3438

SÄRTRYCK ur

EKONOMI POLITIK SAMHÄLLE

Festskrift till Bertil Ohlins 60-årsdag

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The Principle of Recurrent Planning

By RAGNAR FRISCH

THE MOST STRAIGHTFORWARD way to introduce intertemporal relations in economic planning is to consider a sequence of "years" or "stages", say $\tau = 1, 2 \ldots T$—which would mean a T-year plan—and for each of these years introduce a complete set of variables with all the equations and bounds that go with it. We could then formulate a complete programming problem—including a preference function—in all these variables. This may be called the *simultaneous multistage method.*

In this approach the number of variables will be very great. To avoid this difficulty attempts have been made at developing a *recurrent method.* The main idea in this direction stems from the work of P. Massé (1946). The sequential element was introduced by A. Wald (1950 and earlier), the generality of the approach was forcefully brought out by R. Bellman (Dynamic Programming, Princeton, 1957). Compare also the Presidential Address by Kenneth Arrow. Econometrica 1957.

The logical principle of the recurrent method is simple but the application to a full-fledged macroeconomic decision problem is, I believe, not a practical proposition in the present state of the theory. The principle itself, is, however, so interesting and may offer so many possibilities of improvement—leading perhaps finally to practical applications—that an exposition in macroeconomic terms may be useful.

I shall formulate the exposition in terms of capital, income, consumption, saving and total utility. The principle itself is, however, independent of the terminology used.

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First consider only two years, Nos. 1 and 2. Let $K_{\tau-1}$ be the capital at the beginning of year τ ($\tau = 1, 2, 3$) with K_0 given. Let Y_t ($\tau = 1, 2$) be income in year τ, and let

$$Y_\tau = rK_{\tau-1} \qquad (\tau = 1, 2) \qquad (1)$$

where r is a constant (usually $r < 1$). In other words we assume that there is plenty of labour available and that capital is the only scarce production factor. Let

$$Y_\tau = C_\tau + I_\tau \qquad (\tau = 1, 2) \qquad (2)$$

and

$$K_\tau = K_{\tau-1} + I_\tau \text{ hence } Y_{\tau+1} = Y_\tau + rI_\tau \qquad (\tau = 1, 2) \qquad (3)$$

Finally let $U_\tau(C_\tau)$ be the total utility function for consumption in the year τ, and $V(K_2)$ the total utility function for terminal capital, all the utility functions being viewed from a given point of time, say the beginning of year 1. Instead of total utility functions we may, if we like, speak of total preference functions.

Up to this point there are two degrees of freedom, which may be represented by I_1 and I_2. These two variables should satisfy the conditions

$$0 \lesseqgtr \bar{I}_\tau \lesseqgtr I_\tau \qquad 0 < \bar{C}_\tau \lesseqgtr C_\tau \qquad (\tau = 1, 2) \qquad (4)$$

where $\bar{I}_\tau$ and $\bar{C}_\tau$ are given lower bounds.

If I_1 and I_2 are chosen (and $Y_1 = rK_0$ given) the total utility gained over the two years will be

$$\Phi(I_1, I_2) = U_1(Y_1 - I_1) + U_2(Y_1 + rI_1 - I_2) + V(K_0 + I_1 + I_2) \qquad (5)$$

The decision problem—as viewed from the beginning of the first year—consists in maximizing (5) over I_1, I_2 under the constraints (4). Even in this over—simplified example the problem is difficult unless we replace the utility functions by segments of linear functions. If this is actually done, the problem is simple even if we use many breakdowns for Y_τ and I_τ i.e. interpret these two symbols as vectors.

