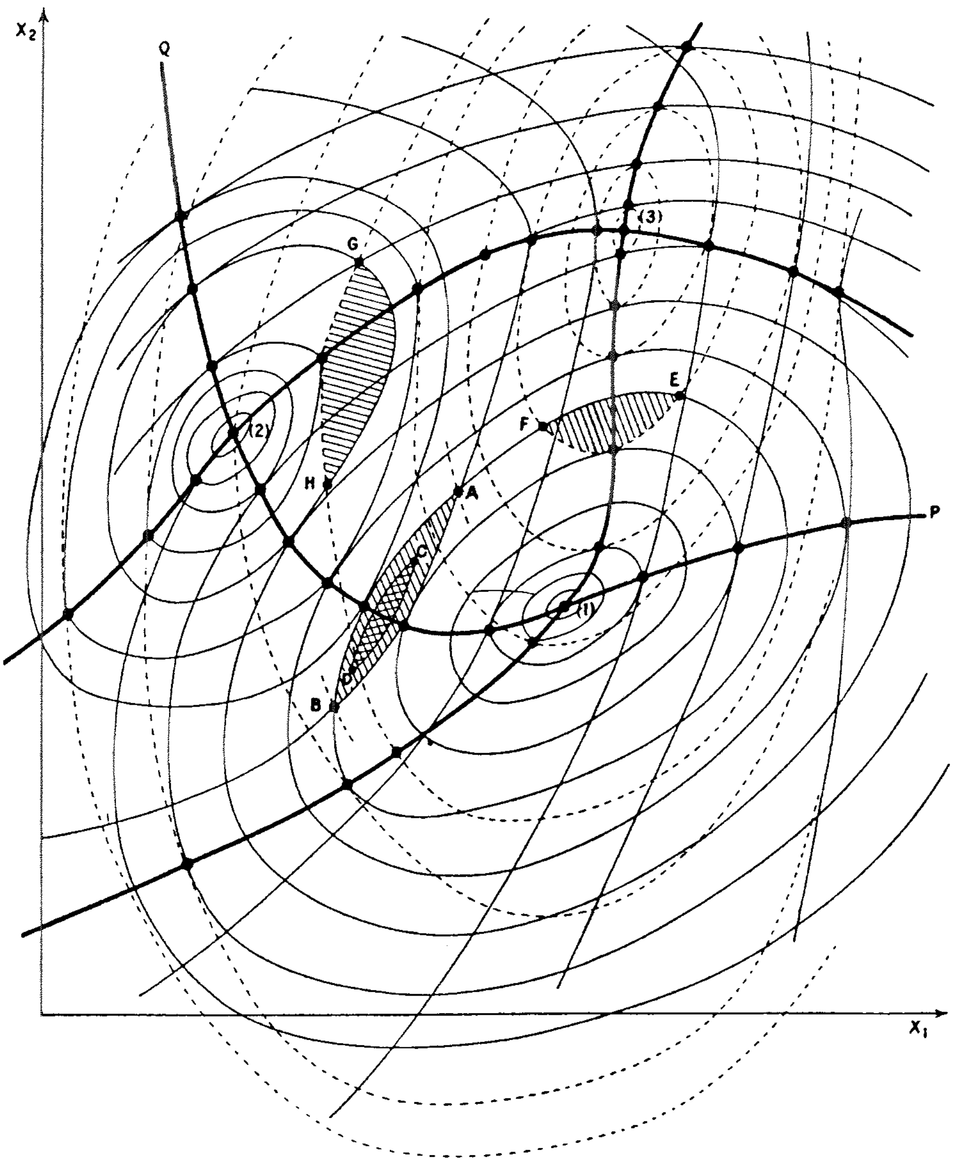

FIG. 4 (3.14)

tangency between a preference line of the individual 1 and a preference line of the individual 2 (assuming continuous variation of the partial derivatives of the preference functions). The locus of points where such a tangency exists is the heavily drawn curve $P(1)(2)Q$. This curve is defined as the set of points satisfying (3.3) for $i=1, 2$ (with $n=2$) *and no condition* put on the Θ's. Introducing the additional condition that the two Θ's involved shall be of the same sign, we get the points of the curve $P(1)(2)Q$ that are situated on the *finite segment* between the points (1) and (2). To the right of the point (1) and to the left of the point (2) the two Θ's are of opposite signs. Hence: any point belonging to the finite segment of the heavily drawn curve between the points (1) and (2) is unconditionally Pareto-optimal for the two individuals 1 and 2. This optimality is to be taken in the general sense if we adopt the viewpoint of a locally linear preference structure, but it is even a strong optimality from the exact curvilinear viewpoint. Furthermore, no other point in space has this property which the points on the finite segment (1)(2) have. In other words, any point on this segment—the endpoints included—is such that there is no other point in space which would be Pareto-preferred to it by the two individuals 1 and 2. And no other point in space has this property.

Similar interpretations are possible for the finite segments of the heavily drawn curves between the points (1) and (3) and between (2) and (3), respectively. The construction of the curve between (1) and (3) is suggested by the shaded area *EF* and that of the curve between (2) and (3) by the shaded area *GH*.

The region of points which are Pareto-optimal simultaneously for all three individuals—when no constraints are imposed on the points—is the two-dimensional area enclosed by the three curves (1) (2), (1) (3) and (2) (3). If we adopt the exact curvilinear viewpoint, we may add: boundary included. Indeed, take, for instance, the point *A*. Divide the total space in two parts: the one outside and the one inside of the closed indifference curve for the individual 3 that passes through *A*. The points that are Pareto-preferred by 1 and 2 are situated entirely in the outside region, namely in the shaded area *AB* (endpoint *B* not included), while those points that are preferred by 3 are situated entirely in the inside region. The only point that is indifferent to both 1 and 2 is *B* and this point is for individual 3 on a lower level of satisfaction than *A*. Since no other point can be Pareto-preferred to *A* by the group of the three individuals than points that are either Pareto-preferred to *A* by the two individuals 1 and 2 or indifferent to *A* for these two individuals, we see that there does not exist any point which is Pareto-preferred to *A* by the three individuals 1, 2, 3. Hence *A* is unconditionally globally Pareto-optimal for these three individuals.

On the other hand, take the point *B*. Again divide the space in two parts: the one inside and the one outside the closed indifference line for 3 that passes through *B*. There are points in the inside region, namely the points in the shaded area *AB*, that are preferred by all three individuals. Hence *B* is not Pareto-optimal.

Considering in this way one by one all the 6 regions: SW. of the line (1) (2), NW. of the point (2), NW. of the line (2) (3), NE. of the point (3), E. of the line (3) (1), S. and SE. of the point (1), we see that each of these regions will be characterized in a certain way with respect to the preferences of the individuals and always such that a point in the region becomes non-Pareto-optimal for the group of the three individuals. Hence: a necessary and sufficient condition for a point to be Pareto-optimal for the group of three individuals 1, 2, 3, is that it is situated in the closed area (1), (2), (3)—boundary included if we adopt the exact curvilinear viewpoint of the indifference lines.

By changing the nature of the indifference lines, for instance by moving the point (3) towards (2), one may produce a situation where the curve segment (1) (3) coincides with the line segment (1) (2). This would reduce the dimensionality of the Pareto-optimal region to unity, *i.e.* the region which is Pareto-optimal for all the three individuals would now become a curve segment. This and similar situations would represent degenerate cases.

4. The Region which is Pareto-optimal under a Set of Conditions expressed by Equations

Consider first the *dimensionality* (the number of degrees of freedom) of the Pareto-optimal region. A full discussion of all possible limiting and degenerate cases would be a tedious and, perhaps, not a very important task. Apart from degenerate cases a simple rule may be formulated. We notice first that in the case where no constraints are imposed on m variables to be preference-evaluated by n individuals, the Pareto-optimal region will—apart from degenerate cases—be of dimensionality (number of degrees of freedom) $min[m, n-1]$, where $min[\quad]$ indicates the smaller of the two numbers written in brackets. From what has been mentioned, this follows by noticing that (3.3) gives m equations. If we eliminate from these m equations, the $(n-1)$ *ratios* between the Θ's, we are left with $m-(n-1)$ equations, hence $(n-1)$ degrees of freedom. This assumes that $(n-1)$ does not exceed m. If it does, the number of degrees of freedom is, of course, limited by m.

In the case where s constraints are imposed, we may re-formulate the problem by expressing everything in terms of $(m-s)$ independent parameters. In the new setting of the problem we can apply the above reasoning. Hence we have the proposition:

Proposition (4.1). The region which is Pareto-optimal under a set of constraints exposed by s independent equations between m variables that are preference-evaluated by n individuals has a dimensionality (number of degrees of freedom) which—apart from degenerate cases—is equal to $min[m-s, n-1]$.

Thus, so far as the dimensionality of the Pareto-optimal region is concerned, the addition of constraints in the form of equations tends to reduce the content of the region. But when it comes to the *volume* of the region within its given dimensionality, the addition of constraints tends to *widen* the region—just as the addition of more individuals tends to widen the Pareto-optimal region (see Section 3). The precise meaning of the above statement is given in Figure 5 (4.2) and Proposition (4.3) and in the examples discussed in connection with the proposition. Before proceeding to this more detailed discussion, I would like to state that in my opinion the fact here considered is of fundamental importance for the whole welfare analysis. It has far-reaching consequences for our interpretation of *what is really meant* when we say that a point is "good" because it is Pareto-optimal. A slipslop way of passing over the basic logical principle here involved is, as I see it, the main cause for so much of the unprecise and unwarranted reason that has been given about the superiority of the régime of free competition—amongst others by Pareto himself and his immediate followers.

Let C and c be two sets of constraints such that c is included in C. That is to say, any point X that satisfies C must necessarily satisfy c, but the inverse is not true. For shortness we may call C the strong and c the weak condition. The general argument leading to Proposition (4.3) applies to any type of condition whether expressed by equations or inequalities, but the subsequent example is concerned with the case where the conditions are expressed by equations.

Since C is the stronger of the two sets of conditions, the points X that satisfy C form a region $X[C]$ that is entirely included in the region $X[c]$ of the points that satisfy c. This is illustrated in Figure 5 (4.2). For simplicity the region

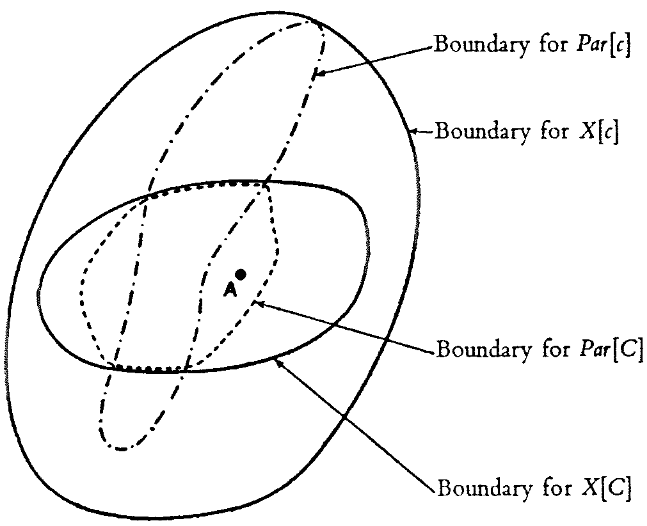

Fig. 5 (4.2)

$X[c]$ is here drawn as one continuous region, but the argument is general and applies no matter what the structure of the point set $X[c]$ is. Similar remark on $X[C]$.

The region $Par[c]$ must, of course, be situated in $X[c]$ and the region $Par[C]$ in $X[C]$. The region $Par[c]$ may, or may not, have points in $X[C]$. If it does have points in $X[C]$, these points must belong to $Par[C]$. On the other hand, there are, in general, points that are Pareto-optimal under C without being Pareto-optimal under c, as, for instance, the point A in Figure 5 (4.2). That is to say, if we confine our attention to points in $X[C]$, we find that the points that are Pareto-optimal under the stronger condition C form *a more inclusive* set than the points that are Pareto-optimal under the *weaker* condition c. This idea that a stronger condition leads to a more inclusive set is a bit unfamiliar and it takes a little attention to incorporate it correctly in a reasoning about "good" points and "good" régimes.

We may formulate the fact referred to in the following proposition:

Proposition (4.3). *Par*[*c*] . *X*[*C*] is included in *Par*[*C*] but not inversely, *i.e.* those points that are strongly Pareto-optimal under the weaker condition *c* and lie within the region of points that satisfy the stronger condition *C*, must be included in the points that are strongly Pareto-optimal under the stronger condition *C*. But the inverse is not true.

