STUDIES RELATING TO PLANNING FOR NATIONAL DEVELOPMENT

No. 3

PLANNING FOR INDIA : SELECTED EXPLORATIONS IN METHODOLOGY

By

RAGNAR FRISCH

INDIAN STATISTICAL INSTITUTE

CALCUTTA : 1960

FOREWORD

Shri Jawaharlal Nehru, Prime Minister of India, inaugurated in November 1954 in the Indian Statistical Institute studies on planning for national development. Since then the Institute has been actively engaged in studies on planning in collaboration with the Planning Commission, the Central Statistical Organization and other government agencies and a considerable amount of work has already been done. A part of the work was done either directly by foreign experts who visited the Institute during the last four or five years or in cooperation with them. Indian scholars were responsible for the remaining part of the work. Some of these papers, particularly those by the visiting experts, are of considerable interest from the methodological and theoretical point of view. Some others are of value because they contain interesting estimates and analyses. A close collaboration of the Planning Commission, the Central Statistical Organization and the Economic Wing of the Finance Ministry was available for the preparation of a number of these studies and the findings are something more than abstract speculations by isolated research workers. On the other hand there are papers which are mere exercises by junior staff.

All these studies were incorporated in our mimeographed series. "Studies Relating to Planning for National Development" as well as in other mimeographed reports and papers. As there has been continuing demand for the mimeographed papers it has been decided to bring them out in a printed form to make them available to all research workers interested in this matter.

This should be of help in promoting studies on planning in the country. Experience of the last few years has shown that for progress of planning it is necessary that persons outside government agencies should participate in thinking on planning in concrete terms. In India it is now generally accepted that planning is necessary and desirable. We have now the task of putting across to the public what planning really implies, and this has to be done at a technical and technological level. Wider dissemination of ideas is, therefore, essential.

The present publication is the third in the printed series. We propose bringing out in this series selected papers based on both the older and current studies. We hope this series will add something of value to the growing body of literature on planning in India.

22 February 1960 — *P. C. Mahalanobis*

PREFACE

During my stay at the Indian Statistical Institute from the end of 1954 to the spring of 1955 I wrote a number of memoranda on various technical aspects of national planning.

The selections and excerpts from these memoranda that are presented here, have been made by the editors at the Indian Statistical Institute. What to include and what to leave out is definitely a matter of taste, and the selection here made is probably as good as it would have been if I had made it myself.

On reading the general parts of this material in galley proofs, I find that the gist of the planning philosophy, as I developed it at that time, is still valid. It is my firm conviction that a substantial development in this direction is bound to come, not least in underdeveloped countries.

Since my stay in Calcutta I have devoted most of my time to the further development and streamlining of the relevant methodology, and a number of memoranda describing these efforts have been widely circulated from the Institute of Economics at the University of Oslo, and to some extent from the National Planning Committee, Cairo. A systematic presentation of the methodological results and practical experience is in preparation.

For the account of my logarithmic potential method of linear programming which is contained in the present material, the Editor Shri S. Sankar Sengupta is responsible.

I have had reports from various quarters that the method has been successfully used on medium sized problems. The method depends, however, very much on the skill of the computor and therefore, is not suitable for automatic computations, at least not in the form which I have been able to give it so far.

For big problems on electronic computers I put great faith in my multiplex method. In its original form it was first published in Sankhya, the Indian Journal of Statistics 1957. Subsequently it has been streamlined and shaped into a form well suited for automatic computation (Memorandum of 12 September 1958 from the Institute of Economics at the University of Oslo). It has been very successfully coded for the Norwegian electronic Mercury computer Frederic by Mr. O.e-Johan Dahl, research associate at the Norwegian Defence Research Establishment (Report F-375, March 1959). A flow diagram is included as an appendix. A sufficient number of problems have been solved on Frederic to verify that the coding works. The computing time seems to be proportional to n^3 or $n^{3.5}$ where n is the number of degrees of freedom.

Work is going on to code the method for the Deuce machine of the Central Bureau of Statistics, Oslo.

I believe that the multiplex method compares favourably with the simplex method in the case where there is a moderate number of degrees of freedom (say not larger than 32) and a large number of equations, and many variables are bounded upwards as well as downwards and there are structural relations which make many of the variables nearly linearly dependent.

April 1959

Ragnar Frisch

CONTENTS

PLANNING IN UNDERDEVELOPED ECONOMY

1. Can peacefulness and speed be combined ?

Peaceful planning for national development is the burning question of the day in a large number of underdeveloped countries in Asia and Africa and it is also an important issue in many countries of Europe and Latin America.

At the same time, speed is essential in all the underdeveloped countries.

Can these two objectives - peacefulness and speed be combined ? The question is pertinent because basically the two objectives are antagonistic. They can be combined, however, on one condition, a condition which is a sine qua non : A streamlined rational methodology for the planning work must be developed, a methodology that utilizes deep-ploughing scientific procedures not only for the gathering of technical and statistical data but also for *selecting the optimum way* in which to combine the various kinds of development.

It can safely be said that the First Five Year Plan of India has been an unquestionable success in the sense that great things have been accomplished under the Plan. However, a constructive question may be raised whether it would not have been possible to achieve still more or to achieve the same results by means of a smaller amount of human sufferings if the individual parts of the plan had geared better into each other,—if more scientific analyses had gone into finding out which was the best possible combination of specific targets to be found amongst the virtually infinite number of combinations which were realistically conceivable at the time when the Plan was worked out.

What is of crucial importance today is to insist that a definite attempt be made to assure that the next plan shall come as close to the optimum solution as is possible to get with the available statistical and technical information. In order to do this, it is necessary to apply existing scientific techniques fully. But this must be done in such a way that the responsible political authorities maintain complete control over the whole thing at every stage of the work. It is possible to assure this if the successive phases of the work are arranged in an appropriate way.

2. Planning in real terms, but not ad hoc estimates

Many orthodox economists have lived so long in an atmosphere of banking, money and money-cost way of thinking, that they seem to be unable to see that all these concepts only pertain to a special case of institutional set up. Behind these symbols there is a world of realities which will persist no matter what sort of economic institutions man devises. We must begin by planning in terms of these real things. Of course, the monetary aspects of the problem do come in; but they form only a system of tools which we may find convenient to use in implementing the plan. We have considerable freedom of choice in shaping and reshaping this system of tools, but we have very little freedom with respect to these physical means. It is, therefore, in the physical systems that the essence of the problem lies.

Scientific planning means coordinating everything in one simultaneous (and integrated) piece of analysis and doing it on some optimum basis. It is solving the whole nexus as one simultaneous problem where everything determines everything else, much in the same way as all the unknowns in a *system* of equations determine each other simultaneously.

This is the only way in which the existing physical possibilities can be fully utilized with optimum speed.

This means that not only are all sorts of demand effectively balanced against one another, but the quantitative ratios between these demands (and more generally, the quantitative ratios between any of the projects) are determined in an optimum manner, i.e., in that particular way which permits us to fulfil the desired ends most effectively. For instance, there will be one particular ratio between the rates of expansion in the producer goods and consumer goods industries, and again one particular ratio between the rates of expansion in the heavy and light producer goods industries and particular ratio between the rates of expansion of individual industries in these categories which are such that if this particular set of ratios is chosen we shall be able to realize the desired ends more closely than by choosing any other set of ratios of expansion. It is the optimal determination of these ratios that constitutes the essence of scientific as distinguished from guess-work planning.

3. Logical steps in formulating the plan

3A. *Statement of desired ends*

The first phase of any reasoned economic plan is to state explicitely *what one wants to obtain.*

But it is equally important to emphasize that to plan an economic development is not simply to write out a list of things we like : More rice, more cotton and wool, more cement and machine tools, etc. To take action on the basis of such a list may lead to enormous losses because the list may be lacking in inner consistency or may lead to unexpected results which may force us to make last minute improvised changes. In this manner we would learn it the hard way, by trial and error.

To do real planning means to *foresee* as far as possible all repercussions, to take account of them beforehand in a consistent way and amongst alternative courses of action to choose that one which comes nearest to realizing what we really want. It may, for instance, well be that everything taken into account, this optimal solution does not include the use of tractors in Indian agriculture in the first years to come although even a layman will understand that a tractor is generally a good thing in agriculture. How far to expand the sectors A, B, C,... etc., must in general be a matter of *compromise.* And this compromise should *not* be made by *an a priori* guess at how far and how rapidly the expansion in these sectors should be for best total results, but by formulating certain requirements *of a very general sort*, and then leave it to the analytical machinery to find a solution that satisfies simultaneously these various requirements and at the same time tells us how far and how rapidly to expand sectors A, B, C,...etc., in order to comply with the requirements formulated.

For instance, we may formulate the requirements in some such terms as these :

1) In no single year in the planning period should consumption decrease. In the course of the first 10 years it should increase by at least 7 per cent, in the course of the next 10 years by 15 per cent.

2) Unemployment, open or disguised, as measured by some statistical indicator, should never increase. And it should never reach a level higher than such and such a figure.

3B. *Specifying the list of activities*

The preparatory work for planning in a democratic society will consist in making up a list of possible activities. The inclusion of any activity in this list does not mean that one has as yet made any commitment to make this activity a part of the final plan. And still less is any commitment made regarding the *level* at which this activity might be operated under the final plan. This list of contemplated activities should be looked upon as a list of *alternatives*. The meaning of this can be best explained by some illustrations. For each specific development project, the viewpoint should not be that we first estimate how large the effective demand for this sort of product will be and then work out the detailed capacity-plans accordingly. To proceed in such a way is really not to be planning, but to start by a specific sort of guessing and subsequently computing the details that correspond to the overall guess adopted. The list should contain not only such conspicuous and much-talked-about *labour saving* and capital intensive projects as modern steel production, aluminium production, heavy fertiliser production, etc. Also a number of *labour intensive* activities should be included such as road building, slum clearing, minor irrigations, construction of open air schools, handloom weaving, etc. Only by including a sufficiently large spectrum of such activities will it be possible at a later stage to find optimal solutions that can really satisfy the desired conditions on employment in a transition period.

It is, of course, not enough to have *a list* of the possibilities. To actually incorporate an activity in the analysis it must be *technically described.*

As one goes further and *increases the list of projects* considered, there is, however, another aspect of the problem that comes more and more into the foreground and ends by becoming an overwhelmingly important part of the whole nexus : as the list of projects increases it becomes more and more impossible by mere common sense to follow the innumerable ways in which the various things mutually depend on each other and influence each other. In particular it becomes impossible by mere common sense to find out what the total effects will be on *employment, consumption* and the *balance of payment situation.*

There exists, however, a well-worked-out technique for laying bare all these repercussions and for doing it in way which gives automatic checks, much in the same way as automatic checks are furnished by the double entry book-keeping system in an individual enterprise. *The planning chart* is the formal frame around which this technique centres. The chart is an expansion and generalisation of the now classical inter-industry table.

The planning chart may be applied for various purposes. One is to accumulate technical coefficients that describe the established production methods in the various sectors and also new projects that have been proposed. Another is to balance the final plan out in rupees.

For the production sectors this balancing is important because it will check that the economy can proceed in a harmonized way with no single sector as a bottleneck.

For consumer demand the balancing is even more important. If political and social unrest is to be avoided, the development of consumer goods and producer goods industries must be harmonized with each other in such a way that effective consumer demand can be satisfied. In checking this one must take account not only of the technical relations within the production sectors but also of the way in which changing levels of production in the production sectors change the flow of purchasing power going to the consumers, thereby affecting consumer demand in a specific way which depends on the nature of the price, wage, tax, subsidy policies that are followed and on the behaviour of the consumers. The planning chart is a framework where information about these relations is organized and collected.

3C. *Determining the optimum combination of targets*

When the desired ends have been formulated by the appropriate political organs, and the list of contemplated activities—that is of alternatives of production—have been fixed through the work of the technical and economic experts—finally passed on by the political organs—an entirely different aspect of the problem emerges, namely how to find which particular *combination* of the rates of development of the various activities that should be chosen in order to realize the desired ends to the highest possible degree.

To find this optimum combination of targets is a work which in its character is entirely different from that involved in the other phases of the planning.

Indeed, the number of possible combinations is virtually infinite and it would be a hopeless task to proceed by trial and error. For each combination that could be tested in this way there would be hundreds or thousands of other combinations that still remained untested.

Fortunately the field of scientific analysis which has now come to be known as operational analysis—to a large extent based on what is called linear programming —provides us with tools by which this search for an optimum solution can proceed in an orderly and systematic fashion.

The work involves very large computations. When a large capacity electronic computer becomes available in the Indian Statistical Institute or in some other Indian institutions, these problems can be handled with comparative ease. For the time being it will be necessary to use more man.power and more machine-power of the classical punched card sort. In any case the necessary man-power and machine-power is available so that the problems can be handled.

To get an idea of how sensitive the solution is to the inaccuracy of the data the calculations can be carried through on slightly different set of data and the results compared.

In view of the special character of this work it must be organized in a special unit. It is imperative that its administrative affiliation is so organized that the unit can have the closest possible connection with the centre where the relevant scientific technique is being cultivated.

3D. *Some typical problems*

Some typical problems which can be solved by the approach outlined earlier are indicated below.

Type I question. If an *additional investment* of a given size is made *in* any of the following 22 production sectors :

1. *Primary products :*
    1. Agriculture
    2. Animal husbandry, fishery and forestry
    3. Mining

B. *Large-scale manufactures :*

4. Metal and engineering
5. Chemicals
6. Building materials and wood manufacture
7. Fuel oil and power
8. Food, drink and tobacco
9. Textile and textile products
10. Ceramics and glass
11. Leather and rubber
12. Paper and printing

C. *Small-scale manufactures :*

13. Textile and textile products
14. Metal
15. Food, drink and tobacco
16. Building materials and wood manufacture
17. Miscellaneous

D. *Others :*

18. Railways and communication
19. Trade and other transport
20. Banks, insurance, professions
21. Construction
22. House property

what will be the probable effects on any of the basic magnitudes that characterize the economy, provided the following assumptions are made :

(a) The input coefficients for materials produced in domestic sectors and the labour input and import input coefficients (all expressed as percentages of the total production in the sector in question) are not appreciably affected by the additional investment considered.

(b) The *taxes* levied on the various sectors remain constant in relation to the total output of the sectors and similarly taxes on households remain constant in relation to the income of the households. Similar assumption is made for the *gross savings* within the various production sectors and within the households.

(c) The proportions in which *households current use of goods and services* is distributed, over the domestic production sectors and over imports, are not appreciably effected by the additional investment considered.

(d) Similar assumption as in (c) is made for *government use of goods and services on current account, and*

(e) For the proportion in which *exports* are distributed over the various production sectors.

Type II question. If the pattern of investment—that is to say the proportions in which gross investment is distributed over the sectors, remains constant, how will all the basic magnitudes that characterize the economy most likely be affected by a given increase or decrease in *total gross savings ?*

Other assumptions are the same as specified under (I).

Type III question. If the pattern of investment—that is to say the proportions in which gross investment is distributed over the sectors—remains constant, how will all the basic magnitudes that characterize the economy most likely be affected by a given increase or decrease in *total exports?* Or in total *factor income?* Or in *total government expenditure?*

Other assumptions are the same as specified under (I).

However the analysis is shaped, a fairly exhaustive list of assumptions must *always* be made explicity for the analysis to have a satisfactory standard of clearness and precision. It is easy to produce simplicity of exposition by not facing the assumptions, but such a procedure is not conducive to the clearness and precision which is needed in discussing numerical conclusions pertaining to the ecnomy as a whole.

4. A MODEL OF BALANCED EXPANSION

1. *The meaning of balanced expansion*

Some fundamental factors which must claim attention under a policy of expansion in an underdeveloped country are: *The capacities* in the various production sectors as determined by the capital outfit of the sectors, the need for *investment* in these sectors in order to cover the *depreciation* on the capital outfit and to *expand* the capacities, the *maturity lags* for the investments in the various sectors, i.e., the length of time that will elapse between the moment when the actual investment input is made and the moment when the corresponding capital goods are ready to function as factors of production (this lag is very different for different types of investment); finally and most important of all *the employment* in the various sectors, and the production of *consumer goods.*

These various factors are connected in a great number of ways which must be clarified and taken account of if the development is to proceed in an orderly and *balanced* fashion.

Two conditions are essential in order that we shall be able to say that the development proceeds in a balanced way:

In the first place, the *capacity* in each sector must at any time be neither too large nor too small but just in harmony with the total production which is needed from this sector at that time. This total production will be used for three purposes: (a) *input* in other sectors, that is cross-deliveries on current account, (*b*) *investment* in other sectors in order to maintain in the future years the exact capacity balance in these other sectors, and (c) *consumption goods* delivered on current account to cover final demand. For simplicity we shall disregard exports and imports. It would not be difficult to include these items, but the formula would become a little unwieldy.

In the second place, the *labour force* attached to each sector should at any time be neither too large nor too small but just in harmony with the total production needed from this sector at that time.

These conditions of balance express an ideal which will of course never be reached completely in practice, but it is well to keep in mind that in point of principle these conditions must be fulfilled if we are to characterize the development as "balanced".

It is entirely impossible to carry through an analysis of these complicated relations by a verbal argument alone. We must base the reasoning on a mathematical theory.

2. *Derivation of the conditions*

We shall make certain assumptions. They may not cover all the variety of details that could be found by a meticulous investigation of the economy but they do give all the essentials needed in a first approach.

We use the following notations

$i = 1, 2 \ldots$ or $j = 1, 2 \ldots$ or $k = 1, 2 \ldots$ or $h = 1, 2 \ldots$ are used to denote sectors.

X^t_{ij} = cross-deliveries from sector i to sector j on current account in year t. The "year" may be a time period of conventional length. It is convenient to choose it in such a way, (for instance, six months or a quarter) that all the the maturity lags can be indicated by integer numbers.

C^t_i = consumption goods on current account delivered from sector i in year t

J^t_{ij} = investment goods delivered from sector i to sector j in year t

$J^t_{i.} = \Sigma_j J^t_{ij}$ = total of investment goods delivered *from* sector i in yeart t.

$J^t_{.j} = \Sigma_i J^t_{ij}$ = total of investment goods delivered *to* sector j in year t

$X^t_i = \Sigma_j X^t_{ij} + C^t_i + J^t_{i.}$ = total production, i.e., total level of activity in sector i in year t, ... (1)

$$X^t_{ij} = A_{ij} X^t_j \qquad \ldots (2)$$

where the cross-delivery coefficient A_{ij} is assumed to be a *constant*, independent of t, and also independent of the magnitude of X^t_j.

Inserting (2) into (1) we get

$$\Sigma_j (e_{ij} - A_{ij}) X^t_j = C^t_i + J^t_{i.} \qquad \text{for all } i \qquad \ldots (3)$$

where

$$e_{ij} = \text{unit numbers} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} \qquad \ldots (4)$$

The formula (3) express a system of linear equations, equal in number to the number of sectors. Solving it for the levels of activity X^t_1, X^t_2... we get

$$X^t_i = \Sigma_j\, a_{ij}(C^t_j + J^t_{j.}) \qquad \text{for all } i \qquad \dots \quad (5)$$

where matrix a_{ij} = inverse of matrix $(e_{ij} - A_{ij})$. ... (6)

By (5) the total levels of activities X^t_1, X^t_2 ... are expressed in terms of the *final* deliveries ("the bill of goods") from the various sectors, that is $(C^t_1 + J^t_1)$. Inversely by (3) the final deliveries are expressed in terms of the total levels of activities.

For policy making purposes the investment items classified by *receiving* sectors, i.e., the magnitudes $J^t_{.j}$ are even more important than the investment items classified by *delivering* sectors, i.e., the magnitudes $J^t_{i.}$. To bring $J^t_{.j}$ into the formulae we introduce the *investment coefficients* B_{ij} defined by

$$J^t_{ij} = B_{ij}\, J^t_{.j} \qquad \text{for all } i \text{ and } j. \qquad \dots \quad (7)$$

The coefficients B_{ij} have a similar meaning as the cross-delivery coefficients A_{ij}, the B_{ij} indicate how large a fraction of the investment in sector j will have to come from sector i. We assume the B_{ij} to be constants.

From (7) follows

$$J^t_{k.} = \Sigma_h\, B_{kh}\, J^t_{.h} \qquad \text{for all } k \qquad \dots \quad (8)$$

which can also be written

$$J^t_{k.} = \Sigma_h(c_{kh} - \Sigma_i\, A_{ki}\, c_{ih})J^t_{.h} \qquad \text{for all } k. \qquad \dots \quad (9)$$

Introducing (8) into (5) we get

$$X^t_i = \Sigma_k\, a_{ik}\, C^t_k + \Sigma_h c_{ih}\, J^t_{.h} \qquad \text{for all } i \qquad \dots \quad (10)$$

where

$$c_{ij} = \Sigma_k\, a_{ik}\, B_{kj} \qquad \text{for all } i \text{ and } j. \qquad \dots \quad (11)$$

We assume that the *capacity* K^t_h, when fully used, stands in a certain proportion to the total production in sector h, that is

$$K^t_h = b_h\, X^t_h \qquad \text{for all } h \qquad \dots \quad (12)$$

where the b_h are constants.

We also assume that the *depreciation* D_h^t stands in a certain proportion to the total poduction in sector h, that is

$$D_h^t = d_h X_h^t \qquad \text{for all } h \qquad \ldots \quad (13)$$

where the d_h are constants.

Finally we assume that the *labour requirement* N_h^t stands in a certain proportion to the total production in sector h, that is

$$N_h^t = n_h X_h^t \qquad \text{for all } h \qquad \ldots \quad (14)$$

where the n_h are constants.

The way in which the capacity in any sector evolves over time is given by

$$K_h^t = K_h^{t-1} + J_{.h}^{t-s_h-1} - D_h^{t-1} \qquad \text{for all } h \qquad \ldots \quad (15)$$

where s_h is the maturity lag in sector h.

Note the way in which the time affixes are used. The time affix on the *stock* concept K indicates the exact point of time to which the stock is associated and the time affix t on a flow concept, i.e., a current account concept, indicates that the flow concept in question measures what has happened *between* the points of time t and $t+1$.

Equation (15) shows how the capacity at time t is determined by what happened *before* t, while equation (12) shows how the capacity at time t determines what will happen after t. This follows from the assumption about a balanced development.

From (12) and (14) follows

$$K_h^t = \frac{b_h}{n_h} N_h^t \qquad \text{for all } h. \qquad \ldots \quad (16)$$

And from (13) and (14) follows

$$D_h^t = \frac{d_h}{n_h} N_h^t \qquad \text{for all } h. \qquad \ldots \quad (17)$$

Inserting (16) and (17) into (15) we get

$$J_{.h}^t = \frac{b_h}{n_h} N_h^{t+s_h+1} - \frac{b_h - d_h}{n_h} N_h^{t+s_h} \qquad \text{for all } h. \qquad \ldots \quad (18)$$

This is a remarkable formula. It shows that if the *population growth curves* are given, and if one decides to maintain certain given proportions of the working population in the various sectors, then there *is no more room for policy decisions about investment.* If the development goes on in a balanced fashion, the total investment which has to be made in any year is uniquely determined. It depends on some of the technical coefficients in a well defined way as given by (18). If this is not followed there must sooner or later develop unemployment of men and/or capital capacity, or labour will have to be moved rapidly so as to produce a population distribution over the sectors different from the one originally envisaged.

The cross-delivery coefficients do not enter the formula (18), nor do they come in when we ask how the total levels of activities depend on the distribution of the working population. But they do come in when we ask the all important question of the *consumer goods production* that will follow from a given distribution of the working population under a balanced development.

This consumer goods production can be computed in different ways. The simplest is perhaps to use the expression

$$C^t_k = \frac{N^t_k}{n_k} - \Sigma_j A_{kj} \frac{N_j}{n_j} - \Sigma_h B_{kh} J^t_{.h} \qquad \text{for all } k. \qquad \ldots (19)$$

The first term in the right member of (19) expresses the total production in sector k in year t, the second represents all cross-deliveries on current account from sector k in year t, the last term being expressed by means of the investment items classified by receiving sectors. These items themselves are given in terms of the distribution of the working population by (18).

Instead of the matrix B_{kh} in (19) we can also, if we like, use

$$B_{kh} = c_{kh} - \Sigma_i A_{ki} c_{ih} \qquad \text{for all } k \text{ and } h. \quad \ldots (20)$$

The latter expression is more complicated to use if we start from scratch, but it may be advantageous if certain intermediate results are already available, c_{ih} having perhaps been estimated independently. The formula (20) may also be used as a check on the numerical work.

3. *The conditions in relation to operational planning*

The above argument has a special application in formulating the *preference function* in an operational planning problem.

First suppose that the preference function is tentatively taken in the form

$$f = 16u + 4v + w \qquad \ldots (21)$$

where u = millions of new jobs created annually, v = annual rate of investment, and w = net annual increase in India's net foreign assets (liquid or non-liquid), expressed as a percentage of the national income.

In a situation where unemployment exists and immobility of labour and other practical and humanitarian reasons prevent a rapid redistribution of the population over the production sectors, one will have to be satisfied with a *gradual* evolution towards a solution giving balanced development. For such a transition period the preference function[1] (21) expresses *some* of the considerations to make. If the analysis is made by including simultaneously a number of years, the optimal solution based on this preference function will have gone a long way towards solving explicitly the investment policy problem. But it is practically impossible to include so many years as to cover the complete transition to a balanced development. Therefore it may be indicated to add to the preference function still one more term that would express *the ideal investment* activity as given by (18). One may, for instance, consider a term expressing an aggregation of the ratios of actual investment achieved in the various sectors to those expressed by (18), where the distribution of the working population is taken according to some *pattern* which for social and humanitarian reasons is accepted as desirable[2].

How much *emphasis* should be put on this term, that is to say, how large a weight it should have as compared to the other weights would have to be decided by the top level political authorities.