If a T-year (a T-stage) problem is considered, we have

$$\begin{aligned}
Y_1 &= \text{given} &&= C_1 + I_1 \\
Y_2 &= Y_1 + rI_1 &&= C_2 + I_2 \\
Y_3 &= Y_2 + rI_2 &&= C_3 + I_3 \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
Y_T &= Y_{T-1} + rI_{T-1} &&= C_T + I_T
\end{aligned} \qquad (6)$$

and (5) will be replaced by

$$\begin{aligned}
\Phi(I_1, I_2 \ldots I_T) &= U_1(Y_1 - I_1) + U_2(Y_1 + rI_1 - I_2) + \\
&+ U_3(Y_1 + r(I_1 + I_2) - I_3) + \ldots + U_T(Y_1 + r(I_1 + I_2 + \ldots + I_{T-1}) - I_T) + \\
&+ V(K_0 + I_1 + I_2 + \ldots + I_T)
\end{aligned} \qquad (7)$$

There are here T decision parameters to choose, namely $I_1, I_2 \ldots I_T$. The variables will be bounded by (4), now applied for $\tau = 1, 2 \ldots T$.

The same problem can be attacked by the recurrent method in the following way.

We start a recurrent reasoning beginning in the *last* period. The basic idea of the recurrent method is that we say: *Suppose that Y_T has already been determined.* How should *then* I_T be chosen? The past history of the system up to and including the year $T - 1$ will now be of no interest, and we are only concerned with what *addition* to utility we can achieve by choosing I_T in the best possible way. To investigate this question we consider the partial utility function

$$\Phi_T(I_T) = U_T(Y_T - I_T) + V\left(\frac{Y_T}{r} + I_T\right) \tag{8}$$

This represents the addition to utility which is achieved by choosing the value I_T, when Y_T is given. In particular we are interested in the maximum of (8) over I_T, when Y_T is given. This maximum will itself be a function of the given value of Y_T. We denote it

$$\hat{\Phi}_T(Y_T) = \underset{I_T}{\text{Max}} \left[U_T(Y_T - I_T) + V\left(\frac{Y_T}{r} + I_T\right)\right] \tag{9}$$

where the variable I_T is subject to the conditions

$$\bar{I}_T \leqq I_T \leqq Y_T - \bar{C}_T \tag{10}$$

We will look a little closer into the maximization of (8). We assume that the functions U_T and V have continuous derivatives and consider the derivative of Φ_T with respect to I_T

$$\Phi'_T(I_T) = - U'_T(Y_T - I_T) + V'\left(\frac{Y_T}{r} + I_T\right) \tag{11}$$

We can assume that the marginal utilities U' and V' are monotonically *decreasing* as their arguments increase, hence $\Phi'_T(I_T)$ will be constantly *decreasing* as I_T increases, i.e. $\Phi(I_T)$ will be a concave function of I_T. There is consequently a unique value of I_T that makes (11) vanish and it will produce a maximum, not a minimum. Let this value be I_T^*.

Since the variables I_τ and C_τ are bounded by (4)—now applied for $\tau = 1, 2 \ldots T$—we see that the optimum value $\hat{I}_T$, i.e. the value that maximizes (8) is given as the following function of Y_T (whose value we have for a moment assumed to be given)

$$\hat{I}_T = \hat{I}_T(Y_T) = \begin{cases} \bar{I}_T & \text{if } \Phi'_T(\bar{I}_T) \leqq 0 \\ I_T^* & \text{if } \Phi'_T(\bar{I}_T) \geqq 0 \text{ and } \Phi'_T(Y_T - \bar{C}_T) \leqq 0 \\ Y_T - \bar{C}_T & \text{if } \Phi'_T(Y_T - \bar{C}_T) \geqq 0 \end{cases} \tag{12}$$

We will assume that $\hat{I}_T(Y_T)$ has been worked out *as a function of* Y_T *over a whole range of* Y_T, not only as a number corresponding to some specific value of Y_T. This is an essential feature which distinguishes the recurrent method fundamentally from the simultaneous multistage method. On a computing machine one would have to choose a grid of values for Y_T and the tightness of the grid would be a delicate question depending on the rapidity of change of I_T as a function of Y_T. I shall make some further comments on this functions aspect in the sequel.

For a moment suppose that the function $\hat{I}_T(Y_T)$ has been worked out over a Y_T-range which is sufficiently large for the uses we are going to make of the function. Enlightened guesses about the needed Y_T-range can, perhaps, be made from previous knowledge of the problem.