To give a formal proof of (4.3) let *X* be any point in *X*[*C*] and consider the following three cases—which form an exhaustive list of all possibilities:

1. There exists a point in *X*[*C*]—and consequently in *X*[*c*]—which is Pareto-preferred to *X*.	Then *X* is *not Par*[*C*] and *not Par*[*c*].
2. There exists no point in *X*[*C*], but a point in *X*[*c*] which is Pareto-preferred to *X*.	Then *X is Par*[*C*] but *not Par*[*c*].
3. There exists no point in *X*[*c*]—and consequently no point in *X*[*C*]—which is Pareto-preferred to *X*.	Then *X is Par*[*C*] and *is Par*[*c*].

In other words, for any given point *X* which belongs to *X*[*C*] it is true that *if* it is *Par*[*c*], it must be *Par*[*C*], but it may be *Par*[*C*] without being *Par*[*c*]. In this formulation *Par* stands for "strongly Pareto-optimal".

As a special case *all* the points that are Pareto-optimal under the weak condition *c* may fall in the region of points that satisfy the strong condition *C*, but that is of no particular importance. In any case will Proposition (4.3) hold.

As an example consider the case where *c* is a production constraint given, say, by the limitation of natural resources and the technical situation, in short a set of what might be called *obligatory* conditions, while *C* is a more inclusive set of conditions, consisting of the obligatory conditions and a set of *facultative* conditions expressing, say, conditions to the effect that the total output shall be distributed amongst the individuals in certain proportions. In short, we may term these conditions the production and distribution constraints, respectively.

If we are to use the concept of Pareto-optimality on this set-up, we must keep in mind that Pareto-optimality under the production constraint is not the same thing as Pareto-optimality under the production and distribution constraints. For instance, if we want to say something about the distribution system from the viewpoint of Pareto-optimality, the pertinent question is whether some point *X* which shows the prescribed distribution ratios, is Pareto-optimal under a constraint which consists of the *production constraint only*. If a choice is to be made between different systems of distribution, we must, of course, not evaluate a given point in terms of an optimality definition that imposes one particular distribution as a necessity. In other words, if we ascertain that some point *X*, which has been reached in some way or another, has the property of being

Pareto-optimal under the inclusive set of constraints consisting of the production and distribution constraints, this does not say anything about whether this point is "good" in the sense of giving an "effective" utilization of the resources. If we only know that our point X is Pareto-optimal under the inclusive condition C, it may, perhaps, fall in A of Figure 5 (4.2) and thus not be Pareto-optimal under the production constraints. In all cases we must be very careful to impose only the *obligatory* conditions and not any condition pertaining to the alternatives between which we want to select.

The nature of the pitfall might perhaps be illustrated by an analogy in partial derivatives.[6] Suppose we have a function $f(x, y, z)$ of three variables and we want to find the value of its partial derivative $f_x'(x, y, z)=\frac{\partial f(x, y, z)}{\partial x}$ (under constant y and z) in a point (x_0, y_0, z_0) that satisfies $z_0=g(x_0, y_0)$ where g is a given function of two variables. The value sought is

$$f_x'(x_0, y_0, z_0)$$

and *not* the value that would be obtained by imposing the relation $z=g(x, y)$ *under the derivation process.* This latter value would be

$$f_x'(x_0, y_0, z_0)+f_z'(x_0, y_0, z_0)\, g_x'(x_0, y_0)$$

One way to approach the problem correctly would be first to seek all points that are Pareto-optimal under the production constraints only, and then within this set to pick out those points that show some desired distribution ratios. It may then, of course, happen that it is *impossible* to realize the desired distribution ratios by confining the point to lying in the region that is Pareto-optimal under the production constraint. If so, this simply shows that the goals put up are inconsistent.

A simple example will illustrate the situation.

Example (4.5)

Consider two goods Nos. $i=1, 2$ and two individuals Nos. $j=1, 2$. Let each good be produced in an independent production process with only one factor of production which we may call labour. We use the following notation:

N = total quantity of labour.
N_1 and N_2 = quantities of labour used in the two industries.
X_1 and X_2 = quantities produced in the two industries.
X_{1j} and X_{2j} = volumes of these goods received by the two individuals ($j=1, 2$).
$\Omega_j=\Omega_j(X_{1j}, X_{2j})$ = preference function of individual j ($j=1, 2$).
$X_i=f_i(N_i)$ = production function of industry i ($i=1, 2$).

[6] *Note added in March* 1959. The essence of the matter is that the stronger the conditions, the easier will it be to find points which do *not* satisfy these conditions. My friends and colleagues Professors Trygve Haavelmo and B. Thalberg have produced another example to illustrate the nature of the above pitfall:

Given a function $F(x,y,p,q)$ of four variables. Consider three subsets in space (x,y).

(1) Maximize F over x and y under fixed p and q. To a given p,q region there will correspond a well-defined (x,y) region. Let this be subset S_1.

By definition we have

(4.6) $$N_1+N_2=N$$

(4.7) $$X_i=X_{i1}+X_{i2} \quad (i=1,\ 2)$$

The marginal preferences and the marginal productivities will be denoted

(4.8) $$\omega_{ji}=\omega_{ji}(X_{1j},\ X_{2j})=\frac{\partial\Omega_j(X_{1j},\ X_{2j})}{\partial X_{ij}} \quad (i=1,\ 2;\ j=1,\ 2)$$

(4.9) $$f_i'=\frac{df_i(N_i)}{dN_i} \quad (i=1,\ 2)$$

Marginal labour cost in industry No. i is $\frac{1}{f_i'}$

The inverse production functions will be denoted

(4.10) $$N_i=f_i^{-1}(X_i) \quad (i=1,\ 2)$$

Reducing the model to its lowest terms by eliminating all unnecessary variables, we can say that we have a problem in the four variables

(4.11) $$X_{11},\quad X_{12},\quad X_{21},\quad X_{22}$$

on which is imposed the single condition

(4.12) $$f_1^{-1}(X_{11}+X_{12})+f_2^{-1}(X_{21}+X_{22})=N$$

where N is given. The shapes of the two functions f_1^{-1} and f_2^{-1} are also assumed to be given.

We shall first determine the region that is Pareto-optimal for the two individuals, one constraint in the form of an equation being imposed on the four variables ($m=4$, $n=2$, $s=1$).

We take X_{11}, X_{12} and X_{21} as free variables and consider X_{22} as a function of the three free variables, the relationship being defined through (4.12).

Letting now ∂ indicate partial derivation in the model of three degrees of freedom, we can formulate the necessary and sufficient condition for a Pareto-optimal point as

(4.13) $$\frac{\dfrac{\partial\Omega_1}{\partial X_{11}}}{\dfrac{\partial\Omega_2}{\partial X_{11}}}=\frac{\dfrac{\partial\Omega_1}{\partial X_{12}}}{\dfrac{\partial\Omega_2}{\partial X_{12}}}=\frac{\dfrac{\partial\Omega_1}{\partial X_{21}}}{\dfrac{\partial\Omega_2}{\partial X_{21}}}=\text{negative (compare (3.3))}$$

This is a set of conditions formed by two equations and a sign condition. In other words, the dimensionality of the solution will be one, and out of this one-dimensional pointset there will, in general, be a finite segment that answers the requirement. This checks with $min[m-s,\ n-1]=min[3,\ 1]=1$.

(2) Add the condition $g(x,y)=0$. This gives S_2. It is a subset of S_1.

(3) Maximize F over x and y under the constraint $g(x,y)=0$, the values of p and q being given. To the same p,q region as in case (1), there will correspond a well-defined (x,y) region. Let it be S_3.

It is *not* true that S_3 will in general be identical with S_2. In general it will be larger than S_2.

By implicit derivation of (4.12) we find

$$\frac{\partial X_{22}}{\partial X_{11}} = \frac{\partial X_{22}}{\partial X_{12}} = -\frac{f_2'}{f_1'} \qquad \frac{\partial X_{22}}{\partial X_{21}} = -1 \tag{4.14}$$

and hence

$$\begin{array}{ccc} \dfrac{\partial \Omega_1}{\partial X_{11}} = \omega_{11} & \dfrac{\partial \Omega_1}{\partial X_{12}} = 0 & \dfrac{\partial \Omega_1}{\partial X_{21}} = \omega_{21} \\ \dfrac{\partial \Omega_2}{\partial X_{11}} = -\dfrac{\omega_{22} f_2'}{f_1'} & \dfrac{\partial \Omega_2}{\partial X_{12}} = \omega_{21} - \dfrac{\omega_{22} f_2'}{f_1'} & \dfrac{\partial \Omega_2}{\partial X_{21}} = -\omega_{22} \end{array} \tag{4.15}$$

The conditions (4.13) can therefore be written

$$\frac{\omega_{11} f_1'}{\omega_{22} f_2'} = \frac{0}{\omega_{22} f_2' - \omega_{21} f_1'} = \frac{\omega_{12}}{\omega_{22}} = \text{positive} \tag{4.16}$$

Rearranging the terms and writing out all the variables, we get (the central denominator must be zero)

$$\frac{\omega_{11}(X_{11}, X_{21})}{\omega_{12}(X_{11}, X_{21})} = \frac{\omega_{21}(X_{12}, X_{22})}{\omega_{22}(X_{12}, X_{22})} = \frac{f_2'[f_2^{-1}(X_{21}+X_{22})]}{f_1'[f_1^{-1}(X_{11}+X_{12})]} \tag{4.17}$$

ω_{12} and ω_{22} of the same sign (in the regular case both positive).

The result can be summarized in the formula

$$Par[12] = X[17, 12] \tag{4.18}$$

In words: the region (of the four-dimensional space X_{11}, X_{12}, X_{21}, X_{22}) which is Pareto-optimal for the two individuals under the constraint (4.12) (with given N), consists of the points where (4.17) and (4.12) are satisfied.

Since the region which is Pareto-optimal under the production constraint has one degree of freedom, there is still room for adding a condition without giving up the requirements that the point shall be Pareto-optimal under the production constraint. Suppose that we try to dispose of the remaining degree of freedom by requesting that a certain *distribution ratio* shall prevail between the two individuals. This way of disposing of the remaining *degree of freedom is purely conventional* and not derived by any sort of consideration on Pareto-optimality.

In order to be able to compare the part that one of the individuals gets of the total product with the part that the other individual gets, we must in some way or another introduce a principle for comparing quantities of the two goods. Suppose we do it by using labour costs as value coefficients. *This too is a purely conventional criterion not derived by any consideration on Pareto-optimality.*

If labour costs are used as value coefficients, the relative part which the individual 2 gets of the total product will be

$$(4.19)\qquad \frac{\dfrac{X_{12}}{f_1'}+\dfrac{X_{22}}{f_2'}}{\dfrac{X_{11}+X_{12}}{f_1'}+\dfrac{X_{21}+X_{22}}{f_2'}}=L(X_{11}, X_{12}, X_{21}, X_{22})$$

If we insert in the left member of (4.19) the expressions for f_1' and f_2' in terms of the four variables X_{11}, X_{12}, X_{21}, X_{22} (the same expressions as those written in the right member of (4.17)), we get a certain function L of the point in four-dimensional space. The shape of this function L is uniquely determined through the shape of the production functions. Hence, the requirement that the distribution ratio shall have a preassigned value λ is expressed by

$$(4.20)\qquad L(X_{11}, X_{12}, X_{21}, X_{22})=\lambda$$

This is simply a one-dimensional condition on the point and may consequently be considered as a means of taking out the one degree of freedom that remained after imposing the condition that the point shall be Pareto-optimal under the production constraint.

Is it possible to assign an arbitrary value to λ ? Certainly not if the point shall be Pareto-optimal in the above sense. Since the value of the function $L(X_{11}, X_{12}, X_{21}, X_{22})$ in any point of the four-dimensional space is given directly from the shape of the two production functions and from nothing else, we may follow the variation of this function as the point moves over the one-dimensional pointset *Par*[12] defined by (4.18). For simplicity assume continuous variation and let L_{max} and L_{min} be the maximum and minimum respectively, assumed by the function L under this variation. The two numbers L_{max} and L_{min} depend on the shapes of the two production functions and the two preference functions and on nothing else. Obviously, if we require that the distribution coefficient shall have a value between L_{max} and L_{min}, the problem of finding a point that gives this ratio and at the same time is Pareto-optimal *under the production constraint*, has a solution (at least one). But if we choose a distribution ratio that falls *outside* of the interval (L_{max}, L_{min}), no point can be found that is Pareto-optimal under the production constraint and has the distribution ratio chosen.

This simplified example illustrates the principle according to which a *selection problem—i.e.* a problem of *how to formulate the goal* for economic policy—may be handled. The goal in the example has been formulated in such a way as to conform to Pareto-optimality *under those constraints that it is not possible to change simply by a human decision.* And the remaining degrees of freedom have been handled by a *postulate* in the form of a social value judgement. This social value judgement is something which the economist as scientist and technician simply has to take as a datum. But *all the rest* is within his sphere of competence. It would seem that even with the above limitation of the economist's field, there is more than enough for him to do.