---

[1] For a fuller treatment of the preference function, *see* section 7, p. 56.

[2] Or a compromise made between this population pattern and a desired consumption pattern. Such a compromise would involve the cross-delivery coefficients as is seen from (19).

ORGANIZATION OF INFORMATION AND THE MATHEMATICAL METHOD

1. The programming matrix

The usual input-output analysis gives fundamental and indispensable data for the programming work. But the input-output table itself does not give us *all* that is needed. In realistic planning a number of other things must be taken account of. In particular it is necessary to take explicit account of the *capacity limitations* due to restricted fixed capital in the various sectors. To insert a certain "bill of goods" in the usual input-output equations and to figure out the levels of activities in the various sectors, does not give a *concrete* solution, but only a *hypothetical* solution, namely the solution that would emerge if there were no capacity limitations. There are also other important limitations which must be taken account of and which forms a fundamental part of the problem in an expanding economy like India : limitation on the available amount of foreign capital, for instance. We also have to focus attention on the optimal determination of the ratios between the expansion of producer goods and consumer goods industries and other ratios. And we must consider the monetary and fiscal aspects of planning in order not to run into an uncontrolled inflation.

What is needed is a framework that can include *all these various factors* in one compact and complete exposition which is built up logically in such a way that we know exactly *where to find* any given relevant figure and where we can read off immediately the relation between any figure and the others. Experience has shown that in programming work very much depends on such an effective and logical way of arranging the figures. The programming matrix is a tool for obtaining this. An illustrative chart of this type for India is appended at the end of the section.

In the production sectors of the programming chart a distinction must be made between established enterprises and contemplated enterprises. For the former the levels of activity cannot be changed as easily as they can be for the latter. In programming terms : An extra set of linear inequalities will have to be imposed for the established enterprises.

For programming purposes it is probably best to have all established enterprises placed in a first main part of the programming chart (which will then have very much in common with a usual inter-industry table), and the contemplated enterprises placed as a second part of the programming chart. This means that the nomenclature for the production sectors in the production chart will have to be—partly or wholly—reproduced twice in the enumeration of the rows of the table.

For machine computation work the items will have to be put on punched cards so that the concepts of "row" as distinguished from "column" do not have any strict meaning. In the explanation for the principles involved it is, however, a help to think in terms or "rows" and "columns" as exhibited in the accompanying draft of the chart.

PROGRAMMING CHART FOR INDIA

					gross deliveries current account (each column represents a receiving sector)				gross domestic investment in real capital goods (each column represents a receiving sector)			
					1	2	..	n	1	2	..	n
domestic transactions	domestic deliveries of goods and services	delivering sectors		1								
				2								
				3								
				..								
				n								
		Physical investment at home. by investment receiving sector	Gross domestic investment in fixed capital goods	101								
			Net increase in domestic inventories	102								
		Remuneration of primary factors by income paying sectors	Wages	103								
			Salaries	104								
			Dividends and ownership income distributed to nationals	105								
	domestic transfers	Domestic interests paid *minus* domestic interests received (row total is zero)		106								
		Indirect taxes *minus* subsidies		107								
		Direct taxes *minus* income support		108								
		Social security contributions *minus* social security payment		109								
		Accumulated non-distributed profits at home owned by nationals		110								
		Net increase in domestic liquid assets		111								
foreign transactions	All imports of goods and services (also capital goods)			112								
	foreign transfers	Dividends and ownership income distributed to abroad *minus* the same from abroad		113								
		Foreign interests paid *minus* foreign interest received		114								
		Unilateral transfers to abroad *minus* unilateral transfers from abroad		115								
		Accumulated non-distributed profits abroad, owned by India		116								
		Net increase in foreign liquid assets		117								
		Grand total : Total purchasing power given out		118								

row no.	rural by income groups			ur[illegible]ups by incom[illegible]			govt. consumption of goods and services current account other than those specified in special columns	net increase in domestic inventories whether govt. or private by delivering sectors	net domestic loans by loan obtaining sectors, the column sum is zero	all exports [illegible]f goods and [illegible]vices [illegible]ether required (to be paid for or not) by exporting sectors	net foreign loans by loan obtaining sectors	grand total : purchasing power taken in	row no
	low	medium	high	low	medium	high							
1													1
2													2
3													3
..													..
..													..
..													..
n													n
101													101
102													102
103													103
104													104
105													105
106													106
107													107
108													108
109													109
110													110
111													111
112													112
113													113
114													114
115													115
116													116
117													117
118													118

When the data are organized in the programming matrix, the specific equations and inequalities that will enter into the operational analysis problem (linear programming) will have to be worked out through which the Plan-frame will emerge.

It might not be feasible to attack the problem at once in the complete form given by the programming matrix. One may start by *aggregating* the programming matrix more or less heavily and solve the operational analysis problem in the reduced size which thus emerges. In a next step more details can then be included. In any case it is essential to have a programming matrix that can describe the problem in its real complexity.

2. The double gradient method : an introduction*

The problem of Linear Programming is the problem of solving a system of equations $x_j = b_{j0} + \sum_{k=1}^{n} b_{jk}x_k$, $(j = n+1, \ldots, n+m)$, subject to the conditions that (a) $x_j \geqslant 0$, (b) some function of the form $f(x_1, x_2, \ldots, x_n)$ shall be maximum. The system, together with the condition (a) generates a convex set. This convex set defines the admissible region, and the function f is known as the preference function. In most practical instances, this function can be written as $f = p_1x_1 + p_2x_2 + \ldots + p_nx_n$. The weights p_i are called price-coefficients. The price coefficient p_k attached to the basis variable x_k denotes the change in the value of the preference function as one x_k is slightly altered while other x_k's are held constant. (Each concrete problem will have a special operational identification of these price coefficients.) The x_k are called basis variables, and the x_j are called dependent variables. It should be clearly understood that an optimum solution is a set of x_i $(i = 1, 2, \ldots, n+m)$ which remains within the admissible region defined by the condition (a), and fulfil (b) along with the given system of equations. It is equally fruitful to remember that one does not know which variables are to be *proper* basis and which the *proper* dependent variables, i.e., basis and dependent variables respectively in the optimum situation. In fact, it is the object of any method of solution to proceed step by step, to locate the *proper* basis (and, hence, *proper* dependent) variables. The classical simplex method elaborated by Dantzig and the Double Gradient method of Professor Frisch both purport to hit at these proper variables. The scope of the present note forbids any reference to the comparative merits of these two methods and, therefore, we proceed to state precisely what the Double Gradient method is.

Suppose that we assign two sets of magnitudes for the x_k, viz., x_k^0, x_k^1. Then, the difference in the values of the preference function at x_k^0 and x_k^1 is given by $f^1 - f^0 = \sum_k p_k(x_k^1 - x_k^0)$ which reduces to $\sum_k p_k x_k^1$ when all the $x_k^0 = 0$. This shows that if all the price coefficients are negative we cannot choose any set of non-negative x_k^1 that can make $f^1 - f^0 > 0$. That is to say, at the point where f attains its maximum, the price coefficients of x_k ($k \subset$ basis set) shall be $\leqslant 0$.

* by editor, S. Sankar Sengupta.

This argument implies that in the optimum we have to have certain coefficients linearly connecting the *proper* dependent variables with the *proper* basis variables, such that the proper prices, p'_k and $p'_j < 0$. In particular, these *proper* prices are for the proper dependent and proper basis variables respectively.

$$p'_h = p_h + \sum_{k=A,B,\dots} p_k b^{-1}_{kh} = p_h - \sum_{j=\alpha,\beta,\dots} p'_j b_{jh}, \qquad h = 1, 2, \dots n. \quad k = A, B, \dots .$$

$$p'_j = \sum_{k=A,\,B,\dots} p_k b^{-1}_{kj}, \quad j = \alpha, \beta, \dots \text{ of the dependent set.}$$

To pick up the h and j is to solve the problem of linear programming which is to determine which of the $n+m$ variables will be *proper* basis and, hence, zero in the optimum and which will be the *proper* dependent and, therefore, positive non-zero. The essence of the Double Gradient method lies precisely in furnishing an algorithm for determining the *proper* basis and dependent vairables.

As we make a move away from the starting point, (i.e., the point defined by the initial values of x_i obtained from any concrete problem) towards the point where f is maximum, we come up against three sorts of difficulties:

(A) In making such a move, the directions of normals make sudden jumps at the corners of the polyhedron;

(B) One does not know if the move is within the prescribed bounds ; and

(C) One does not know if the value of f is increasing (or, at the worst, not diminishing).

To tackle with the first problem, Professor Frisch devises his logarithmic potential function in the free variables $x_1, \dots, x_n$.

$$V(x_1, \dots, x_n) \equiv \sum_{j=1}^{n+m} \log x_j, \quad (\text{or}, \sum_{j=1}^{n+m} p_j \log x_j),$$

continuous for $x_j > 0$, (i.e., away from the boundary and towards the interior of the polyhedron).

To deal with the second and the third difficulties, Professor Frisch considers the gradient components $V_k = \frac{\partial V}{\partial x_k}$ and the price vectors respectively. In fact, these two issues are but conflicting desiderata and, therefore, some compromise has to be struck. This is done by taking the projection

$$p_k + \nu V_k, \quad \nu = -\frac{p_1 V_1 + \dots + p_n V_n}{V_1^2 + \dots + V_n^2},$$

of the price vector on the tangent-plane to the equipotential surface through, say, the initial point. This idea is further refined by insisting that the move shall be confined within the plane

$$d_k = w(p_k + \nu V_k) + (1-w) p_k,$$

unfolded by the vectors $p_k + \nu V_k$ and p_k. Taking $-\infty < w < +\infty$, one obtains $d_k = p_k + \mu V_k$, where μ is a multiple of w. This μ can be interpreted as a magnitude indicating the average projection of the price vector on the tangent-plane to the equipotential surface through the initial point.

A journey from one point to another in the space of the basis variables has, then, the components $x'_k = x_k + \lambda d_k$ and corresponding to these, we shall have

$$x'_j = x_j + \lambda d_j = x_j + \lambda \sum_{k=1}^{n} b_{jk} d_k = p_j + \mu V_j, \ (V_j = \Sigma\, b_{jk} V_k).$$

When a move is made with direction-components $d_k = p_k + \mu V_k$ and is defined by $x'_k = x_k + \lambda d_k$, the magnitude of the preference function changes by $f' - f = \lambda(\Sigma p_k^2 + \mu \Sigma p_k V_k) = \lambda \Sigma p_k (p_k + \mu V_k)$. We are free to choose that value of μ which will make

(1) the line $x'_k = x_k + \lambda d_k$ pass through the admissible region and

(2) the move to continue until f attains its maximum.

The first of these two considerations implies that λ shall be made as large as possible, provided that $x_j + \lambda d_j \geqslant 0$ for all j. This again involves a restriction, $\lambda \geqslant \frac{x_j}{-d_j}$, for those j for which $d_j < 0$. The optimal value of λ, say, λ_{opt} is given by

$$\lambda_{opt} = \begin{cases} \min\limits_j \dfrac{x_j}{-d_j} = \min\limits_j \dfrac{x_j}{-(p_j + \lambda V_j)} & x_j > 0, \quad p_j + \mu V_j < 0 \\[2ex] -\min\limits_j \dfrac{x_j}{p_j + \mu V_j} & x_j > 0, \quad p_j + \mu V_j > 0. \end{cases}$$

To deal with the second consideration, let us write

$$f' - f = |\lambda_{opt}| \cdot |\Sigma p_k^2 + \mu \Sigma p_k V_k|,$$

and substitute for λ_{opt}. We thus obtain

$$F(\mu) = \frac{\max\limits_j \dfrac{p_j + \mu V_j}{x_j}}{|\Sigma p_k^2 + \mu \Sigma p_k V_k|} \quad \begin{array}{l} x_j > 0 \\ p_j + \mu V_j < 0; \ \Sigma p_k^2 + \mu \Sigma p_k V_k > 0 \\ p_j + \mu V_j > 0; \ \Sigma p_k^2 + \mu \Sigma p_k V_k < 0. \end{array}$$

This expression stands for the largest increase in f attainable when we choose a μ for determining the direction of movement and like to make sure that all the x_j which were positive in the initial point do not become negative.

The next step consists in observing how the $F(\mu)$ changes with μ. This is done in the so-called Boundary-line Diagram drawn with the lines $y_j = \frac{p_j}{x_j} + \mu_0 \frac{V_j}{x_j} \Big| x_j > 0$, where μ_0 is the separation point of the two domains

$$\Sigma p_k^2 + \mu \Sigma p_k V_k < 0, \quad \Sigma p_k^2 + \mu \Sigma p_k V_k > 0 \qquad (\text{or,} \ \lambda < 0, \ \lambda > 0).$$

This value of μ_0, given by $0 = \Sigma p_k^2 + \mu_0 \Sigma p_k V_k$, is chosen for the reason that by choosing μ so as to make $0 = \Sigma p_k^2 + \mu \Sigma p_k V_k$ we cannot improve the value of f. Consider, now, any two consecutive joints defined by the interesction of two of these lines. Let μ_1, μ_2, be the values of μ at these joints and let $\alpha_1 = |y_1|$, $\alpha_2 = |y_2|$ be the corresponding absolute values of the ordinates at these joints; also write $K = \frac{\alpha_2 - \alpha_1}{\mu_2 - \mu_1}$. Now, differentiating $F(\mu)$ w.r.t μ, we obtain

$$\frac{d}{d\mu} F(\mu) = -\text{Sgn}(\Sigma p_k^2 + \mu \Sigma p_k V_k) \frac{K \Sigma p_k V_k}{(\Sigma p_k^2 + \mu \Sigma p_k V_k)^2} \left\{ \frac{\alpha_1}{K} - (\mu_1 - \mu_0) \right\}.$$

Since this expression retains constant sign throughout the interval, $\mu_1 \leqslant \mu \leqslant \mu_0$, therefore, in our search for Min. $F(\mu)$ it will suffice to restrict attention to the joints determined by the intersections of these lines. The optimum value of μ, say μ_{opt}, is given by the minimum joint where $F(\mu)$ is minimum, i.e., $f' - f$ is maximum. Professor Frisch makes this rest upon the convexity of the function, $\underset{j}{\text{Max}} \left| \frac{p_j + \mu V_j}{x_j} \right|$.

With the determination of $\mu_{opt} = \dfrac{\frac{p_\alpha}{x_\alpha} - \frac{p_\beta}{x_\beta}}{\frac{V_\beta}{x_\beta} - \frac{V_\alpha}{x_\alpha}}$ (α and β being the suffixes for the variables defining the minimum joint) we come to know at least two variables that must be *proper* variables (basis or dependent, as the case may be). Just as we ascertain μ_{opt}, we can also ascertain the corresponding y_{opt} which is nothing but $-\frac{1}{\lambda_{opt}}$. In particular, we now suspect that probably some more variables (besides the two by which the minimum joint was defined) might be given the status of *proper* variables. This suspicion is not unfounded because the function $F(\mu)$ is, in practice, not continuous, so that other variables too may compete with the two in defining the μ_{opt}. The final step in the search for *proper* dependent and *proper* basis variables lies in looking out for some measure of the degree of relative optimality,

$$r_j = 1 - \frac{y_j}{y_{opt}} = 1 - \frac{\frac{p}{x_j} + \mu_{opt} \frac{V_j}{x_j}}{y_{opt}}.$$

Those j for which this magnitude will be nearer zero are expected to be *proper* variables (dependent or basis, as the case may be).

3. A SEQUENCE OF CALCULATIONS FOR DETERMINING AN OPTIMAL PLAN–FRAME BY MEANS OF LINEAR PROGRAMMING

The place which an optimally determined plan-frame holds in a complete national planning system, is described earlier. In the memorandum "Principles of Linear Programming" of 18 October 1954 from the University Institute of Economics, Oslo, are described various tools that can be used for solving different types of linear programming problems. References in the following are to the formulae in this work.

Until extensive experimental computations have been made it is not possible to say for certain which of the various linear programming tools will prove to be the most advantageous ones to use on the type of data which one will confront in Indian national planning. A certain estimate can, however, be made. On the basis of present knowledge of the type of data one will most likely encounter, and of the nature of the linear programming tools referred to, I shall make a suggestion of what I believe will be the most advantageous sequence of calculations. This suggestion will apply equally to the case where the work is to be done on desk machines, multiplying punches or small or medium sized electronic computors.

The following suggestion is based on the assumption that the successive rounds of the Double Gradient method used *without* freedom truncations, do not turn out to require such a rapidly increasing number of digits that an approach without freedom turncations becomes impracticable. If this assumption holds, it ought to be possible to carry the work to final completion, including the final test for optimality, without making any *inversions* of the order of magnitude of the number of degrees of freedom in the system. Only one way solutions of this order will be necessary.

This is computationally extremely important, particularly, if the system of coefficients contains many zeros—which it seems safe to assume that it will in the data one is likely to encounter in national planning work. Indeed, in this case the most labour saving method for a one way solution (or even for an inversion if it should be necessary) is, as far as my experience goes, the one indicated in (16.1)* to (16.12). Here the forward solution involves only multiplications and no divisions. In the back solution there will come in one division for each variable. If the system of coefficients contains many zeros, the number of multiplications and divisions for a complete one way solution by the method will be of the order of n, when n is the number of unknowns. If the data are put on punch cards, it is easy to sort out the cards that do not lead to any multiplications, so that one can *profit fully* by the fact that there are many zeros.

For a straight forward application of, say, the Gaussian method of solution, described in (17.1) to (17.23) the number of multiplications and divisions involved

* These references relate to formulae given in the Principles of Linear Programming by the author.

in a one way solution is in principle of the order of n , and there does not seem to be any easy way of profiting by the great number of zeros in the original matrix. Since the back solution will in any case consist in the solution of a triangular system, regardless of which method is used for the forward solution, the back solution will involve a number of multiplications and divisions which is of the order n^3 for an inversion and n^2 for a one way solution. For an inversion we will, therefore, always have to face a work of the order n^3 even though we use the method (16.1) to (16.12) for the forward solution. For that part of the work which consists in the forward solution the saving by (16.1) to (16.12) will always exist, even in an inversion, but it is only in the case of a one way solution (or a solution with a small number of right members) that the method (16.1) to (16.12) is capable of changing the picture fundamentally.

I. *Bringing the equations over into a basis form*

This should be done by the method of (3.1) to (3.8). A real effort must be made to get through by the simple procedures suggested in that section. If that is not possible, one faces an inversion of the order of the number of degrees of freedom, and this would increase enormously the magnitude of the task.

As basis variables should be selected a set of variables which we guess will be an *optimum set*, i.e., a set of variables that are all zero in the optimum point, and can be used as a basis.

II. *Eliminating redundant bounds*

This should be done immediately after the equations have been brought over into a basis form. The method to be used is indicated in (5.1) to (5.8).

III. *How to get into the admissible region*

If a point in the admissible region, i.e., a point where all variables are non-negative, cannot be found by a simple inspection of the figures (as, for instance, in the case where all the constant terms in the expressions are non-negative or even positive), the method of (6g.4) to (6g.33) should be used, starting from an arbitrary initial point where the variables may have any signs.

To use the method (6g.4) to (6g.33) one must first chose a set of direction numbers $d_k (k=1,2...n)$ for the basis variables, and from these compute all the d_j $(j=1,2,...(n+m))$ by $d_j = \sum_{k=1}^{n} b_{jk} d_k$.

If one is prepared to spend in each round the computational cost involved in a one way solution, one should determine the d_k by (6e.1). In this case it does not matter whether the initial point contains variables that are zero.

If one wants to avoid the work involved in a one way solution for each round, one may determine the d_k for any round by (III.1)

$$d_k = \sum_{j/x_j<0} b_{jk} \qquad (k = 1,2,...n) \qquad \dots \quad \text{(III.1)}$$

where the x_j $(j=1,2,\ldots,(n+m))$ are the values of the variables in the initial point of the round. In the formula k may be the affix of any of the basis variables and the summation over j runs through all the values $(j=1,2,\ldots,(n+m))$ for which x_j is effectively negative. This formula is developed heuristically on the assumption that the initial point only contains variables that are effectively different from zero (positive or negative). While the formula as it stands may formally be applied to any case, it would be well to confine its use to the case where the initial point does not contain any variable that is zero.

It is to be expected that one round using (III.1) will be much less effective than one round using (6e.1). Whether this is more than compensated by the expediency of determining d_k by (III.1) instead of by a one way solution, will depend on the circumstances. One may perhaps start by using (III.1) and revert to (6e.1) if the convergence by (III.1) turns out to be too slow.

Whatever method is used, it will be an advantage to move to a point in the *interior* of the admissible region, that is, to a point where all the variables are effectively positive, not zero. The subsequent application of the Double Gradient method will then be simpler.

IV. *One round of the double gradient method starting from a point in the interior of the admissible region*

The computational steps in one round of the Double Gradient method is explained in (13.1)—(13.32), and the use of the boundary line diagram in the case where there is no side condition on μ is explained in (12.29)—(12.33) and (12.41).

The explanations in (13.1)—(13.32) are given so as to cover the general case where one or more of the variables may be zero in the initial point from which the Double Gradient method is applied. In this general case the explanations must necessarily be more complicated than in the case where all the variables are effectively positive in the initial point. Since this latter case will be the usual one in practice—it is indeed possible and may be advisable to go through the whole work in this way —it will be useful to give a version of the explanations of (13.1)—(13.32) in a reduced form where one makes use of the assumption that all the variables are effectively positive in the initial point. The following is such a reduced explanation.

A table of the basis coefficients b_{jk} $(j = 1, 2 \ldots (n+m);\ k = 0, 1 \ldots n)$ is supposed to be given with the row sums

$$b_{j\cdot} = \sum_{k=1}^{n} b_{jk} \qquad (j = 1, 2 \ldots (n+m)) \qquad \ldots \quad \text{(IV.1)}$$

and the column sums

$$b_k = \sum_{j=1}^{n+m} b_{jk} \qquad b_{\cdot k} = \sum_{j=n+1}^{n+m} b_{jk} \quad (k = 0, 1, 2 \ldots n). \qquad \ldots \quad \text{(IV.2)}$$

A single dot is by convention used to denote a summation either over the basis variables as in (IV.1) or over the dependent variables as in the expression to the right in (IV.2). A double dot is used to denote a summation over all the variables.

Table (IV.4) indicates the form of the main work sheet to be used for one round of the Double Gradient method when the work is done on desk machines (calculators supplemented by listing-adding machines).

In the first cell in the bottom appendix is indicated the value of the preference function in the initial point. This value is useful amongst others as an indication of which point we are working with. This initial value of the preference function is copied from a previous work sheet, or it is computed from the definition of the preference function.

The values of the variables $x_j (j = 1, 2 \ldots (n+m))$ for the initial point are recorded in column (1). All these values x_j are supposed to be effectively positive. The first n of them —supposed given— are sufficient to define the point. The following m of them are—if they are not easily available from previous computing—computed by (3.6) and listed in the lower part of column (1). They are checked by the zero test

$$b_{\cdot 0} + \sum_{k=1}^{n} b_{\cdot k} x_k - x_{\cdot} = 0. \qquad \ldots \quad \text{(IV.3)}$$

This check should be applied even in the case where the figures in column (1) are simply copied from some other work sheet. It is essential that these basic data for the following work are verified beyond doubt.

TABLE (IV.4). MAIN WORK SHEET FOR ONE ROUND BY THE DOUBLE GRADIENT METHOD WHEN ALL THE VARIABLES ARE EFFECTIVELY POSITIVE IN THE INITIAL POINT

(0)	(1) initial point x_j	(2) $\frac{1}{x_j}$	(3) V_j	(4) p_j	(5) $\frac{V_j}{x_j}$	(6) $\frac{p_j}{x_j}$	(7) $1-\frac{y_j}{y_{opt}}$	(8) ranking order r
$j=1$								
2								
.								
:								
n								
$n+1$								
$n+2$								
.								
:								
$n+m$								
S								
appendix	f_{init}	P	M	μ_0	$\mu_0 approx$	μ_{opt}	y_{opt}	f_{opt}

The reciprocal figures $\frac{1}{x_j}$ are computed, listed in column (2) and checked by

$$\sum_{j=1}^{n+m} x_j \left(\frac{1}{x_j}\right) = n+m. \qquad \text{... (IV.5)}$$

Next the basis gradient components V_k $(k = 1, 2 \ldots n)$ are computed by (12.5), each of them being simply the product-sum of the figures in column (2) by b_{jk}. The V_k $(k = 1, 2 \ldots n)$ are listed in the upper part of column (3). They are checked by

$$V_1 + V_2 + \ldots + V_n = \frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n} + \sum_{j=n+1}^{n+m} \frac{b_j}{x_j}. \qquad \text{... (IV.6)}$$

The V_j for $j = n+1, n+2 \ldots n+m$ are computed by (12.15) are checked by

$$V_{n+1} + V_{n+2} + \ldots + V_{n+m} = \sum_{k=1}^{n} b_{\cdot k} V_k. \qquad \text{... (IV.7)}$$

The prices $p_k (k = 1, 2 \ldots n)$ associated with the basis variables are also supposed to be given. If they are not already easily available on some other sheet, they are listed in the upper part of column (4). The prices p_j for $j = n+1, n+2 \ldots n+m$ are computed by (12.17) and checked by

$$p_{n+1} + p_{n+2} + \ldots + p_{n+m} = \sum_{k=1}^{n} b_{\cdot k} p_k. \qquad \text{... (IV.8)}$$

The ratios $\frac{V_k}{x_k}$ for $k = 1, 2 \ldots n$, are computed, entered in the upper part of column (5), and checked by

$$\sum_{k=1}^{n} x_k \left(\frac{V_k}{x_k}\right) = V_1 + V_2 + \ldots + V_n. \qquad \text{... (IV.9)}$$

The ratios $\frac{V_j}{x_j}$ for $j = n+1 \; n+2 \ldots n+m$ are computed, entered in the lower part of column (5) and checked by

$$\sum_{j=n+1}^{n+m} x_j \left(\frac{V_j}{x_j}\right) = V_{n+1} + V_{n+2} + \ldots + V_{n+m}. \qquad \text{... (IV.10)}$$

For subsequent checking purposes one may also compute the complete column sum

$$\left(\frac{V}{x}\right) = \sum_{j=1}^{n+m} \frac{V_j}{x_j} \qquad \text{... (IV.11)}$$

and list it in the bottom line of table (IV.4).