Inserting in (8) the function $\hat{I}_T(Y_T)$, we get the optimal value of Φ_T as a function of Y_T. We denote it

$$\hat{\Phi}_T(Y_T) = U_T(Y_T - \hat{I}_T(Y_T)) + V\left(\frac{Y_T}{r} + \hat{I}_T(Y_T)\right) \tag{13}$$

We assume that also the function $\hat{\Phi}(Y_T)$ has been computed over the Y_T—range considered.

Next let us move one step further back and now assume that Y_{T-1} is known. The past history of the system up to and including the year $T-2$ will then be of no interest to us, as long as Y_{T-1} is known. But we are greatly interested in the question of how I_{T-1} and I_T should be chosen.

Suppose that tentatively I_{T-1} has been chosen. Then $Y_T = Y_{T-1} + rI_{T-1}$ is also known. It is clear that *whatever value we fixed for* I_{T-1}, the best thing we can do *with respect to* I_T is to put it equal to $\hat{I}_T(Y_T)$, where now Y_τ is the value that emerges after the tentative selection of I_{T-1} (when Y_{T-1} is given). So it looks as if it would be of considerable interest to consider the following function of I_{T-1} (when Y_{T-1} is given)

$$\Phi_{T-1}(I_{T-1}) = U_{T-1}(Y_{T-1} - I_{T-1}) + \hat{\Phi}_T(Y_{T-1} + rI_{T-1}) \tag{14}$$

The first term in the right member of (14) expresses the consumption utility that is *actually* obtained in year $T-1$ by choosing the value I_{T-1} and the second term expresses by (13) the *maximum* utility $(U_T + V)$ in the terminal year that can be achieved when I_{T-1}—and hence $Y_T = Y_{T-1} + rI_{T-1}$—is already fixed (and we make the optimal choice of I_T that corresponds to the given Y_T).

All this applies *whatever* value of I_{T-1} we have chosen. It must consequently also apply if we choose I_{T-1} in the best possible way. Therefore the maximum total utility that can be realized *altogether* in the last two years $T-1$ and T, must be equal to the maximum of (14) considered as a function of I_{T-1} (with Y_{T-1} given and I_{T-1} satisfying (4) for $\tau = T-1$ and Y_T following I_{T-1}). This maximum value of Φ_{T-1} will be a function of Y_{T-1} and so will the value of I_{T-1} that maximizes (14). We denote these functions

$$\hat{\Phi}_{T-1}(Y_{T-1}) = \underset{I_{T-1}}{\text{Max}} [U_{T-1}(Y_{T-1} - I_{T-1}) + \hat{\Phi}_T(Y_{T-1} + rI_{T-1})] \qquad (15)$$

$$\hat{I}_{T-1}(Y_{T-1}) = \text{the value of } I_{T-1} \text{ that produces the maximum of (15).} \qquad (16)$$

If the function $\hat{\Phi}_T(Y_T)$ defined by (13) has previously been computed for a sufficiently large range of values of its argument Y_T, it is in principle possible to carry out the maximization indicated in (15). In general it will be a much more complicated task than the maximization of (8) which led to (9). And we must face the problem of carrying out the maximization of (15) for a whole range of values of Y_{T-1}. On the extension of this range we may, perhaps, make enlightened guesses using previous knowledge of the problem. But these are practical and computational questions to which I shall revert. In principle it is possible to construct the function $\hat{\Phi}_{T-1}(Y_{T-1})$ defined by (15). And in the process of constructing it, the function $\hat{I}_{T-1}(Y_{T-1})$ will come out as a by-product.

In this way we can continue. Assuming that Y_{T-2} is given, we may disregard the history of the decision process up to and including the year T—3. For any given choice of I_{T-2} the additional utility produced altogether in the years T—2, T—1 and T, on the assumption that I_{T-1} and I_T are to be chosen in the way that is the best possible one when I_{T-2} (and Y_{T-2}) are given, will be

$$\Phi_{T-2}(I_{T-2}) = U_{T-2}(Y_{T-2} - I_{T-2}) + \hat{\Phi}_{T-1}(Y_{T-2} + rI_{T-2}) \qquad (17)$$

Maximizing (17) over I_{T-2} (with given Y_{T-2}), we get the functions $\hat{\Phi}_{T-2}(Y_{T-2})$ and $\hat{I}_{T-2}(Y_{T-2})$. And so on.