So far I have not discussed the problem of *the régime*, that is the problem of how actually to organize production and trade in order to make the equilibrium point fix itself in the desired position. Before proceeding to a discussion of this I shall work through the above example once more and now determine explicitly the region that is Pareto-optimal under a set of conditions that consist simultaneously of the production constraint *and* the distribution constraints. The resulting region will prove to be very different from the one that is Pareto-optimal under the production constraint only.

We now have to consider a model with two degrees of freedom, namely the same four variables X_{11}, X_{12}, X_{21}, X_{22} as before, but now with two constraints (4.12) and (4.20) with N and λ given.

We take X_{11} and X_{21} as free variables and determine first the elements of the Jacobian $\frac{\partial(X_{12}, X_{22})}{\partial(X_{11}, X_{21})}$. Letting ∂ indicate partial derivation within the two degrees of freedom now considered, we get

$$\text{(4.21)} \qquad \begin{aligned} \frac{\partial f_1^{-1}}{\partial X_{11}} &= \frac{df_1^{-1}}{dX_1} \cdot \frac{\partial(X_{11}+X_{12})}{\partial X_{11}} = \frac{1}{f_1'}\left(1+\frac{\partial X_{12}}{\partial X_{11}}\right) \\ \frac{\partial f_2^{-1}}{\partial X_{11}} &= \frac{df_2^{-1}}{dX_2} \cdot \frac{\partial(X_{21}+X_{22})}{\partial X_{11}} = \frac{1}{f_2'}\frac{\partial X_{22}}{\partial X_{11}} \end{aligned}$$

and hence by implicit derivation of (4.12)

$$\text{(4.22)} \qquad \begin{aligned} \frac{\partial X_{12}}{\partial X_{11}} + \Phi\frac{\partial X_{22}}{\partial X_{11}} &= -1 \\ \frac{\partial X_{12}}{\partial X_{21}} + \Phi\frac{\partial X_{22}}{\partial X_{21}} &= -\Phi \end{aligned}$$

where

$$\text{(4.23)} \qquad \Phi = \frac{f_1'}{f_2'}$$

Since

$$\text{(4.24)} \qquad \begin{aligned} \frac{\partial}{\partial X_{11}}\left[\frac{1}{f_1'}\right] &= -\frac{1}{(f_1')^2}\frac{\partial f_1'}{\partial X_{11}} = -\frac{1}{(f_1')^2}\frac{df_1'}{dN_1}\frac{dN_1}{dX_1}\frac{\partial(X_{11}+X_{12})}{\partial X_{11}} = -\frac{f_1''}{(f_1')^3}\left(1+\frac{\partial X_{12}}{\partial X_{11}}\right) \\ \frac{\partial}{\partial X_{11}}\left[\frac{1}{f_2'}\right] &= -\frac{1}{(f_2')^2}\frac{\partial f_2'}{\partial X_{11}} = -\frac{1}{(f_2')^2}\frac{df_2'}{dN_2}\frac{dN_2}{dX_2}\frac{\partial(X_{21}+X_{22})}{\partial X_{11}} = -\frac{f_2''}{(f_2')^3}\frac{\partial X_{22}}{\partial X_{11}} \end{aligned}$$

we get by implicit derivation of (4.20)

$$\delta_1\frac{\partial X_{12}}{\partial X_{11}} + \Phi\delta_2\frac{\partial X_{22}}{\partial X_{11}} = 1-\delta_1$$

(4.25)
$$\delta_1 \frac{\partial X_{12}}{\partial X_{21}} + \Phi\delta_2 \frac{\partial X_{22}}{\partial X_{21}} = \Phi(1-\delta_2)$$

where

(4.26) $$\delta_i = \delta_i(\lambda;\ X_{i1},\ X_{i2}) = 1-\lambda+\frac{f_i''}{(f_i')^2}\,[\lambda X_{i1}-(1-\lambda)X_{i2}] \quad (i=1,\ 2)$$

Hence

(4.27)
$$\frac{\partial X_{12}}{\partial X_{11}} = -(1-\Delta) \qquad \frac{\partial X_{22}}{\partial X_{11}} = -\frac{\Delta}{\Phi}$$
$$\frac{\partial X_{12}}{\partial X_{21}} = \Phi\Delta \qquad \frac{\partial X_{22}}{\partial X_{21}} = -(1+\Delta)$$

where

(4.28) $$\Delta = \Delta(\lambda;\ X_{11},\ X_{12},\ X_{21},\ X_{22}) = \frac{1}{\delta_1-\delta_2}$$

In the two degrees of freedom model now considered we have

(4.29)
$$\frac{\partial \Omega_1}{\partial X_{11}} = \omega_{11} \qquad \frac{\partial \Omega_1}{\partial X_{21}} = \omega_{12}$$
$$\frac{\partial \Omega_2}{\partial X_{11}} = \omega_{21}\frac{\partial X_{12}}{\partial X_{11}} + \omega_{22}\frac{\partial X_{22}}{\partial X_{11}} \qquad \frac{\partial \Omega_2}{\partial X_{21}} = \omega_{21}\frac{\partial X_{12}}{\partial X_{21}} + \omega_{22}\frac{\partial X_{22}}{\partial X_{21}}$$

Furthermore, the condition for Pareto-optimality is now

(4.30) $$\frac{\dfrac{\partial \Omega_1}{\partial X_{11}}}{\dfrac{\partial \Omega_2}{\partial X_{11}}} = \frac{\dfrac{\partial \Omega_1}{\partial X_{21}}}{\dfrac{\partial \Omega_2}{\partial X_{21}}} = \text{negative}$$

Inserting in (4.29) from (4.27), and further inserting the expressions thus obtained in (4.30), we finally get

(4.31) $$\frac{\omega_{11}}{\omega_{12}} = \frac{\omega_{21}(1-\Delta)+\omega_{22}\cdot\dfrac{\Delta}{\Phi}}{\omega_{22}(1+\Delta)-\omega_{21}\Phi\Delta}$$

ω_{11} and $\left[\omega_{21}(1-\Delta)+\omega_{22}\frac{\Delta}{\Phi}\right]$ of the same sign (in the regular case both positive).

The formula (4.31) expresses a constraint consisting of one equation and a sign condition.

The result may be summarized in the following proposition:

Proposition (4.32)

$$Par[12,\ 20] = X[31,\ 12,\ 20]$$

In words: the region (of the four-dimensional space X_{11}, X_{12}, X_{21}, X_{22}) that is Pareto-optimal simultaneously under the production constraint (4.12)—

with given N—and the distribution constraint (4.20)—with given λ and the function L being defined by (4.19)—is the one-dimensional segment of points satisfying (4.31), (4.12) and (4.20).

Already a glance at the conditions (4.31) and (4.17) shows that we have here two fundamentally different kinds of regions. For instance, (4.31) depends on the second order derivatives of the production function while (4.17) depends only on the first order derivatives of these functions.

Any point that satisfies (4.17)—the Pareto condition under the weak constraint—must satisfy (4.31). Indeed, noticing that the right member of (4.17) is $\frac{1}{\Phi}$, we see that if (4.17) holds, so that $\omega_{22}=\Phi\omega_{21}$, (4.31) reduces to the first equation in (4.17). This applies no matter what magnitude has been fixed for λ. That is to say, any point that is *Par*[12] must satisfy *that* part of the conditions for *Par*[12, 20] which is expressed by (4.31). In other words, if we take any point in *Par*[12] and verify that it gives a certain value λ of the distribution ratio defined by (4.19), this is sufficient to make sure that the point in question is also *Par*[12, 20] under *this particular* value of λ.

Or again we may say: let λ be any given value of the distribution ratio (between the maximum and minimum assumed by the function L over *Par*[12]). Consider any point in *Par*[12] that gives this value of the distribution ratio. This point must also be *Par*[12, 20] with this value of λ. This checks with the general structure exhibited in Figure 5 (4.2).

On the other hand, let λ be any given number and consider the set of points which satisfy (4.31), (4.12) and (4.20) with this particular value of λ. We have no guarantee that this point will satisfy (4.17). In other words, within the set of points that give a certain distribution ratio, any point that is Pareto-optimal under the production constraint will be so also under the more inclusive set of constraints which consist of the production and distribution constraint with the given distribution ratio. But it is not true that any point which is Pareto-optimal under this more inclusive set of constraints is also Pareto-optimal under the production constraint. This checks with Figure 5 (4.2), where we may have a point such as A. No importance should be attached to the dimensionality of the regions in Figure 5 (4.2), they do not correspond to the dimensionalities in the example under consideration. Only the inclusion or non-inclusion of one region in another is important in Figure 5 (4.2).

In the discussion on Pareto-optimum, one way of reasoning which is frequently used is this: suppose that the individuals act on the assumption that prices are constant and let us determine Pareto-optimality under this condition. We can do it by introducing a system of prices and be careful to specify that they constitute *any* system of relative prices, the precise magnitude of them being determined *afterwards* through the equilibrium process itself.

As I see it, this specification does not help in the least to correct the fundamental logical fallacy involved. Any analysis of this sort means that the assumption of some sort of free competition is smuggled into the analysis already from the outset so that the conclusion can only be formulated by saying that "free competition is the best possible of all régimes in the class of régimes which consists of the régime of free competition". In other words: in the course of the analysis Pareto-optimality is determined under the *stronger* condition, while it should have been determined under the weaker condition. The example under consideration can illustrate what I have in mind.

Suppose that we do not begin by attributing any *specific* value to the distribution ratio λ, but let it have *any* value. Afterwards we may dispose of this value in some way which we find convenient, for instance we may just put it equal to the value which the function L assumes in the final point to which we may be led possibly by some sort of régime.

Let us determine the points that are Pareto-optimal under the set of conditions which consist first of the production constraint where the total labour input N is some *given* magnitude not to be determined by the equilibrium point, and second, *some* value λ of the distribution ratio, this value, however, to be determined afterwards as the value of the function L in the point in which we arrive, no matter what it might be. This formulation of the problem resembles, I think, in all logical essentials a type of reasoning frequently encountered in the discussion on Pareto-optimality.

As our example will show, the above proviso that the magnitude of the distribution ratio is to be taken as it emerges afterwards, does *not* correct the fundamental fallacy. Indeed, the solution to the present problem is obtained simply by inserting in (4.31) instead of λ the explicit expression for the function L in terms of its four variables, and *drop* the condition (4.20) that the distribution ratio shall have a given magnitude λ. This will lead to a two-dimensional point-set, namely the region in the four-dimensional space $(X_{11}, X_{12}, X_{21}, X_{22})$ which satisfies (4.12) and the single equation obtained from (4.31) by inserting $L(X_{11}, X_{12}, X_{21}, X_{22})$ instead of λ.[7]

Let $\delta_1(L)$ and $\delta_2(L)$ be the functions of $X_{11}, X_{12}, X_{21}, X_{22}$ obtained when we replace λ in (4.26) by the left member of (4.19). We get

$$\text{(4.33)} \quad \begin{aligned} \delta_1(L) &= (1-\lambda) + \frac{\Phi \,.\, (X_{11}X_{22} - X_{12}X_{21})}{(X_{11}+X_{12}) + \Phi \,.\, (X_{21}+X_{22})} \cdot \frac{f_1''}{(f_1')^2} \\ \delta_2(L) &= (1-\lambda) - \frac{(X_{11}X_{22} - X_{21}X_{12})}{(X_{11}+X_{12}) + \Phi \,.\, (X_{21}+X_{22})} \cdot \frac{f_2''}{(f_2')^2} \end{aligned}$$

Inserting these expressions for δ_1 and δ_2 in the right member of (4.28), we get Δ as a function of the point $(X_{11}, X_{12}, X_{21}, X_{22})$, hence the whole right member of (4.31) becomes a function of the point.

[7] The fact that we now have a two-dimensional Pareto-optimal region is not in contradiction with the formula *min*$[m-s, n-1]$ because this formula is to be applied *before* the optimization takes place. Any additional degrees of freedom that come in afterwards by letting certain previously fixed parameters become variable, must be added.

It is obvious that the pointset determined by the production constraint (4.12) and the constraint expressed by the new form of (4.31) will, in general, not belong to the pointset determined by the production constraint and (4.17), while on the other side any point in the latter set must belong to the first. In other words, if we ascertain that a point is Pareto-optimal in the special meaning now considered, *that does not tell us anything about whether it is Pareto-optimal under the production constraint or not.* Consequently it gives no criterion for whether the point represents an "effective" utilization of resources or not.