The ratios $\frac{p_k}{x_k}$ for $k = 1, 2 \ldots n$ are computed, listed in the upper part of the column (4), and checked by

$$\sum_{k=1}^{n} x_k \left(\frac{p_k}{x_k}\right) = p_1 + p_2 + \ldots + p_n. \qquad \ldots \text{(IV.12)}$$

The ratios $\frac{p_j}{x_j}$ for $j = n+1, n+2 \ldots n+m$ are computed and checked by

$$\sum_{j=n+1}^{n+m} x_j \left(\frac{p_j}{x_j}\right) = p_{n+1} + p_{n+2} + \ldots + p_{n+m}. \qquad \ldots \text{(IV.13)}$$

For subsequent checking purposes one also computes the column sum

$$\frac{p}{x} = \sum_{j=1}^{n+m} \frac{p_j}{x_j} \qquad \ldots \text{(IV.14)}$$

and lists this sum in the bottom line of table (IV.4).

If the operator has confidence in his ability to work correctly, he may cut short the checks (IV.9)—(V.9)—(IV.10) and (IV.12)—(IV.13) by first forming on a listing-adding machine the sums $\left(\frac{p_j}{x_j} + \frac{V_j}{x_j}\right)$ for each $j = 1, 2 \ldots (n+m)$ and then check that

$$\sum_{j=1}^{n+m} x_j \left(\frac{p_j}{x_j} + \frac{V_j}{x_j}\right) = \sum_{k=1}^{n} b_{\cdot k} \, (p_k + V_k). \qquad \ldots \text{(IV.15)}$$

If the number of variables—basis or dependent—is very great, it may be necessary to split the checks up into *parts* by introducing subtotals for different groups of the variables. The test principles for such groups are exactly the same as those expressed above.

When the figures in the body of the table (IV.4) are computed and checked as described above, the first coefficients in the appendix below the table are computed. P and M are computed by (12.18) and (12.19) respectively and μ_0 is computed by (12.20). A round approximation to μ_0, say, with one or two digits only, is also computed and entered as μ_0 approx.

All the data for performing an optimal movement from the initial point are now collected. This work proceeds as follows.

We want to draw the straight lines (12.30). For any such line we need the ordinate of two points.

One of the ordinates may be taken as the ordinate for $\mu = 0$. This ordinate is simply $\frac{p_j}{x_j}$. The other we take at μ_0^{approx}. To do this we compute for all $j = 1, 2, \ldots, (n+m)$ the ordinates

$$Y_j = \frac{p_j}{x_j} + \mu_0^{approx} \frac{V_j}{x_j}. \qquad \ldots \quad \text{(IV.16)}$$

For each $j = 1, 2 \ldots (n+m)$ we draw in a (y, μ) diagram the straight line whose ordinate for $\mu = 0$ is $\frac{p_j}{x_j}$, (read off from column (6) of the main sheet) and whose ordinate for $\mu = \mu_0^{approx}$ is (IV.16) (read off from the adding machine tape). Each line is drawn as one continuous line over *both* parts of the diagrams, i.e., to the left of μ_0 as well as to the right of μ_0. Immediately after a straight line is drawn, it is marked with the number j. The marking is done before one proceeds to drawing the next line. ... (IV.17)

Occasionally one might for better orientation need to compute also the abscissa of the point where the line intersects the μ-axis. This point is given by

$$\mu_j = -\frac{\frac{p_j}{x_j}}{\frac{V_j}{x_j}} = -\frac{p_j}{V_j}. \qquad \ldots \quad \text{(IV.18)}$$

By means of the straight lines thus drawn, we first determine graphically the value μ_{opt}. The method used is that of (12.41).

When the value μ_{opt} has been determined approximately by the graphical method just described, the value is computed exactly and with a number of digits sufficiently large to utilize all the accuracy carried in the preceeding steps. The formula to be used is

$$\mu_{opt} = \frac{\frac{p_r}{x_r} - \frac{p_s}{x_s}}{\frac{V_s}{x_s} - \frac{V_r}{x_r}} \qquad \ldots \quad \text{(IV.19)}$$

where $j = r$ and $j = s$ are the affixes of the two straight lines that give the optimum point in fig. (12.31), or any two of the lines that pass through this point if there are more than two. There must always be at least two. In the case of a horizontal optimum segment the optimum point is conventionally taken as the end-point of this segment that comes nearest to μ_0. The case of horizontal optimum segment is an initial case in this connection because the optimum in fig. (12.31) cannot be horizontal unless it coincides with a part of the μ-axis which means that the value of the preference function can be rendered arbitrarily great.

In fig. (12.31) a vertical straight line should be drawn for the abscissa $\mu' = \mu_{opt}$ determined by (IV.19). The intersection point between this vertical and the broken boundary line of fig. (12.31) will determine the ordinate y_{opt} which by definition is the ordinate of the two straight lines, nos. r and s, that intersect in this point. (Compare (IV.23).) Looked upon from the view point of the admissible region as a whole this ordinate characterizes the *breaking out point*, that is to say the point where we hit the boundary when we move in a direction given by μ_{opt}. The value y_{opt} is the negative of the inverse of the value λ_{opt} in (12.12) that gives the largest possible increase in the preference function which is compatible with the condition that d shall be of the form (12.16) (where μ is a parameter to be disposed of) and the point reached shall belong to the admissible region; i.e., we have ;

$$\lambda_{opt}^{cont} = -\frac{1}{y_0^{cont}} . \qquad \ldots \quad (IV.20)$$

Indeed the optimal λ considered is determined by the formula for one or any number of the variables that become zero in the breaking out point. We have (at least) $x_r = x'_s = 0$ where r and s have the same meaning as in (IV.19). By (12.12) this gives

$$\begin{aligned} x_r + \lambda_{opt}(p_r + \mu_{opt} V_r) &= 0 \\ x_s + \lambda_{opt}(p_s + \mu_{opt} V_s) &= 0 \end{aligned} \qquad \ldots \quad (IV.21)$$

where $x_r > 0$ and $x_s > 0$. Except for the trival case $\lambda_{opt} = 0$ (which would mean no movement at all away from the initial point) (IV.21) is equivalent with

$$\frac{1}{\lambda_{opt}} = -\frac{p_r + \mu_{opt} V_r}{x_r} = -\frac{p_s + \mu_{opt} V_s}{x_s}. \qquad \ldots \quad (IV.22)$$

Since any of the two expressions to the right in (IV.22) is equal to $-y_{opt}$, that is,

$$y_{opt} = \frac{p_r + \mu_{opt} V_r}{x_r} = \frac{p_s + \mu_{opt} V_s}{x_s} \qquad \ldots \quad (IV.23)$$

we have (IV.19).

Once the affixes r and s are determined from the graph, the value y_{opt} should be computed from (IV.23) with as high an accuracy as the data will permit.

Next we compute

$$y_j = \frac{p_j}{x_j} + \mu_{opt} \frac{V_j}{x_j} \qquad (j = 1, 2 \ldots (n+m)). \qquad \ldots \quad (IV.12)$$

If only a *ranking order* of the variables is wanted for making a guess about which one of them will most likely be zero in the optimal point, (IV.24) need only be computed for those values of j for which the ordinate y_j of the intersection point of the boundary line j with the vertical through $\mu = \mu_{opt}$, are the *largest in absolute values* and of the *same sign* as y_{opt}. These values of j are the only ones that can compete for high ranking order.

In most cases it is, however, wanted not only to determine a ranking order for the variables of highest priority but it is wanted to compute the complete set of values $x'_j (j = 1, 2 \ldots (n+m))$ in the new point, and then it is most convenient to do it by means of all the y_j. We have indeed.

$$x'_j = x_j \left(1 - \frac{y_j}{y_{opt}}\right) \qquad (j = 1, 2, \ldots (n+m)). \qquad \ldots \quad \text{(IV.25)}$$

In the main work sheet (IV.4) the multipliers to be used in (IV.25) are listed in column 7.

An ordinal number r is entered in the last column of table (IV.4) in the work sheet to indicate the ranking order. The variables that turned out to be zero in the new point, are given the ordinal number 0 and the others a ranking number according to the value of y_j.

In the new point x'_j the value f' of the preference function is equal to

$$f' = f - \frac{P + \mu_{opt} M}{y_{opt}} \; ; \qquad \ldots \quad \text{(IV.29)}$$

this new value $f' = f_{opt}$ is entered as the last item in the bottom appendix to the main work sheet.

V. What to do after a round of the double gradient method

The course to follow after a round of the Double Gradient method is to some extent a matter of subjective judgement.

The most radical step is to start from the top of the ranking order and proceed downwards the exact number of places to get a number of variables equal to the number of degrees of freedom, and such that these variables form a basis set, i.e., a set such that when the values of these variables are fixed, all the other variables are uniquely determined. Having done that, we might *guess* that this set of variables is an optimum set, that is to say, is such that when these variables are put equal to zero we get a value of the preference function which cannot be exceeded in any point in the admissible region or on its boundary. To test whether the set of variables considered is actually an optimum set, one has to apply the optimality test described in section VI below. (A test which only involves two one way solutions and no inversions.) Such a procedure may in certain cases be completely successful, leading in one step to the final solution. An example of such a happy situation is given in (13.49).

In other cases the structure of the problem may be more difficult and several rounds of the Double Gradient method may be needed. A first idea would then be simply to *recede* a little from the point where one hits the boundary when using the Double Gradient method (following the straight line from the initial point) and then take the point thus obtained as a new starting point for the Double Gradient method. Experience has shown, however, that this does not lead to a practical solution because the convergence is too slow. An entirely different procedure must be used.

We must try to change as much as possible the *locality* from which we make a shoot by the Double Gradient method.

There are several ways feasible. One is to determine the *tentative* optimum point. This is the point where the n first ranking variables that form a basis set, (i.e., determines the point uniquely) are equal to zero. This involves a one way solution. The work to be done in this connecton is at the same time part of the work that has to be done if we want to test for optimality. We can, therefore, play on two possibilities: If the tentative optimum point turns out to fall outside the admissible region, we can proceed as described in section VII below and thus reach a new initial point for the Double Gradient method, and if the tentative optimum point falls on the boundary of the admissible region, we can proceed to test for optimality (possibly after one round of the inward method as explained in section VIII). The testing for optimality will involve another one way solution. By following such a procedure in the example (13.49)* we will be led straight to the optimum point, and no part of the computations will have been wasted.

If we do not want to spend the cost of a one way solution involved in the method of the tentative optimum, we can instead use the *inward* method described in section VIII below. From a point obtained by one round of the inward method we can again start with the Double Gradient method.

If we can continue in this way, using between each round of the Double Gradient method either the method of tentative optimum or the inward method, or a combination, without having to use too great a number of decimal places, we ought to approach the optimum point at a speed roughly of the order indicated in (13.37).

If the above procedures do not lead to a satisfactory result, one will have to use *freedom truncations* as described in (13.33)—(13.53). Each freedom truncation necessitates an inversion, but this inversion is only of the order of the number of variables truncated or, rather, something like half of this because in general some of the variables to be truncated will be in the basis set, and the truncation of such variables does not involve any computational cost. If truncations are made in stages we can very roughly assume that at each stage $\frac{n}{2T}$ variables are truncated which

* (13.49) contains a typographical error; the figure 8 shall be omitted so that the list contains exactly 15 figures. (The variable no. 8 actually becomes zero in the optimum point but it does not come within the first 15 ranking places.)

give a computational cost of the order $\left(\frac{n}{2T}\right)^3$ for the inversion, so that a total inversion work of the order $T\left(\frac{n}{2T}\right)^3 = \frac{1}{T^2}\left(\frac{n}{2}\right)^3$, will be involved. Apart from the factor $\frac{1}{8}$ this can be written $\left(\frac{n}{T}\right)^2 n$. If T increases with n, the cost item here considered will not be excessive, but we must also reckon with the other parts of the work involved and this will in general increase with T. In any case freedom truncations will involve an additional amount of thinking.

VI. Testing for optimality

In "Principles of linear programming" different ways to approach the problem of testing a given point for optimality are discussed. The differences between these approaches are only formal. When it comes to the actual computations, the methods are practically the same. I shall here only discuss what is essential from the computational viewpoint.

Consider a set of n variables such that there is nothing in the structural equations that prevents these variables from being zero simultaneously and further such that when these n variables are arbitrarily given all the other variables are determined uniquely.

Such a selected set of n variables is, by definition, an *optimum set* if and only if the point obtained by putting these n variables equal to zero, is a point such that there are no other points in the admissible region (or on its boundary) that can produce a higher value of the preference function.

One method of testing whether the point considered is an optimal one, is to express the preference function as a linear function of the n variables that are suspected to form an optimum set, and to test whether the n "prices" that then appear, that is to say the n coefficients of the variables in the linear expression obtained for the preference function, are all *non-positive*. Furthermore it must be verified (if it has not been done previously) that the point obtained by putting the n selected variables equal to zero, actually belongs to the admissible region (its boundary), that is to say gives non-negative values for all the variables. These conditions form a set of conditions that are necessary and sufficient for the n variables considered to be an optimum set.

If all the prices obtained are effectively negative, the optimum is *unique*, that is to say, any other point than the one obtained by putting the n variables considered equal to zero, must give an *effectively* smaller value of the preference function. If N of the n prices are zero, the optimum is a linear manifold of a dimensionality N. This is seen from the fact that by changing any of the N variables we do not change the value of the preference functions. The condition that these N variables must only be changed in such a way that the point obtained remains in the admissible region, does not take out any of the N degrees of freedom (except in extreme cases).

If the preference function was already previously expressed in terms of the n variables that are suspected of forming an optimum set, and all the other variables were also expressed in terms of these n variables, the problem is simple. We then simply have to read off the prices appearing in the linear function that expresses the preference function and also read off the constant terms in the expressions for all the other variables. The condition is then that all these prices shall be non-positive and all the constant terms referred to shall be non-negative.

If one or more of the n variables, which we suspect of forming an optimum set, were not previously used as basis variables to express the preference function and all the other variables, some computation will be necessary to test whether these n variables really form an optimum set.

We will first discuss the case where we know already from previous computations that the point considered—where all of the n suspected variables are zero—belongs to the admissible region.

Let numbers $r, s, \dots t$ be those of the variables in the suspected set, that were *not* amongst the original basis variables. There will then be an equal number of variables, let it be numbers, $A, B \dots C$, that belong to the original basis set but *not* to the set which we suspect of forming an optimum set. Indeed, both sets contain exactly n variables.

The prices $p_k (k = A, B \dots C)$ that appear in the original expression for the preference function are supposed to be known.

This being so, consider the linear system

$$\sum_{j=r,s\dots t} p'_j b_{jk} = p_k \quad (k = A, B \dots C) \qquad \dots \quad \text{(VI.1)}$$

where the b_{jk} are the coefficients of the original equations. (Compare (3.1), as exemplified in (3.4)).

This is a linear system in the magnitudes $p'_j (j = r, s \dots t)$. To solve it we only have to perform a one way solution. In view of the special nature which can be foreseen for the matrix b_{jk}, the best method to use for this one way solution is probably (16.1)—(16.12).

A first condition that must be fulfilled in order that the set of variables considered shall be an optimum set, is that all the prices p'_j $(j = r, s \dots t)$ determined by VI.1) are non-positive.

If this condition is *not* fulfilled, there is no use to go any further, because we then know that the set of n variables which we suspected of forming an optimum set, can *not* in fact be such a set. Possibly there may be only one or two variables which we have guessed incorrectly, but at any rate the set in its totality cannot be correct.

If this first criterion *is* fulfilled, we proceed to compute the new prices of the other variables in the new basis set, that is the new prices of those variables that were not taken out of the old basis set. We have to test that these prices also are non-positive. This is done by the last formula in (4.20).

In this formula we use the new prices p'_j $(j = r, s \ldots t)$ that were just computed.

If *not* all these new prices p_h are non-positive, there is no use to go any further, but if all of them are non-positive, we can conclude that the point considered is an optimum point, provided we know already that it belongs to the admissible region (its boundary).

If this is not known— in the case where all the p'_h turned out non-positive so that it is worthwhile to proceed further—we check it in the following way.

The values $\bar{x}_k(k = A, B \ldots C)$ of those of the old basis variables that do not belong to the suspected optimum set are, at the point considred, the same as the values of the constant terms b'_{k0} which we would get if we computed the expresss-sions for these variables in terms of the new basis set (which contains the variables $r, s \ldots t$, but not $A, B \ldots C$). These values are determined by the system of equations

$$\sum_{k=A,B,\ldots} b_{jk}b'_{k0} = -b_{j0} \qquad (j = r, s \ldots t). \qquad \ldots \quad \text{(VI.3)}$$

To solve (VI.3) we only have to perform a one way solution, and in view of the nature of the matrix b_{jk} which we can foresee, it is again the method (16.1)-(16.12) that will probably be the most advantageous.

When the values $\bar{x}_k = b'_{k0}$ $(k = AB \ldots C)$ are determined, we have the values of all the n old basis variables in the point where all the n suspected variables are zero. For numbers $A.B\ldots$C these values are given by $b'_{A0}, b'_{B0} \ldots b'_{C0}$, and all the other of the old basis variables are 0 in the point considered. Therefore, from the *old* expressions of all the variables in terms of the old basis variables, we can, without computing any other of the new coefficients b'_{kj} verify whether all the varibles are non-negative in the point now considered. We simply use the formula

$$x_g = b_{g0} + \underset{k=A,B,\ldots}{S} b_{gk}b'_{k0} \; (g = 1, 2 \ldots (n+m)\text{, except the affixes in the} \ldots \quad \text{(VI.4)}$$

(suspected optimum set).

If all the x_g determined in this way turn out to be non-negative, we can conclude that the point considered is actually an optimum point. But if one or more of them turn out to be effectively negative, the point is not an optimum point.

If we are at a point on the boundary where the n variables that could form a basis set are equal to zero we can proceed directly to test for optimality by the above method of (VI.1). But we can also proceed a little differently: We can first use one round of the inward method described in section VIII. If such a round increases the value of the preference function, we know that the original point was *not* an optimum point and we have at the same time proceeded further. On the other hand the fact that the inward method does *not* succeed in increasing the preference function is not in itself a proof that the point is optimal. The only safe verification of an optimum is through the prices as explained above.

VII. The method of the tentative optimum

The method of the tentative optimum consists in *guessing* at a certain optimum set, and then computing the consequences of this guess. This may, in particular, be done if we have applied the Double Gradient method and reached a breaking out point and used the ranking order in this point to form a guess about the optimal set. This is illustrated in fig. (10.1). If we move from x to the breaking out point 4, tentative optimum is A', and if the breaking out point is 4″, the tentative optimum is A″.

The calculations involved in the method of the tentative optimum are very much the same as in the testing for optimality, the only difference being that we may now perform the operations in another order.

In the method of the tentative optimum we first determine the values of the old basis variables that are *not* included in the set which we suspect of being an optimum set. These variables are the ones we denoted by A, $B...C$ in section VI. Let $\bar{x}_k(k = A, B...C)$ be the values which these variables assume in the point that is determined by the condition that all the variables in the suspected optimum set are zero. These values $\bar{x}_k(k = A, B...C)$ are the same as $b'_{k0}(k = A, B...C)$ and will, therefore, be determined by the system,

$$\sum_{k=AB...C} b_{jk}\bar{x}_k = -b_{j0} \qquad (j = r, s \ldots t). \qquad \ldots \quad \text{(VII.1)}$$

Compare (VI.3). In other words the values $\bar{x}_k$ $(k = A, B \ldots C)$ are determined by a one way solution, and again it will, because of the particular nature of the matrix b_{jk}, most likely be advantageous to use the method (16.1)—(16.12).

When the $\bar{x}_k$ $(k = A, B \ldots C)$ are determined by (VII.1), the values of all the old basis variables are known in the point now considered, because all of these old variables that are *not* contained amongst the $\bar{x}_k$ $(k = A, B \ldots C)$, are zero. Therefore the values of all the variables in the point considered can be computed by (VI.4).

If the point turns out to be situated in the admissible region (on its boundary) one should proceed to testing by the method of section VI whether the point is optimal.

If this test is negative, i.e., if the point is situated on the boundary without being optimal, one should move into the interior by the method of section VIII and then make a round by the Double Gradient method.

If the tentative optimum point falls outside the admissible region as, for instance, A' or A'' in fig. (10.1), one should move along a straight line towards the breaking out point (4′ or 4″ in fig. 10.1) that served to determine the tentative optimum point. Moving along this line one should *stop* when the boundary of the admissible region is reached. In fig. (10.1) this will lead directly to the optimum whether one starts from A' or A''.

In formulae this procedure is expressed by

$$x''_k = \bar{x}_k + L(x'_k - \bar{x}_k) \qquad (k = 1, 2 \ldots n) \qquad \ldots \text{(VII.2)}$$

x'_k being the coordinates of the moving point on the line, i.e., the values of the basis variables along the line x'_k are the values of the basis variables in the breaking out point from which the tentative optimum was determined, and $\bar{x}_k$ the values of the basis variables in the tentative optimum.

The $\bar{x}_k$ $(k = A, B \ldots C)$ were determined by the one way solution and the $\bar{x}_k$ for the other values of the affixes $k = 1, 2 \ldots n$, are zero.

In the course of the movement from $\bar{x}_k$ to x'_k none of the variables $x_j(j = 1, 2 \ldots (n+m))$ that were positive or zero in $\bar{x}_k$, can become negative. Indeed all along the line considered the change is a *monotonic* one, and all the variables are actually non negative in the point x'_k. The point where we hit the boundary is consequently determined as the point where the last one of the variables x_j that are negative in $\bar{x}_k$, becomes zero.

For all the variables the movement is defined by

$$x''_j = \bar{x}_j + Ld_j \qquad (j = 1, 2 \ldots (n+m)) \qquad \ldots \text{(VII.3)}$$

where

$$d_j = \sum_{k=1}^{n} b_{jk}(x'_k - \bar{x}_k) \qquad (j = 1, 2 \ldots (n+m). \qquad \ldots \text{(VII.4)}$$

For $L = 0$ we get $x''_j = \bar{x}_j$ whatever be the affix j. Further

$$x''_k = \bar{x}_k + L(x'_k - \bar{x}_k) \qquad (k = 1, 2, \ldots n) \qquad \ldots \text{(VII.5)}$$

i.e., $L = 1$ corresponds to $\bar{x}''_k = x'_k$. In other words the movement from the tentative optimum point back to the breaking out point from which the tentative optimum was determined, is generated when the parameter L varies from 0 to 1. This being so we have for any $j = 1, 2 \ldots (n+m)$

$$x'_j = \bar{x}_j + 1.d_j, \qquad \ldots \text{(VII.6)}$$

i.e.,

$$d_j = x'_j - \bar{x}_j. \qquad (j = 1, 2 \ldots (n+m)). \qquad \ldots \text{(VII.7)}$$

In other words, all the direction numbers—and not only the basic ones are simply given by the difference between the value which the variable in question assumed in the breaking out point and in the tentative optimum point. This shows amongst others that d_j is *positive* for all the affixes j for which $\bar{x}_j$ is negative.

The value of L for which we hit the boundary, is determined by

$$L_{hit} = \text{Max} \left. \frac{-\bar{x}_j}{d_j} \right| \bar{x}_j < 0 \qquad \ldots \text{(VII.8)}$$

In the maximum form of (VII.8) j runs through all the affixes $j = 1, 2 \dots (n+m)$ for which $\bar{x}_j$ is effectively negative.

The point x''_j determined by (VII.3)—(VII.8) is a point on the boundary. In this point at least three of the variables are zero if x was determined as the breaking out point of the Double Gradient method and the two leading variables, (i.e., two variables that were zero in the breaking out point) are included amongst those that were put equal to zero in order to determine the tentative optimum point. Indeed both these variables will then be zero all along the line leading from the tentative optimum point back to the breaking out point so that there must be at least one other variable that determines the point where we hit the boundary when moving back from the tentative optimum point towards the breaking out point.

In all cases the above procedure leads to a point on the boundary.

An alternative procedure which does not necessarily involve one round by the inward method for each round of the tentative optimum method, consists in receding a certain distance—let us provisionally say 10 per cent—from the breaking out point and then let this recession point take the place of the breaking out point in the procedure described above. In formulae, this alternative procedure is characterized as follows:

The recession point is defined by

$$x_k^0 = x'_k - b(x'_k - x_k) = (1-b)x'_k + bx_k \; (k = 1, 2 \dots n) \qquad \dots \quad \text{(VII.9)}$$

$$x_j^0 = \Sigma_{k=1}^n b_{jk} x_k^o \qquad (j = n+1,\; n+2 \dots (n+m)) \qquad \dots \quad \text{(VII.10)}$$

where x'_j and $x_j (j = 1, 2 \dots (n+m))$ are respectively the breaking out point and the initial point from where this breaking out point was reached, b is a constant, say $b = 0.1$. Since (VII.10) is linear in the variables, the two formulae (VII.9)—(VII.10) are equivalent with

$$x_j^0 = (1-b)x'_j + bx_j \qquad (j = 1, 2 \dots (n+m)). \qquad \dots \quad \text{(VII.11)}$$

If the initial point had all the variables effectively positive, the same will be true of the recession point provided b is positive and less than unity. Therefore, if we let the recession point x'_k take the place of the breaking out point x_k $(k = 1, 2 \dots n)$ in the above formulae, which means using (VII.3) with

$$d_j = x_j^0 - \bar{x}_j \qquad (j = 1, 2 \dots (n+m)) \qquad \dots \quad \text{(VII.12)}$$

we get into the interior of the admissible region provided we replace the value L_{hit} which results from (VII.8)—where now $d_j = x_j^0 - \bar{x}_j$—by a *somewhat larger* value, let us provisionally say a value obtained by adding to L_{hit} 10 per cent of the increment that would lead from the boundary to the recession point. Since this increment is $(1-L_{hit})$, the procedure amounts to determining the new point x''_j by

$$x''_j = \bar{x}_j + (a + (1-a)L_{hit})(x_j^0 - \bar{x}_j) \quad (j = 1, 2 \dots (n+m)). \qquad \dots \quad \text{(VII.13)}$$

where a is a constant, say, 0.1.