To summarize: We consider the recurrent definition of the two sets of functions $\hat{\Phi}_\tau(Y_\tau)$ and $\hat{I}_\tau(Y_\tau)$ by the relations

$$\hat{\Phi}_\tau(Y_\tau) = \underset{I_\tau}{\text{Max}} [U_\tau(Y_\tau - I_\tau) + \hat{\Phi}_{\tau+1}(Y_\tau + rI_\tau)] \qquad (\tau = T-1, T-2, \ldots 1) \quad (18)$$

$$\bar{I}_\tau \gtreqless I_\tau \gtreqless Y_\tau - \bar{C}_\tau \qquad (\tau = T-1, T-2, \ldots 1) \quad (19)$$

$$\hat{I}_\tau(Y_\tau) = \text{value of } I_\tau \text{ that maximizes (18).} \qquad (20)$$

As an initial condition the function $\hat{\Phi}_T(Y_T)$ is determined by (9), the solution of which is indicated in detail in (12) and (13).

Continuing this process we will end up with the function $\hat{I}_1(Y_1)$. And from this point the whole solution can be unfolded because we assume that K_0 is given, which is the same as to say that Y_1 is given. The optimal value $\hat{I}_1$ follows then simply by inserting the given Y_1 in the function $\hat{I}_1(Y_1)$. From this follows the optimal value $\hat{Y}_2 = Y_1 + r\hat{I}_1$, and hence the optimal value $\hat{I}_2$ by inserting $\hat{Y}_2$ for Y_2 in the function $\hat{I}_2(Y_2)$. And so on.

In the more complicated cases this method leads to problems that are at present not solved, neither mathematically nor computationally. It may nevertheless be of interest to indicate here the directions in which it would be necessary to develop this theory in order to make it macroeconomically applicable in practice.

In the first place generalizing without introducing higher dimensionality, we may replace (6) by

$$Y_\tau = aY_{\tau-1} + bI_{\tau-1} \tag{21}$$

where a and b are constants.

Secondly, in (18) there is only a function U_τ of the single variable $(Y_\tau—I_\tau)$. We may add another function depending on I_τ or, more generally, we may replace U_τ in (18) by a function U_τ of the two variables Y_τ and I_τ so that (18) takes the form

$$\hat{\Phi}_\tau(Y_\tau) = \underset{I_\tau}{\text{Max}} \, [U_\tau(Y_\tau, I_\tau) + \Phi_{\tau+1}(Y_\tau + rI_\tau)] \tag{22}$$

A drastic simplification would occur if we assumed U_τ in (22) or U_τ in (18) to be independent of τ. Such a simplification would hardly be acceptable in macroeconomics. If for no other reasons, it would be made impossible by the "perspective shortening" of the evaluation of future goods and possibilities. This could be expressed approximately by a discount factor, but the particular form in which a dependency on τ is introduced is of lesser importance. The essential fact is that some sort of dependency on τ must be introduced.

From a theoretical viewpoint it may be of interest to investigate the situation that arises if the number of years, i.e. stages, tends towards infinity. Intuitively one would guess that in the special case where U in (18) or (22) does not depend on τ, convergence would be produced by the fact that the coefficients a and b in (21) are non negative and less than unity ("resources are used up"). If U depends on τ, a still more important element in the convergence will be the way in which U depends on τ ("the perspective shortening"). In all practical situations the limiting case of an infinite planning period would, however, seem to be of little practical interest in macroeconomics. The way in which U depends on τ would be such that after a relatively small number of years the difference between a T year plan and a $T+1$ year plan would be much smaller than could be revealed by the data, which will always be more or less inaccurate.

A much more important direction in which to perfect the theory would be to introduce higher dimensionality in the decision process of each year, i.e. at each stage. Let

$$Y^1_\tau, Y^2_\tau, \ldots, Y^n_\tau \text{ and } I^1_\tau, I^2_\tau \ldots, I^m_\tau \tag{23}$$

be variables characterizing the situation in the year τ. Let $I^1_\tau, \ldots, I^m_\tau$ be variables to be chosen, and let all the $Y^\nu_{\tau+1}$ grow out of the Y^ν_τ and the decision variables I^ν_τ by a rule of the form

$$Y^\nu_{\tau+1} = \eta^\nu_{\tau+1}(Y^1_\tau \ldots Y^n_\tau, I^1_\tau \ldots I^m_\tau) \tag{24}$$

where the $\eta^\nu_{\tau+1}$ are given functions.