The question may be raised whether it might not be possible to escape the above antagonism between Pareto-optimality under different kinds of constraints by making special sorts of assumptions about the *shape of the production functions.*

In particular, it might be interesting to look into a special case which so frequently produces a particular sort of situation, namely the case where the production functions are homogeneous of the first order. In the present example this simply means that the marginal productivities f_1' and f_2' are *constant, i.e.* that f_1'' and f_2'' are zero. It will turn out that this particular case does not introduce any modification in the conclusions reached about the antagonism between the two types of Pareto-optimality.

The solution in the case now considered is obtained from (4.31) simply by letting Δ tend towards infinity. In this case the condition for Pareto-optimality under the production and distribution constraint reduces to

$$(\omega_{11}\Phi - \omega_{12})\,(\omega_{21}\Phi - \omega_{22}) = 0 \tag{4.34}$$

If the first parenthesis in (4.34) is different from zero, the second must be equal to zero, and *vice versa.* In other words, the optimality condition in the present case can be expressed by saying that *at least* one of the two expressions $(\omega_{11}\Phi - \omega_{12})$ and $(\omega_{21}\Phi - \omega_{22})$ must be equal to zero. When the condition is expressed in this form, we get a simple and very illustrative comparison with the condition (4.17) for Pareto-optimality under the production constraint only. The condition (4.17) states that *both* expressions $(\omega_{11}\Phi - \omega_{12})$ and $(\omega_{21}\Phi - \omega_{22})$ must be equal to zero. Thus, also in the case of homogeneous production functions of the first degree, it is true that a Pareto-optimality determined under the production *and* distribution constraint does *not* entail a Pareto-optimality determined under the production constraint only, while the inverse is true (for all points having the desired distribution ratio). Pareto-optimality under a stronger condition leads to a *more inclusive* pointset than Pareto-optimality under a weaker condition.

5. The Region which is Pareto-optimal under a Set of Conditions expressed partly or wholly by Inequalities. The Pareto-pressure

The examples so far have only been concerned with conditions expressed in the form of equations. We now turn to the more complex case where some of, or all, the conditions are expressed by inequalities. By introducing one more variable for each inequality, any system involving inequalities can, of course, be reduced to a system involving only equations and *sign conditions* on one or more variables. (This re-formulation of the problem is, for instance, sometimes found useful in linear programming.)

We will illustrate the case now considered by studying rather carefully a simplified example.

Let the quantities of two goods X_1 and X_2—say "public health service" and "public radio entertainment"—be evaluated by two individuals Nos. 1 and 2. Let the indifference lines be as indicated in Figure 6 (5.1). The figure indicates a

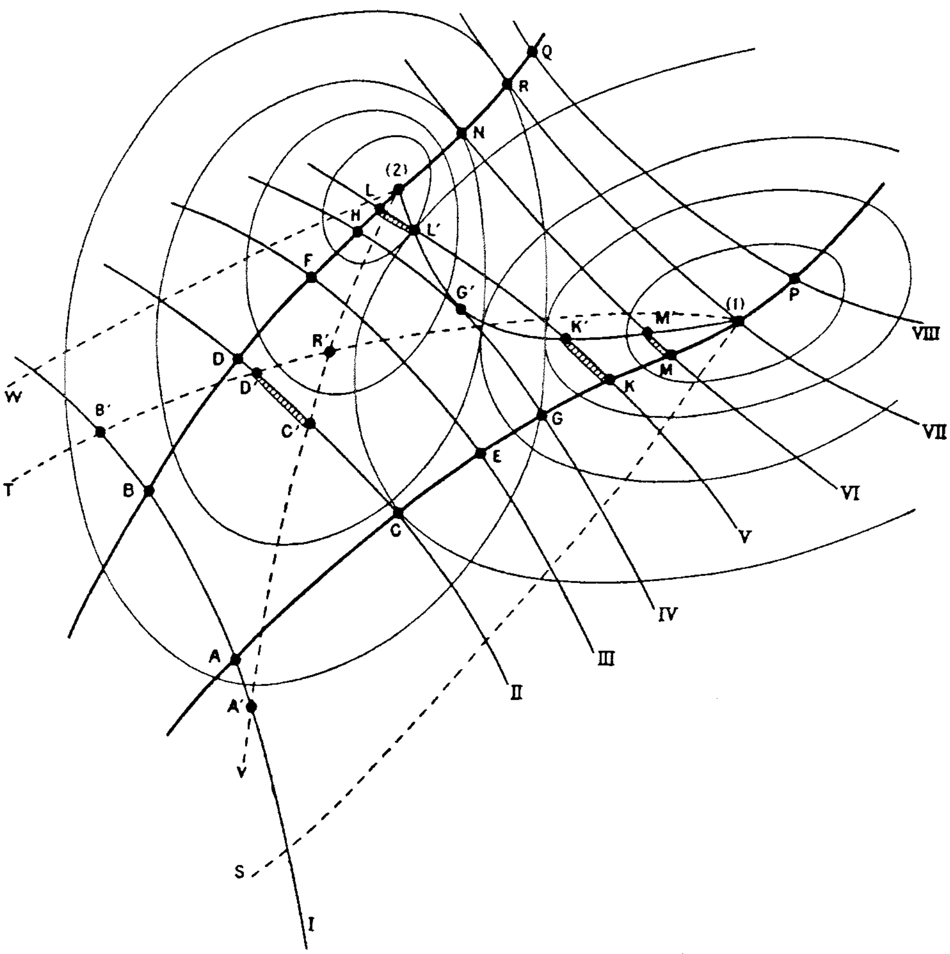

Fig. 6 (5.1)

situation where the individual No. 1 attributes greater weight to health service and smaller weight to radio entertainment than No. 2 does. Let $\Omega_1(X_1, X_2)$ and $\Omega_2(X_1, X_2)$ be the indicators of choice, and suppose there exists between X_1 and X_2 a production constraint such that for given magnitudes of some common factor of production—say "labour"—one gets the isoquants Nos. I, II, ... VIII. Let the curve *ACE* ... *P* be determined by the tangency of isoquants with the indifference curves of individual No. 1. Similarly we get the curve *BDF* ... Q for individual No. 2. Let (1)*S* and (1)*T* be the locus of points where the indifference lines for No. 1 are horizontal and vertical, respectively. The curve (2)*V* and (2)*W* has a similar meaning for individual No. 2. Thus: in the region *S*(1)*T* the preference direction for individual No. 1 has both its components positive, in the region *V*(2)*W* the preference direction for individual No. 2 has both its components positive, and in the region *TR′V* all four preference components are positive. The finite curve segment (1)*M′K′G′L′*(2) is—as follows from the reasoning in connection with Figure 4 (3.14)—the region that is Pareto-optimal under no constraint.

We start by considering the case where it is *prescribed* that exactly so much of the common factor of production shall be used that the (X_1, X_2) point falls somewhere on the isoquant marked I—*i.e.* somewhere on the curve *A′ABB′*. The finite segment *AB* of this isoquant represents the points that are Pareto-optimal for the group of the two individuals 1 and 2 under the constraint that exactly so much of the common factor of production shall be used as is indicated by the isoquant I. Indeed any point on this segment—and no point outside of it—has the property that it is impossible to depart from it without making one of the individuals worse off. Along this segment the interests of the two individuals are directly opposed. No. 1 wants to go in the direction of *A* and No. 2 in the direction of *B*. The segment *AB* is situated entirely in the region *TR′V* where all four preference components are positive.

If the quantity used of the common factor is increased so much that the isoquant becomes II, the region which is Pareto-optimal under the production constraint becomes the finite segment *CD*. Of this segment only the part *C′D′* is situated in the region where all the four preference components are positive. On the isoquant III the Pareto-optimal segment is *EF*. Here there are *no points* where all preference components are positive. The requirement that exactly so much *shall* be used of the common factor, entails that at least one of the preference components for one of the individuals must become negative. We must either produce so much public radio-entertainment that it becomes a nuisance to individual No. 1, or so much public health service that it becomes a nuisance to No. 2.[8]

On the isoquant IV—where the conditional Pareto-optimal region is the finite segment *GH*—there is *one point*, namely *G′*, which is at the same time Pareto-optimal under no constraint. This gives another example of the fact that Pareto-optimality under a weak condition plus the additional requirement that the point shall satisfy a certain constraint, is sufficient to ensure that the point is Pareto-optimal under the stronger condition obtained by appending the additional requirement to the original weak condition.

[8] A supplement to the Pareto-criterion might perhaps be to *require* that all the preference components of all the individuals shall be non-negative. I shall, however, not go into this question here.

On the isoquant V—where the conditional Pareto-optimal region is the segment *LK*—there are two points, namely L' and K', that are at the same time unconditionally Pareto-optimal.

On the isoquant VI the conditional Pareto-segment is *MN*. This isoquant is different from the isoquants so far considered in one respect: there is one individual, No. 2, who can find no point on the Pareto-optimal segment of VI, which is as preferable to him as the most preferable point on the preceding isoquant V.

The isoquant VII—where the conditional Pareto-optimal region is *PQ*—is even such that *none* of the individuals can find a point in the Pareto region on this isoquant, which to him is more preferable than the most preferable point on the preceding isoquant VI.

On the isoquant VIII we can make a similar statement—in relation to VII—and we can even replace the term "more preferable" by "equally preferable".

Each of the above Pareto-optimal regions was defined under the condition that the quantity used of the common factor should be *exactly* the quantity represented by the isoquant in question.

Let us now go through the same alternatives, but each time replace the condition exactly equal to by *not larger than*. This means that in the first case—isoquant I—the condition is that the point may be anywhere on this isoquant or *South-West of it*. Which points in this area are Pareto-optimal for the two individuals? Obviously, again the segment *AB*. Similar reasoning for all the isoquants II-IV.

For the isoquant V the situation becomes different. The region which is Pareto-optimal under the condition that the quantity used of the common factor shall be equal to, or less than, that represented by the isoquant V, is the curve made up of the segments LL' and $L'G'K'$ and $K'K$. This is easily seen by a method of analysis similar to the one used in connection with Figure 4 (3.14).

For the isoquant VI the Pareto region will be made up of the two segments $(2)L'G'K'M'$ and $M'M$. Here again we see an example of the fact that the points (namely, the segment $K'G'L'$) that are Pareto-optimal under the weak condition (VI) and satisfy the strong condition (V) must also be Pareto-optimal under the strong condition (V). And we see that there are points (namely, the segments KK' and LL', endpoints K' and L' not included) that are Pareto-optimal under the strong condition (V) without being Pareto-optimal under the weak condition (VI).

For the constraint defined by the isoquant VII the Pareto-optimal region will be the segment $(2)L'G'K'M'(1)$, *i.e.* the same as the region which is Pareto-optimal under no constraint. The same will be true for any of the higher isoquants, such as, for instance, VIII.

We will now study these various situations from the point of view of the *sequence* which they form. That is, we consider the sequence of situations that is characterized by the isoquants I, II, III ... VIII, each of these isoquants now being interpreted not as a locus of admissible points, but as the North-East boundary of a set of admissible points. This means that the conditions considered form a sequence of conditions that become weaker and weaker. We shall say that a condition C_2 is effectively weaker than a condition C_1 if the set of points that satisfies C_2 contains all the points satisfying C_1 and at least one other point. The sequence of conditions now considered obviously satisfies this criterion.

In the sequence now considered, the isoquant IV—or more precisely the region whose North-East boundary is isoquant IV—occupies a special position. There is indeed a distinct difference between the isoquants *lower* than IV and those *higher* than IV. Any isoquant in the range below IV has the property that if we pass from this isoquant to a higher one in the sequence, we find that the Pareto region corresponding to this higher isoquant *contains no point* that belongs to the Pareto region corresponding to the lower isoquant. This means that to any point X in the Pareto region obtained by taking the lower isoquant as a constraint, there corresponds at least one point X' situated within the boundary given by the higher isoquant, but not within that given by the lower isoquant, which is Pareto-preferred to X under the new (and weaker) conditions. We can therefore say that from the Pareto viewpoint *an unquestionable gain is obtained by weakening the condition.*

This does *not* apply to isoquant IV itself or to any of the higher isoquants. True enough, *something* is gained by passing, say, from IV to V, namely *a larger variety* of points that are Pareto-optimal, but, to use a simplified expression, "none of these points are any Pareto-better than the best point attainable under the old (and stronger) condition". More precisely expressed: each point X' obtainable under the new (and weaker) condition is such that we can indicate at least one point X which was attainable already under the old (and stronger) conditions and which is such that X' is not Pareto-preferred to X under the new (and weaker) conditions.