The last formulae can also be written

$$x''_j = (1-a)((1-L_{hit})\bar{x}_j+(a+(1-a)L_{hit})x^0_j \quad (j = 1, 2 \dots (n+m)). \quad \dots \quad \text{(VII.14)}$$

This shows that the value of each variable in the new interior point can be computed as a positively weighted arithmetic average (sum of weights equal to unity) of the values which this variable assumed in the recession point and in the tentative optimum point respectively. A similar interpretation applies to (VII.9) and (VII.11).

Inserting from (VII.11) into (VII.14) we get

$$x''_j = (1-a)(1-L_{hit})\bar{x}_j+(1-b)(a+(1-a)L_{hit})x'_j+b(a+(1-a)L_{hit})x_j$$

$$(j = 1, 2 \dots (n+m)). \quad \dots \quad \text{(VII.15)}$$

This is a positively weighted arithmetic average (with weight-sum unity) of the values in the initial point, in the breaking out point and in the tentative optimum point.

Heuristically it seems natural to choose a and b *all the smaller* the smaller L_{hit} is in comparison to unity. Indeed unity is a measure of the distance from the tentative optimum point to the recession point and L_{hit} is a measure of how large a part of this distance we have to go before we hit the boundry. If this latter distance is small (even exactly zero) the tentative optimum is not far from the boundary, and being not far from the boundary it might not be very far from the actual optimum point. In other words only small and refined adjustments should be considered.

Therefore, it might even seem quite plausible simply to put

$$a = b = L_{hit} \quad \dots \quad \text{(VII.16)}$$

If this is done (VII.15) reduces to

$$x''_j = (1-b)^2\,\bar{x}_j+b(1-b)(2-b)x'_j+b^2(2-b)x_j \quad (j = 1, 2 \dots (n+m) \quad \dots \quad \text{(VII.17)}$$

If b is small, as compared to unity, the three terms of (VII.17) are of descending order of magnitude, which is very plausible.

For any value of the coefficient b in (VII.17) which is effectively positive and not larger than all the values x''_j determined by (VII.17) will be effectively positive provided all the values x_j have this property.

According to the above reasoning the value b should be equal to L_{hit} and will, therefore, have to be determined implicitely from (VII.8), that is by solving the equation in b :

$$b = \underset{j}{\text{Max}} \; \frac{-\bar{x}_j}{(1-b)x'_j+bx_j-\bar{x}_j} \; \bar{x}_j < 0. \quad \dots \quad \text{(VII.19)}$$

If we are not too far from the optimum point, b will be small as compared to unity. We may, therefore, use (VII.18) for an iteration process by inserting first $b = 0$ as an initial value in the right member of (VII.18) and determining the value of the right member which this will lead to and then taking this value as a new improved value of b. This is the same as to use (VII.8). The ensuing value of b is already a value which would make all the x''_j effectively positive if it is inserted in (VII.17). This simply follows from the fact that this value of b is effectively positive.

At least one more step in the iteration process might, however, be performed, inserting the newly found value of b in the right number of (VII.18) and determining the new value of b which this will lead to. Perhaps even a third step may be used, but as a general rule it will not pay to go very far since any positive value of b (less than unity) inserted in (VII.17) will produce a point in the interior of the admissible region.

If it is fairly apparent that the affix j for which the maximum occurs, is relatively stable as b changes the exact value of b can be computed from the equation

$$(x'_j - x_j)b^2 + (\bar{x}_j - x'_j)b = -\bar{x}_j,\ j = \text{affix corresponding to maximum.} \quad \ldots \quad \text{(VII.19)}$$

Since the solution of such an equation—or even one more step in the iteration by means of (VII.18)—only makes up a negligible per cent of the whole work when n and m are large, it will in general be worthwhile to perform the computation here discussed. It is more satisfactory to determine the coefficients by some principle than just conventionally.

To summarize: If the method of the tentative optimum is worked out on the lines here suggested, the steps will be as follows:

From a point x in the interior of the admissible region we have performed a round of the Double Gradient method and have thus reached the breaking out point x'. From this the tentative optimum point $\bar{x}$ is determined. If $\bar{x}$ falls in a corner of the admissible region, we test for optimality. If it falls on the boundary but not in a corner we use the inward method. If it falls outside the admissible region we determine the coefficient b using (VII.18) as an iteration formulae, and taking $b = 0$ as an initial value. Perhaps also (VII.19) may be found useful. A great accuracy in the determination of b is not needed. With the value of b thus determined we compute a new point x'' in the interior of the admissible region by (VII.17). This point may be taken as a new starting point for the Double Gradient method.

VIII. The inward method

Suppose we are in a point on the boundary of the admissible region, i.e., a point where all the variables are non-negative and at least one of them is zero. We want to make a move from this point towards the *interior* of the admissible region utilizing only data belonging to the boundary point from where the movement starts. And we want to do it in a way which brings us as far as possible to an entirely new locality without, however, decreasing the preference function too much, rather increasing it if that is feasible.

We shall assume that there is picked out a set of variables; let it be the variables, $r, s \ldots t$ such that we are particularly interested in making the movement in a direction which will *increase*, or at least not *decrease* these variables. A typical case where such a set of variables $r, s \dots t$ is given, is the case where we have performed one round by the Double Gradient method and have reached a breaking out point on the boundary, with the determination of a ranking order of the variables. We may then take as the set $r, s \dots t$ the first n variables of this ranking order, or more precisely the first n of the variables which are such that they can be used as a basis set, i.e., a set such that all the variables can be expressed uniquely in terms of these n variables.

Amongst the first variables in the ranking order there will always be some that are exactly zero in the point considered. As a matter of fact when the ranking order has been determined by a round of the Double Gradient method at least the two first variables in the ranking order will be exactly zero. We must, therefore, be prepared to handle the case where some of the variables $r, s \dots t$ are exactly zero and some are effectively positive. None of them can be effectively negative since we are in a point belonging to the admissible region.

Let x_k with $k = 1, 2 \dots n$ denote the basis variables and x_j with $j = 1, 2 \dots n, n+1, n+2 \dots n+m$ denote all the variables.

Determine the inward basis components

$$b_k = \frac{b_{rk}}{B_r} + \frac{b_{sk}}{B_s} + \dots + \frac{b_{tk}}{B_t} \quad (k = 1, 2 \dots n) \qquad \dots \quad \text{(VIII.1)}$$

where

$$B_j = \sqrt{b_{j1}^2 + b_{j2}^2 + \dots + b_{jn}^2} \quad (j = r, s \dots t) \qquad \dots \quad \text{(VIII.2)}$$

the b_{jk} being the coefficients by which all the variables are expressed in terms of the basis variables. Compare (3.6).

By means of the inward basis components determine the *inward dependent components*

$$b_j = \sum_{k=1}^{n} b_{jk} b_k \quad . \quad (j = r, s \dots t). \qquad \dots \quad \text{(VIII.3)}$$

The definitions (VIII.2) and (VIII.3) are valid for any $j = 1, 2 \dots (n+m)$, but at this stage it is only necessary to perform the computations for $j = r, s \dots t$.

It should be checked that all those of the components $b_r, b_s \dots b_t$ that correspond to variables that are exactly zero in the boundary point considered, have *the same sign*, either all of them positive or all of them negative.

Let ϕ_{opt} be determined by

$$\phi_{opt} = \begin{cases} \min\limits_{j} \dfrac{b_j}{-p_j} \quad p_j < 0 & \text{if those of the components } b_r, b_s \dots b_t \text{ that correspond to variables that are exactly zero in the boundary point, are } \textit{positive} \\[2ex] -\min\limits_{j} \dfrac{b_j}{p_j} \quad p_j < 0 & \text{if those of the components } b_r, b_s \dots b_t \text{ that correspond to variables that are exactly zero in the boundary point, are } \textit{negative} \end{cases} \qquad \dots \quad \text{(VIII.4)}$$

In the minimum forms, $\underset{j}{\text{Min}}$ in (VIII.4) the affix j only runs through the values corresponding to the variables that are zero in the boundary point from which we start.

When ϕ_{opt} is determined by (VIII.4), we reduce it by a conventional fraction, say 10 per cent so as to get an effective value

$$\phi_{eff} = 0.9\, \phi_{opt}. \qquad \ldots \text{(VIII.5)}$$

With this effective value we determine the basic direction numbers

$$d_k = b_k + \phi_{eff}\, p_k \qquad (k = 1, 2, \ldots n). \qquad \ldots \text{(VIII.6)}$$

By means of these we determine the dependent direction numbers.

$$d_j = \sum_{k=1}^{n} b_{jk} d_k \qquad (j = n+1, n+2 \ldots (n+m)).$$

This gives a total set of direction numbers $d_j (j = 1, 2 \ldots (n+m))$. ... (VIII.7)

If all those of the $b_r, b_s \ldots b_t$ that correspond to variables which are zero in the boundary point from which we start, had the same sign, then all the corresponding direction numbers d_j will also have the same sign, namely, the sign of the corresponding b_j.

Next determine the parameter

$$L_{opt} = \begin{cases} \underset{j}{\text{Min}} \dfrac{x_j}{-d_j} \quad d_j < 0 & \text{if we have the } upper \text{ alternative in (VIII.4)} \\ -\underset{j}{\text{Min}} \dfrac{x_j}{d_j} \quad d_j > 0 & \text{if we have the } lower \text{ alternative in (VIII.4)} \end{cases} \qquad \ldots \text{(VIII.8)}$$

In the minimum forms of (VIII.8) j runs through all the values $j = 1, 2 \ldots (n+m)$, for which d_j has the sign indicated.

When L_{opt} is determined by (VIII.8), we reduce it by a conventional fraction, say 50 per cent, so as to get an effective value

$$L_{eff} = 0.5\, L_{opt}. \qquad \ldots \text{(VIII.9)}$$

With this effective value we determine the new point

$$x'_j = x_j + L_{eff}\, d_j \qquad (j = 1, 2 \ldots (n+m)). \qquad \ldots \text{(VIII.10)}$$

These computations are checked by

$$\sum_{j=1}^{n+m} x'_j = \sum_{j=1}^{n+m} x_j + L_{eff} \sum_{j=1}^{n+m} d_j. \qquad \ldots \text{(VIII.11)}$$

TABLE (IX.1). WORK SHEET FOR FORWARD ONE WAY SOLUTION BY THE ELIMINATION METHOD

	primary equations										derived equations				
	$i=13$	14	15	17	18	19	21	22	23	24	25	26	27	28	29
$k=12$						8.0		−13.6	19.22						
11					17.0		−13.6			18.21					
9			−1.0			−13.0		2.9	−153.60		15.23				
8			−1.0		−4.0		2.9			−5.10	−153.60	24.25			
6		−1.0				−7.0		−1.1	−104.00		104.00	−530.40	14.26		
5		−1.0			2.0		−1.1			8.50		1305.60	−1836.00	17	
4		−1.0		3.0									−530.40	.0	13
3	−1.0					7.5		1.9	117.20		−117.20	597.72	−597.72		−1.0
constant term	4	−7	9	6	2	−1	3	4	18.40	78.20	1364.00	5055.12	8767.22	6.0	5:0
G_i	−4.0	10.0	−7	−9	−17	5.5	8.8	5.9	1122.0	−81.60	−1197.20	−6428.04	11732.04	−9.0	−4.0
forward check	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

The point x_j determined by (VIII.10) will be a point in the interior of the admissible region.

The increase in the preference function from the boundary point to the interior point will be

$$f'-f = \sum_{k=1}^{n} p_k(x'_k - x_k) = L_{eff}\left(\sum_{k=1}^{n} p_k b_k + \phi_{eff} \sum_{k=1}^{n} p_k^2\right). \quad \ldots \quad \text{(VIII.12)}$$

This formula may be used independently or as an extra check (which will however, involve only the affixes $k = 1, 2 \ldots n$).

The interior point thus obtained may be taken as a starting point for another of the Double Gradient method.

IX. The elimination method

The elimination method may be used either for a one way solution or for an inversion. As stated previously the method is advantageous if the matrix contains many zeros, and it is particularly advantageous if only a one way solution is needed. Here some suggestions will be made for how to handle the eliminations.

In (16.9) is given an example where the method is used for an inversion. In a one way solution the lower part of the table contains only one row, (apart from the rows G_i and forward check) but this one row may contain numbers all through. An example is given in (IX.1). The upper part of this table is identical with (16.9).

Before one starts the regular elimination work, it will be well to investigate whether there are some simple *partitions* that can be made, that is to say, whether there are some sub-systems that can be separated. A sub-system is a system consisting of a certain number of the equations which are such that these equations only contain a set of variables *equal in number* to the number of equations in the sub-system, and further such that these variables are uniquely determined by the sub-system. In general it will only pay to look for the simplest forms of partitions. When looking for partitions one should scrutinize the data *column by column* (when the data are organized as in (16.9) and (*IX*.1)).

First of all one should locate the columns—if any—that contains only *one* item (apart from the constant term and the check terms), i.e., equations which contain only one variable each. Each such equation can be solved separately for the variable in question.

Second, one may try to locate a pair of two columns which are such as to contain only two variables and the *same* two variables in both columns. Suppose, for instance, that the two equations r and s contain only the variables, A and B, besides constant terms. The sub-system then is as indicated in table (IX.2).

TABLE (IX.2). SUB-SYSTEM OF TWO EQUATIONS

		equations	
		r	s
variables	A	a_{rA}	a_{sA}
	B	a_{rB}	a_{sB}
	constant term (transferred to same member as the variables)	a_{r0}	a_{s0}

If the determinant is non vanishing, the solution of this sub-system can be written in the following explicit form :

$$x_A = D(a_{so}a_{rB} - a_{ro}a_{sB}) \quad \text{... (IX.2)}$$

$$x_B = D(a_{ro}a_{sA} - a_{so}a_{rA})$$

where

$$D = \frac{1}{a_{rA}\, a_{sB} - a_{sA}\, a_{rB}} \quad \text{... (IX.3)}$$

This form is applicable for machine computations directly as it stands.

When all separate systems of one or two variables each have been partitioned and solved separately, each of the remaining equations is *rearranged* so that the values of the variables which are now already known are included in the constant term of this equation (after multiplication by the coefficients with which these known variables occur in the equation in question). This being done, the regular elimination work starts on the remaining reduced system.

Even the case where a partitioning *can* be done to start with, there is no *necessity* to perform it. In the tables (16.9) and (IX.1), for example, the variables 3 and 4 could have been found by partitioning to start with, but it was not done in this small example. When the number of variables is great and there are many variables that can be determined by a partitioning to start with, it should be done for economy of computation.

When the elimination work starts, one considers the data *row by row* (when the data are organized as in (16.9) or (IX.1)). In the example (IX.1) we start from the top and note that the structure of the equations happens to be such that on the row $k = 12$ there are only two items. These items occur in columns 19 and 22. This means that all the equations are free from the variable no. 12, *except* the equations

19 and 22. Therefore, if we combine the equations 19 and 22 in such a way as to eliminate the variable no. 12, we will obtain an equation which together with the $(n-2)$ remaining unused equations 13, 14, 15, 17, 18, 21 ($n = 8$ in the example) form a system of $(n-1)$ equations not containing the variable no. 12.

To eliminate no. 12 from the two equations 19 and 22, we multiply the former by $+13.6$ and the latter by 8.0 and add, (the former of these two numbers being the *negative* of the coefficient of no. 12 in equation 22 and the latter the coefficient of no. 12 in equation 19). This operation is carried through on the two columns 19 and 22 clear down to the bottom, including the row G_i. The figures obtained are recorded in the new column 23 (which is also marked 19.22 to indicate that it has been obtained by combining the columns 19 and 22). As a check on the computations recorded in the new column 23 we only have to verify that the sum of the elements in the column is zero. This column 23 together with the $(n-2)$ columns 13, 14, 15,16 18, 21 form a set of $(n-1)$ columns not containing the variable no. 12.

Looking at these $(n-1)$ equations we see that not only are they free from the variable no. 12, but the structure of the system happens to be such that all these $(n-1)$ equations are also free from the variable no. 11 *except* the two equations 18 and 21. Therefore, we combined the equations 18 and 21 in such a way as to eliminate the variable no. 11, that is to say we multiply equation 18 by $+13.6$ and equation 21 by 17.0 and add. The result is recorded in column 24. Again a check on the computation is obtained by verifying that the sum of the figures in the new column is zero. This new equation 24 together with equation 23 and the $(n-4)$ remaining unused primary equations 13, 14, 15, 17 form a set of $(n-2)$ equations containing neither the variable no. 12 nor the variable no. 11.

Looking at these $(n-2)$ equations 24, 23, 13, 14, 15, 17, we see that not only are they free from the variables no. 12 and 11, but the structure of these equations happen to be such that they are also free from the variable no. 9, *except* the equations 15 and 23. So we combine the equations 15 and 23 in such a way as to eliminate the variable I. The result is recorded in column 25 and checked by the zero sum in this column. This equation, together with the remaining $(n-4)$ equations 24, 13, 14, 17 form a set of $(n-3)$ equations, namely 25, 24, 13, 14, 17 that do not contain either of the three variables 12, 11, 9.

In this way we continue. In the example the structure of the system happened to be so simple that as we proceed to eliminate one variable at a time we always found that it was sufficient to handle *one* pair of equations to get rid of one more variable. In other words in the right part of the table—where the derived equations are recorded—there is only *one* column for each of the successing *levels of height.*

Sometimes we may need to handle two or perhaps a small number of pairs in order to get completely rid of one more variable. In other words in the right part of the table—where the derived equations are recorded—there will be one or more *levels of height* for which there is more than one column. In the extreme case where the original system of equations contains no simplification at all, there would be $(n-1)$ columns of height $(n-1)$, $(n-2)$ columns of height $(n-2)$ and so on. In this extreme case the elimination method would *not* be an advantageous method to use. Then some other method must be relied upon, for instance, the Gaussian algorithm. But if we can get along with one or a small number of new columns for each variable that is eliminated, the method expounded above is decidedly superior so far as machine time is concerned.

The above only concerns the forward solution. The *back* solution will consist in solving a triangular system and this is done in the standard way which is common to all methods that lead to a triangular system. It is performed by successive insertions: First one variable is found from an equation that only contains this variable. Then this value is inserted in an equation that contains this variable and one other variable and so on.

The final check is performed by insertion in the original equations.

SOME APPLICATIONS: THE EXPERIMENTAL PLAN-FRAME NO. 1

1. The interflow matrix.

The set up is based on a 26×26 interflow matrix. The first 22 sectors of the matrix represent the productive enterprises. The remaining 4 rows and columns used in the present set up—also termed sectors—are organized so as to conform as much as possible to the principles underlying the interflow matrix produced by the University Institute of Economics, Oslo.

For all sectors, also the four additional ones, the following two principles have been applied strictly :

1) The sum in any row shall be equal to the sum in the corresponding column when the figures are expressed as Rupee values.
2) The diagonal elements shall be zero, i.e., only the *net* input-output of each sector is considered.

The four additional sectors are defined as follows :

sector no.	designation of the row	designation of the corresponding column
23	primary factor input (wages, salaries, distributed ownership income), the total of this is by definition the income of the households	households current outlays (consumption of goods and services, domestic or imported, gross savings, disregarding depreciation, and tax payments)
24	taxes	government current outlays (use of goods and services on current account, domestic or imported, and gross savings, disregarding depreciation)
25	gross savings (of enterprises, households and government)	gross investment, private and government (gross investment in fixed capital whether from domestic sources or imported, plus net investment in inventories, plus the balancing item export surplus)
26	import (including—under column 25—the balancing item export surplus)	export

From the matrix of Rupee values are determined the interflow *coefficients* by dividing each element by the corresponding column sum (which is equal to the row sum). In the matrix of coefficients thus obtained each column adds up to unity, but the sum on any column is not necessarily equal to the sum in the corresponding row.

The complete 26×26 table of coefficients is given in table (1)*. The data pertain to the period 1950-51.

2. THE CASE OF CONSTANT COEFFICIENTS EXAMINED.

The most important feature which characterizes the structure of the model is the set of assumptions describing which one of these coefficients we take as *constants*.

To assume *all* of them constant would mean that the table of Rupee values would have only one degree of freedom. All the figures in this matrix can then be multiplied by an arbitrary common factor, but otherwise the whole content of the table of Rupee values will be fixed. This is the case which is some times called the case of a *closed* model.

Indeed, let the Rupee value element in row i and column j be denoted by X_{ij}, the number of sectors n, the ith. row sum $X_i = \sum_{j=1}^{n} X_{ij}$ —which by definition is equal to the ith. column sum—and let the input-output coefficients A_{ij} be defined by

$$X_{ij} = A_{ij} X_j. \qquad \dots \quad (1)$$

We then have $X_i = \sum_{j=1}^{n} A_{ij} X_j$, i.e.,

$$\sum_{j=}^{n} (e_{ij} - A_{ij}) X_j = 0 \qquad \text{where } e_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \qquad \dots \quad (2)$$

In other words the values $X_1, X_2 \dots X_n$ would have to satisfy a system of *homogenous* linear equations. The matrix of this system is singular because all the row sums are zero. We have indeed $\sum_{i=1}^{n} (e_{ij} - A_{ij}) = 1 - \sum_{i=1}^{n} \frac{X_{ij}}{X_j} = 0$. Therefore, values of X_1 $X_2 \dots X_n$ not all zero, exist such that the system is satisfied. If the matrix $(e_{ij} - A_{ij})$ is not of lower rank than $n-1$, these values $X_1, X_2 \dots X_n$ are determined apart from a common multiplier, because by the theory of linear equations all these values must be proportional to any row in the adjoint of the matrix $(e_{ij} - A_{ij})$. If the matrix is singular, all the rows of the adjoint are proportional to each other. If the matrix A_{ij} has the properties of an input-output matrix it can even be proved that there exists a set of *non-negative* values $X_1, X_2 \dots X_n$ that satisfy the equations (2).

The concept of a closed model —as defined by (2)—will not be applied in the present case; it would mean petrifying the whole structure of the system. We just want to investigate what possibilities there exist of making *some changes* in the system.

---

* This table which appears at the end of the paper together with the estimates of investment coefficients and the formulations for the upper and lower bounds are due to T. P. Chowdhury and his associates.

3. The working hypotheses.

For simplicity in working out the first experimental computations on a plan-frame we shall here assume that a fairly large part of the system is constant. We shall assume that all the interflow coefficients are constant except those in the *investment* column. This means that we can take the Rupee-figures in column 25 as *free variables* and consider all the other elements as functions of the 25 gross investment figures that occur in column 25. In the column 25 there are, of course not 26 but only 25 figures because by definition the diagonal element is zero.

Assuming the coefficients in all the 25 columns to be constant involves not only constant relations in the flow from any production sector to any other, but it also involves a constant percentage *input of primary factors*—labour, services of salaried personnel and ownership work (all reckoned on a value basis); further a constant percentage of *imports* and also of *taxes*, and it involves that *consumption* (and savings) in the households changes proportionally to income, that is in such a way that the percentage expenditure pattern remains the same; finally, it involves that a similar percentage expenditure pattern holds for *government expenditure* and also for *exports*. Several of these assumptions are of course very unrealistic, and in a realistic approach to the computation of a plan-frame the *change* in some of these patterns—optimal changes in them—are exactly what we are after. When these assumptions are made here, it is only in order to approach the complexities by steps, considering in this first experimental computations the isolated effect of an *investment* policy, and an optimal determination of this policy.

The fact that all the elements in the complete 26×26 Rupee values table are determined when the 25 gross investment figures are given, is seen as follows.

We use a single dot(.) to designate a summation over the 25 sectors 1,2..., 26 *except* no. 25 and a double dot(:) to designate a summation over the 26 sectors 1, 2..., 26 without exception. The inverted parenthesis is used to denote "exclusion of".

By definition we have

$$X_i = X_{i.} + X_{i,25} \qquad \text{where } X_{i.} = \sum_{j)25(} X_{ij} \ (i = 1, 2, \ldots, 26). \qquad \ldots (3)$$

This equation holds even for $i = 25$ (in which case the last term will be $X_{25,25} = 0$), but we now only use it for $i = 1, 2, \ldots)25(, 26$.

Inserting from (1) into the last equation in (3) we get

$X_{i.} = \sum_{j)25(} A_{ij} X_j$, and hence we have the linear system in 25 variables $X_1, X_2, \ldots)X_{25}(, X_{26}$

$$\sum_{j)25(} (e_{ij} - A_{ij}) X_j = X_{i,25} \qquad i = 1, 2, \ldots)25(, 26 \qquad \ldots (4)$$

The 25 rowed (not 26 rowed) matrix $(e_{ij} - A_{ij})$ is in general *not singular*. Therefore, the system (4) has a well determined solution in the 25 magnitudes X_1, X_2...)X_{25}(X_{26}. When these 25 magnitudes are determined as functions of $X_{1,25}$...)$X_{25,25}$ all the $n(n-1)$ elements in the corresponding 25 columns nos. 1, 2 ...)25(26 are determined by (1). This even applies to the elements in these columns which are listed in the row 25. Indeed, if we assume that all the 25 coefficients A_{1j}, A_{2j} ...)$A_{25,j}$ ($A_{26,j}$ in the row no. j are constant, the coefficient $A_{25,j}$ must also be constant because it is equal to $1-[A_{1j}+A_{2j}+$...)$A_{25,j}$ ($+A_{26j}$]. The elements in the column 25 are, of course, given by the free variables themselves. Thus, the complete 26×26 table of the Rupee values are determined as homogenous linear functions of the 25 free variables $X_{1,25}$, $X_{2,25}$...)$X_{25,25}$($X_{26,25}$.

