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NUMERICAL DETERMINATION OF A QUADRATIC PREFERENCE FUNCTION FOR USE IN MACROECONOMIC PROGRAMMING

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1. Introduction.

Before one can entertain any hope of being able to apply successfully linear or quadratic (or more general forms of) programming to macroeconomic problems, one must have available a method of actually determining in concrete numerical form the preference function that will express the desires of the responsible political authority at the top level where the ultimate decision regarding the macroeconomic programming must rest.

A method for determining such a function must not only be based on sound theoretical principles in harmony with the analytical model and the programming technique, but it must be worked out in a form which is practical enough to catch in a reasonably short time and with a reasonable burden of interviewing the desires of the responsible political authority. I have more and more come to believe that an effective method of organizing the co-operation that is needed between the political authorities and the analytical technicians for an effective determination of the preference function is one of the most important aspects — if not *the* most important aspect — of macroeconomic programming.

At the Oslo University Institute of Economics considerable effort has been directed towards working out a technique for preference function determination in the macroeconomic field [1]. In the sequel a brief account will be given of some of the most promising possibilities. On the basis of our experiences with interviewing of responsible politicians according to our technique and the actual construction of the ensuing preference function, I am confident that the problem can be given a practical workable solution if a sufficiently careful analysis is made in each case.

I am responsible for the theoretical framework and for drawing up the main lines of the interviewing technique. In the working out of details, performing the actual interviewing and supervising the numerical computations three men of the Institute staff have given their wholehearted co-operation: assistant professor Hans Heli, research associate H. J. A. Kreyberg and chief of the computing and technical unit of the Institute Mr. Sven Vigger. Without their aid it would have been impossible to carry the work to completion.

The following are the main lines of reasoning that have been followed.

First as to the nature of the preference function (2). In principle the preference function is *an indicator of choice* and so we are facing the same problems as in the general theory of choice. Incidentally, this means that whatever methods we can develop for actually determining the preference function to be used for programming purposes, can very likely be applicable more or less in the same form to the determination of utility indicators for instance in a consumer market.

In order to make headway it is necessary to make some sort of simplifying assumptions of the mathematical form of the preference function. In the processing of the interview data it is impossible to assume that the preference function is linear, i.e. the marginal preferences constant. In the processing some form of non linearity must be assumed. We have assumed a *quadratic* and *partitioned* function. Some remarks are also made about a cubic preference function. When we afterwards are going to use this function for programming purposes, we may perhaps decide to approximate it by a linear function as long as we move in the narrow confines of a given locality. But that is a different question which has nothing to do with the numerical determination of the preference function itself.

Second, as to a general principle for determining the numerical characteristics of the function. We may use *dichotomic* questions. These are questions where the alternative sets of values of two of the variables are compared with a view to fixing indifferent sets of values. We adopt the rule that no answer « indifferent » shall ever be utilized. Only clear answers « yes » or « no » should be taken into consideration. By changing successively the value of one of the variables in the question, we can obtain an indifference range or threshold range (a « just noticeable difference ») with a « yes » answer in one end and a « no » answer in the other end such that it is impossible further to decrease the range without getting « indifference » answers. This gives an illuminating expression for what might be called the perceptibility of the variable from the preference viewpoint. The midpoint of the indifference range will enter into the computations, and alternative expressions for the preference function obtained by using the upper and lower limits of the indifference range will indicate how much the preference function is to be relied upon. Systems of dichotomic questions are worked out with or without checks. These systems have certain points in common with the systems of Latin squares, and this device may indeed be of some help in a certain phase of the work on dichotomic questions. These aspects of the problems are discussed in detail in a number of special memoranda (in Norwegian) from the Oslo Institute.

Third, while it is in principle possible to cover the whole field by a system of dichotomic questions, the type of reasoning which is in-

(1) The details are available only in mimeographed form, mainly memoranda (in Norwegian) from the Oslo University Institute of Economics and some memoranda (in English) from the Institute of Statistics, Calcutta, produced during my stay there from the fall of 1954 to the spring of 1955. Certain aspects of the problem are discussed briefly in *Formulazione di un piano di sviluppo nazionale come problema di programmazione convessa. (Il caso del piano indiano)*, « L'Industria », n. 3, 1956, Milano, and in *Programme convexe et planification pour le développement national*, Colloque International d'Econometrie, 23-28 Mai 1955. Publications du CNRS, Paris 1956, and in *Macroeconomics and Linear Programming. 25 Economic Essays in honour of Erik Lindahl*, Stockholm 1956. Also memorandum of 10 January 1956.

(2) Other names that might be proposed are: (1) « Welfare function ». This I do not like in this connection because the function in question does not measure the welfare of the population but the whishes of the politician. (2) « Objective function ». This I do not like because it indicates that certain *targets* are set (somewhat in the manner of targets in a national budget). The essence of programming is just that to begin with targets for individual variables are not set at all, but the viewpoint is « what would you prefer » if such and such things were possible. (3) « The pay-off function ». This I do not like because it attaches the problem too much to the idea of purely pecuniary gains.

volved in such questions is rather different from the interviewed individual's own way of thinking in daily life, and so this type of question will put quite a burden on him, and great care must be exercised in the formulation of such questions. As far as possible one should therefore try to replace the dichotomic questions by *distribution* questions. In such a question the assumption is that a certain total (for instance the net national product, or the total of gross investment etc.) is given and one wants to know how the interviewed person would like to see this total distributed over a number of items that together should make up the given total. It turns out that data of this sort are not sufficient for building up the complete preference function. Even if all the variables that enter into the preference function are summative [1] in the sense that they are items in a list whose total has a good meaning (and may, if we like, be taken as a datum in the interview questions), there is one constant in the preference function which remains undetermined. To fix it at least one dichotomic question must be used. And if two dichotomic questions are used, we get a check (and more checks if several dichotomic questions are used). If not all the variables in the preference function form one summative whole, but fall into a number of distinct groups, all items within a group being summative to a group total, but no summativity relation existing between groups, at least one dichotomic question is needed for each group in order to link it to the other groups.

Fourth, the method may be applied in a *pyramidal* way. This means that if a preference function is first constructed for certain macro variables, and if any of these macro variables may be looked upon as a sum of a number of items, then we can through an additional set of distribution questions concerning these items, introduce them into the preference function to replace the macro variable we had before. In other words we can dis-aggregate the preference function as we feel the need for it and see fit to collect the necessary distribution data.

These various aspects of the method will be considered in the sequel.

(1) I use the term summative rather than summable in order not to cause confusion with summability in the sense of infinite series or functions (summability in the Borel sense or in the Cesaro sense etc.).

2. Basic notation and the element of arbitrariness in the preference function.

The preference function contains an element of arbitrariness in two directions: first with respect to the formulation of a programming problem and second with respect to the amount of information which a set of interview data contains.

First with regard to the formulation of a programming problem.

Let there be n independent variables — basis variables — $x_u, x_v \dots x_w$ and m dependent variables x_j given by [1]

$$x_j = b_{j0} + \sum_{k=uv\dots w} b_{jk} x_k \qquad (j = 1,2 \dots N \text{ where } N = n + m) \tag{2.1}$$

Let the preference function considered as a function of all the N variables $x_1 \dots x_N$ be

$$F(x_1 \dots x_N) \tag{2.2}$$

with the free marginal preferences

$$F_j = \frac{\delta F(x_1 \dots x_N)}{\delta x_j} \qquad (j = 1,2 \dots N) \tag{2.3}$$

Let

$$f(x_u, x_v \dots x_w) \tag{2.4}$$

— apart from a constant term — be the function of the n basis variables, which is obtained from (2.2) by inserting from (2.1). Finally let the conditional marginal preference be

$$f_k = \frac{\delta f(x_u \dots x_w)}{\delta x_k} \qquad (k = u, v \dots w) \tag{2.5}$$

In what follows we shall most of the time consider the preference function (2.2) and the free marginal preferences (2.3). They are the relevant concepts to use in interviewing with the purpose of constructing the preference function numerically.

(1) The notation here is the same as in (1.1) - (1.10) in the memorandum « The Multiplex Method for Linear and Quadratic Programming » of 21 January 1957 from the University Institute of Economics, Oslo.