Starting with the isoquant VII, we see that this and the higher isoquants even have the property that the Pareto-optimal region remains *unchanged* as the condition is weakened. We may generalize these considerations in the form of the following definition :

Definition (5.2). Let $C(\alpha)$ be a one-dimensional pencil of conditions, that is, a sequence of conditions depending on a parameter α, such that an effective increase in α means an effective *weakening* of the condition, *i.e.* if $\alpha_2 > \alpha_1$, any point that satisfies $C(\alpha_1)$ will also satisfy $C(\alpha_2)$ and there exists at least one point which satisfies $C(\alpha_2)$ but not $C(\alpha_1)$. If the pointset $Par[C(\alpha_2)]$ contains *no point* belonging to $Par[C(\alpha_1)]$, when $\alpha_2 > \alpha_1$, we shall say that there exists a positive Pareto-pressure on α from α_1 to α_2.

An essential feature of this definition is that $Par[C(\alpha_2)]$ shall contain *no point* belonging to $Par[C(\alpha_1)]$. We do not use the weaker formulation that $Par[C(\alpha_2)]$ shall contain *some points* that do *not* belong to $Par[C(\alpha_1)]$. If we want to use only the kind of criteria that are genuine to a Pareto-optimality way of thinking, we can indeed not say that any point within a given Pareto region is any better than another point in this region. Therefore, if by weakening the condition we have found a new Pareto region that contains one or more points that belonged already to the old Pareto region, we cannot say that we have found something that is actually better than what we had before.

Nor can we approach the definition of a Pareto pressure, for instance, by saying that there exists a positive Pareto pressure on the parameter that generates the conditions, if at least one of the points that satisfy the new (weaker) condition, but not the old (stronger) condition, is Pareto-preferred—under the new conditions—to all points that satisfy the old condition. If such a situation should occur, we would certainly be led to say that there exists a pressure of an extremely heavy sort. But this idea of looking for a point that *is actually Pareto-preferred to all other points* (in a certain admissible region) is not genuine to the way of thinking in Pareto-optimality analysis. Such a point would exist only in cases that are so extremely particular as to be virtually trivial. The definition (5.2) on the contrary builds on ideas that are natural in Pareto-optimality analysis.

An obvious generalization is to consider a set of conditions depending on two or more parameters. This would lead to the concept of *directional* Pareto pressure, that is Pareto pressure defined by changing one of the parameters while keeping all the others constant. The concept of the region of the parameter space where all the directional Pareto pressures are positive would then indicate a situation where all the parameters are far from being "free goods" in the Pareto sense.

6. The Problem of the Régime

All the above arguments pertained to the *selection* problem, that is, to the definition of the point X which it is considered "desirable" to reach. Having decided on such a point or on a class of such points, the next question is to indicate a *régime* or a class of régimes which will achieve this goal.

From the analysis of the previous section we know that a point X that shall be Pareto-optimal under the production constraint (when a change in quantities is possible) must have the marginal preferences proportional for all the individuals and proportional to marginal costs. Contrary to what is commonly believed, such a point *cannot* be reached under a system of production and distribution possessing the characteristics which are commonly associated with free competition and a system of income taxes and capital taxes of the usual type and a "neutral" monetary system.

I shall discuss this by considering a number of *special cases* that are constructed on several different assumptions. From these special cases I shall extract certain conclusions that will seem to hold under very general conditions.

In all the subsequent discussions it is vitally important to be fully aware of the nature of the *constraints* under which Pareto-optimality is defined. When it comes to discussing the optimality-conditions of a régime, any reasoning that is not built on a terminology and mathematical symbolism that express the *kind* of

Pareto-optimality involved in the argument (the nature of the constraints), is *a waste of time*. Just as it would be a waste of time to discuss the "derivative" of a function of several variables without indicating the variable with respect to which the derivative is to be taken.

Can abstract discussions on Pareto-optimality be of any practical significance at all for questions of economic policy? Yes, I am convinced that they do have great practical significance. Even though this reasoning involves many quantitative concepts that cannot be measured *directly* through actual observation, certain important *conclusions* can be drawn which are of paramount practical importance. These conclusions show amongst others that most present-day taxation systems are in one particular respect extremely questionable—to say the least.

In a general way we may say that the useful result that can be deduced from discussions of this type, consists in asserting that certain economic régimes do *not* satisfy certain optimality criteria which would seem to be of such a nature that we *must* accept them. In other words, Pareto-optimality is a principle of *negation*, not one of affirmation. In this it has the same logical structure as the statistical testing of hypotheses. What statistical testing can do, is to *reject* a hypothesis.

Logically the first step in a discussion of Pareto-optimality is to formulate the *obligatory conditions*, that is certain conditions which we will accept as the basis of the analysis. For instance, certain fundamentals regarding the technical aspects of production, or certain basic human rights, etc. The formulation of these obligatory conditions has logically much in common with the circumscription of the class of *admissible* hypotheses in statistical testing.

Once the obligatory conditions are formulated, we are by pure logic, without the intervention of any social value judgements, led to formulate the criterion that no régime can be accepted as satisfactory if it is *not* Pareto-optimal under the obligatory constraints.

This criterion may exclude a great number of particular régimes, but it will, in general, not lead to the selection of *one* particular régime. In general, there will be a whole class of régimes which are Pareto-optimal under the obligatory constraints. Here is where the spectrum of social value judgements comes in.

The economist as scientist and technician simply has to take these judgements as *data*. As citizen he has as much as, but nothing more, to say about these value judgements than other citizens. When the supplementary conditions derived from social value judgements are given, all the rest is within the sphere of competence of the economist. This limitation of the economist's field is necessary for the maintenance of his objectivity and self-respect *as a scientist*. It would seem that even with this healthy limitation of the economist's field, there is more than enough for him to do.

With this general philosophy in mind I now turn to a study of some models of régimes.

Case 1

Consider first the model defined by (4.6)-(4.12). If it is humanly possible to impose a sufficiently strong *control-system* and one is willing to take the inconveniences that go with this régime, one could produce a preassigned combination of X_{11}, X_{12}, X_{21} and X_{22}. By choosing this combination in an appropriate way, it would seem that one could always reach a point that has a *preassigned* distribution ratio (within certain objectively given limits) and at the same time satisfies (4.12) and (4.17), which by (4.18) means that the point is Pareto-optimal under the production constraint. The following will illustrate this and also illustrate that such a situation cannot be realized by decontrolling the market completely.

Suppose, for instance, that forced labour is used so as to ensure a given magnitude of N. Suppose that a certain sum of money R_1 is given to the first individual and another sum R_2 to the second individual and that prices of the two goods are not controlled but left to be determined by the play of forces in the market. Suppose that each individual acts as a quantity adapter under the conjectural assumption that prices are constant. Finally, suppose that the central production authority has full knowledge of the preference functions of the individuals so that it is able to calculate what the prices will become under a given set of conditions, and also suppose that the authority distributes the available labour force between the two industries so as to achieve proportionality between marginal labour costs and prices. Under this régime we will have

(6.1) For individual 1: $$\frac{\omega_{12}}{\omega_{11}}=\frac{p_2}{p_1} \qquad p_1X_{11}+p_2X_{21}=R_1$$

(6.2) For individual 2: $$\frac{\omega_{22}}{\omega_{21}}=\frac{p_2}{p_1} \qquad p_1X_{12}+p_2X_{22}=R_2$$

(6.3) Production policy equation: $$\frac{p_2}{p_1}=\frac{f_1'(f_1^{-1}(X_{11}+X_{12}))}{f_2'(f_2^{-1}(X_{21}+X_{22}))}$$

Finally we have the production constraint (4.12). This gives a total of six independent equations which determine the six variables X_{11}, X_{12}, X_{21}, X_{22}, p_1 and p_2. In the present case where the absolute magnitudes of R_1 and R_2 are given, the absolute prices—not only the relative prices—will be determined. Thus, the régime considered leads, in general, to a well-defined equilibrium point. The equilibrium point now considered satisfies the conditions (4.12) and (4.17), hence is Pareto-optimal under the production constraint (4.12). This applies whatever the magnitudes of R_1 and R_2 (within admissible limits); hence the equilibrium point will show a prescribed distribution coefficient.[9]

The pure exchange market *without* production but with given initial quantities can also easily be analysed in terms of this example with a central authority

[9] By (6.1)-(6.3) the ratio between R_1 and R_2 is a technically given function of the four quantities X_{ij}. Maximizing and minimizing this expression, we *may* get upper and/or lower limits for the ratio R_1/R_2.

that gives certain amounts R_1 and R_2 to the two individuals. This is a distribution condition expressed by

$$(6.4) \qquad R_1 = \text{given} \qquad R_2 = \text{given}$$

The "production" constraint is now expressed by (6.18). Necessary and sufficient for Pareto-optimality under this condition is easily seen to be simply the proportionality of marginal utilities (see (6.21)), *i.e.* the first member of (4.17), with the appropriate sign condition. A point of this character is alway reached if the central authority gives arbitrarily given amounts R_1 and R_2 to the two individuals. The equilibrium point will then be given by the six equations (6.1), (6.2) and (6.4). This determines the four X and the two p. In this case the absolute prices are determined. On the other hand, if the central authority fixes the *real* expenditures R_1/p_1 and R_2/p_2, only the relative prices are determined. The quantity combination is determined and is the same as before. The exchange market with given initial quantities is only a verbal reformulation of this case.

This case has not much in common with "free competition" under realistic circumstances. Indeed, who shall determine the *given* initial quantities that characterize the real income distribution? Some dictator?

The crucial question in a modern society is, of course, just this: does the excessively skew income distribution that is produced by the "free play of the forces", represent a state of affairs that in some sense of the word can be considered as optimal within a certain class of states *between which we choose*? The case with given initial quantities has therefore no realistic relevance whatsoever for a discussion of whether the régime of free competition can be considered an "optimal" system or not.

But suppose that the income distribution is corrected, say, by an *income tax* system? Would it not then be an "optimal" procedure to leave the market to the free play of the forces? It would not. As will appear from the examples below, *no régime* of "free competition" with an income tax on labour can produce an equilibrium point that is Pareto-optimal under the production constraint.

Case 2

We will now change the model somewhat in the direction of a free economy, but still maintain it within the framework of (4.6)-(4.12), for which we have determined the region that is Pareto-optimal under the production constraint. We assume that there has by custom or by regulation of working hours or in some other way been fixed certain amounts of work N_{11} and N_{21} which the individual No. 1 performs in the industries 1 and 2, respectively. Similarly we assume certain fixed amounts N_{12} and N_{22} for individual No. 2. These four magnitudes, which we prefer to write in the order

$$(6.5) \qquad N_{11}, \quad N_{12}, \quad N_{21}, \quad N_{22}$$

are for the time being assumed given. That is to say, no adaptation of the working hours is supposed to take place as a consequence of changes in prices and wages, etc. Consequently the total amounts of work performed in the two industries are also given, namely

$$N_i = N_{i1} + N_{i2} \qquad (i = 1, 2) \tag{6.6}$$

Suppose that we now let the prices of the two goods, *i.e.* p_1 and p_2, as well as the wage rates in the two industries, q_1 and q_2, be determined by the play of the forces in the market, each of the two industries acting as a quantity adapter trying to maximize profit.[10] Each of the two individuals will, as before, try to maximize his preference function, but the budget equation is now obtained by expressing that the value of the goods purchased is equal to the value of the wages received—after correction for taxes.

We consider a *tax* T imposed as a certain constant—a *per capita* tax—on individual No. 1, the proceeds of the tax to be given to individual No. 2. The magnitude T may be positive or negative, it may be looked upon simply as an *income redistribution parameter.* Under this régime the following equations will hold:

$$\frac{\omega_{12}}{\omega_{11}} = \frac{p_2}{p_1} \qquad p_1 X_{11} + p_2 X_{21} = q_1 N_{11} + q_2 N_{21} - T \tag{6.7}$$

$$\frac{\omega_{22}}{\omega_{21}} = \frac{p_2}{p_1} \qquad p_1 X_{12} + p_2 X_{22} = q_1 N_{12} + q_2 N_{22} + T \tag{6.8}$$

$$f_1' = \frac{q_1}{p_1} \tag{6.9}$$

$$f_2' = \frac{q_2}{p_2} \tag{6.10}$$

(6.7) and (6.8) are the adaptation equations of the first and second individuals respectively, (6.9) and (6.10) are the adaptation equations of the two industries. In addition we have to reckon with the two production equations

$$X_{11} + X_{12} = f_1(N_{11} + N_{12}) \qquad X_{21} + X_{22} = f_1(N_{21} + N_{22}) \tag{6.11}$$

This gives a total of eight independent equations.

We may attribute to the nominal value of the income redistribution parameter T an *arbitrary* value, with the proviso, however, of distinguishing fundamentally between a zero and a non-zero value of T as explained below. To any such given value of T there will correspond an equilibrium point under the régime (6.7)-(6.11), and a comparison of the form of these equations with those of (4.18) shows that the equilibrium point now reached is Pareto-optimal under the single production constraint (4.12). Since (4.12) is a weaker constraint than (6.11) and we know that the point now considered satisfies this stronger condition (6.11), we can conclude that it is Pareto-optimal under the stronger production constraint (6.11).