Finally let

$$U_\tau(Y^1_\tau \ldots Y^n_\tau, I^1_\tau \ldots I^m_\tau) \tag{25}$$

be the utility of the state (23).

By analogy with the reasoning above, we would be lead to consider a recurrence scheme of the form

$$\hat{\Phi}_\tau(Y^1_\tau \ldots Y^n_\tau) = \underset{I^1_\tau \ldots I^m_\tau}{\text{Max}} [U_\tau(Y^1_\tau \ldots Y^N_\tau, I^1_\tau \ldots I^m_\tau) + \hat{\Phi}_{\tau+1}(\eta^1_{\tau+1} \cdots \eta^n_{\tau+1})] \tag{26}$$

where the $\eta^\nu_{\tau+1}$ are to be replaced by their expressions (24). A definition of the admissible region in the Y^ν_τ and I^μ_τ would have to be added.

The recurrent decision functions, i.e. the functions that express the policy to be adopted in a year characterized by $Y^1_\tau \ldots Y^n_\tau$ would follow as a by-product of the maximalization (26) and would be of the form

$$\hat{I}^\mu_\tau = \hat{I}^\mu_\tau(Y^1_\tau \ldots Y^n_\tau) \qquad (\mu = 1, 2 \ldots m) \tag{27}$$

Now for an evaluation of the practical possibilities. It is true that by a recurrent method the dimensionality of the problem is greatly reduced as compared to the dimensionality involved in a simultaneous formulation of a multistage problem. *But there is a very heavy price to be paid for this reduction.* Instead of a problem concerning the fixation of a certain number of variables, we will have to compute the numerical shapes of a certain number of *functions*, namely the $\hat{\Phi}_\tau$ and the $\hat{I}^\mu_\tau$, each depending on n variables, namely the Y^ν_τ. The storage of information about such numerical function shapes represents very serious problems. The tightness of the Y^ν_τ-grids needed would have to be decided on in the light of computational experience, and even assuming that the grids have been well fixed, the storage would represent practically unsurmountable difficulties as soon as the number of variables Y_τ^ν on which the functions depend, is somewhat large, as it always will be in macroeconomic problems.

Considering all these difficulties, I must confess that at the present state of affairs I do not see much hope that the computational task can in a general way be drastically reduced by turning from the simultaneous multistage formulation to the recurrent formulation.

A thorough study of the special *structure* of the problem may, of course, lead to special reformulations that may mean an enormous saving. This is perhaps at present the most hopeful aspect. But this possibility is open not only in the recurrent but also in the simultaneous formulation.

The formulation (26) is closely related to the general theory of constructing sequential decision functions. In the language of this theory Y_τ^ν will represent "the information available" at time τ, $U_\tau(Y_\tau^1 \ldots Y_\tau^n, I_\tau^1 \ldots I_\tau^m)$ the "pay-off" in period τ and the functions $\eta_{\tau+1}^\nu(Y_\tau^1 \ldots Y_\tau^n, I_\tau^1 \ldots I_\tau^m)$ "the information transmitted" to the next period. The stochastic element may be introduced by letting U_τ depend on certain stochastic variables and introducing in the recurrence formulae (26) the mathematical expectation of U_τ instead of U_τ itself. Formal theories expressed in this language may be extremely valuable in special problems of low dimensionality, but—for the reasons stated above—the immediate application to macroeconomic problems of high dimensionality does not seem possible and we must look for other more drastic simplifications.

Since we need to consider different types of approaches to the problem of intertemporal planning, I do not think it would be a happy terminology to use the term "dynamic programming" only for methods built on the recurrence idea. The term dynamic programming should be used for all forms of intertemporal planning—crude or refined—while the special approach through (27) or some similar equations, should be designated by a more special term.