The optimum point set for a programming problem will not be changed if instead of the preference function (2.2) we consider the preference function

$$(2.6) \qquad \phi(x_1 \dots x_N)$$

which emerges after making a monotonically increasing transformation on the function F. That is to say a transformation of the form

$$(2.7) \qquad \phi = \Omega(F)$$

where Ω is a continuous and everywhere increasing function of one variable, but otherwise arbitrary. It is clear that any point that produces the maximum of F under certain constraints will also produce the maximum of ϕ under the same constraints, and vice versa. For a given programming problem it is therefore immaterial whether we use F or ϕ. The only thing we need to know is the indifference map and the numbering of the indifference surfaces. Obviously this will be the same for F and ϕ. The reasoning regarding F and ϕ is exactly the same as in the classical theory of choice.

When we are attempting to determine a preference function from interview data, we must therefore remember that it is not necessary to reach any special function in the class of functions with the indifference map in question. We only need to focus our attention on those properties in the preference function which remain invariant under an arbitrary monotonically increasing transformation.

In particular we may notice that if we assume a linear preference function, that is a preference function with constant free marginal preferences, or even if we assume a marginal preference function with free marginal preferences that are linear functions of the variables, we have assumed much more than is strictly necessary to define the optimum. It may nevertheless be a computational advantage to make such specifications because this may allow us to process the interview data in an easier way.

3. The nature of distribution data.

Tab. (3.1) gives an idea of the kind of data one will get by asking distribution questions for a *main group* such as the four items indicated in the gross national product.

TAB. (3.1) - Distribution of the gross national product of Norway.

	Actual distribution in 1948		Preferred distribution had gross nat. prod. been 96.5%, 100.0% etc. of the actual 1948 value 27.5			
	Milliards of Kroner	Percentages	(I)	(II)	(III)	(IV)
1. Private consumption	15.6	56.7	55.7	57.8	58.9	60.0
2. Government consumption (including defence)	2.8	10.2	10.2	10.6	11.1	11.5
3. Gross investment, private and government (exclusive of changes in inventories)	9.5	34.5	32.0	33.0	34.2	35.5
4. Export surplus	—0.4	—1.4	—1.4	—1.4	—0.7	0.0
TOTAL	27.5	100.0	96.5	100.0	103.5	107.0

The data in tab. (3.1) were obtained by actually interviewing a prominent Norwegian politician who has carried a geat responsibility in shaping the economic policy of Norway. In the two first columns of tab. (3.1) are given the actual distribution in 1948. In the column (I) is recorded the distribution which the interviewed politician would prefer in case the gross national product were only 96.5 per cent of what it was in 1948. Similarly in the column (II) is given the distribution which he would prefer if the total were to be equal to what it actually was in 1948, and so on for the two remaining columns.

Tab. (3.2) gives the result of similar interviews for the special group gross investment distributed over 9 sectors (1).

In principle there is no immediate connection between the alternatives in tab. (3.1) and tab. (3.2). The preferred expenditures for gross investment may, of course, not change proportionally to the given changes in gross national product and even if they did, we need not use these percentages in the interviewing on distribution of gross investment. On the other hand it may be an advantage first to formulate and have answers to the question regarding distribution of gross national product as exhibited in tab. (3.1) and then afterwards

(1) These 9 sectors are the same as those given in the memorandum « Main features of the Oslo Median model » of 10 October 1956.

TAB. (3.2) - Distribution of gross investment in the following sectors.

	Actual distribution in 1948 Percentages	Preferred distribution had total gross investment been the indicated percentages of the actual 1948 value 9.5.			
		(1)	(2)	(3)	(4)
1. Investment in extractive industries excluding mining	9.0	8.5	9.0	9.0	9.0
2. Investment in sea transport and whaling	28.0	25.0	25.0	25.0	26.5
3. Investment in manufacturing industries producing for the export market	5.0	6.0	6.0	6.5	7.0
4. Investment in land and air transport, post and telecommunications	8.0	9.5	10.0	10.5	10.5
5. Investment in mining and in manufacturing industries producing capital goods for the home market (capital goods used by Norwegian sectors)	8.0	10.0	10.0	10.5	11.0
6. Investment in manufacturing industries, producing consumer goods for the home market	3.0	2.5	3.0	3.0	3.0
7. Investment in housing, power plants and gas works	20.0	17.0	18.0	18.0	18.5
8. Other investments (in wholesale and retail trade, in equipment for the building and construction activity, in services etc.). Exports of existing fixed capital goods are also included here	10.0	8.0	8.0	8.0	8.0
9. Government investment	9.0	10.5	11.0	12.5	12.5
TOTAL	100.0	97.0	100.0	103.0	106.0

formulate specific questions regarding the distribution of gross investment, working now with the exact distribution figures from tab. (3.1) as the *given* totals in tab. (3.2) (1). It may be that this would make it easier for the interviewed person to bring order into his own thoughts when answering the questions.

Regardless of whether this has been done or not, it is, of course, possible by interpolation and redistribution to produce a new table where the condition just mentioned is fulfilled. And it is also possible to produce a new complete table where the item 3 in tab. (3.1)

(1) This method is not used in working out the items in tab. (3.2).

is specified as indicated in a table of the type (3.2). If this is dc we would have only one single table with a total of 12 items (i.e. variables). Since one of the sub items appears as constant, we incl it in the (unessential) constant term of the preference function ε thus get only 11 items (i.e. 11 variables).

For computational purposes afterwards it may be an advant to pool the two tables in this way. In the example the group gr investment is summative in the sense that the total of its items m up the exact figure that appears as one item in the main table (3

A table similar to (3.2) was also obtained for the distribution total disposable income over 4 income categories (1). If the sum these four categories as an approximation is put equal to priv consumption, which appears as item 1 in tab. (3.1), we have anotl summative group that can replace one item in the main group. this is done, we are finally left with one pooled table of 14 ite (The preference function constructed for this group is given in S tion 13).

From tables of the above kind we immediately derive the *l gel-functions*

(3.3) $$x_j = \phi^S_j \qquad (j = 1,2 .$$

where S indicates the total in question and ϕ^S_j the chosen distril tion numbers, so that

(3.4) $$\Sigma_j \phi^S_j = S$$

where the summation runs over all affixes j indicating the indiv ual items that together make up the total sum S.

The question of units of measurement is important. In order avoid confusion we have found it useful to work as much as pos ble in the standard unit of millions of kroner, not in percentag In other words the data from interviewing results of the form (3. and (3.2) will be recomputed and expressed in millions of kroner l fore the tables are pooled. If nothing is said to the contrary, we : sume the units of measurements to be comparable throughout.

(1) Namely those given in equations (13.2) - (13.5) in the memorandum 21 October 1956 « Supplementary remarks on the Oslo Median model ».

In (8.4)-(8.19) are considered the special questions that arise when such normalizations to a common unit have not been made, but the data from a partial table are nevertheless to be built into a pooled table.

The question is now how we can from the observed Engel-functions derive information about the preference function and the free marginal preferences.

Before we proceed to this work it may be useful to warn against the following error of reasoning.

The interviewed person has expressed that he will prefer a private consumption of 57.8 and a gross investment of 33.0 when the gross national product is 100 - see tab. (3.1) - while he will prefer a private consumption of 58.9 and a gross investment of 34.2, when the gross national product is 103.5. We must not interpret this to mean that the increase in private consumption from 57.8 to 58.9, that is to say an increase of 1.1, is *equivalent to* (in utility equivalent to) the increase in gross investment from 33.0 to 34.2, that is, an increase of 1.2. *If* such an interpretation were correct, values of the preference coefficients (applicable, say, in the middle of the intervals in question) could be derived simply as the reciprocal values of the observed changes, that is to say, a marginal coefficient of $\frac{1}{1.1} =$ 0.909 with respect to private consumption and $\frac{1}{1.2} = 0.833$ with respect to gross investment. If the marginal preferences were assumed constant, the preference function would then simply be $F = 0.909\ x_1 + 0.833\ x_3$ where x_1 and x_3 indicate the size of the two items No. 1 and No. 3 in tab.(3.1).

Such an interpretation is, however, completely erroneous when the data have emerged as explained in connection with tab. (3.1). When the assumptions behind the interviewing are fulfilled, it takes a much more complicated computation in order to determine the preference coefficients.