[10] The fact that total labour in the industries is *objectively* given does not prevent bidding by the entrepreneurs under a *conjectural* variation where they take account of marginal productivities.

This fact, however, must not be interpreted to mean that under the régime considered it is possible to reach an equilibrium point which is Pareto-optimal under the production constraint (6.11) and gives a preassigned *real*-income distribution as defined by the distribution coefficient λ in (4.20). To believe this would be an error of the "homogeneity-kind", that is an error which has emerged by not being sufficiently aware of the particular form of the equations with respect to homogeneity.

The situation under the régime (6.7)-(6.11) is as follows: *by changing the value of* T *one would simply change proportionally all prices and wage rates.* Hence the *real value* of T, that is, its *deflated* value, would remain unchanged. This deflated value of T is determined by the régime (6.7)-(6.11). That is to say, under the régime considered it is *necessary to perform a specific real-income redistribution, in order that the system of equations shall have a solution.* The amount of this income redistribution, reckoned in real, *i.e.* deflated, values *must* be a specific number in order that the equations shall be solvable. In other words, if the production functions and the preference functions have arbitrarily given shapes, we can *not* influence the real income distribution.

This follows from the nature of the solution. First consider the equation obtained by taking the sum of the two budget equations for the two individuals. This equation is

$$(6.12) \qquad p_1(X_{11}+X_{12})+p_2(X_{21}+X_{22})=q_1(N_{11}+N_{12})+q_2(N_{21}+N_{22})$$

In other words, the two industries taken together must produce with a zero profit. That this emerges as a consequence of the two budget equations of the individuals shows that the régime now considered is a plausible one for discussing distribution as a problem *between the two individuals.* If we should introduce the concept of a surplus going to the industries, *i.e.* a part of the national income which did *not* go to one or the other of the two individuals, the whole problem of the Pareto region would have to be reconsidered, now with (at least) one third party besides the two individuals (compare Case 7). What sorts of concrete arrangements one could apply in the industries in order to assure the fulfilment of (6.12) is not relevant in this connection.

Equation (6.12) together with the first two equations of (6.7) and (6.8) and the equations (6.9)-(6.11) give a total of seven independent equations in the seven variables $X_{11}, X_{12}, X_{21}, X_{22}$ and $\frac{p_2}{p_1}, \frac{q_1}{p_1}, \frac{q_2}{p_2}$. Eliminating the last three of these variables, we get the following set of four independent equations between the four variables $X_{11}, X_{12}, X_{21}, X_{22}$: the two production equations (6.11) and the two equations

$$(6.13) \qquad \frac{\omega_{11}}{\omega_{12}}=\frac{\omega_{21}}{\omega_{22}} \text{ which may also be written } \begin{vmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{vmatrix}=0$$

and

$$(6.14) \qquad \omega_{i1}(X_{11}+X_{12}).(1-\epsilon_1)+\omega_{i2}(X_{21}+X_{22}).(1-\epsilon_2)=0 \text{ (for an arbitrary } i\text{)}$$

The equation (6.14) may be taken for an arbitrary i. If it holds for one i, it will by (6.13) hold also for the other i.

The functions ϵ_1 and ϵ_2 are the passus-coefficients of the two production functions, *i.e.*

$$(6.15) \qquad \epsilon_i = \frac{f_i' \, . \, N_i}{X_i} \qquad (i=1, 2)$$

ϵ_i, like f_i, is a function of one variable. As its argument we may take N_i or X_i. Over a region where the production function $f_i(N_i)$ is monotonic, it does not matter whether we take N_i or X_i as the argument of ϵ_i.

The four equations (6.11), (6.13) and (6.14) contain—when the four N are given—only the four variables X_{11}, X_{12}, X_{21}, X_{22}. This set of equations will therefore—apart from degenerate cases—define the point X in four-dimensional space that will be realized under the régime (6.7)-(6.11). By means of the four equations mentioned, this definition of the point X is achieved *directly* without any reference to how prices and wages will work out or how the income transfer T is fixed. The point X determined by these four equations is not subject to any argument of the form: "it cannot hold because then this price—or this wage rate—would change". It is a great advantage to have the point X fixed by such a system of equations.

When the point X in four-dimensional space is fixed, we derive from, say, the first of the budget equations of the individuals

$$(6.16) \qquad \frac{T}{p_1} = X_1\left[\frac{X_{11}}{X_1} - \epsilon_1 \frac{N_{11}}{N_1}\right] + \frac{\omega_{12}}{\omega_{11}} X_2 \left[\frac{X_{21}}{X_2} - \epsilon_2 \frac{N_{21}}{N_2}\right]$$

The right member of (6.16) depends *only on the point* X (and the given values of the N_{ij}). Hence, since the point X is fixed by the régime, the real (deflated) value of T is also fixed.

This argument shows immediately what conditions must be put on T from the viewpoint of zero or non-zero values. If the nature of the preference functions and the production functions is such that under the given values of the N_{ij} the equilibrium point in X_{ij}, as determined by the four equations mentioned, turns out to make the right member of (6.16) zero, then only zero-values of T are permissible. That is to say, if under these conditions we tried to impose any form of income transfers, we would run into an incompatible system. That is to say, the model could not give a realistic picture of what would actually happen. We could only say that some of the relations considered would in the world of realities have to break down. The most effective way to discuss this situation might be the introduction of the pressure coefficients of the decision-model—they can describe the tension towards "instability" and "crisis"—but I shall not enter into details on this here.

On the other hand, if in the equilibrium point the right member of (6.16) is different from zero, then only non-zero values of the income transfer T are permissible. More precisely: only values of T are permissible which are of the *same sign* as the value which the right member of (6.16) assumes in the equilibrium point (and it is wanted to have a regime with a positive price p_1).

The wage-rate ratio $\frac{q_1}{q_2}$ is also determined as a function of the equilibrium point. From (6.7), (6.9) and (6.10) we get indeed

$$\frac{q_1}{q_2}=\frac{\omega_{11}}{\omega_{12}}\cdot\frac{f_1'}{f_2'} \tag{6.17}$$

The price ratio $\frac{p_2}{p_1}$ is given as a function of the equilibrium point by any of the equations (6.7) or (6.8).

To see whether the equilibrium point now reached is Pareto-optimal under the production constraint, let us determine the region of points that now have this property. The production constraint can now be written

$$X_{11}+X_{12}=X_1 \text{ (given)} \qquad X_{21}+X_{22}=X_2 \text{ (given)} \tag{6.18}$$

Taking X_{11} and X_{21} as independent variables, we have

$$\begin{aligned} \frac{\partial\Omega_1}{\partial X_{11}}&=\omega_{11} & \frac{\partial\Omega_1}{\partial X_{21}}&=\omega_{12} \\ \frac{\partial\Omega_2}{\partial X_{11}}&=-\omega_{21} & \frac{\partial\Omega_2}{\partial X_{21}}&=-\omega_{22} \end{aligned} \tag{6.19}$$

The condition for Pareto-optimality under the production constraint in the present case, namely

$$\frac{\dfrac{\partial\Omega_1}{\partial X_{11}}}{\dfrac{\partial\Omega_2}{\partial X_{11}}}=\frac{\dfrac{\partial\Omega_1}{\partial X_{21}}}{\dfrac{\partial\Omega_2}{\partial X_{21}}}=\text{negative,} \tag{6.20}$$

can therefore be written

$$\frac{\omega_{11}}{\omega_{12}}=\frac{\omega_{21}}{\omega_{22}} \qquad \omega_{12} \text{ and } \omega_{22} \text{ of the same sign.} \tag{6.21}$$

Thus Pareto-optimality in the present case does *not* require proportionality between marginal utilities and marginal productivities as, for instance, in (4.17).

The fulfilment of (6.21) is assured by (6.7) and (6.8) (apart from the sign condition, which we may assume to hold except in pathological cases).

To summarize: if the *real* value of the income transfer is fixed at the magnitude which it must have in order to make equilibrium possible, the equilibrium point produced by the régime considered will be Pareto-optimal under the production constraint. If the real value of the income transfer is not fixed at this magnitude, no point satisfying the régime exists.

In the *pari passu* case, that is, in the case where both production functions are homogeneous of the first order, *i.e.* both ϵ_1 and ϵ_2 identically equal to 1, the equation (6.14) *drops out* and the régime considered has a solution with one degree of freedom. In this case a certain leeway is left in the fixation of the real value of the income transfer. If we note the maximum and minimum assumed by the right member of (6.16) over the one-dimensional region that now gives the solution, the real value of the income transfer (the deflation being done by p_1) may be fixed arbitrarily at any magnitude between this maximum and minimum.

If a model of this sort is to be reasonably realistic, we will have to assume that there is only a small span between the maximum and minimum referred to. The *pari passu* case is, indeed, not a realistic one, as witnessed by the fact that so much of the discussion in other sectors of distribution theory proceeds on the assumption that output is *not* for all values of the input proportional to input. In the present case we must therefore take *arbitrarily given* shapes of the production functions and discuss what consequences will follow from such a general assumption. These consequences are contained in the above remarks on the real value of the income transfer.

Case 3

One might perhaps think that if the amounts N_{ij} of the labour inputs were considered as *variables* to be adapted by the individuals according to the price and wage structure, the situation would be radically changed and then it would be possible to achieve an equilibrium point which is Pareto-optimal under a condition which simply consists of the production constraint, and at the same time gives a preassigned distribution of the national income. Closer examination will reveal, however, that this is not so. It will turn out that the above conclusions about the real value of the income transfer are valid also under the new régime now considered.

Indeed, consider a model with X_{11}, X_{12}, X_{21}, X_{22} and N_{11}, N_{12}, N_{21}, N_{22}, as basic variables. Suppose that each individual has a preference function

$$\Omega_j(X_{1j}, X_{2j}, N_{1j}, N_{2j}) \qquad (j=1, 2) \tag{6.22}$$

with continuous partial derivatives

$$\omega_{ji}=\frac{\partial\Omega_j}{\partial X_{ij}} \qquad \omega_{j(i)}=\frac{\partial\Omega_j}{\partial N_{ij}} \qquad \begin{pmatrix} i=1, 2 \\ j=1, 2 \end{pmatrix} \tag{6.23}$$

As before, let the production functions be given. The production constraints are then given by the two equations (6.11). Since the distribution of the labour force between the two industries is now to be considered explicitly in the optimization process, we have to take the production constraint in the form (6.11), not only in the form (4.12).

Let us first determine the region which is Pareto-optimal under the production constraint (6.11). We have $n=2$, $m=8$, $s=2$, hence $min[m-s, n-1]=1$, so that the region to be looked for is one-dimensional.

Disregarding for a moment the sign conditions for derivatives, we make a simple analysis by means of Lagrange multipliers. This leads to maximizing the function

$$(6.24)\qquad (\Theta_1\Omega_1+\Theta_2\Omega_2)+\mu_3[X_{11}+X_{12}-f_1(N_{11}+N_{12})]+ \\ +\mu_4[X_{21}+X_{22}-f_2(N_{21}+N_{22})]$$

on the assumption that the four X and the four N are free variables while the four parameters Θ_1, Θ_2, μ_3 and μ_4 are constants. Equating the eight partial derivatives to zero, we get

$$(6.25)\qquad \begin{array}{ll} \Theta_1\omega_{11}+\mu_3=0 & \Theta_1\omega_{1(1)}-\mu_3 f_1'=0 \\ \Theta_1\omega_{12}+\mu_4=0 & \Theta_1\omega_{1(2)}-\mu_4 f_2'=0 \\ \Theta_2\omega_{21}+\mu_3=0 & \Theta_2\omega_{2(1)}-\mu_3 f_1'=0 \\ \Theta_2\omega_{22}+\mu_4=0 & \Theta_2\omega_{2(2)}-\mu_4 f_2'=0 \end{array}$$

That is

$$(6.26)\qquad \begin{array}{ll} \dfrac{\mu_3}{\Theta_1}=-\omega_{11}=\dfrac{\omega_{1(1)}}{f_1'} & \dfrac{\mu_3}{\Theta_2}=-\omega_{21}=\dfrac{\omega_{2(1)}}{f_1'} \\[2ex] \dfrac{\mu_4}{\Theta_1}=-\omega_{12}=\dfrac{\omega_{1(2)}}{f_2'} & \dfrac{\mu_4}{\Theta_2}=-\omega_{22}=\dfrac{\omega_{2(2)}}{f_2'} \end{array}$$

If we eliminate from the system the ratios between the multipliers (this takes out three equations), we get the following five equations:

$$(6.27)\qquad \frac{-\omega_{1(i)}}{\omega_{1i}}=\frac{-\omega_{2(i)}}{\omega_{2i}}=f_i' \qquad (i=1,\ 2)$$

$$(6.28)\qquad \frac{\omega_{11}}{\omega_{12}}=\frac{\omega_{21}}{\omega_{22}}$$

These five equations together with the two equations (6.11) determine the one-dimensional region which is Pareto-optimal under the production constraint in the present case. For the argument which follows it is not necessary to discuss the sign condition on the derivatives.