The coefficients of the linear forms that express X_1, X_2 ...)X_{25}(X_{26} in terms of the free variables are determined by inverting the 25×25 matrix written in the left member of (4).[1]

The result of the inversion of the 25×25 matrix to the left in (4), that is, the matrix obtained by leaving out row 25 and column 25 from the 26×26 matrix given in left (1), is given in table (2)appended at the end of the paper.

4. The investment matrix.

From the economic-political viewpoint it is not the investment subdivided by *delivering* sectors, i.e., the elements of column 25 in the interflow table of Rupee values, that are the pertinent figures, but it is the investment subdivided by *receiving sectors*, i.e., by the sectors into which investment is made, that we must focus attention on. In order to exhibit the connections that exist between investment by delivering sectors and investment by receiving sectors we must consider another matrix: *the investment matrix* which in so far as formal appearance is concerned, resembles the interflow matrix given in table (1).

The delivering sectors in the investment matrix are exactly the same as those in table (1), except for no. 25, which is absent in the investment matrix. The receiving sectors are also the same as in table (1) except for the absence of no. 25. In other words the investment matrix has the same rows and columns as the matrix in the left member of (4). Or we may segregate export surplus as in table (3).[2]

Let J_{ij} denote the elements of the investment matrix—when it is worked out in Rupee values—which indicate the amount delivered from sector i to be used as investment in sector j, where i and j may be any of the numbers 1, 2,...)25(,26. Or in table (3), $i = 1, 2,$ ...)25(,26; $j = 1, 2 ... 24$.

[1] The work of inverting this matrix in the present case was done by Messrs. D. Bose, A. Roy, A. Halder, M. Bhattacharya and associates of the Machine Section, using two IBM 602 calculating punches, together with tabulators, etc.

[2] See section 5.

We have to consider in some detail the logical implications of the investment matrix, and in so doing we shall provisionally take the elements of column 25 in the current account interflow table, i.e., the investment items classified according to *delivering* sectors as representing the degrees of freedom of our model. Subsequently, the transformation to the investment items classified according to receiving sectors will be made.

When the investment matrix is conceived of as containing the rows and columns 1, 2, ...)25(,26 (the sectors being the same as in the current account interflow table), it will include not only investment in domestic sectors, but also investment in the rest of the world, this last item being conceived of as export *surplus* (included in the item in cell 26 of column 25 in the current account table).

There is a perfect formal consistency in this. Indeed, when the items in the production sector cells of column 25 in the current account table are designated as *gross* investment, it is only to indicate that they express figures in which no deduction has been made for depreciation. It does *not* mean that these items express a total output from the sector in question before deduction of the input into the sector. On the contrary, in the case now considered where the coefficients of the 25 columns 1, 2,...,) 25 (,26 are considered as constants and therefore, the flows *within* the system of these 25 sectors are considered an *internal* matter described by the solution of the 25×25 system (4), the items in the twenty-two first cells (the production sectors) of the column 25 of the current account table express the *net* deliveries that go *outside* the current account interflow system definied by (4). This applies to any of the 25 sectors that belong to this system, and consequently, it also applies to the sector 26, i.e., the 'rest of the world' (which is now through our assumption of the constancy of the coefficients in column 26 and the inclusion of this column in the system of equations (4), included in the interflow system itself). That is to say it is the export *surplus* that should logically appear in the cell 26 of the column 25. As an additional element in this cell is entered the import of investment goods (exclusive of import duties on investment goods which are entered in investment column as taxes).

This viewpoint is also in keeping with the idea that we now consider the Rupee value elements of column no. 25—designated gross investment—in the current account table as the free variables. This applies also to sector no. 26 (the rest of the world). This means that *in the present set up the export surplus (added to the import of investment goods) is one of the free variables that is to be disposed of with a view to achieving whatever goals we may want to formulate*. The other variables to be used for this purpose are the other elements in the investment column of the current account table, namely the investment parts of the output of the 22 production sectors as well as no. 23 : The *primary factors input* to be used directly for investment purposes, (for instance, labour used to install machines which are themselves delivered from some production sector) and no. 24, the *taxes on investment*, (for instance, import duties on investment goods).

This way of handling the export surplus as in investment in the rest of the world does not only give a scheme that is satisfactory from the view point of principles but it has also a plausible practical justification. Indeed from the view point of an individual production sector, it will in many cases not matter very much whether its goods are sold for investment purposes at home or they are exported, provided only that they *are* sold.

5. The matrix multiplier

The home investment matrix is given below :

TABLE 3. HOME INVESTMENT MATRIX IN RUPEE VALUES J

		production sectors into which investment is made $h=1$	2	..	21	22	investment in households 23	investment in govt. admn. 24	total home investment from delivering sectors
	$k=1$	$J_{1,1}$	$J_{1,2}$	..	$J_{1,21}$	$J_{1,22}$	$J_{1,23}$	$J_{1,24}$	$J_{1\cdot}=X_{1,25}$
	2	$J_{2,1}$	$J_{2,2}$	..	$J_{2,21}$	$J_{2,22}$	$J_{2,23}$	$J_{2,24}$	$J_{2\cdot}=X_{2,25}$
production sectors delivering investment goods	. . .	—	—	—	—	—	—	—	—
	21	$J_{21,1}$	$J_{21,2}$	..	$J_{21,21}$	$J_{21,22}$	$J_{21,23}$	$J_{21,24}$	$J_{21\cdot}=X_{21,25}$
	22	$J_{22,1}$	$J_{22,2}$	..	$J_{22,21}$	$J_{22,22}$	$J_{22,23}$	$J_{22,24}$	$J_{22\cdot}=X_{22,25}$
use of primary factors for investment purpose	23	$J_{23,1}$	$J_{23,2}$	..	$J_{23,21}$	$J_{23,22}$	$J_{23,23}$	$J_{23,24}$	$J_{23\cdot}=X_{23,25}$
taxes on investment or investment goods	24	$J_{24,1}$	$J_{24,2}$	..	$J_{24,21}$	$J_{24,22}$	$J_{24,23}$	$J_{24,24}$	$J_{24\cdot}=X_{24,25}$
imports for investment purposes	26	$J_{26,1}$	$J_{26,2}$	..	$J_{26,21}$	$J_{26,22}$	$J_{26,23}$	$J_{26,24}$	$J_{26\cdot}=X_{26,25}-E$
total investment at home by receiving sectors		$J_{\cdot 1}$	$J_{\cdot 2}$	..	$J_{\cdot 21}$	$J_{\cdot 22}$	$J_{\cdot 23}$	$J_{\cdot 24}$	

E=export surplus, $J_{k\cdot} = \sum_h J_{kh}$, $(h = 1, 2, \ldots, 24)$, $J_{\cdot h} = \sum_k J_{kh}$, $(k = 1, 2, \ldots, 24) 25 (26)$

Each row sum in table (3) involves 24 items and each column sum involves 25 items.

If the elements of each *column* in table (3) is divided through by the column sum, we get a table of *investment coefficients* B_{kh} defined by

$$J_{kh} = B_{kh}J_{\cdot h} \quad (k = 1,2,\ldots,24)25(26) \quad (h = 1,2,\ldots,24). \qquad \ldots \quad (5)$$

Rough numerical estimates of the coefficients defined by (5) are given in table (4) which is attached at the end of the paper.

Inserting the expressions (5) in the definition of $J_{k\cdot}$ given at the right side of the table (3), we get

$$J_{k\cdot} = \sum_{h = 1,2,\ldots,24)} B_{kh}J_{\cdot h} \qquad (k = 1, 2, \ldots,)\ 25(26) \qquad \ldots \quad (6)$$

By this formula the investment items classified by *delivering* sectors are expressed in terms of the investment items classified by *receiving* sectors.

The solution $X_1, X_2, \ldots X_{24})X_{25}(X_{26}$ of (4) can be written in the form

$$X_i = \sum_{(k = 1,2,\ldots)25(26} (e_{ik}-A_{ik})^{-1}X_{k,25} \qquad (i = 1,2,\ldots)25(26) \qquad \ldots \quad (7)$$

where $(e_{ik}-A_{ik})^{-1}$ is the inverse matrix whose elements are given numerically in table (2).

For instance, if such a change in the economy occurs as will increase the use of investment goods from the sector no. 4 Metal and engineering by an amount $X_{4,25}$ and the use of investment goods from the sector no. 21 Construction by an amount $X_{21,25}$—while the use of investment goods from other sectors are not changed, the increase in total *imports* (denoted as sector no. 26) which this will entail (provided the coefficients in all the columns of the 26×26 interflow matrix given in table (1) remain constant except those in column 25), is estimated at

$$X_{26} = 0.3517X_{4,25} + 0.3239X_{21,25} \qquad \ldots \quad (8)$$

the coefficient 0.3517 is recorded in the cell (26,4) and 0.3239 in the cell (26,21) of the inverse given in table (2). The increase (8) in imports includes not only the direct import needs of the sectors immediately involved namely nos. 4 and 21, but also all import needs created indirectly by the fact that many of the other sectors in the economy will have to increase their production for current account use when sectors 4 and 21 are to increase their total production.

Inserting into (7) the expressions for the $X_{k,25}$ in terms of the $J_{k\cdot}$ as given by the last column of table (3) we get

$$X_i = \sum (e_{ik}-A_{ik})^{-1}J_{k\cdot} + (e_{i,26}-A_{i,26})^{-1}\,E \ (i = 1,2,\ldots)25(26) \qquad \ldots \quad (9)$$

where E is the export *surplus* (positive, negative or zero).

Finally, inserting into (9) the expression for $J_{k\cdot}$ taken from (6), we get

$$X_i = \Sigma M_{ih} J_{\cdot h} + M_{i,26} E \quad (i = 1,2,\ldots)25(26) \qquad \ldots \quad (10)$$
$$h = 1,2,\ldots,24$$

where

$$M_{ih} = \Sigma (e_{ik} - A_{ik})^{-1} B_{kh} \qquad (i = 1,\ 2,\ldots)25(26)$$
$$(h = 1,\ 2,\ldots,\ 24) \qquad \ldots \quad (11)$$
$$k = (1,\ 2,\ldots)25(26)$$

$$M_{i,26} = (e_{i,26} - A_{i,26})^{-1} \qquad (i = 1,2,\ldots)25(26). \qquad \ldots \quad (12)$$

The coefficients M_{ih} are *multipliers* that indicate what change can be expected in the total levels of activity in the various sectors, as a consequence of a given investment into any *individual* investment-receiving sector or as a consequence of any given total *pattern* of investment defined by the way in which the investments are distributed over the investment-receiving sectors. The values of these multipliers are basic data for any reasoned investment policy.

If the investments *into* investment-receiving sectors $J_{\cdot h}$ are given arbitrary values, the corresponding investment deliveries *from* delivering sectors $J_{k\cdot}$ can be computed unambigously by (6), but the inverse will in general not be true. Not only will by (6) the magnitudes $J_{k\cdot}$ in general be linearly dependent amongst themselves, but even if they are attributed values compatible with these linear dependencies, it may not be possible to determine the corresponding magnitudes $J_{h\cdot}$ uniquely because of the special structure which the matrix B_{kh} may have. In general it will therefore, be more satisfactory to formulate any given problem in terms of the $J_{\cdot h}$ than in terms of the $J_{k\cdot}$. This means, amongst others, that it will in general be more satisfactory to use the equations (10) than the equations (7). The only weak point in (10) as compared to (7) is that (10) depends on *more data* and data that may be even more difficult to obtain than those involved in the current account interflow matrix of table (1). This is only an expression for the obvious fact that the more and better information we have, the more satisfactory will be the analytical machinery. Since the first purpose of the present study is to exemplify a rational type of analysis, we will build on (10) rather than on (7).

In the linear programming approach discussed below, the non-negativity of all the $J_{k\cdot}$ is assumed through the non-negativity of all the $J_{\cdot h}$ because all the coefficients B_{kh} in (6) are non-negative. This is another reason why it is an advantage to use the $J_{\cdot h}$.

The fact that the expression to the right in (10) is a homogeneous one, i.e., not involving any term independent of the $J_{\cdot h}$ and E, shows that according to the present set up no activity on *current* account is possible without some *home investment* or *export surplus* being present. This is a realistic aspect of the scheme. It exhibits a basic feature in a capitalistic economy where consumption has to be disposed of through a consumer demand mechanism which operates by means of purchasing power distributed as remuneration to primary factors of production.

6. The admissible region : upper and lower bounds

The equations (7) or (10) express what *would* emerge as solution for the levels of activity in the various sectors *if* the capital capacities, the labour force and the import items were available in sufficient magnitudes to make these levels of activity *possible*. These various items are in the present set up determined through (1) as uni-valued functions of the levels of activity. For a realistic approach we must take account of the fact that the capital capacities in the various sectors may be limited so that the levels of activities have certain upper bounds. Therefore, the solution (7) or (10) may be entirely unrealistic if arbitrary values are inserted for the variables in the right members. Similarly immobility of the labour force and the ensuing limit to the particular kind of labour available for use in a given sector, may create an upper bound to the total output from that sector. Also the restrictions on the imports to a given sector, introduced by considerations of a limited supply of foreign exchange, may put an upper bound to the product of a given sector. And there may be other bounds introduced for technical, financial or political reasons. The introduction of such linear *bounds* to be considered simultaneously with the structural linear *equations*—for instance, equations of the form (10)—is an essentially new element in the analysis. It is the first step towards linear programming.

Some of the bounds introduced may be redundant. For instance, the existing capital capacity in sector i may be such that the level of activity X_i is subject to the upper bound $\bar{X}_i$ and considerations on the immobility of labour may lead to the upper bound $\bar{\bar{X}}_i$ for this output, the latter is redundant if $\bar{\bar{X}}_i \geqslant \bar{X}_i$, and the former redundant if $\bar{\bar{X}}_i \leqslant \bar{X}_i$.

In the present study of the *current year* programming problem, the upper bounds indicated in table (5) were introduced by taking account of the existing capital capacities in the various sectors, and the upper bounds indicated in table (6) were introduced by taking account of the immobility of labour.

Finally, a condition was introduced to the effect that the consumption in the *current* planning year, should not fall below the total 1950-51 level of consumption. Similar bounds were specified for each of the three consumption groups : 1) agriculture plus animal husbandry, 2) textile consumption (from large-scale and small-scale production together), 3) services of house property. These bounds are specified in table (7).

Now for the way in which the structural variables and the bounds are to be expressed in terms of basis variables. In our set up government consumption on current account is not related in any *explicit* way to the productive activity, as are the other sectors (through the introduction of input coefficients of a more or less technical nature). It will, therefore, be natural to leave the variable X_{24} out of the optimality consideration of the present set up. We shall simply consider X_{24} as a constant set at the 1950-51 value. That is we put

$$X_{24} = X^0_{24} \qquad \dots \quad (13)$$

where the superscript o denotes 1950-51.

In doing this we have taken out one degree of freedom from our system. We choose to express this fact by *not* considering E as a basis variable but as a dependent variable. E will be expressed in terms of the remaining 24 basis variables $J_{.1}, J_{.2}, \dots, J_{.24}$ by writing out the equation (10) for $i=24$, which, in conjunction with (13), gives,

$$E = \frac{X^0_{24} - \sum\limits_{h=1,2\,..\,24} M_{24,h}\, J_{.h}}{M_{24,26}} \, . \qquad \dots \quad (14)$$

This expression for E will then have to be introduced into all the expressions for the dependent variables $X_1, X_2, \dots, X_{23})X_{24}, X_{25}(X_{26}$ which gives

$$X_i = M'_{io} + \sum_{h=1,2\dots24} M'_{ih}\, J_{.h} \qquad (i=1,2,\dots,23)24,25(26) \qquad \dots \quad (15)$$

where

$$M'_{io} = \frac{X^0_{24}}{M_{24,26}}\, M_{i,26} \qquad (i = 1,2,\dots,23)24,25(26) \qquad \dots \quad (16)$$

$$M'_{ih} = M_{ih} - \frac{M_{24,h}}{M_{24,26}}\, M_{i,26} \qquad \begin{matrix}(i=1,2,\dots,23)24,25(26)\\ h=1,2,\dots,24\end{matrix} \qquad \dots \quad (17)$$

We need not consider (14) separately as part of our system because we do not impose any condition of non-negativity on E.

The equations (15) form the fundamental part of the total system. We now have to add to (15) a set of new equations expressing the bounds listed in table (5), (6) and (7).

In order to do so we must express also the consumption items $C_1, C_2, \dots, C_{22}$, where $C_i = X_{i,23}$, in terms of the same 24 basis variables which are used in (15). This is done simply by noting that we have

$$C_i = A_{i,23} X_{23} \qquad (i = 1, 2, \dots, 22). \qquad \dots \quad (18)$$

Inserting into (18) the expression for X_{23} taken from (15), we get

$$C_i = (A_{i,23} M'_{23,o}) + \sum_{h=1,2,\dots24} A_{i,23} M'_{23,h} J_{.h} \quad (i = 1, 2, \dots, 22) \qquad \dots \quad (19)$$

This is the expression for all the consumption items in terms of the 24 basis variables now considered, viz., $J_{.1}, J_{.2}, \dots, J_{.24}$.

Each of the bounds considered can be expressed by introducing a slack variable equal to the linear function whose non-negativity expresses the bound in question. For instance, the upper bound $\bar{\bar{X}}_i \geqslant \bar{X}_i$ may be expressed by introducing the slack variable

$$Y_i = \bar{\bar{X}}_i - \bar{X}_i \qquad \ldots \quad (20)$$

and requiring that Y_i shall be non-negative.

In this way all the bounds considered in table (5), (6), (7) may be reduced to a standard form consisting in requiring the non-negativity of all the variables considered, structural as well as slack.

In order not to give rise to confusion the slack variables have been designated by numbers above 70.

It should be remembered that the inclusion of *bounds* does not change the number of degrees of freedom. Indeed each definition of such slack variable brings in one more variable (namely this slack variable) and one more equation (namely the equation expressing this slack variable in terms of the selected basis variables).

The fact that we have assumed total government outlay (including the balancing item government gross savings) fixed at the level X^0_{24}, as expressed in (13), which is the same as to assume that total government *intake* (including the taxes on investment or investment goods) is fixed at the level X^0_{24}, together with the fact that we have assumed (6) to hold with non-negative coefficient B_{kh} (and non-negative $J_{.h}$) also for $k = 24$, which means that $X_{24,25} (= J_{24})$ must be non-negative, put a strong limitation on the total levels of activity in the economy. Indeed, considering the items in row 24 in the current account table we see that when (13) is fulfilled we have

$$A_{24,1} X_1 + A_{24,2} X_2 + \ldots + A_{24,22} X_{22} + A_{24,23} X_{23} + X_{24,25} + A_{24,26} X_{26} = X^0_{24}. \quad \ldots \quad (21)$$

In this equation all the terms in the left number are expressed by means of non-negative current account coefficients $A_{24,h}$, except the term $X_{24,25}$ for which no current account coefficient is assumed as a constant. By the above reasoning about the investment coefficients B_{kh} for $k=24$, the term $X_{24,25}$ becomes, however, non-negative. Therefore, a general and simultaneous increase in the levels $X_1, X_2, \ldots, X_{22}$ can only occur if either X_{23} = total consumer outlay (including consumers' gross savings) or total exports X_{26} become very small.

It would have no realistic economic meaning to consider a negative value of total exports. If the current consumption coefficients had not been assumed as non-negative constants, we might have admitted a negative X_{23}. It would have meant a great negative gross savings in the households or great negative taxes on the households. The assumption about the constancy of total government intake in taxes, etc., could then have been maintained even under a great general expansion of the activity levels in the production sectors. Government could, through the negative taxes on household have left the *necessary spending* to the consumer instead of doing part of it itself. The equation in (15) that expresses the non-negativity of X_{23}, i.e., the equation (15) for $i + 23$, would then be left out as an element in the determination of the admissible region in $J_{\cdot 1}, J_{\cdot 2}, \ldots, J_{\cdot 24}$. Such an approach is, however, realistically impossible when the current account consumption coefficients $A_{k,23}$ $(k = 1, 2, \ldots, 22)$ are assumed given as non-negative constants, because a negative X_{23} would then mean many negative consumption levels. The approach through (13) and (15)-(17) with given non-negative current account coefficients $A_{24,h}(h = 1, 2, \ldots, 24)25(26)$ and given non-negative investment coefficients $B_{24,h}$ $(h = 1, 2, \ldots, 24)$, only serves to show how far it is possible to go when government outlay does *not* follow suit with other developments in the economy. This problem in itself is interesting, although, of course, it does not cover all aspects of the planning.

Modifications in the set of assumptions which may be made to analyse the probable results of *other types* of action, if all the current account coefficient except those in column 25 of table (1) are still assumed constant—can be exemplified as follows.

One modification may be to drop (13) and consequently consider the system (10)—(12) with 25 degrees of freedom instead of the system (15)—(17) with 24 degrees of freedom.

Another modification may be to drop the 24 equations (5) for $k = 24$, which implies dropping the equation (6) for $k = 24$, so that $X_{24,25}$ in (21) may become positive negative or zero. All the 24 variables $J_{24,1}, J_{24,2}, \ldots, J_{24,24}$ in the row 24 in table (3) could not be used as additional basis variables to express the additional degree of freedom that comes in through the dropping of the 24 equations (5) for $k = 24$. This approach is interesting because it will throw light on what can be achieved through changing taxes on investment, possibly using negative taxes on investment.

All the above considerations pertain to the definition of an admissible region. The non-negativity of the basis variables $J_{\cdot 1}, J_{\cdot 2}, \ldots, J_{\cdot 24}$ in (15) or (10) (but not necessarily a non-negativity condition on E if it is retained as a basis variable) and the non-negativity of the dependent variables X in (15) together with the non-negativity of certain slack variables will together define the admissible region.

7. The preference function in the coming year problem

Now for the definition of the *preference function*. At this point we must state explicitly that we now consider the *coming year* planning-problem as distinguished from the problem of the asymptotic pattern (the future pattern) of the economy. This latter problem is considered in detail below. Here it suffices to indicate that as a result of an optimality analysis of the asymptotic pattern we will have reached the conclusion that there are certain capital capacities $\bar{K}_k$ $(k = 1, 2, \ldots, 22)$ in the various production sectors which *would be* the ideal ones at the end of the coming year or period, i.e., the year or period for which plans are *now* being made. I stress the expression "would be" because the capacity K_k do not express an actual target set for the coming year but only something that would fit in with a balanced expansion in line with the population growth and with exactly full employment for men as well as for machines as explained below.

In principle the coming period may be something different from a year, perhaps half a year or perhaps a little more than a year, but in practice a year might be a fair compromise. The period in this connection should be so short that the target contained in the plan for this period can be considered as something definite on which *action is to be taken* more or less immediately after the beginning of the period, without awaiting new information or analysis. This means that the "coming period" cannot be something as long as, say, five years. Realistic planning in the changing world of today must be a continuous process. It must be influenced by the newest and best information that is available at any time. A plan cannot be petrified for, say, five years in advance. The future possibilities are taken account of in the asymptotic pattern, while the *immediate action* is decided on by means of the "coming year" plan.

The asymptotic capital capacities $\bar{K}_k$ $(k = 1, 2, \ldots 22)$ are defined as *real capital*, in the sense of fixed real capital as well as whatever inventories may be deemed necessary for a smooth running of the production.

We only consider the capital capacities $\bar{K}_k$ for the 22 *production* sectors. This means that we do not consider any capital capacity bound for sector 23 (the import of primary factors), or sector 24 (the levy of taxes), or sector 24 (gross savings), or sector (26 exports.) We may have to consider bounds for any of these items but these bounds will not be in the nature of a capital capacity that can be increased by *investment.*

Let $K_k^0 (k = 1, 2, \ldots, 22)$ be the capital capacities that actually exist in the beginning of the year for which planning is made. Consequently,

$$K_k = K_k^0 + J_k - D_k \qquad (k = 1, 2, \ldots, 22) \qquad \ldots \quad (22)$$

is the capital capacity that will actually exist in sector k at the end of the coming year, where J_k and D_k and respectively the gross investment and the depreciation in sector k in the coming year.

In general K_k will be different from the asymptotic value $\bar{K}_k$ because it will be impossible in the course of one single year to pass from the actual situation to one that is compatible with the asymptotic pattern.

The discrepancy between the two patterns $K_1, K_2, \ldots K_n$ and $\bar{K}_1, \bar{K}_2, \ldots, \bar{K}_n$ can be measured in different ways, for instance, by the square deviation

$$\sum_{k=1,2,\ldots,22} (\bar{K}_k - K_k)^2 = \sum_{k=1,2,\ldots,22} [(\bar{K}_k - K_k^0) - (J_k - D_k)]^2 \qquad \ldots \quad (23)$$

In order to be able to work with linear expressions we approximate (23) by

$$\sum_{k=1,2,\ldots,22} [(\bar{K}_k - K_k^0)^2 - 2(K_k - K_k^0)(J_k - D_k)] \ldots \qquad \ldots \quad (24)$$

which can also be written

$$\sum_{k=1,2,\ldots,22} (\bar{K}_k - K_k^0)^2 - 2 \sum_{k=1,2,\ldots,22} (\bar{K}_k - K_k^0)(J_k - D_k). \qquad \ldots \quad (25)$$

This approximation is all the better the further the initial situation is from being equal to the asymptotic pattern.