4. Necessity of assuming a non linear preference function in processing the data contained in the preferred distributions (the Engel-functions).

Regarding the linearity of the preference function — which is the same as the question of the *constancy* of the marginal preferences — we have the following rule.

Rule (4.1).

If *more than one* of the distribution numbers that are determi under a given constant sum S are different from what will, by meaning of the interview questions, be upper or lower bounds for t variable (for instance lower bound zero for private consumption), a if the distribution numbers are well defined (without degrees of fr dom in the answers), then the interviewed person's preference funct cannot have been linear.

This simply follows from the fact that by the theory of linear p gramming the optimum point — if it contains no degrees of freed — will be characterized as a point where at least as many of the riables are at a bound as there are degrees of freedom in the proble In an interview question all variables are free except for the fact t there is one equation (the given total).

The following special case is of particular interest.

Rule (4.2).

If the interview bounds on the variables only consist in requiri that all the variables, or all the variables except one (for instan export surplus) shall be non negative, and if the optimum point is s uated at a limited distance from origin and is well defined (without grees of freedom), then the preference function of the interviewed p son cannot be linear unless he — under the given positive sum S has chosen all the variables with non negativity condition equal to ze except possibly one of them.

From this rule follows immediately that the preference functi which is behind the distribution choice exhibited in tab. (3.1) and (3 cannot have been linear.

If such is the situation, we must consequently, under the *p cessing of the data* in order to determine the numerical form of t preference function, not assume a linear preference function.

This may not prevent us from working in certain later pha of the programming with a linear approximation to the non linear pr erence function derived from the interview data, but this is a d ferent matter. In the processing of the data we must reckon with a n linear preference function.

In what follows we will assume a quadratic or cubic preferen function. In addition we will assume it to be partitioned in the ser that a given marginal preference depends only on the variable wi respect to which the marginal increment is taken. I. e. we will assume one of the following two forms.

$$(4.3) \qquad F(x_1 \dots x_N) = \sum_{j=1}^{N} (P_j x_j + \frac{1}{2} Q_j x_j^2)$$

or

$$(4.4) \qquad F(x_1 \dots x_N) = \sum_{j=1}^{N} (P_j x_j + \frac{1}{2} Q_j x_j^2 + \frac{1}{3} R_j x_j^3)$$

where P_j, Q_j and R_j are constants.

The corresponding marginal preferences will, in the case of (4.3), be

$$(4.5) \qquad F_j = P_j + Q_j x_j \qquad \text{(for all } j\text{)}$$

and in the case of (4.4) be

$$(4.6) \qquad F_j = P_j + Q_j x_j + R_j x_j^2 \qquad \text{(for all } j\text{)}$$

Inversely it is easy to see that if the partial derivatives are of the form (4.6), the preference function must — apart from an arbitrary constant — be of the form (4.4). Indeed, integrating (4.6) partially with respect to x_j, we see that we must have

$$(4.7) \qquad F = P_j x_j + \frac{1}{2} Q_j x_j^2 + \frac{1}{3} R_j x_j^3 + \phi_{)j(}$$

where $\phi_{)j(}$ is a function not depending on x_j. If (4.7) is differentiated with respect to one of the other variables, say x_g, we see from (4.4) that we must have

$$(4.8) \qquad \frac{\partial \phi_{)j(}}{\partial x_g} = P_g + Q_g x_g + R_g x_g^2$$

Integrating this with respect to x_g, we get by (4.7)

$$(4.9) \qquad F = P_j x_j + \frac{1}{2} Q_j x_j^2 + \frac{1}{3} R_j x_j^3 + P_g x_g + \frac{1}{2} Q_g x_g^2 + \\ + \frac{1}{3} R_g x_g^3 + \phi_{)jg(}$$

where $\phi_{)jg(}$ is a function neither depending on x_j nor on x_g. Contin in this way, we get — apart from an arbitrary constant — (4.4). form (4.5), is, of course, only a special case of (4.6).

Most of the time we will work with (4.3) and assume Q_j to be n tive or at least non positive (principle of decreasing marginal util

5. PROCESSING DISTRIBUTION DATA WHEN ALL THE VARIABLES IN THE QU RATIC PREFERENCE FUNCTION FORM A SUMMATIVE GROUP.

If any ϕ_j^S should appear as a constant (independent of S) in group of distributed items — as for instance in tab. (3.2) — the a j will be called irregular and the corresponding « variable » will sin be omitted from F, it being assumed that this variable will ren constant also in other combinations and consequently only contrib to the constant term of F (which is without significance for the p gramming problem). We let $\sum_{\text{reg } j}$ indicate a summation over the regu affixes so that the preference function can now be written

$$(5.1) \qquad F = \sum_{j \text{ reg}} (P_j x_j + \frac{1}{2} Q_j x_j^2)$$

If none of the distribution numbers are at their interview boun we must have for any value of S

$$(5.2) \qquad F_1 = F_2 = \dots)F_j \text{ irreg } j(\dots = F_N$$

These are the tangency condition for choice equilibrium. They are same as in general choice theory, except that we now have all « price equal to 1. The inverted parenthesis in (5.2) indicates « exclusion o In addition to the substitumal condition (5.2) we have the bud condition

$$(5.3) \qquad \sum_{j \text{ reg}} x_j = S - \sum_{j \text{ irreg}} \phi_j^S$$

If the P_j and Q_j are given, the equations (5.2) and (5.3) will det mine the equilibrium point. This is the usual reasoning in the thec of choice. We have now to adopt the opposite viewpoint: that the qua tities are given as function of S and the P_j and Q_j are the unknow For this purpose we write the equations in the form

$$(5.4) \qquad P_1 + Q_1\phi_1^S = P_2 + Q_2\phi_2^S = \dots)P_j + $$
$$+ \; Q_j\phi_j^S \; j \text{ irreg}(\dots = P_N + Q_N\phi_N^S$$

Picking out the equation where the left member has the affix α and the right member the affix β, we get

$$(5.5) \qquad (P_\alpha - P_\beta) + Q_\alpha \phi_\alpha^s - Q_\beta \phi_\beta^s = 0 \quad (\alpha \text{ and } \beta \text{ any two regular affixes})$$

This equation can be interpreted as a requirement that there shall exist a *linear relation* between the two variables ϕ_α and ϕ_β over the range of variation S. The coefficients Q_α and Q_β and the difference $(P_\alpha - P_\beta)$ can be affected by an arbitrary factor of proportionality (different from zero). In addition the P coefficients can be affected by an arbitrary level term, that is a constant additive term, the same for P_α and P_β. All this applies when the φ_j^s are known.

In order to determine the coefficients $(P_\alpha - P_\beta)$, Q_α and Q_β of (5.5.), we can construct some sort of *regression equation* over the range of variation S, the coefficients of this empirically determined regression can then be put proportional to the three numbers $(P_\alpha - P_\beta)$, Q_α and Q_β. Since there is nothing to choose between α and β, the regression in question *would necessarily have to be one that treats the two affixes symmetrically*. There can be no question of using, say, the elementary regression of ϕ_α on ϕ_β or inversely. In other connections I have worked successfully with the diagonal mean regression and I will use it also in the present case [1].

The diagonal regression in the present case is

$$(5.6) \qquad \frac{\phi_\alpha - \bar{\phi}_\alpha}{\sqrt{m_{\alpha\alpha}}} = \frac{\phi_\beta - \bar{\phi}_\beta}{\sqrt{m_{\beta\beta}}}$$

where the $\bar{\phi}_j$ and m_{jj} are defined as in (5.11) and (5.10). From (5.6) follows

$$(5.7) \qquad \left(\frac{\bar{\phi}_\alpha}{\sqrt{m_{\alpha\alpha}}} - \frac{\bar{\phi}_\beta}{\sqrt{m_{\beta\beta}}}\right) + \left(\frac{-1}{\sqrt{m_{\alpha\alpha}}}\right)\phi_\alpha - \left(\frac{-1}{\sqrt{m_{\beta\beta}}}\right)\phi_\beta = 0$$

[1] See Additional Note 1 at the end of the paper.