Now let us consider the following régime: as in Case 2, we assume prices p_1 and p_2 as well as wage rates q_1 and q_2 to be determined by the free play of the forces in the market, each of the two industries acting as a quantity adapter trying to maximize profits, and each of the two individuals trying to maximize his preference function under the same kind of budget equations as in (6.7) and (6.8), but now not only with the X, but also the N as free variables. This régime leads to the following equilibrium equations:

$$(6.29)\qquad \frac{q_i}{p_i}=\frac{-\omega_{1(i)}}{\omega_{1i}}=\frac{-\omega_{2(i)}}{\omega_{2i}}=f_i' \qquad (i=1,\ 2)$$

$$(6.30)\qquad \frac{p_1}{p_2}=\frac{\omega_{11}}{\omega_{12}}=\frac{\omega_{21}}{\omega_{22}}$$

These eight equations together with the two budget equations—the same as those written to the right in (6.7) and (6.8)—and the two production constraints (6.11) give a total of twelve independent equations to determine the eight basic variables X_{ij} and N_{ij} together with the four variables $\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{p_2}{p_1}$ and $\frac{T}{p_1}$. In other words, we have exactly the same kind of solution as in Case 2. Also in the present case will the equilibrium values of X_{ij} and N_{ij} be independent of the choice of the nominal value of T, a change in T only entailing a corresponding proportional change in the prices and wages rates. The formula for the *real value* of the income transfer will also now be (6.16). All the rest of the discussion can be repeated practically word by word as in Case 2.

Case 4

This case will illustrate a general proposition which can best be brought out by a still more simplified example.

Consider the case where there is only one individual (one typical individual representing the whole working population) and one industry (representing the whole production activity of the nation). Let X be the quantity produced and N the input of labour. Let $\Omega(X,N)$ be the preference function of the representative individual and $f(N)$ the production function of the representative industry.

The region which is Pareto-optimal under the production constraint in the present case is simply the social optimum point defined by seeking the unconditional maximum of the function $\Omega(f(N), N)$ of the single variable N. This point is obviously determined by

$$\text{(6.31)} \qquad \frac{-\omega_N}{\omega_X}=f'$$

where $\omega_X=\frac{\partial\Omega}{\partial X}$, $\omega_N=\frac{\partial\Omega}{\partial N}$ and $f'=\frac{df}{dN}$. Both members of (6.31) may be looked upon as depending on the single variable N. Its optimum value is determined as a root of this equation.

What sorts of *régimes* will lead to this social optimum point?

We shall assume that the representative worker is not necessarily remunerated at a fixed wage rate, but at a wage rate that may depend on the input of labour (a progressive or degressive wage rate). This means that the total wage bill $B(N)$ will be *some* function of the total input N, not necessarily a magnitude proportional to N as in the case of a fixed wage rate. We assume that B is measured in terms of the good produced, *i.e.* the price of the good is conventionally put equal to 1.

Consider the following régime:

(6.32) *Consumption equals production, i.e.* the magnitude X that enters into $\Omega(X, N)$ is equal to $X=f(N)$.

(6.33) *The representative worker consumes all he earns, i.e.* $X=B(N)$ (which is only another way of saying that the profit of the industry is zero).

(6.34) *The representative worker tries to maximize his preference function under the constraint* (6.33), *i.e.* he will try to maximize $\Omega(B(N), N)$ with N as a free variable.

(6.35) *The representative enterprise will try to maximize profit, i.e.* it will try to maximize $f(N)-B(N)$.

The two institutionally determined types of behaviour (6.34) and (6.35) lead to

$$\frac{-\omega_N}{\omega_X}=B'(N) \tag{6.36}$$

$$f'(N)=B'(N) \tag{6.37}$$

(6.32), (6.33), (6.36) and (6.37) give four conditions on the two variables X and N. Hence, in order that the system shall have a solution (under arbitrarily given shapes of the preference function Ω and the production function f), the wage-bill function $B(N)$ must satisfy two point conditions. One of these may be thought of as corresponding to the fixing of the wage rate in the orthodox theory, the other is a condition which in the orthodox theory is generally overlooked.

The necessary and sufficient condition which the wage-bill function must satisfy, can be obtained as follows.

If $B(N)$ does have a shape such that a solution is possible, the equilibrium point must satisfy (6.31). This is seen by combining (6.36) and (6.37). In other words, if a solution exists, it must be the same as the social optimum defined by (6.31). In this social optimum point—which can be characterized by the corresponding magnitude N_0 of N—it is by (6.32), (6.33), (6.36), (6.37) *necessary* that we have

$$B(N_0)=f(N_0) \quad \text{and} \quad B'(N_0)=f'(N_0) \tag{6.38}$$

The fulfilment of (6.38) in the point N_0—where $f'(N_0)$ by (6.31) is equal to $\frac{-\omega_N}{\omega_X}$—is obviously also *sufficient* for the existence of an equilibrium point.

We can summarize this in the following proposition:

Proposition (6.39). In order that the régime (6.32)-(6.35) shall be *consistent* and lead to an equilibrium point, it is necessary and sufficient that the wage-bill function $B(N)$ satisfies (6.38), *i.e.* that it has *first order contact with the production function in the special input point* N_0 *that corresponds to the social optimum.* If the wage-bill function has this property, the equilibrium point of the régime will coincide with the social optimum.

From this proposition we deduce, for example, immediately that if the production function in the vicinity of the social optimum point is homogeneous of the first degree, *i.e.* gives an output that is proportional to the input, the wage-bill function must have the same property, that is, it must be on the basis of a fixed wage rate in the vicinity of the social optimum point. If the production function does *not* have this character, the remuneration of the representative worker can *not* be on the basis of a fixed wage rate.

Although this example is simplified to the extreme, it does illustrate something of considerable importance. What is here taken account of in the form of a wage-bill function would, of course, in the world of realities have to be worked out in the form of social transfers, an appropriate form of a non-proportional (positive or negative) income tax or some similar device.

Case 5

Let us reconsider Case 3, but now assume that the income transfer T is a *function* of the four variables N_{11}, N_{12}, N_{21}, N_{22}. This might illustrate the case where the income transfer is achieved through some sort of *income tax* system. To indicate this, T may now be called the tax function. The other assumptions about the régime remain the same as in Case 3.

Assuming autonomous adaptation on the part of individual No. 1, we have to maximize $\Omega_1(X_{11}, X_{21}, N_{11}, N_{21})$ under the budget constraint written to the right in (6.7) where now T is a function of the four variables N_{11}, N_{12}, N_{21}, N_{22} of which N_{12} and N_{22} are conjecturally constant for individual No. 1. *Vice versa* for individual No. 2. This gives

(6.40) Adaptation of individual 1: $$\frac{\omega_{11}}{p_1}=\frac{\omega_{12}}{p_2}=\frac{-\omega_{1(1)}}{q_1-T_{(11)}}=\frac{-\omega_{1(2)}}{q_2-T_{(21)}}.$$

(6.41) Adaptation of individual 2: $$\frac{\omega_{21}}{p_1}=\frac{\omega_{22}}{p_2}=\frac{-\omega_{2(1)}}{q_1-T_{(12)}}=\frac{-\omega_{2(2)}}{q_2-T_{(22)}}$$

where

(6.42) $$T_{(ij)}=\frac{\partial T(N_{11}, N_{12}, N_{21}, N_{22})}{\partial N_{ij}} \qquad \begin{pmatrix} i=1,2 \\ j=1,2 \end{pmatrix}$$

We further have

(6.43) Adaptation of the two enterprises: $$\frac{q_i}{p_i}=f_i' \qquad (i=1, 2)$$

In addition we have the two budget equations (6.7)-(6.8) and the two production constraints (6.11). This gives a total of twelve equations on the 11 unknowns X_{ij}, N_{ij}, $\frac{q_1}{p_1}$, $\frac{q_2}{p_2}$ and $\frac{p_2}{p_1}$. Hence the function T must satisfy one point condition if there shall exist a point that satisfies the régime. If T is assumed constant, we get back to Case 3.

So much for the consistency of the régime. Now let us see what additional conditions this function T of the four variables N_{ij} must satisfy in order that the point produced by the régime shall be Pareto-optimal under the production constraint. This means that the function T must satisfy four additional point conditions, namely (6.27). This entails

(6.44) $T_{(11)}=T_{(12)}=T_{(21)}=T_{(22)}=0$ in the equilibrium point.

In other words, it is necessary that the tax function is *constant* in the vicinity of the equilibrium point, *i.e.* it must have the character of a *per capita* tax in this vicinity. This can be expressed in the following proposition:

Proposition (6.45). No régime of the type (6.40)-(6.43) with a tax function T that *actually depends* on the labour inputs N_{ij} in the vicinity of the equilibrium point, can lead to a point that is Pareto-optimal under the production constraint.

In particular, any régime that *includes an income tax* (which in the vicinity of the equilibrium point depends effectively on the income) is ruled out. Only a régime with a tax which in the vicinity of the equilibrium point has the character of a *per capita* tax, can produce an equilibrium point that is Pareto-optimal under the production constraint. If the tax *has* this property and its real value has the precise magnitude which it must have in order that a point satisfying the régime shall exist, then this point *is* Pareto-optimal under the production constraint. The magnitude of the real value of the tax can also in the present case be discussed in the same way as in Case 3.

Case 6

Now consider the case of m goods Nos. $i=1, 2 \ldots m$, and n individuals Nos. $j=1, 2 \ldots n$. Each good is assumed to be produced in a single process using only labour. Let

(6.46) $X_{i1}+X_{i2}+\ldots+X_{in}=f_i(N_{i1}+N_{i2}+\ldots+N_{in}) \qquad (i=1, 2 \ldots m)$

be the production functions, N_{ij} being the amounts of work furnished by individual j to industry i, and X_{ij} being the output of industry i which is received by individual j. Also in this case will the main conclusions of Case 5 apply.

Pareto-optimality under the production constraint (6.46) is now—with an analogous notation as before—expressed by

(6.47) $$\frac{-\omega_{j(i)}}{\omega_{ji}}=f_i' \text{ for all } i \text{ and } j$$

(6.48) $$\frac{\omega_{ji}}{\omega_{jk}}=\text{independent of } j \text{ for all } i, j \text{ and } k$$

Not all the equations (6.47)-(6.48) are independent, but all of them must be satisfied in a point that shall be Pareto-optimal under (6.46).

This being so, consider a régime where each individual and each industry behaves as in Case 5, the individual j being subject to a tax of the form

(6.49) $T_j(N_{1j}, N_{2j} \ldots N_{mj}) \qquad (j=1, 2 \ldots n)$

and a budget equation of the form

(6.50) $q_1N_{1j}+q_2N_{2j}+\ldots+q_mN_{mj}=p_1X_{1j}+p_2X_{2j}+\ldots+p_mX_{mj}+T_j \qquad (j=1, 2 \ldots n)$

The prices $p_1, p_2 \dots p_m$ and the wage rates $q_1, q_2 \dots q_m$ are assumed constant from the viewpoint of any individual or any industry.

This régime leads to

(6.51) Adaptation of the individuals: $\frac{q_i - T_{j(i)}}{p_i} = \frac{-\omega_{j(i)}}{\omega_{ji}}$ for all i and j

(6.52) Adaptation of the industries: $\frac{q_i}{p_i} = f_i'$ for all i

where

(6.53) $$T_{j(i)} = \frac{\partial T_j(N_{1j}, N_{2j} \dots N_{mj})}{\partial N_{ij}} \qquad \begin{pmatrix} i=1, 2 \dots m \\ j=1, 2 \dots n \end{pmatrix}$$

From (6.47), (6.51) and (6.52) follows

(6.54) $T_{j(i)} = 0$ for all i and j in the equilibrium point.

This shows that if the régime shall lead to a point that is Pareto-optimal under the production constraint, it is *necessary* that the taxes have the character of *per capita* taxes, *not income taxes*, in the vicinity of the equilibrium point. In other words, the essence of proposition (6.45) holds good also in the case of m industries and n individuals.

This conclusion is rather far-reaching. It is independent of whether or not the sum of the taxes is zero or not. That is, we need not assume

(6.55) $$T_1(N_{11}, N_{21} \dots N_{m1}) + \dots + T_n(N_{1n}, N_{2n} \dots N_{mn}) = 0$$

If the taxes have the character of *per capita* taxes in a certain region, will an equilibrium point of the régime exist in this region? More specifically: under what further conditions on the income transfers $T_1 \dots T_n$ will such a point exist?