We are interested in minimizing the difference (23) when the $\bar{K}_k$ and K_k^0 are given and the $(J_k - D_k)$ are variables. This is the same as to *maximize*

$$\sum_{k=1,2,\ldots,22} (\bar{K}_k - K_k^0)(J_k - D_k). \qquad \ldots \quad (26)$$

There are, however, also other goals to be considered. The rate at which *unemployment* is reduced in the coming year and the amount by which the country draws on its *foreign exchange reserves*.

In approaching this problem we begin by considering provisionally the following three variables :

u = millions of new jobs created annually. For instance, if the optimal solution gave $u = 2.5$, this would be considered satisfactory because 1.8 million jobs annually would be needed to cover the population growth, so that one would absorb 0.7 millions annually of the unemployed. ... (27)

v = annual rate of investment, that is the annual increase in India's real capital, expressed as a percentage of the national income (or of the net national product) regardless of whether the investment is achieved through foreign loans or not. For instance, if the optimal solution gave $v = 10$ per cent, this would be considered quite satisfactory. ... (28)

w = net annual increase in India's net foreign assets (liquid or non-liquid), expressed as a percentage of the national income (or of the net domestic or national product). This is identically equal to the foreign balance net export surplus) of goods and services defined in an exhausitive way including not only the trade balance in the classical sense, this balance expressed as a percentage of national income (or of net domestic or national product). ... (29)

Regarding the definition (29) we note that the measurement of the net annual increase in India's net foreign assets can be approached either from the *financial* or from the *physical* side, and the two measurements should, in principle, give the same result (apart from unilateral international transfers, if any). For instance, the visible balance of trade would be one of the most important items in w. A negative balance of trade, that is, an import surplus, would give a negative item in w. If there are no other items, the trade balance would simply be equal to w (when expressed as a percentage of national income or of the net domestic or national product). In this case the correspondence between the financial and the physical measurement is seen very clearly. Indeed, an import surplus must be paid for in *some* way or another. Either by depleting the holdings of liquid assets or by increasing indebtedness (or decreasing non-liquid assets). Perhaps sometimes the import surplus manifests itself financially simply as an overdraft on a certain account or by some foreign firm not receiving settlement as early as they had figured on, but in all cases where there are no other items to take account of than the trade balance, it must be true that if net foreign assets (liquid or non-liquid) are defined exhaustively, the net change in net foreign assets is equal to the export surplus, i.e., is negative in the case of an import surplus.

If a steel plant is built in India by a foreign firm, we can reason as follows. Viewed from the financial side that part of the annual deliveries towards the completion of the steel plant which is *paid in cash,* will deplete the holdings of liquid assets, and that part of the annual deliveries which is made *on credit,* will increase India's indebtedness. In all cases it is true that the *total* deliveries will go in as a negative item in w. In other words if we include total deliveries together with the other items in the trade balance we will again have correspondence with the financial definition of w. Quite generally if we define the foreign balance (net export surplus) of goods and services in an exhaustive way including not only the trade balance in the classical sense, then this balance will be identical with the physical w as defined in the first part of (29).

It will be plausible to let all the three variables u, v, w as defined by (27)—(29) —or some modifications of these variables—enter into the preference function for the coming year plan. How large should be the corresponding weights be ?

Consider first only the relative importance to be attributed to u and v, disregarding temporarily w.

Since 2.5 million annually would be a satisfactory achievement in u and 10 per cent a satisfactory achievement in v, one may say that a given increase in $\frac{u}{2.5}$ would be considered roughly equivalent to an equally large increase in $\frac{v}{10}$. In other words $\frac{u}{2.5} + \frac{v}{10}$ would be a preference function in the two variables u and v. Since the preference function can be multiplied by an arbitrary positive factor we can tentatively take

$$f = 4u + v \qquad \dots \quad (30)$$

as a preference function in u and v.

In order to consider the weight to be attributed to w we consider some examples.

An annual rate of investment of 10 per cent achieved by means of big foreign loans would not be looked upon with so much satisfaction as an annual rate of investment of 10 per cent achieved *without* foreign loans. That, of course, does not mean that an investment carried out by means of foreign loans is not in itself *a very good thing*. Perhaps after all not *very much* less satisfactory than as investment carried out without foreign loans.

The problem is to find some sort of *evaluation* which can define the ratio of preference between an investment carried out with or without foreign loans. For instance, would the nation rather have *one* steel plant built entirely by Indian means without any foreign loans (or, which amounts to the same, without any easy terms of payment) than two steel-plants (of the same capacity) built *with* loans (or easy terms of payment). Certainly one would rather have the two steel plants.

To make the questioning more precise let us consider two steel plants of different capacities, say one of a million ton annually and another of a million and a half annually. Would one be more satisfied by a million ton plant built entirely with Indian means than by a million and half tons plant built by means of foreign loans ? One probably would prefer the million and a half tons plant, but the decision now would not be so quick as in the first question above.

If the choice is between a million ton plant built entirely with Indian means and a million and a quarter tons plant built by foreign loans one would probably prefer the former. In this comparison one would, of course, disregard any thought of what could be obtained by using these foreign loans in *other* sectors. If thoughts of that sort enter, one would probably in any case prefer a plant built by means of loans to a plant of the same capacity built entirely by Indian means. In thinking of the above alternatives, one should concentrate on the specified alternatives, assuming *everything else to be the same*. One should, so to speak, look upon the matter as one would when reading a progress report *after* the end of the year. And in so doing one should imagine two alternatives of this progress report and the choice would be between the two alternatives.

My guess is that the coefficient of equivalence is somewhere in the neighbourhood of 1.3. That is to say, one would be equally satisfied with a progress report telling of a million ton steel plant built entirely by Indian means, and a progress report telling of a 1.3 million ton steel plant built entirely by means of foreign loans, everything else being the same.

These questions are not simply academic questions without practical importance. They express the crux of the matter also from a political and practical view point. The fixation of the weights cannot easily be made the object of general political discussions—at least not before politicians and the public have been educated to be more "operational planning" minded—but the meaning and essence of the decision about how to distribute the emphasis on the several good ends must be stated as clearly as possible *beforehand*, otherwise it will not be possible to avoid *afterwards* endless discussions about whether a given result has been a success or not. When the goals have been described as clearly as possible in general terms by the politicians, the programming expert will be also to translate the opinion in quantitative terms.

If the above evaluation coefficient 1.3 is accepted, the result is that instead of having simply v in the preference function, we will have

$$v + 0.24w. \qquad \dots \quad (31)$$

In (31) the figure 0.24 is the approximate value of $\left(1 - \frac{1}{1.3}\right)$.

That (31) is in accordance with the above reasoning can be tested, for instance, by the following example. Consider first the alternative of one steel plant of a million ton capacity built entirely with Indian means. Using conventionally in this example the capacity of producing one million ton of steel annually as the unit of measurement both for v and w, the case just considered would be characterized by $v = 1$ and $w = 0$, hence the expression (31) would be equal to 1. As another example consider a plant of a capacity of 1.3 million tons, built entirely by means of foreign loans. This would be characterized by $v = 1.3$ and $w = -1.3$. In this case the expression (31) becomes $\left[1.3 - \left(1 - \frac{1}{1.3}\right)1.3\right] = 1$, that is to say, the same as in the first alternative, which checks with our assumption that these two alternatives should be equally satisfactory.

The last term in (31) indicates that if India is able to have some export surplus this is considered with satisfaction because it means that she is able to work up a *foreign reserve* which can be used "on a rainy day". But the emphasis on this reserve aspect is not very great, indeed only of such a magnitude that an *investment* would be looked upon with satisfaction even if it were achieved entirely by increasing India's indebtedness correspondingly. This is expressed by the fact that in (31) the coefficient of v is greater than that of w.

Introducing (31) instead of v into the provisional expression (30) we get $f = 4u + v + 0.24w$, or roughly,

$$f = 16u + 4v + w. \qquad \dots \quad (32)$$

The use of v in the above argument was only provisional. Instead of v we now want to introduce the weighted aggregate (26) with a coefficient that will roughly maintain the order of magnitude of the coefficients of (32). Similarly u and w will have to be translated into the symbols used in our general analysis. These transformations are done as follows.

The 1950-51 employment is estimated at 142.339 million and X_{23} is estimated at 8422.45 crores of rupees. That is to say, the rupee values of the interflow table must be multiplied by $\frac{142.339}{8422.45}$ in order to give the corresponding employment figures. We will consequently have to put

$$u = \frac{142.339}{8422.45}(X_{23}-X_{23}^0). \qquad \dots (33)$$

To determine the equivalance of v in terms of the interflow variables we first note that the national product can be measured by total factor remuneration—which in 1950-51 was 8422.45 crores of rupees—plus the gross saving that took place *outside* the households (because the total income of the households including their gross savings is the same as total factor remuneation). Total gross savings was estimated at Rs.1259.57 crores in 1950–51, out of which Rs.30.00 crores was estimated to take place in the households, so that the figure for total gross savings outside the households, can be put at Rs.1229.57 crores, giving a national product of Rs.9652.02, crores. Total net investment expressed as a percentage of national output will, therefore, be of the order of

$$100\frac{\sum_{k=1,2,..,22}(J_k-D_k)}{9652.02}. \qquad \dots (34)$$

If we retain in the preference function net investment taken literally according to the definition (38), the expression (34) would have to be inserted for v in (32). Instead of doing that we now want to aggregate the items $(J_k-D_k)(k = 1, 2 \dots 22)$ in a *weighted* manner so as to take account of the discrepancies $(\bar{K}_k-K_k^0)$ $(k = 1, 2 \dots 22)$, between capacities K_k^0 at the beginning of the year for which planning is made, and the ideal asymptotic capital capacities $\bar{K}_k$. [Compare (26).] We can do this by introducing in (34) the *weighted* sum

$$\frac{\sum_{k=1,2..22}(\bar{K}_k-K_k^0)(J_k-D_k)}{\sum_{k=1,2..22}(K_k-K_k^0)} \qquad \dots (35)$$

instead of $\sum_{k=1,2..22}(J_k-D_k)$. If we do this, the expression to be inserted for v in (32) becomes

$$v = \frac{\sum_{k=1,2..22}(\bar{K}_k-K_k^0)(J_k-D_k)}{96.5202\sum_{k=1,2..22}(\bar{K}_k-K_k^0)}. \qquad \dots (36)$$

From the view point of planning for the coming year all the differences $(\bar{K}_k-K_k)$ are given constants, therefore, the expression (36) is linear in the planning variables (J_k-D_k) $(k = 1, 2, \dots 22)$.

Theoretically a difficulty may arise if the sum in the denominator of (36) should approach zero, as it may in principle do because no assumption is made about all the differences $(\bar{K}_k-K_k^0)$ $(k = 1, 2 \dots 22)$ being non-negative. The case of a vanishing denominator in (36) has, however, only an academic interest, it would indeed mean a state of affiairs where there is surplus capacity not only in *some* sectors but *on the average* in the economy. Such a situation may perhaps arise under an acute liquidity and underconsumption crisis in a laissez-faire economy, but certainly not in a rationally planned welfare state.

If w as defined literally by (29) is expressed in terms of the variables used for our general analysis, we would get a non-linear expression. A rough linear approximation sufficiently exact in the present model where export coefficients are assumed constant, will be obtained by putting w proportional to total exports X_{26} and determining the coefficient of proportionality in such a way that the expression obtained gives the export surplus as actually observed in the basis year 1950-51. This leads to

$$w = 100 \frac{88.72}{675.33} X_{26}. \quad \dots \quad (37)$$

When the three expressions (33), (36) and (37) are introduced into (32) we get the prefeence function in the linear programming problem for the coming year.

This expression will finally have to be transformed into a linear function of the basis variables. If we want to formulate the coming year problem as a problem with 24 degrees of freedom as expressed by the 24 basis variables in the right member of (15), the procedure will be as follows.

X_{23} in the right member of (33) is expressed in terms of $J_{\cdot 1}, J_{\cdot 2} \dots J_{\cdot 24}$ by using (15) for $i = 23$.

D_k in the right member of (36) is expressed in terms of the corresponding X_k by the proportionality

$$D_k = d_k X_k \qquad (k = 1, 2, \dots 22) \quad \dots \quad (38)$$

where d_k is the depreciation coefficient for the sector k. This coefficient is supposed to be given. Next X_k in (38) is expressed in terms of the 24 basis variables by means of (15) used for $i = k$ ($k = 1, 2 \dots 22$). In this way all the D_k are expressed in terms of the basis variables. Further we note that J_k in (36) is the same thing as $J_{k\cdot}$ which means that it is immediately expressed in terms of the 24 basis variables by (6).

In this way all the three variables u, v, w are expressed linearly in terms of the 24 basis variables, and inserting these expressions into (32) we get the preference function expressed in terms of the basis variables.

8. The asymptotic problem

We shall now consider the way in which the population and the economy may *develop* in the course of time partly under the impact of objective natural forces and partly as the result of a planned activity.

Many aspects of this problem are similar to corresponding aspects of the coming year problem which we considered in the first part of this paper but there are also may aspects that are different. We must always remember that in principle the two problems are different, and the *assumptions* we make when studying the problem of development, for instance, assumptions about the constancy of certain coefficients —need not have any counterpart in or be equal to corresponding assumptions in the current year problem.

Suppose that the population increases at a certain constant rate, say 1 per cent a year —a rate which is not very far from that which can be forcasted as the average rate of increase of the Indian population over the next decades—and further suppose that the expansion of the economy should be steered in such a way that it tends towards a *balanced* state, i.e., a state where there is neither under-employment nor over-employment of men in any sector of the economy and where the capital capacity in each sector is at any time exactly equal to that which is needed for the actual production that is carried on in this sector at that time. Finally, suppose that in this balanced state that consumption pattern remains constant, i.e., the amount that are consumed of the various goods remain in the same proportion to each other, and similarly the production pattern remains constant as well as all the technical coefficients that characterize production and all the behaviouristic coefficients that describe the way in which the consumers and producers react. In this state all the various consumption and production activities must increase at exactly the same constant rate per unit of time as the population so that these various items are *constant* when reckoned per head of the population. This state we shall call the *asymptotic state*.

While there are many things in the asymptotic state that are determined by the technical and behaviouristic coefficients which are assumed as given constants, the state may also have many *degrees of freedom*. For instance, if the ratios of consumption, that is the *ratios* which the production of the several consumer goods bear to each other, are not assumed as given, these ratios represents degrees of freedom. They must be constant over time but their magnitudes may be chosen in different ways. Therefore a problem of policy decision arises. Which particular combination of these ratios shall be considered *the best* and what shall be done in order to achieve this best combination? This leads to a linear programming problem which computationally is of exactly the same sort as the linear programming problem of determining a plan for the coming year, only the concrete meaning of the variables involved in the computations are different.

(i) *Definitions and assumptions*

In what follows we shall designate by K^t_h the size of the capital that is present *at the point of time t*, and by $J^t_{.h}$ the (total) gross investment into the sector h in the year *between* t and $t+1$. Similarly D^t_h will be used to designate the total depreciation that takes place between t and $t+1$, and X^t_{ij}, X^t_i etc., will indicate flows in the period extending from t to $t+1$ etc. The period from t to $t+1$ will for shortness be called "the period t".

The use of the superscripts is such that K^t_h designates a capacity that *is needed for* the production of X^t_h, while D^{t-1}_h is the depreciation that *contribute to* determining K^t_h.

We shall consider an asymptotic problem characterized by the following assumptions

$$K^t_h = b_h X^t_h \qquad (h = 1, 2 \ldots 22 \text{ but } \textit{not } 23, 24, 25, 26) \qquad \ldots \quad (39)$$

$$D^t_h = d_h \; X^t_h \qquad (h = 1, 2 \ldots 22 \; \textit{but not } 23, 24, 25, 26) \qquad \ldots \quad (40)$$

$$N^t_h = n_h \; X^t_h \qquad (h = 1, 2 \ldots 22)23(24, 25, 26) \qquad \ldots \quad (41)$$

where b_h, d_h, n_h are given constants independent of time, and N^t_h is the employment in sector h in the period t. This employment can be measured in number of people employed or in man-hours or in some other physical terms.

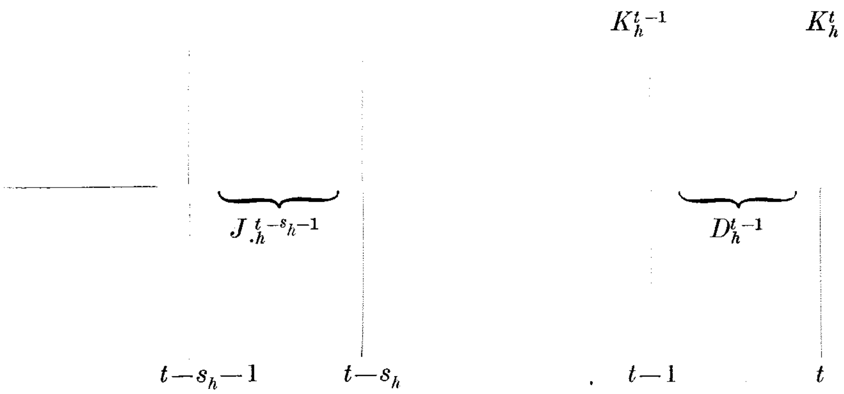
When the assumption (41) is made, there is no need for considering N^t_h *explicitly* in the formulae, we may let N^t_h be represented everywhere by its equivalent in X^t_h, but much of the concrete considerations will still involve references to the labour-force aspect of the problem.

Let s_h $(h = 1, 2 \ldots 22)$ be the *investment lag* in sector h, that is, the average length of time which can be expected to pass between the point of time where an investment is made in this sector and the point of time where the resulting capital goods are in operating order and can be used in the production in this sectors. For a steel factory this delay is perhaps 3 or 4 years, for textile machinery perhaps 12 or 18 months and for an handloom 3 or 6 months. We assume that these investment lags are given constants.

In terms of these lags the time relation between capital capacity, gross investment and depreciation is

$$K^t_h = K^{t-1}_h + J^{t-s_h-1}_{.h} - D^{t-1}_h \qquad (h = 1, 2, \ldots 22). \qquad \ldots \quad (42)$$

The relation (42) is graphically depicted in the following figure for a case where $s_h > 1$.



A simple rearrangement of the terms in (42) and the substitution of $t+1$ for t gives

$$J^{t-s_h}_{.h} = K^{t+1}_h - K^t_h + D^t_h \qquad (h = 1, 2 \ldots 22) \qquad \ldots \quad (43)$$

and hence by (39) and (40)

$$J^{t-s_h}_{.h} = b_h \, (X^{t+1}_h - X^t_h) + d_h X^t_h \qquad (h = 1, \ldots 22) \qquad \ldots \quad (44)$$

9

The equations (39)—(42)—together with the rearrangement (44)—from starting point for an analysis of the asymptotic problem.

We shall take the levels of activity X_1^t, $X_2^t \ldots X_{22}^t$ amongst our basis variables in this problem and will, therefore, here not need to invert any matrix—as we had to in the coming year problem, [compare (10)-(12) and (15)-(17)]. We can now formulate the basis equations by building on certain computations performed directly on the current account production coefficients, the investment coefficients, etc. These coefficients themselves can be looked upon as *the same* in the asymptotic and the coming year problem, when these two problems are worked out *approximately at the same moment* so that the state of factual knowledge is approximately the same.

If the various coefficients in the asymptotic problem are not taken as the values of the coefficients as they *exist* at the moment when the computations are made, but taken with values that are *forcasted* by scientific or technological experts, the period for which the coefficients are forcasted being roughly the time horizon that enters into the mind of the planner who is to use the results of the programming analysis for the asymptotic structure—which may indeed by a very reasonable procedure —then the coefficients of the asymptotic and those of the coming year problem need not be the same.

We now want to express the consumption delivery from sector k, namely C_k^t, in terms of $X_1^t, X_2 \ldots X_{22}^t$ and some other basis variables. To do this we note that according to the definition of the total product X_k^t in sector k,($k = 1, 2, \ldots, 22$)1 which can also be written $X_k^t = \sum_{h=1, 2 \,..\, 22} e_{kh} X_h^t$ where the e_{kh} are the unit numbers—we have

$$\sum_{h=1, 2, \ldots, 22} e_{kh} X_h^t = \qquad \ldots \ldots \quad (45)$$

cross-delivers from k: $\sum_{h=1, 2 \,..\, 22} A_{kh} X_h^t$

+consumption deliveries from k : C_k^t (which is the same as $X_{k,23}^t$)

+government's current use of goods and services from k : $A_{k,24} X_{24}^t$

+investment deliveries from k : $\sum_{h=1, 2 \,..\, 22, 23, 24} B_{kh} J_{.h}^t$

+exports from k : $A_{k,26} X_{26}^t$

In the above equation (45) we have assumed as given constants all the coefficients in the 26×26 current account interflow table (1), except the coefficients in columns 23 (household outlays) and 25 (investment by delivering sectors). That is to say in the present asymptotic set up we have not only refrained from any assumption about the constancy of the coefficients in column 25 of the interflow table (1), but we have also dropped the assumption of constant consumption coefficients. To assume constant consumption coefficients would have been too unrealistic in the asymptotic problem. It would—through the conditions we have put on the asymptotic pattern to make the development a balanced one—have fixed the whole structure of the asymptotic economy in such a way that practically no scope would be left for relevant policy decisions about this structure.

In the investment delivery term in (45) we take out separately the two last items $J^t_{.23}$ and $J^t_{.24}$ because they do not correspond to an activity which increases one of the 22 *capacities* K^t_k which we are considering. The investment delivery term in (45) we shall, therefore, write

$$X_{k,25} = \sum_{h=1,2..22} B_{kh} J^t_{.h} + B_{k,23} J^t_{.23} + B_{k,24} J^t_{.24}. \qquad \dots (46)$$

If the development is a balanced one in the sense described above, so that all the various items of production and consumption activity in the asymptotic state will increase in the same rate as the population increases, we have

$$X^t_k = X_k e^{ct} \qquad (k = 1, 2 \dots 22,)23,(24, 25, 26) \qquad \dots (47)$$

where e^c is the factor by which the total population is multiplied each year, say

$$e^c = 1.01 \qquad \dots (48)$$

and X_k is a *constant* that expresses the Rupee value of the total production in sector k in the year $t = 0$, for instance, in the year 1950–51.

Similarly we have for all the other time functions

$$C^t_k = C_k e^{ct} \qquad (k = 1, 2 \dots 22)23(24, 25, 26) \qquad \dots (49)$$

$$J^t_{.h} = J_{.h} e^{ct} \quad \text{etc.} \quad (h = 1, 2, \dots 22, 23, 24) \qquad \dots (50)$$

where C_k is the rupee value of the consumption of goods from sector k in the basis year $t = 0$ and $J_{.h}$ is the Rupee value of the total investment into sector h in the basis year $t = 0$, etc.

Through the assumption of a *balanced* expansion the whole problem of the asymptotic pattern is in this way reduced to a corresponding problem regarding *a theoretical alternative* for the structure of the economy in the basis year $t = 0$. Instead of reasoning in terms of time curves, we can now reason in terms of this theoretical structure of the economy in the basis year as described by the X_k, the C_k, the $J_{.h}$. etc. By this device we have reduced the calculations to the same kind of calculations as would be used for a genuinely *static* problem.

(*ii*) *A suggested generalization for any shape of time-path*

This method may be generalized in such away as to take account of any shape of time curves with a sufficient degree of approximation. We may, for instance, assume that each of the production and consumption activities is an exponential polynomial, that is to say an expression of the form

$$Z^t = Z_1 e^{c_1 t} + Z_2 e^{c_2 t} + \dots + Z_m e^{c_m t} \qquad \dots (51)$$

where $Z_1, Z_2 \dots Z_m$ are m *constants* characterizing the production or consumption level Z^t (which is changing with time) and $c_1, c_2 \dots c_m$ are m constants characteristic of the expansion of the economy *as a whole* (they are independent of which particular production or consumption level we consider). By admitting complex values of $c_1 c_2 \dots c_m$ we may even produce a movement that contains damped or undamped oscillations. In all cases the programming problem can be handled in a static way by formulating it as a problem in the magnitudes $Z_1, Z_2 \dots Z_m$ and the corresponding m magnitudes referring to each of the other production and consumption levels considered. All these magnitudes are constants in the sense of being independent of t, but they may assume an alternative set of values for each particular pattern of the economy which we chose to consider. The linear programming problem would consist in deciding which one of these alternative patterns of the economy is the optimal one. In the present setup we only consider the case $m = 1$, that is the case where each experimental polynomial (51) contains only one single term.

From the above assumptions we get

$$J^t_{.h} = e^{cs_h}(e^c b_h - (b_h - d_h))X^t_h \quad (h = 1, 2 \dots 22) \qquad \dots (52)$$

and hence

$$\sum_{h=1,2..22,23,24} B_{kh} J^t_{.h} = \sum_{h=1,2..22} B_{kh} e^{cs_h}(e^c b_h - (b_h - d_h))X^t_h \qquad \dots (53)$$

$$+ B_{k,23} J^t_{.23} + B_{k,24} J_{.24} \qquad (k = 1, 2 \dots 22).$$

Inserting this in the investment delivery term in the above expresssion for X^t_h, we get

$$C^t_h = \sum_{h=1,2..22} \bar{M}_{kh} X^t_h - B_{k,23} J^t_{.23} - B_{k,24} J^t_{.24} \qquad \dots (54)$$

$$- A_{k,24} X^t_{24} - A_{k,26} X^t_{26} \quad (k = 1, 2 \dots 22)$$

where

$$\bar{M}_{kh} = (e_{kh} - A_{kh}) - B_{kh} e^{cs_h} (e^c b_k - (b_h - d_h)) \quad (k = 1, 2 \dots 22) \qquad \dots (55)$$

$$(h = 1, 2 \dots 22)$$

In (54) we insert the expressions for the various production and consumption activities as exponential functions of time. This being done we can divide through by e^{ct} and get an equation in the *constants* C_k, X_h, etc. This equation is obtained simply by dropping the superscript t in (65). This gives

$$C_k = \sum_{h=1,2..22} \bar{M}_{kh} X_h - B_{k,23} J_{.23} - B_{k,24} J_{.24} \qquad \dots (56)$$

$$- A_{k,24} X_{24} - A_{k,26} X_{26} \; (k = 1, 2 \dots 22)$$

where the coefficients $\bar{M}_{kh}$ are the same as those given in (55).