Comparing the coefficients of 1, ϕ_α and ϕ_β in (5.5) and (5.7) see that $(P_\alpha - P_\beta)$, Q_α and $(-Q_\beta)$ can be put proportional t

$$\left(\frac{\bar{\phi}_\alpha}{\sqrt{m_{\alpha\alpha}}} - \frac{\bar{\phi}_\beta}{\sqrt{m_{\beta\beta}}}\right), \left(\frac{-1}{\sqrt{m_{\alpha\alpha}}}\right) \text{ and } \left(-\left(\frac{-1}{\sqrt{m_{\beta\beta}}}\right)\right).$$

This shows that (5.8) - (5.11) can be accepted as set of special solut P_j^{oo} and Q_j^o that — from a regressional viewpoint — is in confor with the information obtained by the distribution questions. We k that P_j and Q_j cannot be uniquely determined through these data, (5.8) - (5.11) is *one* solution. The square root in (5.8) - (5.9) is t positive. Through (5.9) the sign of Q_j^o is determined so as to exp the principle of declining marginal utility. This sign is an additi piece of information that has to be added because the diagonal regres itself does not provide the sign.

Collecting the results we have

$$(5.8) \qquad P_j^{oo} = \frac{\bar{\phi}_j}{\sqrt{m_{jj}}} \quad (j \text{ a regular affix, that is an affix such tha} \text{ is not a constant independent of } S)$$

$$(5.9) \qquad Q_j^o = \frac{-1}{\sqrt{m_{jj}}} \quad (j \text{ regular})$$

$$(5.10) \qquad m_{jj} = \Sigma_S(\phi_j^S - \bar{\phi}_j)^2 \qquad (j = 1, 2, \dots$$

$$(5.11) \qquad \bar{\phi}_j = \text{the arithmetic average of } \phi_j^S \text{, namely } \frac{\Sigma_S\phi_j^S}{\Sigma_S 1} \quad (j = 1, 2.$$

The most general solution of (5.5) is obtained from a spe solution P_j^{oo} and Q_j^o by the following transformation

$$(5.12) \qquad P_j = G(P_j^{oo} + C) \qquad \text{or if we like } P_j = GP_j^{oo} +$$

$$(5.13) \qquad Q_j = GQ_j^o$$

where G, C and $K = GC$ are constants, $G > O$. The constant G will call the scale factor and C — as already mentioned — the le term. If the observed distribution numbers ϕ_α^s and ϕ_β^s are such t there exist coefficients P_j^{oo} and Q_j^o that satisfy (5.5) identically in the formula (5.8) — (5.11) will produce such a set. If not, (5.8)

(5.11) is taken as a regressional compromise. In any case the numbers P_j and Q_j as defined by (5.12) — (5.13) will satisfy (5.5) with the same right as P_j^{oo} and Q_j^o.

The *scale factor* only has a convertional meaning so long as we consider all the variables in the preference function as forming a single set of summative variables that are handled as a unity in the distribution questions of interviewing. The assumption of summativity only has reference to the fact that we now consider the determination of P_j^{oo} and Q_j^o through (5.8) - (5.11), which are built on the Engel-magnitudes ϕ_j^S obtained through interviews built on summativity. If we had some other determination, say P_j' and Q_j', we could conventionally multiply these by an arbitrary scale factor G (> 0), regardless of whether the variables in the preference function are summative or not. The choice of a particular value of the scale factor can be looked upon as choosing the unit of measurement for the « utility » that is expressed by the preference function.

The *level term* C is of an entirely different sort. It has more than a conventional significance because it will influence the level around which the individual marginal preferences (the derivatives taken with respect to the individual variables in the preference function) are situated. Indeed if we maintain a particular value of the conventional scale factor G, but change the level term C, all the marginal preferences will change by G times the same amount as the level term has changed. A change of the level of the marginal preferences means, however, in general, that we change the indifference map and hence the preference function. Increasing the level constant towards infinity, for instance, would mean that we put all the free marginal preferences not only constant but equal. The determination of the level constant through interview data must therefore be an essential part of our problem (1).

In discussing this problem we put

$$P_j^C = C + P_j^{oo} \tag{5.14}$$

where P_j^{oo} is defined through (5.8) and C is the so far undetermined level term. Assuming the conventional scale factor G to be put equal to unity (Q_j^o being determined by (5.9)), the preference function (5.1) can be written in the form

(1) See Additional Note 2 at the end of the paper.

$$F^C = \sum_{j\ \mathrm{reg}} (P_j^C x_j + \frac{1}{2} Q_j^o x_j^2) = \tag{5.15}$$

$$= C(\sum_{j\ \mathrm{reg}} x_j) + \sum_{j\ \mathrm{reg}} (P_j^{oo} x_j + \frac{1}{2} Q_j^o x_j^2) \quad ,$$

All the Q_j^o in this formula will be negative, not zero, when t are determined by (5.9). The next to last term in (5.15) gives a g expression for the rôle played by the level term C.

Suppose that we have information about two situations that *indifferent* and have the same values for all the variables in the p erence function, except the two variables x_α and x_β, these having values x'_α, x'_β in one of the situations and x''_α, x''_β in the other the two situations are indifferent and all the other preference variat equal in the two situations, we must by (5.14) - (5.15) have

$$(C + P_\alpha^{oo})x'_\alpha + (C + P_\beta^{oo})x'_\beta + \frac{1}{2} Q^o_\alpha x'^2_\alpha + \frac{1}{2} Q^o_\beta x'^2_\beta \tag{5.16}$$

$$= (C + P_\alpha^{oo})x''_\alpha + (C + P_\beta^{oo})x''_\beta + \frac{1}{2} Q^o_\alpha x''^2_\alpha + \frac{1}{2} Q^o_\beta x''^2_\beta$$

Solving this equation for C, we get the special value

$$C^o = \frac{(x'_\alpha - x''_\alpha)\left[P_\alpha^{oo} + Q^o_\alpha \frac{x'_\alpha + x''_\alpha}{2}\right] + (x'_\beta - x''_\beta)\left[P_\beta^{oo} + Q^o_\beta \frac{x'_\beta + x}{2}\right.}{(x''_\alpha - x'_\alpha) + (x''_\beta - x'_\beta)} \tag{5.17}$$

$$= \frac{(x'_\alpha - x''_\alpha)y_\alpha + (x'_\beta - x''_\beta)y_\beta}{(x'_\alpha - x''_\alpha) + (x'_\beta - x''_\beta)}$$

where

$$y_i = \frac{\frac{x_i' + x_i''}{2} - \bar{\phi}_i}{\sqrt{m_{ii}}} \qquad (i = \alpha, \beta \tag{5.18}$$

The nature of the dichotomic interview question that can serv to define the values x'_α, x'_β and x''_α, x''_β is discussed in detail in work ing papers from the Oslo Institute. Here it will suffice to note thai at least one of the two situations x' and x'' must be significantly differ- ent from a situation that is *substitumal*, i. e. such that the interviewed person has himself chosen the variables in this situation (subject to a given total). Indeed, if one of the situations is substitumal and the two are indifferent, the denominator of (5.17) will be approximately zero. The more different from a substitumal situation it is possible to choose one of the two situations, the more computationally significant will the determination of C^{o} by (5.17) be, but, of course, for practical interviewing reasons it is impossible to go to situations that are too far fetched. A wise compromise must be made.

The lower expression in (5.17) holds good if P_j^{oo} and Q_j^{o} are determined by (5.8) - (5.11), while the upper expression holds for the case where P_j^{oo} and Q_j^{o} are determined by any methods such that by a suitable choice of C the two sets of parameters $P_j^{C} = (P_j^{oo} + C)$ and Q_j^{o} become plausible values to use for P_j and Q_j in the preference function (5.1).

For the partial derivatives of (5.15) — which holds good when $G = 1$ — we get

$$F_g^{C} = C + P_g^{oo} + Q_g^{o} x_g = C + Q_g^{o}(x_g - \overline{\phi}_g) \qquad (g \text{ reg}) \tag{5.19}$$

Hence for

$$x_g = \overline{\phi}_g \qquad (g = \text{any affix}) \tag{5.20}$$

$$F_g^{C}(\overline{\phi}_g) = C \qquad (g \text{ reg}) \tag{5.21}$$

The expression to the extreme right in (5.19) holds if P_j^{oo} and Q_j^{o} are determined as above, while the middle expression holds when P_j^{oo} and Q_j^{o} are determined by any method such that $P_j^{C} = (P_j^{oo} + C)$ and Q_j^{o} become plausible values to use for Q_j and P_j in the preference function (5.1).