In the problem we have $2mn$ variables X_{ij} and N_{ij}, further m prices p_i, m wage rates q_i and n income transfers T_j, all reckoned in nominal, *i.e.* undeflated, values. These $2m+n$ nominal magnitudes enter, however, in the equations in such a way that only their *relative* values—expressed, say, in terms of p_1—count as unknowns. That is, the total number of unknowns is

(6.56) $$2mn + 2m + n - 1$$

In order to rewrite the equilibrium equations in the form of a system of *independent* equations, we first put

(6.57) $$\frac{\omega_{j1}}{p_1} = \frac{\omega_{j2}}{p_2} = \dots = \frac{\omega_{jm}}{p_m} = \frac{-\omega_{j(1)}}{q_1} = \frac{-\omega_{j(2)}}{q_2} = \dots = \frac{-\omega_{j(m)}}{q_m} \qquad (j=1, 2 \dots n)$$

This gives $(2m-1)n$ equations. When these are fulfilled, we have (6.51)—in view of (6.54). Adding the m equations (6.52) is sufficient to assure the fulfilment also of (6.47). We further have m production equations (6.46) and budget equations (6.50), in other words, a total of

(6.58) $$2mn + 2m$$

independent equations, hence $(n-1)$ degrees of freedom. If we further add the condition (6.55), the number of degrees of freedom is reduced to $(n-2)$. In the case $n=2$, there is no degree of freedom. An example of this is Case 3 where the real value of the income transfer is determined by the régime.

In the general case, $(n-1)$ of the *per capita* income transfers may, in principle, be fixed arbitrarily. There may, however, by the nature of the preference functions and production functions involved, be very close upper and lower limits to how the choice can be made if an equilibrium point shall exist. Whether the range is small or large, the income transfers must have the character of *per capita* taxes in the vicinity of the equilibrium point of the régime in order that this point shall be Pareto-optimal under the production constraint.

Case 7

So far we have considered only régimes where the industries taken as a totality work with zero profit. Now let us drop this assumption. Consider a single industry—"industry as a whole"—producing a single commodity—"the national output"—with a single factor of production which is supplied by a single individual—"the labour class".

As in Case 4, let

$$X=f(N) \tag{6.59}$$

be the production function and let Y be the amount consumed by labour. This means that the residual $X-Y$ is retained by the industry. It is immaterial whether we call it "profit", "investment" or "consumption of the entrepreneur". The essential point is that by considering the two variables X and Y separately, we get two degrees of freedom. If labour tries to maximize an indicator $\Omega(Y,N)$ subject to the budget equation

$$pY-qN=0 \tag{6.60}$$

with the product price p and the wage rate q constant, and industry tries to maximize the profit $(pX-qN)$ on the same assumption about the constancy of p and q, the equilibrium point is determined by:

$$\frac{\omega_1}{p}=\frac{-\omega_{(1)}}{q} \tag{6.61}$$

$$pf'=q \tag{6.62}$$

where

$$\omega_1=\frac{\partial\Omega}{\partial Y} \qquad \omega_{(1)}=\frac{\partial\Omega}{\partial N} \tag{6.63}$$

The two equations (6.61)-(6.62) in conjunction with (6.59) determine the equilibrium point. In this case there is no over-determinateness because industry takes up whatever may be left over in the equilibrium point as a residuum $X-Y$, positive or negative. In other words, no condition on the *magnitude* of the (real value of the) profit $X-Y$ is imposed besides the *marginal* condition (6.62).

The real value of the profit in the equilibrium point is

$$X-Y=(1-\epsilon)X \tag{6.64}$$

where

$$\epsilon=\frac{d\log X}{d\log N}=\frac{f'N}{X} \tag{6.65}$$

This expression (6.64) for the real value of the profit in the equilibrium point follows immediately by noticing that in this point we have

$$f'=\frac{Y}{N} \tag{6.66}$$

because both sides in (6.66) are equal to $\frac{q}{p}$ in the equilibrium point.

If the production function f and the indicator Ω have arbitrarily given shapes, nothing prevents the equilibrium profit from being negative. If such a situation is considered unrealistic, *another régime must be introduced.* If we proceed more or less on Marshallian neo-classical lines, it would be natural to assume that ϵ is decreasing as N increases, and that there is some institutionally or conventionally fixed "normal profit"

$$\gamma=\frac{X-Y}{Y} \tag{6.67}$$

which "industry as a whole" must be assured in the equilibrium point. If (6.62) is actually *replaced* by (6.67) where γ is given, we have again, in general, a well-defined equilibrium point. But in this point we will, of course, *not* necessarily have (6.62), that is, marginal productivity will not necessarily be equal to the real wage rate. If we would require *at the same time* that (6.62) is fulfilled and that the "normal profit" γ takes on a preassigned value in the equilibrium point, we would get an *over-determinate* system. In a system where labour is assumed to act as a quantity adapter and we do not have any taxes that can be manipulated, it is—under arbitrarily given shapes of the production function f and the preference function Ω—impossible to impose both a *marginal* and a *total* condition on the adaptation of industry.

If we find that it is unrealistic to drop the assumption (6.62) in a market that is reasonably atomistic, we shall have to look for still another régime.

We can always assure a given "normal profit" γ and an atomistic marginal adaptation on the part of the entrepreneurs if we put a *point condition* on the shape of the production function. In our case this point condition will have to be that

$$\epsilon=1-\gamma \text{ in the equilibrium point} \tag{6.68}$$

where γ is institutionally or conventionally given. Personally, I don't think that this is a very realistic approach. If we want to be realistic, we do not have very much freedom in the choice of assumptions about our technical production function.

To me it would seem much more realistic to introduce a *tax system* and shape this tax system in such a way that we can maintain our assumption about atomistic marginal adaptation on the part of industry and still be able to produce an equilibrium point which satisfies certain desiderata about "normal profit". If we do this, and discuss the character which the tax system must have in order that the equilibrium point shall be Pareto-optimal under the production constraint, we will find that we get back to *the same fundamental properties* which we found in the previous examples which were constructed in a rather different way. Also in the present case must taxation of the workers have the character of a *per capita* tax in the vicinity of the equilibrium point. And if the disposable "normal profit" of industry shall have a given rate γ, the real value of the *per capita* tax on labour must have a *specific magnitude* in order that an equilibrium point shall exist. The previous examples illustrated the case $\gamma=0$. For the question of determinateness or over-determinateness it is, of course, immaterial whether we put γ equal to zero or to some other given value.

To verify the above conclusions let $T(qN)$ be the tax function on labour and $t(pX-qN)$ that on industry. We do not assume that we necessarily have

$$T+t=0 \tag{6.69}$$

T is a function of one variable, and so is t.

The budget equation for labour is now

$$qN-pY=T(qN) \tag{6.70}$$

and the equilibrium equation for labour now becomes

$$\frac{-\omega_{(1)}}{\omega_1}=\frac{q}{p}(1-T') \tag{6.71}$$

where $T'=\dfrac{dT}{d(qN)}$ is the marginal tax on labour.

The equilibrium equation for industry is now

$$(1-t')(pf'-q)=0 \tag{6.72}$$

If t' is not greater than unity, that is, if the tax on industry is not "over-progressive", (6.72) reduces to the same condition as before, namely (6.62). Thus, the equilibrium point of the régime is now determined by the three equations (6.59), (6.62) and (6.71) between the three unknowns X, Y and N.

Will the equilibrium thus determined be Pareto-optimal under the production constraint? Pareto-optimality is now determined by maximizing

$$\theta\Omega(Y,N)+\nu\Phi(X-Y)+\mu[X-f(N)] \tag{6.73}$$

where θ, ν, μ are constant multipliers, θ and ν positive. Φ is an indicator of utility to the industries. This leads to

$$(6.74) \qquad \nu\Phi'+\mu=0 \qquad \theta\omega_1-\nu\Phi'=0 \qquad \theta\omega_{(1)}-\mu f'=0$$

where Φ' is the derivative of Φ with respect to its argument. These equations entail amongst others

$$(6.75) \qquad \frac{-\omega_{(1)}}{\omega_1}=f'$$

Comparing this with (6.62) and (6.71), we see that it is *necessary* that

$$(6.76) \qquad T'=0 \text{ in the equilibrium point}$$

That is to say, T must have the character of a *per capita* tax in the vicinity of the equilibrium point of the régime. If T does not have this property, the equilibrium point of the régime cannot be Pareto-optimal under the production constraint. This, of course, is in essence the same as Proposition (6.45). For the tax on industry, t, no condition ensues (apart from $t'<1$).

7. What did Pareto prove on Welfare?

During my stay in Rome in the late fall of 1950, I had several intensely interesting talks with Professor Gustavo Del Vecchio on welfare economics and in particular on how Pareto's work in this connection should be interpreted. There is probably no living economist who knows the way of thinking of Pareto so well as Professor Del Vecchio. Some of Del Vecchio's views have been incorporated in his introductory remarks to Volume IV of the *Nuova collana di economisti* (pp. x-xi). On certain points he gave valuable further comments in the course of our talks. I am authorized to bring these points before the public.

First it might be well to restate Pareto's own definition of what we would now call a Pareto-optimal position. To do this one should not take as a starting-point *Corso di economia politica*, Lausanne, 1896. The approach in these lectures represents precisely what Pareto later tried to get away from. If one shall discuss the Pareto optimum, one should refer to his *Manuale*, 1909. On p. 337 of this work we find the definition which in English translation (checked by Del Vecchio) can be rendered as follows: "Let us begin by defining a term which is very convenient to use in order to save words. We shall say that the individuals (*i componenti*) of a group (*una collettività*) in a given position have *maximum ophelimity* (*massimo di ophelimità*) if it is impossible to depart some small distance (*allontanarsi pochissimo*) from this position in such a way that this departure is useful for all the individuals of the group. Every small displacement from this position would necessarily have the effect of being useful to some of the individuals of the society and detrimental to some others."

Professor Del Vecchio comments that Pareto had the habit of working fast and not to bother too much with limiting cases. Therefore when Pareto here says "useful for all individuals of the group" it should be interpreted "useful to at least one of them and for the rest of them either useful or indifferent".

In the appendix to the French edition of the *Manuel d'économie politique*, 1909, Pareto not only developed the mathematical theory of free competition, but also gave what Pareto himself seems to think is a mathematical demonstration that the régime of free competition leads to maximum satisfaction. In the subsequently published article in the *Encyclopédie des sciences mathématiques** Pareto only develops the mathematical theory of free competition and *stops short*, without adding anything on maximum satisfaction. The *Encyclopédie* was published in instalments, and from all appearances—not least from a comparison between the structure of the *Encyclopédie* article and that of the mathematical appendix to the French *Manuel*—it would seem that a section on maximum satisfaction was to follow in a subsequent instalment of the *Encyclopédie* article. However, this continuation never appeared. Del Vecchio thinks that this was an *advantage*. He thinks that Pareto attributed too far-reaching consequences to his analysis on this point. Pareto wrote in terms which must lead the reader to believe that Pareto has proved a proposition to the effect that a régime of free competition produces a higher level of satisfaction than other régimes. It is difficult to escape the conclusion that Pareto himself believed that he had actually demonstrated this. See, for instance, the phrasing used in the Italian *Manuale*, 1909, p. 544: "Quindi le conclusioni . . ." ("hence the conclusions . . ."). At any rate it is an indisputable fact that many people have thought that Pareto's demonstration was of this general character. This applies certainly to Barone in his "Il ministro della produzione nello stato collettivista",** 1908, and to a great number of mathematical economists.

This being so, a logical explanation of the fact that the continuation of the *Encyclopédie* article never appeared, would be that Pareto—who frequently worked over his manuscripts many times—had in the meantime become aware that the demonstration in his previous works was *not* a general demonstration of the sort he was in search of and had believed that he had produced.

In spite of all efforts made by Del Vecchio to find out definitely the reason for the discontinuation of the *Encyclopédie* article, an authentic answer has not been found. Pareto's own manuscript for a possible continuation has never been unearthed.

As the matter now stands, we must say that Pareto has not established any mathematical proof of a general proposition that free competition produces maximum satisfaction. What Pareto has done is only to give *a particular sort of definition* of maximum satisfaction. This definition in itself is one of great value and we should all be grateful for the analytical tool it has provided.

* Published in English as "Mathematical Economics", *International Economic Papers* No. 5, 1955—Ed.

** Published in English as "The Ministry of Production in the Collectivist State", Appendix A of *Collectivist Economic Planning*, edited by F. A. Hayek, London, 1935—Ed.