The equations (56) can be taken as a set of basis equations through which practically the whole pattern of the economy—as expressed by the time-independent theoretical production and consumption levels—are determined in terms of the 26 basis variables $X_1, X_2 \ldots X_{22}, J_{\cdot 23}, J_{\cdot 24}, X_{24}$ and X_{26}. Indeed, when these variables are determined and all the coefficients in the current account table are supposed to be given except those in columns 23 and 24, we can determine $J_{1\cdot}, J_{2\cdot} \ldots J_{22\cdot}$ as the residual in each of the rows 1, 2, ... 22 after the elements in any such row (in the columns 1, 2, ...23, 24, 26) have been determined through $X_1, X_2, \ldots X_{22}, X_{24}, X_{26}$ and the current account coefficients, and $C_1, C_2 \ldots C_{22}$ have been determined through (56). This gives all the Rupee value elements in rows 1,2,...22 of the current account table. For the complete determination of the elements in the four rows 23, 24, 25, 26 we need three more data, for instance, C_{24}, C_{25}, C_{26}. In rows 23, 24, 25, 26 the elements in columns 1, 2,... 22 will be determined by $X_1, X_2, \ldots X_{22}$ and the current account coefficients, the elements in column 23 will be determined by C_{24}, C_{25}, C_{26}, the elements in column 24 will be determined by X_{24} and the coefficients, and the elements in column 26 are determined by X_{26} and the coefficients. The three remaining elements $X_{23,25}$ $X_{24,25}$ and $X_{26,25}$ ($X_{25,25}$ being by definition zero) are finally determined as follows: $X_{23,25}$ is determined as the residual in row 23, the sum X_{23} in this row now being known as the sum of the elements in column 23. $X_{24,25}$ is determined as the residual in row 24, the sum in this row being one of the basis variables of (54). $X_{26,25}$ is determined as the residual in row 26, the sum in this row being one of the basis variables of (54).

As in the coming year problem we shall also now assume that government administration does not increase. In the present dynamic case this should be specified as "government administration *per head* of the population does not increase". To express this condition in terms of the symbols adopted we note that the condition mentioned is equivalent to saying that X_{24} shall be equal to 792.0 crores of Rupees when t is put equal to 1950-51. This is the same as to say that $X_{24}\,e$ shall be equal to 792.20 when t is put equal to 1950-51. If this period is conventionally denoted $t=0$, we get back to the condition (13). This condition takes out one degree of freedom and this reduction can simply be expressed by putting X_{24} equal to the fixed value $X^0_{24} = 792.20$ in (56), which means that the equation gets the constant term $-A_{k,24}$ X^0_{24}. That is to say we now have

$$C_k = \bar{M}_{k0} + \Sigma \bar{M}_{kh} X_h + \bar{M}_{k,23}\, J_{\cdot 23} + \bar{M}_{k,24}\, J_{\cdot 24} + \bar{M}_{k,26}\, X_{26} \quad (k = 1, 2 \ldots 22) \quad \ldots \quad (57)$$

where $\bar{M}_{kh}$ is given by (55), and

$$\bar{M}_{k0} = -A_{k,24} X^0_{24} \quad \text{with} \quad X^0_{24} = 792.20 \quad \ldots \quad (58)$$

$$\bar{M}_{k,23} = -B_{k,23} \quad \ldots \quad (59)$$

$$\bar{M}_{k,24} = -B_{k,24} \quad \ldots \quad (60)$$

$$\bar{M}_{k,26} = -A_{k,26} \quad \ldots \quad (61)$$

The 22 equations (57) are taken as the structural equations of our problems with $X_1, X_2, \ldots X_{22}, J_{\cdot 23}, J_{\cdot 24}$ and X_{26} as the basis variables. This gives a total of 25 degrees of freedom.

(iii) The problem of bounds

Let us consider the bounds to be imposed. We impose three conditions on consumption, namely:

The sum of agricultural and animal husbandry production shall not be less than in the basis year. This is equivalent to requiring

$$C_1+C_2 \geqslant 4167.67 \text{ (crores of rupees).} \quad \ldots \quad (62)$$

The sum of large-scale and small-scale textile production shall not be less than in the basis year. This is equivalent to requiring

$$C_9+C_{13} \geqslant 445.83 \text{ (crores of rupees).} \quad \ldots \quad (63)$$

The services of house property shall not be less than in the basis year. This is equivalent to requiring

$$C_{22} \geqq 521.70 \text{ (crores of rupees)} \quad \ldots \quad (64)$$

The conditions (62)-(64) can be expressed by introducing the following three slack variables

$$\text{variable no. } 67 = C_1+C_2-4167.67, \quad \ldots \quad (65)$$

$$\text{variable no. } 68 = C_9+C_{13}-445.83, \quad \ldots \quad (66)$$

$$\text{variable no. } 69 = C_{22}-521.70, \quad \ldots \quad (67)$$

and requiring these slack variables to be non-negative. For practical computational reasons these three slack variables are numbered from 67.

Finally, we shall introduce a bound on the sum $(C_{24}+C_{25}+C_{26})$.. This sum is the total amount which the households spend on taxes, gross savings and imported consumption goods. We will require that this sum shall be not less than C_{26}^{0}, i.e., what the households spend on imported consumption goods in the basis year. This means that if government put no taxes on the households, the households should have at disposal after consumption of domestically produced consumer goods, a sum large enough to purchase imported consumer goods to the same extent as in the basis year. Or they may choose to consume less imported goods than in the basis year and set the rest aside as their gross savings. Whatever taxes government may put on the households would correspondingly restrict the amount available for consumption of imported goods and/or for savings.

As we have seen the sum $(C_{24}+C_{25}+C_{26})$ is not determined by the fact that the 26 basis variables in (56) are given. And a similar reasoning would apply to the case of 25 basis variables in (57). The sum in question would, however, be determined if we know the value of X_{23}, that is, of the total income of the households (the sum in row 23). We have indeed

$$(C_{24}+C_{25}+C_{26}) = X_{23}-(C_1+C_2+\ldots+C_{22}) \quad \ldots \quad (68)$$

where $C_1, C_2, \ldots, C_{22}$ are given by (57). As an estimate of the value of X_{23} that will describe the asymptotic pattern—where full employment is assured—we may take the basis year value X_{23}^{0} inflated to correspond to full employment in the basis year. This amount can be estimated at $1.21X_{23}^{0}$, where $X_{23}^{0} = 8422.45$. The condition in question can, therefore, be expressed by saying that $1.21X^0_{23}-(C_1+C_2+\ldots+C_{22})$ shall be not less than $C^0_{26} = 111.95$.

This is the same as to introduce the slack variables

$$\text{variable no. } 70 = 1.21X^0_{23}-C^0_{26}-(C_1+C_2+\ldots+C_{22}) \qquad \ldots \quad (69)$$

and to require this variable to be non-negative.

Finally, as a preference function we introduce

$$f = C_1+C_2+\ldots+C_{22} \qquad \ldots \quad (70)$$

which means that we simply want to maximize total consumption of home produced consumer goods.

When C_{22}—the consumption of services of house property—is required to be not less than what it was in the base year (as expressed by the non-negativity of the variable no. 69) it is superfluous to require C_{22} to be non-negative. The equation (57) for $k = 22$ may, therefore, be dropped from consideration. This leaves us with the 21 equations (57) for $k = 1, 2 \ldots 21$, and the four definitional equations (65)-(67) and (69) , giving a total of 25 dependent variables that is, 50 variables altogether. It is only accidental that the number of dependent variables here become equal to the number of basis variables.

The expressions for (65)-(67) and (69)-(70) in terms of the 25 basis variables, is easily obtained when all the C_1, C_2, ..., C_{22} are expressed in terms of these variables by (57).

Extending the meaning of $\bar{M}_{kh}$ also to $k = 67, 68, 69$ and 70 (the last "variable" being the preference function), we have

$$\bar{M}_{67,0} = -4167.67+\bar{M}_{1,0}+\bar{M}_{2,0} \qquad \ldots \quad (71)$$

$$\bar{M}_{67,h} = \bar{M}_{1,h}+\bar{M}_{2,h} \qquad (h = 1, 2, \ldots 24)25(26) \qquad \ldots \quad (72)$$

$$\bar{M}_{68,0} = -445.83+\bar{M}_{9,0}+\bar{M}_{13,0} \qquad \ldots \quad (73)$$

$$\bar{M}_{68,h} = \bar{M}_{9,h}+\bar{M}_{13,h} \qquad (h = 1, 2 \ldots 24)25(26) \qquad \ldots \quad (74)$$

$$\bar{M}_{69,0} = -521.70+\bar{M}_{22,0} \qquad \ldots \quad (75)$$

$$\bar{M}_{69,h} = \bar{M}_{22,h} \qquad (h = 1, 2 \ldots 24)25(26) \qquad \ldots \quad (76)$$

These coefficients are collected in table 8.

Strictly speaking we should also have introduced non-negativity conditions for $X_{23,25}$, $X_{24,25}$ and $X_{25,25}$ when these variables are expressed in terms of the basis variable. Compare the above argument about the necessity of introducing three more basis variables in order to have all the variables in the asymptotic pattern determined. In the present experimental computation it was decided to carry the computations through without taking account of these three conditions.

The present paper has given the theoretical set up and the basis numerical tables for formulating an asymptotic problem and the corresponding coming year problem on Indian data. The solution of the asymptotic problem will furnish the values $(K_k-K^0_k)$ which enter into the coming year problem through (32) where v is given by (36).

TABLE 1. 26×26 CURRENT ACCOUNT INTERFLOW TABLE OF COEFFICIENTS FOR INDIA 1950-51

List of the 22 production sectors are given at the end of the table. The meaning of sectors 23–26 is explained in section 1 of the present paper.

The inversion of the 25×25 matrix obtained from the present 26×26 matrix by leaving out row 25 and column 25, is given in table (2).

continued on next page

delivering sector no.	receiving sector no.								
	1	2	3	4	5	6	7	8	9
$k=1$	0	.3506	.0000	.0000	.0040	.0002	.0000	.4330	.2434
2	.1694	0	.0000	.0012	.0152	.1866	.0000	.0027	.0005
3	.0000	.0000	0	.0208	.0090	.0125	.0567	.0029	.0042
4	.0001	.0002	.0194	0	.0085	.0059	.0281	.0025	.0000
5	.0004	.0010	.0065	.0049	0	.0171	.0000	.0014	.0051
6	.0000	.0003	.0000	.0019	.0002	0	.0453	.0034	.0008
7	.0000	.0001	.0130	.0083	.0078	.0268	0	.0023	.0062
8	.0004	.0745	.0000	.0000	.0152	.0000	.0000	0	.0000
9	.0002	.0003	.0000	.0105	.0149	.0114	.0000	.0040	0
10	.0000	.0000	.0000	.0024	.0008	.0000	.0040	.0050	.0000
11	.0000	.0000	.0000	.0042	.0014	.0015	.0000	.0018	.0000
12	.0000	.0000	.0000	.0008	.0006	.0110	.0000	.0017	.0021
13	.0000	.0003	.0000	.0000	.0211	.0000	.0000	.0010	.0026
14	.0014	.0003	.0130	.0494	.0035	.0044	.0013	.0000	.0002
15	.0001	.0164	.0000	.0000	.0076	.0000	.0000	.0011	.0000
16	.0000	.0000	.0000	.0020	.0000	.0132	.0025	.0017	.0002
17	.0016	.0006	.0176	.0058	.0243	.0022	.0040	.0003	.0006
18	.0002	.0047	.0389	.0639	.0380	.0988	.0253	.0170	.0185
19	.0064	.0337	.0648	.1081	.1948	.1758	.0253	.1036	.1096
20	.0089	.0092	.0778	.1179	.0760	.0703	.0202	.0535	.0648
21	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
22	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
23	.7479	.4322	.3174	.2795	.1106	.1153	.2124	.1163	.1641
24	.0232	.0200	.1177	.0787	.1186	.0988	.0951	.0835	.0952
25	.0372	.0549	.2207	.1954	.1559	.0926	.1578	.1083	.1394
26	.0026	.0004	.0936	.0446	.1720	.0558	.3220	.0576	.1425
total	1	1	1	1	1	1	1	1	1

TABLE 1 : 26×26 CURRENT ACCOUNT INTERFLOW TABLE OF COEFFICIENTS FOR INDIA 1950-51

List of the 22 production sectors are given at the end of the table. The meaning of sectors 23–26 is explained in section 1 of the present paper.

The inversion of the 25×25 matrix obtained from the present 26×26 matrix by leaving out row 25 and column 25, is given in table (2).

continued

continued on next page

delivering sector no.	receiving sector no.								
	10	11	12	13	14	15	16	17	18
k=1	.0215	.0112	.0080	.0083	.0006	.3683	.0241	.0301	.0465
2	.0019	.0933	.0125	.0019	.0034	.0036	.2794	.0232	.0043
3	.0570	.0018	.0142	.0035	.0034	.0012	.0135	.0003	.0506
4	.0000	.0000	.0000	.0000	.1372	.0006	.0095	.0012	.0757
5	.0355	.0267	.0151	.0028	.0000	.0000	.0011	.0019	.0027
6	.0336	.0018	.0000	.0009	.0001	.0007	.0268	.0000	.0013
7	.0289	.0061	.0144	.0006	.0007	.0001	.0003	.0001	.0052
8	.0000	.0000	.0000	.0000	.0000	.0083	.0000	.0000	.0000
9	.0000	.0000	.0000	.4760	.0000	.0000	.0000	.0000	.0008
10	0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
11	.0000	0	.0000	.0000	.0000	.0000	.0000	.0175	.0005
12	.0327	.0079	0	.0045	.0003	.0007	.0000	.0000	.0062
13	.0000	.0000	.0000	0	.0000	.0000	.0000	.0000	.0000
14	.0047	.0000	.0000	.0026	0	.0006	.0085	.0078	.0000
15	.0000	.0000	.0000	.0000	.0000	0	.0000	.0000	.0000
16	.0047	.0000	.0000	.0046	.0001	.0000	0	.0000	.0000
17	.0131	.0533	.0215	.0039	.0017	.0005	.0000	0	.0008
18	.0467	.0092	.0142	.0100	.0090	.0033	.0133	.0027	0
19	.0467	.1153	.1301	.0077	.0782	.0413	.0398	.0692	.0240
20	.0467	.0562	.0828	.0107	.0337	.0154	.0164	.1167	.0686
21	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
22	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
23	.1494	.1955	.2105	.3914	.5633	.4385	.4132	.6265	.4184
24	.0514	.1065	.1294	.0278	.0317	.0290	.0199	.2080	.0691
25	.0915	.1077	.1353	.0378	.1016	.0605	.0631	.0616	.1643
26	.3343	.2075	.2119	.0049	.0351	.0274	.0713	.0133	.0620
total	1	1	1	1	1	1	1	1	1

TABLE 1 : 26×26 CURRENT ACCOUNT INTERFLOW TABLE OF COEFFICIENTS FOR INDIA 1950-51

List of the 22 production sectors are given at the end of the table. The meaning of sectors 23–26 as explained in section 1 of the present paper.

The inversion of the 25×25 matrix obtained from the present 26×26 matrix by leaving out row 25 and column 25, is given in table (2).

continued

delivering sector	receiving sector no.							
no.	19	20	21	22	23	24	25	26
$k=1$	.0260	.0003	.0878	.0008	.3981	.0067	.0060	.0813
2	.0000	.0000	.0090	.0090	.0968	.0044	.0054	.0811
3	.0000	.0005	.0083	.0018	.0035	.0001	.0000	.0175
4	.0025	.0016	.1215	.0027	.0029	.0040	.0492	.0179
5	.0000	.0005	.0007	.0004	.0054	.0020	.0000	.0056
6	.0006	.0002	.0509	.0106	.0000	.0059	.0028	.0009
7	.0026	.0001	.0009	.0001	.0024	.0030	.0000	.0000
8	.0000	.0010	.0000	.0000	.0431	.0000	.0000	.1307
9	.0051	.0021	.0000	.0000	.0404	.0101	.0000	.3266
10	.0000	.0001	.0007	.0001	.0006	.0038	.0000	.0012
11	.0028	.0001	.0004	.0000	.0007	.0044	.0000	.0358
12	.0008	.0002	.0000	.0000	.0021	.0202	.0000	.0016
13	.0000	.0002	.0000	.0000	.0125	.0124	.0000	.0151
14	.0004	.0002	.0029	.0004	.0006	.0038	.0579	.0007
15	.0000	.0009	.0000	.0000	.0682	.0000	.0000	.0000
16	.0000	.0001	.0000	.0000	.0005	.0000	.0250	.0000
17	.0037	.0004	.0035	.0003	.0249	.0388	.0180	.0012
18	.0016	.0023	.0545	.0263	.0170	.0044	.0189	.0237
19	0	.0050	.0919	.0442	.0908	.0818	.0539	.0388
20	.0429	0	.0348	.0282	.0996	.0401	.0000	.1428
21	.0000	.0000	0	.0000	.0000	.0000	.3657	.0000
22	.0000	.0000	.0000	0	.0619	.0133	.0000	.0000
23	.6317	.8503	.2798	.7345	0	.5617	.1975	.0000
24	.1123	.0412	.0472	.0484	.0111	0	.0303	.0775
25	.1530	.0539	.1846	.0995	.0036	.1792	0	.0000
26	.0139	.0387	.0298	.0001	.0133	.0000	.1935	0
total	1	1	1	1	1	1	1	1

APPENDIX TO TABLE (1)

List of 22 production sectors

A. *Primary products :*

1. Agriculture
2. Animal husbandry, fishery and forestry
3. Mining

B. *Large-scale manufactures :*

4. Metal and engineering
5. Chemicals
6. Building materials and wood manufacture
7. Fuel oil and power
8. Food, drink and tobacco
9. Textile and textile products
10. Ceramics and glass
11. Leather and rubber
12. Paper and printing

C. *Small-scale manufactures :*

13. Textile and textile products
14. Metal
15. Food, drink and tobacco
16. Building materials and wood manufactures
17. Miscellaneous.

D. *Others :*

18. Railways and communication
19. Trade and other transport
20. Banks, insurance, professions
21. Construction
22. House property

TABLE 2 : 25×25 INVERSION OF MATRIX TO THE LEFT IN EQUATION (4), WHICH IS THE SAME AS THE MATRIX OBTAINED BY LEAVING OUT THE ROW 25 AND THE COLUMN 25 IN THE 26×26 CURRENT ACCOUNT INTERFLOW TABLE (1)

List of the 22 production sectors are given at the end of table (1). The meaning of the sectors 23–26 is explained in section 1.

continued on next page

total activity in no.	as a consequence of the use of investment goods from the following sectors								
	1	2	3	4	5	6	7	8	9
1	5.4719	4.6161	3.2989	3.3691	3.4373	3.7237	3.4583	4.2589	3.8958
2	1.7584	2.5757	1.1865	1.2113	1.2490	1.5121	1.2606	1.4441	1.3528
3	0.0572	0.0553	1.0488	0.0707	0.0606	0.0697	0.1121	0.0551	0.0562
4	0.0632	0.0613	0.0754	1.0632	0.0669	0.0713	0.0900	0.0607	0.5800
5	0.0588	0.0569	0.0514	0.0508	1.0473	0.0671	0.0497	0.0527	0.0547
6	0.0164	0.0163	0.0140	0.0159	0.0149	1.0166	0.0600	0.0186	0.0157
7	0.0334	0.0324	0.0394	0.0356	0.0359	0.0570	1.0301	0.0322	0.0351
8	0.5344	0.5772	0.4024	0.4054	0.4417	0.4555	0.4487	1.4622	0.4508
9	0.5215	0.5502	0.4287	0.4330	0.4953	0.4752	0.5242	0.4774	1.4843
10	0.0090	0.0087	0.0075	0.0098	0.0086	0.0083	0.0121	0.0087	0.0081
11	0.0304	0.0295	0.0281	0.0309	0.0341	0.0315	0.0378	0.0313	0.0318
12	0.0363	0.0352	0.0305	0.0314	0.0331	0.0449	0.0330	0.0354	0.0351
13	0.1181	0.1132	0.0911	0.0921	0.1164	0.1004	0.0978	0.1046	0.1031
14	0.0238	0.2224	0.0319	0.0680	0.0230	0.0251	0.0226	0.0208	0.0199
15	0.5794	0.5672	0.4272	0.4372	0.4459	0.4752	0.4393	0.4959	0.4704
16	0.0062	0.0060	0.0047	0.0068	0.0050	0.0184	0.0081	0.0071	0.0053
17	0.2476	0.2365	0.2062	0.1982	0.2203	0.2120	0.2009	0.2167	0.2078
18	0.2115	0.2076	0.2030	0.2307	0.2103	0.2815	0.2098	0.2025	0.1985
19	1.1166	1.1014	0.9156	0.9751	1.0836	1.1261	0.9375	1.0738	1.0440
20	1.1473	1.1047	0.9608	1.0156	1.0081	1.0529	0.9799	1.0575	1.0383
21	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
22	0.5380	0.5139	0.4068	0.4141	0.4189	0.4489	0.4188	0.4667	0.4457
23	8.0391	7.6644	5.9519	6.0924	6.0992	6.5732	6.1109	6.8785	6.5421
24	0.6458	0.6321	0.6153	0.5922	0.6639	0.6741	0.6508	0.6559	0.6529
26	0.3679	0.3563	0.3898	0.3518	0.4886	0.3991	0.6487	0.3872	0.4654

TABLE 2 : 25×25 INVERSION OF THE MATRIX TO THE LEFT IN EQUATION (4), WHICH IS THE SAME AS THE MATRIX OBTAINED BY LEAVING OUT THE ROW 25 AND THE COLUMN 25 IN THE 26×26 CURRENT ACCOUNT INTERFLOW TABLE (1)

List of the 22 production sectors are given at the end of table (1). The meaning of the sectors 23–26 is explained in section 1.

continued

continued on next page

total activity in no.	as a consequence of the use of investments goods from the following sectors								
	10	11	12	13	14	15	16	17	18
1	3.7022	3.7664	3.5994	4.0716	3.8920	4.6014	4.2178	4.2659	3.6272
2	1.3459	1.4429	1.3086	1.4402	1.3992	1.5779	1.7712	1.5457	1.2978
3	0.1187	0.0563	0.0676	0.0008	0.0589	0.0566	0.0698	0.0560	0.1014
4	0.0689	0.0604	0.0594	0.0607	0.1966	0.0619	0.0730	0.0637	0.1314
5	0.0884	0.0781	0.0646	0.0582	0.0526	0.0561	0.0565	0.0585	0.0506
6	0.0503	0.0173	0.0156	0.0169	0.0153	0.0165	0.0425	0.0159	0.0158
7	0.0611	0.0361	0.0435	0.0343	0.0320	0.0323	0.0328	0.0325	0.0344
8	0.4796	0.4730	0.4516	0.4767	0.4652	0.5137	0.5138	0.5041	0.4329
9	0.5620	0.5167	0.5032	0.9705	0.4777	0.5073	0.5094	0.5075	0.4522
10	1.0086	0.0084	0.0083	0.0084	0.0085	0.0087	0.0086	0.0087	0.0087
11	0.0407	1.0361	0.0300	0.0308	0.0296	0.0302	0.0315	0.0475	0.0289
12	0.0670	0.0421	1.0336	0.0399	0.0335	0.0359	0.0348	0.0355	0.0377
13	0.0000	0.1001	0.1007	1.1086	0.1051	0.1132	0.1105	0.1137	0.0976
14	0.0261	0.0206	0.0199	0.0236	1.0269	0.0226	0.0306	0.0295	0.0232
15	0.4669	0.4790	0.4604	0.5126	0.5072	1.5502	0.0000	0.5526	0.4652
16	0.0104	0.0053	0.0051	0.0103	0.0058	0.0059	1.0061	0.0059	0.0052
17	0.2225	0.2655	0.2271	0.2269	0.2211	0.2365	0.2288	1.2376	0.2046
18	0.2424	0.1963	0.1949	0.2108	0.2058	0.2057	0.2163	0.2061	1.1814
19	1.0224	1.0845	1.0609	1.0619	1.0746	1.1034	1.0884	1.1345	0.9459
20	1.0727	1.0676	1.0577	1.0786	1.0667	1.1107	1.0923	1.2146	1.0221
21	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
22	0.4427	0.4542	0.4396	0.4826	0.4757	0.5137	0.4967	0.5164	0.4389
23	6.4885	6.6444	6.4061	7.1388	7.0718	7.6566	7.3915	7.6983	6.4813
24	0.6578	0.6878	0.6913	0.6524	0.6081	0.6381	0.6266	0.6382	0.6049
26	0.7044	0.5470	0.5417	0.4133	0.3728	0.3803	0.4232	0.3743	0.3815

TABLE 2 : 25×25 INVERSION OF THE MATRIX TO THE LEFT IN EQUATION (4), WHICH IS THE SAME AS THE MATRIX OBTAINED BY LEAVING OUT THE ROW 25 AND THE COLUMN 25 IN THE 26×26 CURRENT ACCOUNT INTERFLOW TABLE (1)

List of the 22 production sectors are given at the end of table (1). The meaning of the sectors 23–26 is explained in section 1.