The formula (5.21) holds in the special case and gives a good interpretation of what the level term C stands for.We first note that the point where each variable is equal to its Engel-average, i.e. where (5.20) holds for all g, will be a point on the substitumal. For any value of S we have in fact, when P_j^{C} and Q_j^{o} are correct values to use in the preference function (5.1) and ϕ_j^{s} are the observed Engel-magnitudes

$$P_j^{C} + Q_j^{o}\phi_j^{s} = P_i^{C} + Q_i^{o}\phi_i^{s} \qquad (i \text{ and } j \text{ any two regular affixes}) \tag{5.22}$$

Extending here a summation over S and dividing by $\Sigma_S 1$, we get $P_j^{C} + Q_j^{o}\overline{\phi}_j = P_i^{C} + Q_i^{o}\overline{\phi}_i$ and hence

$$F_j^{C}(\overline{\phi}_j) = F_i^{C}(\overline{\phi}_i) \qquad (i \text{ and } j \text{ any two regular affixes}) \tag{5.23}$$

Which shows that the point (5.20) is on the substitumal when the preference function has the form (4.3).

This being so, the meaning of the level term C can be interpreted by saying that to choose a specific value of C is the same as to prescribe the common value which the free marginal preferences shall have *in that point* on the interviewed person's substitumal, where each preference variable is equal to its Engel-average. (The conventional scale factor G being chosen equal to unity and P_j^{oo} and Q_j^{o} being determined by (5.8) - (5.11)).

6. Normalizations and changes in units of measurement.

If we change the unit of measurement for the variables x_j in the summative group for which the Engel data ϕ_j^{s} are obtained, the level term C^{o} as determined by (5.17) will be *unchanged.* Indeed, if the unit of measurement for all the x_j is, for instance, reduced to one half of what it was, so that the figures measuring all the ϕ_j^{s} and all the x_j are doubled, the P_j^{oo} will by (5.8) be unchanged and the Q_j^{o} will by (5.9) be reduced to one half so that the brackets in the middle expression of (5.17) are unchanged and hence C^{o} unchanged. Another and perhaps more direct way to see this, is to note that in the last expression of (5.17) the y_i are unchanged.

From the above reasoning follows that the marginal preferences (5.19) are unchanged by a change in the unit of measurement for the x_j.

This fact seems to be in contradiction to the general rule that a derivative is changed in proportion to the unit measurement of the argument with respect to which the derivation is made. The explanation is that when we use (5.19) where P_j^{oo} and Q_j^{o} are determined by (5.8) - (5.9) and C by (5.17), we will implicitly make a proportional change in the preference function when the unit of measurement of the x_j changes. To make this perfectly explicit, let the P_j and Q_j in (4.3) be any given fixed numbers, and let us make transformation

$$x_j = \mu\xi_j \tag{6.1}$$

where μ is a given positive constant. This gives $F = \mu\Sigma_j(P_j\xi_j + \frac{1}{2}\mu Q_j\xi_j^2)$. Consider the transformed preference function

$$\phi = \frac{F}{\mu} \tag{6.2}$$

We obviously have in any given point (which may be defined either through the set of values x_j or through the corresponding set ξ_j)

$$\frac{\partial\phi(\xi_1\ldots\xi_N)}{\partial\xi_g} = \frac{\partial F(x_1\ldots x_N)}{\partial x_g} \qquad (g = \text{any affix}) \tag{6.3}$$

Such a transformation of the preference function whose derivative is (5.19), is involved implicitly when P_j^{oo} and Q_j^o are determined by (5.8) - (5.9) and C by (5.17) and we change the unit measurement of the x_j.

We may express this by saying that the marginal preferences (5.19) are in a sense *normalized* marginal preferences. This means for instance that in tab. (13.1) the P_j have a meaning that is independent of the unit of measurement (millions of kroner).

7. The aggregation problem.

Suppose that an interviewing ought to have contained a very great number of variables, let it be the variables x_j $(j = 1, 2, \ldots N)$. For simplicity we have, however, in a first phase of the interviewing used a heavy aggregation such as for instance the one in tab. (3.1). Each item in this table stands as a representative for a great number of individual items. The assumption behind the interviewing in such an aggregated table is that the interviewed person can, if he wants to, subdivide any item in the way he wishes and distribute the individual items in the way he considers best. This distribution within a given main item is not mentioned explicitly in the interviewing and the interviewed person is not asked to explain how he wants this detailed distribution to be made. He only knows that he is free to use breakdowns if he wants to and use any distribution of the breakdown items he likes.

This means that in principle we should look upon the interview data as if they had emerged after a very detailed interviewing and in such a way that the interviewed person had himself — after making the optimum distribution of all individual items — summed up certain individual items and presented the aggregates to the interviewer.

This being the case, the following question arises: If we assume a special analytical form for the preference function in the detailed distribution — for instance a function of the type (4.3) or (4.4) — will it then be possible to interpret the actual interview data — the aggregated data — by a preference function of a similar analytical form?

Consider first the case of a function (4.3) in all the detailed var[iables]. We can assume theoretically that the constants are determin[ed] by formulae of the type (5.8) - (5.11) and (5.17) applied to this detail[ed] case. This means that whenever there is some variation in a variab[le] along the imaginative substitumal now considered, we can assum[e] the corresponding Q_j to be negative, not zero. If any variable shou[ld] occur with a zero Q_j, we can simply skip it as indicated in (5.15) [or] segregate explicitly the terms with zero Q_j as done in (7.3) - (7.5).

The substitumal conditions can in this case be written

$$P_j + Q_j x_j = \lambda(S) \qquad (j = 1, 2, \ldots) \tag{7.1}$$

where $\lambda(S)$ is a function of the total S in the summative group, i. e.

$$x_1 + x^2 + \ldots + x_N = S \tag{7.2}$$

We can imagine that the substitumal is constructed by considering th[e] single equation (7.2) together with the $N-1$ equations that emerg[e] from (7.1) when we leave out the right member $\lambda(S)$. This will dete[r]mine the substitumal. That is to say, for any S we can compute all th[e] x_j for j regular and the common magnitude of the left member in (7.[1]) for this value of S can be computed. It will give the value of λ that cor[]responds to the given S. In other words the nature of the function $\lambda(S)$ is known.

The actual solution can be carried out as follows. From (7.1) w[e] first get.

$$x_j = \frac{\lambda(S) - P_j}{Q_j} \qquad \Big| \quad Q_j \neq 0 \tag{7.3}$$

Performing here a summation over j for the regular j, we get

$$\sum_{j\|Q_j\neq 0} x_j = \lambda(S) \sum_{j\|Q_j\neq 0} \frac{1}{Q_j} - \sum_{j\|Q_j\neq 0} \frac{P_j}{Q_j} \tag{7.4}$$

and hence

$$\lambda(S) = \frac{[S - \sum_{j\|Q_j=0} \phi_j^S] + \sum_{j\|Q_j\neq 0} \frac{P_j}{Q_j}}{\sum_{j\|Q_j\neq 0} \frac{1}{Q_j}} \tag{7.5}$$

This shows that $\lambda(S)$ is a linear function of the difference between S and the sum of those ϕ_j^S for which $Q_j = 0$ (and hence the marginal preference independent of S). Inserting (7.5) in the right member of (7.3), we find that the regular x_j are also linear functions of the same difference. If no Q_j is equal to zero, the above statement can be replaced by saying that each x_j is a linear function of S. The assumption here is (4.3).

In the case where all $Q_j \neq 0$ (assured for instance by the definition (5.1.)), suppose that we divide the variables into a certain number of groups denoted $(J) = (1), (2) \dots$ and for each group define the group total

$$x_{(J)} = x_\alpha + x_\beta + \dots + x_\gamma \tag{7.6}$$

where $x_\alpha, x_\beta \dots x_\gamma$ are the individual variables in the group (J). We suppose that the subdivision is exhaustive and not overlapping so that by pooling all the groups, we get each of the detailed variables once and only once.

Will it in this case be possible to determine constants $P_{(J)}$ and $Q_{(J)}$ such that in every *substitumal* point, that is in any point where (7.1) and (7.2) are fulfilled, we have

$$P_{(J)} + Q_{(J)} x_{(J)} = \lambda(S) \qquad [(J) = (1), (2) \dots] \tag{7.7}$$

$$\sum_{(J)=(1),(2)\dots} x_{(J)} = S \tag{7.8}$$

where $\lambda(S)$ is the same function as in (7.1)?

This is possible. Indeed, along the substitumal the sum $x_{(J)}$ can be computed by dividing (7.1) by Q_j and extending a summation over j, to all items in the group (J). This gives

$$x_{(J)} = \lambda(S) \sum_{j=(J)} \frac{1}{Q_j} - \sum_{j=(J)} \frac{P_j}{Q_j} \tag{7.9}$$

Consequently the aggregate $x_{(J)}$ will satisfy the substitumal condition (7.7), if we put

$$P_{(J)} = \frac{\sum_{j=(J)} \frac{P_j}{Q_j}}{\sum_{j=(J)} \frac{1}{Q_j}} \qquad [(J) = (1), (2) \dots] \tag{7.10}$$

and

$$Q_{(J)} = \frac{1}{\sum_{j=(J)} \frac{1}{Q_j}} \qquad [(J) = (1), (2 \tag{7.11}$$

Since all Q_j are supposed to be negative, not zero, the same apply to $Q_{(J)}$.

Since the function $\lambda(S)$ in (7.7) is the same as in (7.1), we ge same result by computing $x_{(J)}$ substitumally from (7.7) - (7.8) as by considering the detailed substitumal (7.1) and from the detailed x_j pute $x_{(J)}$ according to (7.6).

Since this reasoning applies to any of the groups (J), we can that when the preference function is of the form (4.3), the whole ε ment is invariant under an aggregation.

This is no longer true if the preference function is of a more ‹ plicated form such as (4.4). Examples of this can be constructed ‹

8. DIS-AGGREGATION OF A VARIABLE IN THE PREFERENCE FUNCTION BY M OF DISTRIBUTION DATA.

Since we have the aggregation invariance described in Sectic provided the preference function is of the form (4.3), it would s possible also to dis-aggregate in case it is found that certain of the riables first considered in the preference function ought to be ε up further, each in a summative group.

The safest way to do this is to start by working out from ta of the kind exhibited in (3.1) and (3.2) a joint table where all the it are specified and in the same units of measurement as previously n tioned.

There is, however, also another way to proceed which may useful in a case where a great amount of work has already been d on the preference function, but it is found that some specific pari it needs to be specified further. We can proceed as follows.

Consider the variable $x_\varkappa$ which has already been handled as item in a total summative group that together made up all the va bles in the preference function. Let the Engel interview datum this variable be

(1) Examples of this have been worked out in detail in a working pape

(8.0) $$x_x = \phi^s$$

We want to replace the variable x_x by a sum

((8.1) $$x_r + x_s + \ldots x_t = x_x$$

and at the same time replace in the preference function (5.15)

(8.2) the single term $P^0_x x_x + \frac{1}{2} Q^\circ{}_x x^2_x$ by a sum $\sum_{i=rs\ldots t} (P_i^{0x}\, x_i +$

$$+ \frac{1}{2} Q_i^{\circ x}\, x_i^2)$$

and replace

(8.3) the single marginal preference $F^C_x = P^C_x + Q^\circ{}_x x_x$

by the set of marginal preferences $F_i^C = P_i^{Cx} + Q_i^{\circ x} x_i$

$(i = r, s \ldots t)$

Here P^C_x and Q_x are constants which are already known from the proceeding preference analysis, while P_i^{Cx} and $Q_i^{\circ x}$ are constants which we want to determine by additional information drawn from interviewing regarding the preferred distribution in the subgroup.

Since the new preference function will contain more variables and hence more degrees of freedom than the old, we can in general not expect to find any solution that is in reasonably good harmony with the interviewing data and still produce a preference function that *in all points in space* — including the new dimensions introduced through the breakdown (8.1) — will give exactly the same value as the old preference function. If we could find such a solution, this would just be an indication that we did not need to go to the trouble of making the breakdown (8.1).

While we cannot expect to find identity between the new and old preference functions in all points in space, we can find a solution that gives identity *along the substitumal*. This follows from the analysis of Section 7.

One way to perform the computations of the new preference function — and perhaps the one with the smallest risk of mistakes in handling the data, particularly the smallest risks in handling the various

units of measurement that come in (practical experience has this to be quite a problem) — is, as mentioned in connection wit and (3.2), to pool the two tables of data into one, if necessary by polation, so that we get a datum table where x_x does not occu $x_r, x_s \ldots x_t$ occur. In this pooled table the method of Section 5 (applied. From (5.8) - (5.11) and (5.17) is seen that all the coeffici and Q pertaining to the old variables that are not dis-aggregated, remain the same. For the variable x_x that is to be dis-aggregate replacements (8.2 - (8.3) would take place.

We can, however, also proceed by a *grafting* method which cc in accepting the previously calculated coefficients P and Q for tl riables that are not to be dis-aggregated and determine the new cients by means of (7.10) - (7.11). We shall write these formula in full as they will now appear.

The sum variable (8.1) may contain some sub-variables x_r, x_s that turn out to be kept *constant* by the interviewed person. Th even though the value of x_x is changed as a datum in the questic the interviewed person expresses the preference for maintainin sub-variable in question constant [1]. Such a preference cannc course, be explained by the existence of a preference function o kind (4.3) applicable to all the variables. One could explain this of answer for instance by assuming that the marginal preferenc the sub-variable considered fell abruptly down to zero for a g value of its argument. We shall not discuss these possibilities of i pretation in detail, but simply consider those affixes in the set, r, s, for which such answers occur as irregular, and handle them by si noting the constant values of these sub-variables as they are g through the answers from the interviewed person, and remarking their contribution to the total preference function will only be a stant and we may leave them out by the same sort of reasoning as used in (5.1). It is, however, necessary to be very careful about the the corresponding corrections should be made in the data that eme from the interviewing, so we will carry along all the way an exp notation for the irregular variables.

To make the formulae general we will assume that the unit measurement for the variables in the sub-group — as the data ab them emerge from the interviewing — is not necessarily the sa as in the main group (it may for instance be percentages of a cert

(1) This applies for instance to the item N. 8 in tab. (3.2).

figure instead of simply millions of kroner). The simplifications that take place in the formulae when the unit of measurement in the sub-group is actually the same as in the main group, are easy to write out.

Let μ be the *correspondence* factor by which the interviewing figures for the variables in the sub-group must be multiplied in order to express them in the same unit (for instance millions of kroner) as is used in the main group. This means that if x_i $(i = r, s \ldots t)$ are the variables expressed in the main group unit and ξ_i the same variables measured in the sub-group unit, we have

$$x_i = \mu \xi_i \qquad (i = r, s \ldots t) \tag{8.4}$$

Correspondingly we may write for the numbers expressing the sum

$$x_x = \mu \xi_x \tag{8.5}$$

For simplicity we will denote the sub-group sum expressed in sub-group unit by $\sigma = \xi_x$, so that instead of (8.5), we have

$$x_x = \mu\sigma \tag{8.6}$$

The sub-group sum $\sigma = \xi_x$ contains the regular sub-group variables as well as the irregular ones, if any (those that appeared as constants in the interview data).

By (8.4) and (8.6) we have

$$\xi_r + \xi_s + \ldots + \xi_t = \sigma \quad \text{that is} \quad \sigma = \sum_{i\ \mathrm{reg}} \xi_i + \sum_{\mathrm{irreg}} \xi_i \tag{8.7}$$

Let ϕ_i^σ be the numbers as they appear from the interviewing in the sub-group and measured in the unit of the sub-group. The same numbers measured in the unit or the main group are

$$\Phi_i^\sigma = \mu \phi_i^\sigma \tag{8.8}$$

Since the interview data must comply with the given value of σ, we have

$$\phi_r^\sigma + \phi_s^\sigma + \ldots + \phi_t^\sigma = \sigma \tag{8.9}$$

The variables ξ_i for i regular must satisfy

$$\sum_{i\ \mathrm{reg}} \xi_i = \sigma - \sum_{i\ \mathrm{irreg}} \phi_i^\sigma \tag{8.10}$$

When discussing the adaptation process for the regular variab the sub-group, the figures in the right member of (8.10) must be l upon as *given* while those in the left member are variables to be ε ed substitumally.