continued

total activity in no.	as a consequence of the use of investment goods from the following sectors						
	19	20	21	22	23	24	26
1	3.8441	4.3672	3.4607	4.0883	4.6969	3.6814	4.1473
2	1.3733	1.5650	1.2407	1.4681	1.6826	1.3235	1.5234
3	0.0504	0.0581	0.0600	0.0566	0.0607	0.0496	0.0745
4	0.0581	0.0649	0.1783	0.0636	0.0665	0.0585	0.0804
5	0.0510	0.0585	0.0474	0.0548	0.0619	0.0515	0.0614
6	0.0155	0.0165	0.0645	0.0260	0.0171	0.0205	0.0176
7	0.0320	0.0333	0.0301	0.0317	0.0353	0.0318	0.0339
8	0.4533	0.5215	0.4040	0.4834	0.5549	0.4355	0.6082
9	0.4636	0.5311	0.4144	0.4850	0.5537	0.4545	0.8240
10	0.0082	0.0091	0.0083	0.0085	0.0094	0.0113	0.0099
11	0.0302	0.0316	0.0265	0.0285	0.0319	0.0310	0.0671
12	0.0343	0.0366	0.0305	0.0346	0.0379	0.0508	0.0385
13	0.1036	0.1178	0.0912	0.1099	0.1254	0.1108	0.1228
14	0.0203	0.0224	0.0273	0.0215	0.0235	0.0231	0.0231
15	0.4989	0.5713	0.4378	0.5347	0.6155	0.4794	0.5061
16	0.0054	0.0062	0.0057	0.0059	0.0066	0.0053	0.0059
17	0.2204	0.2444	0.1952	0.2297	0.2668	0.2444	0.2265
18	0.1855	0.2120	0.2311	0.2233	0.2237	0.1840	0.2254
19	1.9668	1.1018	0.9703	1.0732	1.1720	1.0139	1.0966
20	1.0350	2.1316	0.9409	1.0882	1.2078	1.0022	1.2079
21	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000
22	0.4713	0.5334	0.4126	1.5009	0.5723	0.5090	0.4793
23	6.9561	7.9545	6.0955	7.4570	8.5855	6.6825	7.0193
24	0.6530	0.6565	0.5654	0.6296	0.6531	1.5356	0.7184
26	0.3357	0.4025	0.3240	0.3430	0.3864	0.3194	1.4102

TABLE 4. INVESTMENT COEFFICIENTS

continued on the next page

delivering sector no.	investment receiving sector no.								
	1	2	3	4	5	6	7	8	9
1	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
2	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
3	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
4	.0075	.0000	.0699	.1410	.1242	.0997	.0793	.1690	.0389
5	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
6	.0000	.0000	.0000	.0113	.0002	.0044	.0159	.0178	.0176
7	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
8	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
9	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
10	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
11	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
12	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
13	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
14	.2598	.0434	.0000	.0282	.0306	.0061	.0319	.0165	.0181
15	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
16	.0149	.0611	.0349	.0056	.0306	.0437	.0207	.0330	.0671
17	.0125	.0611	.0070	.0206	.0642	.0437	.0096	.0815	.0543
18	.0013	.0031	.1048	.0451	.0306	.0437	.0478	.0165	.0724
19	.0063	.0367	.1048	.0016	.0428	.0525	.0367	.0660	.1267
20	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
21	.0815	.1222	.0839	.0688	.0373	.0464	.0245	.1304	.0371
22	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
23	.5720	.6724	.0699	.2257	.3670	.4374	.1593	.2641	.1810
24	.0191	.0000	.1090	.0731	.0477	.0455	.1156	.0416	.0775
26	.0250	.0000	.4158	.2959	.2159	.1767	.4317	.1634	.3093
total	1	1	1	1	1	1	1	1	1

TABLE 4. INVESTMENT COEFFICIENTS

continued *continued on the next page*

delivering sector no.	investment receiving sector no.								
	10	11	12	13	14	15	16	17	18
1	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
2	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
2	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0508
3	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
4	.0860	.0399	.1571	.1541	.2504	.1172	.1271	.0732	.0462
5	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
6	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0061
7	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
8	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
9	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
10	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
11	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
12	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
13	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
14	.0000	.0689	.0000	.1454	.1745	.1046	.1603	.1003	.0000
15	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
16	.0292	.0413	.0399	.0484	.0436	.0146	.1068	.1003	.0154
17	.0073	.0275	.0080	.0484	.0436	.1046	.0534	.1003	.0031
18	.0437	.0275	.0399	.0194	.0262	.0209	.0321	.0301	.0385
19	.0729	.0689	.0797	.0581	.0873	.0732	.0855	.0802	.0339
20	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
21	.0146	.0482	.1332	.0969	.0611	.1046	.0748	.1013	.4230
22	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
23	.2187	.1377	.1595	.1938	.1920	.2510	.2457	.2608	.1594
24	.1093	.1047	.0797	.0504	.0227	.0272	.0278	.0261	.0461
26	.4184	.4353	.3030	.1851	.0986	.0921	.0865	.1274	.1775
total	1	1	1	1	1	1	1	1	1

TABLE 4. INVESTMENT COEFFICIENTS

continued

delivering sector no.	investment receiving sector no.						
	19	20	21	22	23	24	total
1	.0000	.0000	.0000	.0000	.0000	.0000	0.0000
2	.0000	.0000	.0000	.0000	.0000	.0000	0.0508
3	.0000	.0000	.0000	.0000	.0000	.0000	0.0000
4	.0849	.0371	.0178	.0000	.1755	.0972	2.1936
5	.0000	.0000	.0000	.0000	.0000	.0000	0.0000
6	.0000	.0000	.0000	.0000	.0000	.0000	0.0822
7	.0000	.0000	.0000	.0000	.0000	.0000	0.0000
8	.0000	.0000	.0000	.0000	.0000	.0000	0.0000
9	.0000	.0000	.0000	.0000	.0000	.0000	0.0000
10	.0000	.0000	.0000	.0000	.0000	.0000	0.0000
11	.0000	.0000	.0000	.0000	.0000	.0000	0.0000
12	.0000	.0000	.0000	.0000	.0000	.0000	0.0000
13	.0000	.0000	.0000	.0000	.0000	.0000	0.0000
14	.0235	.0140	.0330	.0000	.0000	.1051	1.3642
15	.0000	.0000	.0000	.0000	.0000	.0000	0.0000
16	.0242	.0701	.1982	.0000	.3511	.0000	1.4849
17	.0235	.1402	.0330	.0000	.0585	.0000	1.0061
18	.0435	.0350	.0396	.0000	.0293	.0000	0.7910
19	.1059	.0701	.0991	.0000	.1755	.0000	1.6744
20	.0000	.0000	.0000	.0000	.0000	.0000	0.0000
21	.0353	.0701	.0000	1.0000	.0000	.5835	3.3787
22	.0000	.0000	.0000	.0000	.0000	.0000	0.0000
23	.2470	.2102	.2642	.0000	.0000	.1556	5.6444
24	.0854	.0399	.0687	.0000	.0193	.0000	1.2364
26	.3267	.3132	.2464	.0000	.1908	.0586	5.0933
total	1	1	1	1	1	1	24.0000

TABLE 5. UPPER BOUNDS FOR THE TOTAL ACTIVITIES X INTRODUCED BY TAKING ACCOUNT OF EXISTING CAPITAL CAPACITIES IN THE VARIOUS SECTORS

		upper limit of labour input Rs. (10^7)	current a/c coeffs. of receiving sectors for the delivering sector 23	upper limit	
1.	agriculture	5003.04	0.6637	7538.1046*	redundant* because of (1) in table (6)
2.	animal husbandry, fishery and forestry				
3.	mining	17.24	0.3174	54.3163	
4.	metal and engineering	99.54	0.2795	356.1360	
5.	chemicals	20.47	0.1106	185.0814	
6.	building materials and wood manufacture	10.58	0.1153	91.7606	
7.	fuel, oil and power	13.34	0.2124	62.8060	
8.	food, drink and tobacco	131.50	0.1163	1130.6965	
9.	textile and textile products	149.24	0.1641	909.4455	
10.	ceramics and glass	2.05	0.1494	13.7216	
11.	leather and rubber	10.65	0.1955	54.4757	
12.	paper and printing	10.38	0.2105	49.3112	
				2907.7508	
13.	textile and textile products	65.21	0.3914	166.6071	
14.	metal	80.45	0.5633	142.8191	
15.	food, drink and tobacco	372.89	0.4385	850.3763	
16.	building materials and wood manufacture	18.72	0.4132	45.3049	
17.	other small-scale manufactures	207.83	0.6265	331.7318	
				1536.8392	
18.	railways and communication	149.40	0.4184	357.0746	
19.	trade and other transport	1016.19	0.6317	1608.6592	
20.	bank, insurance, professions	1288.99	0.8503	1515.9238	
				3481.6576	
21.	construction	613.47	0.5272	1163.6381	
22.	house property				
		9218.18			

*Although the above computations are made *by means of* labour, the *result* is intended to express capital capacities.

TABLE 6. UPPER BOUNDS FOR THE TOTAL ACTIVITIES X INTRODUCED BY TAKING ACCOUNT OF THE LIMITED LABOUR FORCE AVAILABLE FOR USE IN SPECIAL GROUPS OF SECTORS

			estimate for 1950-51
1.	X_1+X_2	<7512.92 (1950-51 plus 16 p.c.)	(6482.09)
2.	$X_3+X_4+\ldots+X_{12}$	<2929.19 (1950-51 plus 66 p.c.)	(1766.09)
3.	$X_{13}+X_{14}+\ldots+X_{17}$	<1570.11 (1950-51 plus 35 p.c.)	(1165.58)
4.	$X_{18}+X_{19}+X_{20}$	<3645.62 (1950-51 plus 25 p.c.)	(2904.88)
5.	$X_{21}+X_{23}$	<1164.06 (1950-51 plus 19 p.c.)	(978.20)

All the above bounds turn out to be redundant because of table (5)—except the above bound (1).

TABLE 7. LOWER BOUNDS IMPOSED ON TOTAL CONSUMPTION AND ON THE CONSUMPTION OF SPECIAL GROUPS OF CONSUMER GOODS

67.	$4167.67\leqslant C_1+C_2$	$\leqslant$	4593.26
68.	$445.83\leqslant C_9+C_{13}$	$<$	490.97
69.	$521.70\leqslant C_{22}$	$\leqslant$	574.50
	$C_i=X_i$, 23		

TABLE 8. MATRIX OF 25 EQUATIONS FOR THE LINEAR PROGRAMMING ANALYSIS OF THE ASYMPTOTIC STRUCTURE OF THE INDIAN ECONOMY ACCORDING TO THE EXPERIMENTAL PLAN-FRAME No. 1

The concrete meaning of the variables is described in the text.

continued on next page

dependent variable no.	constant term $h=0$	basis variable no.							
		$h=1$	2	3	4	5	6	7	8
$k=1$	−5.3000	1.0000	−0.3506	−0.0000	−0.0000	−0.0040	−0.0002	−0.0000	−0.4330
2	−3.5000	−0.1694	1.0000	−0.0000	−0.0012	−0.0152	−0.1866	−0.0000	−0.0027
3	−0.0800	−0.0000	0.0000	1.0000	−00.090	−0.0090	−0.0125	−0.0567	−0.0029
4	−3.2000	−0.0003	−0.0002	−0.0238	0.9911	−0.0153	−0.0107	−0.0356	−0.0087
5	−1.5500	−0.0004	−0.0010	−0.0065	−0.0049	1.0000	−0.0171	−0.0000	−0.0014
6	−4.6400	−0.0000	−0.0003	−0.0000	−0.0026	−0.0007	0.9998	−0.0468	−0.0041
7	−2.3800	−0.0000	−0.0001	−0.0130	−0.0083	−0.0078	−0.0268	1.0000	−0.0023
8	0.0000	−0.0004	−0.0745	−0.0000	−0.0000	−0.0152	−0.0000	−0.0000	1.0000
9	−8.0000	−0.0002	−0.0003	−0.0000	−0.0105	−0.0149	−0.0114	−0.0000	−0.0040
10	−3.0000	−0.0000	−0.0000	−0.0000	−0.0024	−0.0008	−0.0000	−0.0040	−0.0005
11	−3.4500	−0.0000	−0.0000	−0.0000	−0.0042	−0.0014	−0.0015	−0.0000	−0.0018
12	−16.0000	−0.0000	−0.0000	−0.0000	−0.0008	−0.0006	−0.0110	−0.0000	−0.0017
13	−9.8000	−0.0000	−0.0003	−0.0000	−0.0000	−0.0211	−0.0000	−0.0000	−0.0010
14	−3.0000	−0.0089	−0.0012	−0.0130	−0.0511	−0.0052	−0.0047	−0.0043	−0.0006
15	0.0000	−0.0001	−0.0164	−0.0000	−0.0000	−0.0076	−0.0000	−0.0000	−0.0011
16	0.0000	−0.0004	−0.0010	−0.0022	−0.0023	−0.0017	−0.0153	−0.0045	−0.0029
17	−30.7000	−0.0020	−0.0016	−0.0178	−0.0071	−0.0278	−0.0043	−0.0050	−0.0033
18	−3.5000	−0.0002	−0.0048	−0.0455	−0.0667	−0.0397	−0.1009	−0.0298	−0.0176
19	−64.8000	−0.0066	−0.0343	−0.0714	−0.1134	−0.1972	−0.1782	−0.0313	−0.1059
20	−31.8000	−0.0089	−0.0092	−0.0778	−0.1179	−0.0760	−0.0703	−0.0202	−0.0535
21	0.0000	−0.0024	−0.0020	−0.0053	−0.0043	−0.0020	−0.0022	−0.0023	−00.047
27	−4176.4700	0.8306	0.6493	0.0000	−0.0012	−0.0191	−0.1868	0.0000	−0.4357
28	−463.6300	−0.0002	−0.0006	0.0000	−0.0105	−0.0360	−0.0114	0.0000	−0.0050
29	−532.2000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
30	9874.0100	−0.7999	−0.5021	−0.7238	−0.5727	−0.5371	−0.3460	−0.7594	−0.3464
$f = C_1 + \ldots + C_{22}$	0	0.7999	0.5021	0.7238	0.5727	0.5371	0.3460	0.7594	0.3464

TABLE 8. MATRIX OF 25 EQUATIONS FOR THE LINEAR PROGRAMMING ANALYSIS OF THE ASYMPTOTIC STRUCTURE OF THE INDIAN ECONOMY ACCORDING TO THE EXPERIMENTAL PLAN-FRAME No. 1

The concrete meaning of the variables is described in the text.

continued

continued on next page

dependent variable no.	basis variable no.								
	9	10	11	12	13	14	15	16	17
k = 1	−0.2434	−0.0215	−0.0112	−0.0080	−0.0083	−0.0006	−0.3683	−0.0241	−0.0301
2	−0.0005	−0.0019	−0.0933	−0.0125	−0.0019	−0.0034	−0.0036	−0.2794	−0.0232
3	−0.0042	−0.0570	−0.0018	−0.0142	−0.0035	−0.0034	−0.0012	−0.0135	−0.0003
4	−0.0011	−0.0049	−0.0012	−0.0122	−0.0070	−0.1496	−0.0033	−0.0164	−0.0033
5	−0.0051	−0.0355	−0.0267	−0.0151	−0.0028	−0.0000	−0.0000	−0.0011	−0.0010
6	−0.0013	−0.0336	−0.0018	−0.0000	−0.0009	−0.0001	−0.0007	−0.0268	−0.0000
7	−0.0062	−0.0289	−0.0061	−0.0144	−0.0006	−0.0007	−0.0001	−0.0003	−0.0001
8	−0.0000	−0.0000	−0.0000	−0.0000	−0.0000	−0.0000	−0.0083	−0.0000	−0.0000
9	1.0000	−0.0000	−0.0000	−0.0000	−0.4760	−0.0000	−0.0000	−0.0000	−0.0000
10	−0.0000	1.0000	−0.0000	−0.0000	−0.0000	−0.0000	−0.0000	−0.0000	−0.0000
11	−0.0000	−0.0000	1.0000	−0.0000	−0.0000	−0.0000	−0.0000	−0.0000	−0.0173
12	−0.0021	−0.0327	−0.0079	1.0000	−0.0045	−0.0003	−0.0007	−0.0000	−0.0000
13	−0.0026	−0.0000	−0.0000	−0.0000	1.0000	−0.0000	−0.0000	−0.0000	−0.0000
14	−0.0007	−0.0047	−0.0021	−0.0000	−0.0093	0.9913	−0.0030	−0.0172	−0.0107
15	−0.0000	−0.0000	−0.0000	−0.0000	−0.0000	−0.0000	1.0000	−0.0000	−0.0000
16	−0.0021	−0.0063	−0.0012	−0.0031	−0.0068	−0.0023	−0.0024	0.9942	−0.0028
17	−0.0022	−0.0135	−0.0541	−0.0221	−0.0061	−0.0039	−0.0030	−0.0029	0.9972
18	−0.0206	−0.0592	−0.0101	−0.0173	−0.0109	−0.0103	−0.0038	−0.0150	−0.0036
19	−0.1132	−0.0508	−0.1174	−0.1363	−0.0104	−0.0826	−0.0430	−0.0444	−0.0715
20	−0.0648	−0.0467	−0.0562	−0.0828	−0.0108	−0.0337	−0.0154	−0.0164	−0.1167
21	−0.0011	−0.0008	−0.0015	−0.0104	−0.0044	−0.0030	−0.0024	−0.0041	−0.0029
27	−0.2439	−0.0233	−0.0206	−0.0102	−0.0040	−0.3719	−0.3719	−0.3035	−0.0533
28	0.9974	0.0000	0.0000	0.0000	0.5240	0.0000	0.0000	0.0000	0.0000
29	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
30	−0.5290	−0.6121	−0.6075	−0.6515	−0.4358	−0.6975	−0.5407	−0.5328	−0.7128
f	0.5290	0.6121	0.6075	0.6515	0.5458	0.6975	0.5407	0.5328	0.7128

TABLE 8. MATRIX OF 25 EQUATIONS FOR THE LINEAR PROGRAMMING ANALYSIS OF THE ASYMPOTOTIC STRUCTURE OF THE INDIAN ECONOMY ACCORDING TO THE EXPERIMENTAL PLAN-FRAME No. 1

The concrete meaning of the variables is described in the text.

continued

dependent variable no.	basis variable no.							
	18	19	20	21	22	23	24	26
$k=1$	−0.0465	−0.0260	−0.0003	−0.0878	−0.0008	0.0000	0.0000	−0.0813
2	−0.0109	−0.0000	−0.0000	−0.0090	−0.0019	0.0000	0.0000	−0.0811
3	−0.0506	−0.0000	−0.0005	−0.0083	−0.0018	0.0000	0.0000	−0.0175
4	−0.0816	−0.0058	−0.002	−0.0219	−0.0027	−0.1755	−0.0972	−0.0179
5	−0.0017	−0.0000	−0.0005	−0.0007	−0.0004	0.0000	0.0000	−0.0056
6	−0.0021	−0.0006	−0.0002	−0.0509	−0.0106	0.0000	0.0000	−0.0009
7	−0.0052	−0.0026	−0.0001	−0.0009	−0.0001	0.0000	0.0000	−0.0000
8	−0.0000	−0.0000	−0.0010	−0.0000	−0.0000	0.0000	0.0000	−0.1037
9	−0.0008	−0.0051	−0.0021	−0.0000	−0.0000	0.0000	0.0000	−0.3266
10	−0.0000	−0.0000	−0.0001	−0.0007	−0.0001	0.0000	0.0000	−0.0012
11	−0.0005	−0.0028	−0.0001	−0.0004	−0.0000	0.0000	0.0000	−0.0358
12	−0.0062	−0.0008	−0.0002	−0.0000	−0.0000	0.0000	0.0000	−0.0016
13	−0.0000	−0.0000	−0.0002	−0.0000	−0.0000	0.0000	0.0000	−0.0151
14	−0.0000	−0.0013	−0.0004	−0.0037	−0.0004	0.0000	−0.1051	−0.0007
15	−0.0000	−0.0000	−0.0009	−0.0000	−0.0000	0.0000	0.0000	0.0000
16	−0.0020	−0.0009	−0.0012	−0.0046	−0.0000	−0.3511	0.0000	0.0000
17	−0.0012	−0.0046	−0.0027	−0.0043	−0.0003	−0.0585	0.0000	−0.0012
18	0.9950	−0.0033	−0.0029	−0.0554	−0.0263	−0.0293	0.0000	−0.0237
19	−0.0284	0.9959	−0.0062	−0.0942	−0.0442	−0.1755	0.0000	−0.0388
20	−0.0686	−0.0429	1.0000	−0.0348	−0.0282	0.0000	0.0000	−0.1428
21	−0.0545	−0.0014	−0.0012	1.0000	−0.1964	0.0000	−0.5835	0.0000
27	−0.0573	−0.0260	−0.0003	−0.0968	−0.0026	0.0000	0.0000	−0.1624
28	−0.0008	−0.0051	−0.0023	0.0000	0.0000	0.0000	0.0000	−0.3417
29	0.0000	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000
30	−0.6343	−0.8978	−0.9769	−0.5225	−0.6861	0.7899	0.7858	0.9225
$f=C_1+\ldots+C_{22}$	0.6343	0.8978	0.9769	0.5225	0.6861	−0.7899	−0.7858	−0.9225

Memorandum of 23 April 1959 from
University Institute of Economics, Oslo
Ragnar Frisch/10.

APPENDIX

FLOW CHART

for automatic computation according to the multiplex method.

Based on the memorandum of 11 September 1958 from the University Institute of Economics and on the report F-375, March 1959 by Mr. Ole-John Eahl, Norwegian Defence Research Establishment.

1 Input of data. Threshold value ε. Scaling. Independent variables scaled to approximately unity. Dependent variables divided by $\Sigma_k b_{jk}^2$. Initial setting.

2 Definition of the operation set (perhaps an empty set). The operation set consists of linearly independent, bound attained variables that have all regression coefficients sign correct. Bound attained variables are put exactly equal to their bounds. If $|x_j - \overline{x}_j| \leqq \varepsilon$, x_j is put exactly $= \overline{x}_j$, and if $|x_j - \overline{\overline{x}}_j| \leqq \varepsilon$, x_j is put exactly equal to $\overline{\overline{x}}_j$.

3 Compute all direction numbers d_k d_j and $d_o = \sum_{k=u..w} f_k d_k$ outside the operation set. For all variables in the operation set, put d_i = exactly 0, and for all variables for which we find $|d_j| \leqq \varepsilon$ put d_j = exactly 0. (These variables are considered linearly dependent on those in the operation set).

4a **If $d_o \leqq \varepsilon$**

Optimum reached. Stop and give output.

4b **If $d_o > \varepsilon$**

Compute λ_j for all j for which $|d_j| > \varepsilon$. This means omitting all variables in the operation set, and all variables for which we found $|d_j| \leqq \varepsilon$ in operation 3. Compute λ_{min} i.e. make a preference move. Such a move is made even if λ_{min} is zero, i.e. if there is at least one bound attained variable with inadmissible direction number. (This means that cases II and III in the memorandum of 12 Sep. 1958 are not distinguished operationally. On desk machines they should be distinguished.) Bound attained variables are put exactly equal to bound as specified in 2. We denote here by λ_{min} the quantity that was denoted λ_{max} in the memorandum of 12 Sep. 1958.

5a **If λ_{min} is infinite.**

The preference function can be rendered arbitrarily great. Stop and give output. Operationally the machine is told to compute λ_j only for those variables that have upper or lower finite bounds. Case 5a would then be distinguished by no λ_j being left. In case 5b at least one λ_j is left.

5b **If λ_{min} is finite.**

For all j compute $x'_j = x^o_j + \lambda_{min} d_j$. A breaking out variable is one for which $(\lambda_j - \lambda_{min}) \leqq \varepsilon$. Include in operation set one breaking out variable. If there is only one breaking out variable, take this one. If there are several, take the one with the largest $|d_j|$. This applies regardless of whether λ_{min} is zero or not.
Compute the moments of the variable that was now included in the operation set.
If x_k is a basis variable $M_{kj} = M_{jk} = b_{kj}$ and $M_{ko} = f_k$.
If k and h are basis variables $M_{kh} = e_{kh}$ $\begin{cases} 1 \text{ when } k = h \\ 0 \text{ when } k \neq h. \end{cases}$

6 Solve regression equations $M_{ho} + \sum_{i=\alpha..\gamma} M_{hi} B_i = 0$
Use a one way solution or use the single step building up or building down of the inverse.

7a **If all B_i are sign correct:**

Go back to operation No. 2.

7b. **If one or more B_i are sign incorrect.**

Omit from the operation set all variables with sign incorrect B_{ij} and go back to operation No. 6.

<u>Notation</u> Equations: $x_j = b_{jo} + \sum_{k=u..w} b_{jk} x_k$ Bounds: $\overline{x}_j \leqq x_j \leqq \overline{\overline{x}}_j$ Maximize: $f = \sum_{k=u..w} f_k x_k$ Movements: $M_{ij} = \sum_{k=u..w} b_{ik} b_{jk}$.
Regression coefficient B_i sign correct if non-negative when x_i is at its lower bound, but non-positive if x_i is at its upper bound.

$$d_k = f_k + \sum_{i=\alpha..\gamma} B_i b_{ik} \qquad d_j = \sum_{k=u..w} b_{jk} d_k$$

d_k or d_j admissible if non-negative when the variable is at its lower bound, but non-positive when it is at its upper bound.

$$\lambda_{min} = \underset{j}{\mathrm{Min}}\ \lambda_j$$

$$\lambda_j = \begin{cases} \dfrac{\overline{\overline{x}}_j - x^o_j}{d_j} & d_j > \varepsilon \\ \dfrac{x^o_j - \overline{x}_j}{-d_j} & d_j < -\varepsilon \end{cases} \qquad \text{(applies for j or k)} \quad (\varepsilon \text{ pos.})$$

x^o_j is the point reached so far.