In the interview there must be fixed a certain amount (me ed in main group unit) $\check{x}_x$ of the variable that is to be dis-aggreg which by definition *corresponds* to a certain amount $\check{\sigma}$ of the sub-g sum (measurd in sub-group unit). Through the two given fi $\check{x}_x$ and $\check{\sigma}$, the correspondence factor μ will be determined, name

$$\check{x}_x = \mu\check{\sigma} \qquad \text{i. e.} \quad \mu = \frac{\check{x}_x}{\check{\sigma}} \tag{8.11}$$

Using these concepts we can reason as follows. Suppose th terview data ϕ_i^σ for the sub-groups are available measured in sub-g units. To these data we can apply (5.8) - (5.11) (with σ instead and thus determine $P_i^{\circ\circ}$ and Q_i° for the regular affixes in the group. On these parameters we can make a transformation of the (5.12) - (5.13). Any sets of numbers P_i and Q_i which emerge from transformation are equally good as a description of the interview for the sub-group. The two constants G and C of the transform are then determined in order to have (7.10) and (7.11) fulfilled w now $P_{(J)}$ and $Q_{(J)}$ are the already known constants P_x^c and which were determined by processing the data for the main gro shall not go through the details of the calculations here [1], but indicate the results. We get

$$F_{g\ x}^{\ c} = C + Q_g^{\circ x} (x_g - \bar{x}_g) + \frac{\bar{\phi}_x - \mu\bar{\sigma}}{\sqrt{m_{xx}}} \qquad (g \text{ regular in } r, s\,. \tag{8.12}$$

where

$$Q_g^{\circ x} = \frac{-\sum_{i=rs\ldots t} \sqrt{m_{ii}}}{\sqrt{m_{xx}\ m_{gg}}} \qquad (g \text{ regular in } r, s \tag{8.13}$$

[1] Details are given in a memorandum of 15 November 1956 from the Institute.

$m_{\varkappa\varkappa}$ is given by (5.10) for $j = \varkappa$ (the item in the main group that is to be dis-aggregated)

$\bar{\phi}_{\varkappa}$ is given by (5.11) for $j = \varkappa$

(8.14) $$m_{ii} = \Sigma_\sigma (\phi_i^\sigma - \bar{\phi}_i)^2 \qquad (i = r, s \ldots t)$$

(8.15) $$\bar{\phi}_i = \text{arithmetic average of } \phi_i^\sigma \text{ in the sub-group} = \frac{\Sigma_\sigma \phi_i^\sigma}{\Sigma_\sigma 1} \qquad (i = r, s \ldots t)$$

(8.16) $$\bar{\sigma} = \sum_{i=rs\ldots t} \bar{\phi}_i = \sum_{i\ \mathrm{reg}} \bar{\phi}_i + \sum_{i\ \mathrm{irreg}} \bar{\phi}_i \qquad \text{(summation over the sub-group)}$$

μ is given by (8.11).

(8.17) $\bar{x}_g = \mu\bar{\phi}_g$ = arithmetic Engel average of the variable No. g in the sub-group ($g = r, s \ldots t$) when this variable is expressed in the units of the main group.

The analogy between (8.12) and the expression to the right in (5.19) is obvious. The fraction to the right in (8.12) will as a rule be insignificant, and it will be rigorously zero if the average of the observed $\phi^S_{\varkappa}$ figure, extended over all observed S values in the main group, coincides with the average of σ extended over all observed σ values in the sub-group.

If we write (8.12) in the form

(8.18) $$F_g^{C\varkappa} = P_g^{C\varkappa} + Q_g^{o\varkappa}\, x_g \qquad (g \text{ regular in } r, s \ldots t)$$

the constant $P_g^{C\varkappa}$ will be

(8.19) $$P_g^{C\varkappa} = C - Q_g^{o\varkappa}\, \bar{x}_g + \frac{\bar{\phi}_{\varkappa} - \mu\bar{\sigma}}{\sqrt{m_{\varkappa\varkappa}}}$$

while $Q_g^{o\varkappa}$ in (8.18), as before, is determined by (8.13).

9. The level term determined through a dichotomic question within the sub-group.

The level term C which we so far have assumed to be determined by (5.17), can also be determined through a dichotomic question within

the sub-group. Indeed, let the constants P_i^{oo} and Q_i^o (i regular i $s \ldots t$) be determined through distribution questions within the group by an application of formulae similar to (5.8) - 5.11), that i:

(9.1) $$P_i^{oo} = \frac{\bar{\phi}_i}{\sqrt{m_{ii}}} \qquad (i \text{ regular in } r, s\ .$$

(9.2) $$Q_i^o = \frac{-1}{\sqrt{m_{ii}}} \qquad (i \text{ regular in } r, s\ .$$

where $\bar{\phi}_i$ and m_{ii} are given by (8.14) - (8.15). The most general co cients that will produce the same kind of regressional fit to the E: data in the sub-group when these are expressed in the main group of measurement, are

(9.3) $$P_i^{o\varkappa} = \gamma(P_i^{oo} + c)$$

(9.4) $$Q_i^{o\varkappa} = \frac{\gamma}{\mu}\, Q_i^o$$

where γ and c are arbitrary constants ($\gamma > C$) and μ the corresponde factor defined by (8.11). Indeed from the subset substitumal condi

(9.5) $$P_\alpha^{oo} + Q^o_\alpha\ \xi_\alpha = P_\beta^{oo} + Q^o_\beta\ \xi_\beta \qquad (\alpha \text{ and } \beta \text{ regular in } r, s\ ..$$

we easily deduce

(9.6) $$P_\alpha^{o\varkappa} + Q_\alpha^{o\varkappa}\ x_\alpha = P_\beta^{o\varkappa} + Q_\beta^{o\varkappa}\ x_\beta \qquad (\alpha \text{ and } \beta \text{ regular in } r, s, .$$

where the coefficients $P^{o\varkappa}$ and $Q^{o\varkappa}$ are given by (9.3) - (9.4). By a s able choice of γ and c the coefficients $P_i^{o\varkappa}$ and $Q_i^{o\varkappa}$ must theref emerge as exactly as those we have already used in the composite p erence function. On the other hand, if by a dichotomic question in sub-group we have found that the constellation x'_α, x'_β is indiffer with x''_α, x''_β (α and β regular in $r, s \ldots t$) when all the other varial are the same in the two constellations we have

(9.7) $$P_\alpha^{o\varkappa}\ x'_\alpha + P_\beta^{o\varkappa}\ x'_\beta + \frac{1}{2}\,(Q_\alpha^{o\varkappa} x'^2_\alpha + Q_\beta^{o\varkappa} x'^2_\beta)$$
$$= P_\alpha^{o\varkappa}\ x''_\alpha + P_\beta^{o\varkappa}\ x''_\beta + \frac{1}{2}\,(Q_\alpha^{o\varkappa} x''^2_\alpha + Q_\beta^{o\varkappa} x''^2_\beta)$$

Inserting here from (9.3) - (9.4) and solving for c (γ disappears), we get the special value

$$c^{\circ} = \frac{(\xi'_{\alpha} - \xi''_{\alpha})\left[P^{\circ\circ}_{\alpha} + Q^{\circ}_{\alpha}\frac{\xi'_{\alpha} + \xi''_{\alpha}}{2}\right] + (\xi'_{\beta} - \xi''_{\beta})\left[P^{\circ\circ}_{\beta} + Q^{\circ}_{\beta}\frac{\xi'_{\beta} + \xi''_{\beta}}{2}\right]}{(\xi''_{\alpha} - \xi'_{\alpha}) + (\xi''_{\beta} - \xi'_{\beta})}$$

(9.8)

$$= \frac{(\xi'_{\alpha} - \xi''_{\alpha})\eta_{\alpha} + (\xi'_{\beta} - \xi''_{\beta})\eta_{\beta}}{(\xi'_{\alpha} - \xi''_{\alpha}) + (\xi'_{\beta} - \xi''_{\beta})}$$

where

$$\eta_i = \frac{\frac{\xi_i' + \xi_i''}{2} - \bar{\phi}_i}{\sqrt{m_{ii}}} \qquad (i = \alpha,\ \beta) \tag{9.9}$$

The formulae (9.8) - (9.9) are analogous to (5.17) - 5.18) for the determination of c° by a dichotomic question in the main group.

Another way to determine c would be to equate (9.3) to (8.19) (assuming C to be determined through the dichotomic question in the main group) and to equate (9.4) to (8.13). By so doing and using the expressions for $P_i^{\circ\circ}$ and Q_i° from (9.1) - (9.2), we also get γ determined. The result can be written in the form

$$C = \gamma c - \frac{\bar{\phi}_x - \mu\bar{\sigma}}{\sqrt{m_{xx}}} \tag{9.10}$$

where

$$\gamma = \mu \frac{\sum_{i=rs..t} \sqrt{m_{ii}}}{\sqrt{m_{xx}}} \tag{9.11}$$

μ being defined through (8.11).

In (9.10) the last term in the right member will, as already remarked, as a rule be insignificant or even exactly zero, so that C and c should in pratice be proportional with the proportionality factor (9.11).

In (9.10) C is independent of the unit of measurement in the main group (compare the beginning of Section 6). It is, of course, also independent of the unit of measurement for the distribution numbers in the sub-group because in the computation of C by (5.17) - (5.18) we

have not used these sub-group data at all. The parameter c is independent of the unit of measurement in the main group as as of that in the sub-group. Furthermore the proportionality fac in (9.10) will in most cases be close to unity. What is measured is essentially the fact that the relative distribution within the sub-g changes with the sub-group total. Indeed, if this relative distrib is constant and there is exact correspondence between the S—alt tives in the main group and the σ—alternatives in the sub-grour coefficient γ as defined by (9.11) must be equal to 1. To see thi $\phi_i^{\sigma} = u_i \phi_x^{S}$ where the u_i are constants and S uniquely determined We then have $\Sigma^{\sigma} \phi_i^{\sigma} = u_i \Sigma_S \phi_x^{S}$ and hence $\bar{\phi}_i = u_i\bar{\phi}_x$. Fu $m_{ii} = \Sigma^{\sigma}(\phi_i^{\sigma} - \bar{\phi}_i)^2 = u_i^2 \Sigma_S (\phi_x^{S} - \bar{\phi}_x)^2$ and hence $\sqrt{m_{ii}} = u_i$ √ and consequently $\Sigma_i \sqrt{m_{ii}} = (\Sigma_i u_i)\sqrt{m_{xx}}$. On the other hanc have $\Sigma_i \phi_i^{\sigma} = (\Sigma_i u_i)\phi_x^{S}$, that is $\sigma = (\Sigma_i u_i)\phi_x^{S}$, which by (8.6) i same as $\Sigma_i u_i = \frac{1}{\mu}$, so that $\mu\Sigma_i \sqrt{m_{ii}} = \sqrt{m_{xx}}$. Therefore or assumptions made $\gamma = 1$.

By means of (9.10) - (9.11) we can make a double determin of the level term C, one directly in the main group, using (5.17) - and one indirectly using (9.8) - (9.9) and (9.10) - (9.11). The tw terminations should coincide within the threshold ranges. We even use several determinations, partly direct and partly indirect and finally determine the level term by a compromise. In a pra case where much depends on a correct preference function such a promise determination of the level term should always be made.

If a rough independent estimate can be made of the *satu* *point* for one of the variables, say $\hat{x}_{\chi}$, that is the point x_{χ} beyond w a further increase in x_{χ} is detrimental rather than beneficia have — if x_{χ} is a variable in the main group and measured in the group unit:

$$C = -P^{\circ\circ}_{\chi} - Q^{\circ}_{\chi}\ \hat{x}_{\chi} \tag{9.12}$$

C being comparable to the level term that needs to be added to th as indicated in (5.14). If $P^{\circ\circ}_{\chi}$ and Q°_{χ} are determined by (5.8) - ((9.12) can also we written

$$C = \frac{\hat{x}_{\chi} - \bar{\phi}_{\chi}}{\sqrt{m_{\chi\chi}}} \tag{9.13}$$

where the arithmetic average $\overline{\phi}_\chi$ indicates the Engel-average of x_χ. This computation can be made for several χ and the results compared.

One would expect that the assumption of a linear marginal preference is not entirely realistic. It would perhaps be more realistic to assume that the curve indicating how F_χ depends on x_χ is flattening out, i.e. the second derivative of F_χ is positive. In this event one would expect that the straight line estimate of the saturation point, if the straight line is fitted without bias, would indicate too low a saturation point. Consequently a straight line with correct mean slope but translated so as to pass through the saturation point, would be situated too high in the plane, i.e. the formula (9.13) - (9.12) will tend to *overestimate* C.

10. Processing distribution data with a cubic preference function.

If we want to refine the analysis and take account of the curvature that the marginal preference curves most likely have (compare end of 9), we can use a cubic preference function, for instance simplified to the partitioned form (4.4).

The substitumal relations corresponding to (5.5) will then be

$$(10.1)\quad (P_\alpha - P_\beta) + Q_\alpha \phi^s_\alpha - Q_\beta \phi^s_\beta + R_\alpha (\phi^s_\alpha)^2 - R_\beta (\phi^s_\beta)^2 = 0$$

(α and β any two regular affixes)

As in the quadratic case the P_j — if only the data (10.1) are available — are obviously affected by an arbitrary level constant, and all the coefficients by a common multiplier.

Since ϕ^s_α, ϕ^s_β and consequently $(\phi^s_\alpha)^2$ and $(\phi^s_\beta)^2$ are known functions of S, we can for any two sets of affixes α, β determine a mean regression line fitted to a variation over S. We can simplify the problem by imposing the condition that the regression line shall pass through the mean. This is obtained by introducing

$$(10.2)\qquad \overline{\phi}_i = \frac{\Sigma_S\, \phi_i^S}{\Sigma_S 1} \qquad \overline{\overline{\phi}}_j = \frac{\Sigma_S (\phi_j^S)^2}{\Sigma_S 1}$$

extending a summation over S to (10.1), dividing by $\Sigma_S 1$ and subtracting the equation thus obtained from (10.1), which gives

$$(10.3)\qquad \begin{aligned} &Q_\alpha (\phi^s_\alpha - \overline{\phi}_\alpha) - Q_\beta (\phi^s_\beta - \overline{\phi}_\beta) + \\ &R_\alpha [(\phi^s_\alpha)^2 - \overline{\overline{\phi}}_\alpha] - R_\beta [(\phi^s_\beta)^2 - \overline{\overline{\phi}}_\beta] = 0 \end{aligned}$$

(α and β any two regular affixes)

If by some means or another — for instance the diagonal method (1) — a mean homogeneous regression is determined between the four variables (all functions of S) that occur in (10.3), namely $(\phi^s_\alpha - \overline{\phi}_\alpha)$, $(\phi^s_\beta - \overline{\phi}_\beta)$, $[(\phi^s_\alpha)^2 - \overline{\overline{\phi}}_\alpha]$ and $[(\phi^s_\beta)^2 - \overline{\overline{\phi}}_\beta]$, the coefficients in the regression equation can be put proportional to the coefficients of Q_α, $(-Q_\beta)$, R_α and $(-R_\beta)$ of (10.3). We can for instance let the set (α, β) successively run through (1, 2), (2, 3)... etc. Choosing the proportionality factor in the first set arbitrarily Q_1, R_1, Q_2, R_2 are determined. In the set (2, 3) we determine through the regression equation a sequence of four numbers that should be proportional to Q_2, R_2, Q_3, R_3. Choosing the proportionality factor so as to get the best possible fit with the two coefficients Q_2, R_2 already determined, Q_3 and R_3 will be determined, and so on.

For reference purposes I shall give the formula of the diagonal mean regression in a set of variables X_i ($i = 1, 2 \ldots n$) for which observations are available over a certain field of variation. Let

$$(10.4)\qquad x_i = X_i - \overline{X}_i \qquad (i = 1, 2 \ldots n)$$

be the variables measured from their means

$$(10.5)\qquad \overline{X}_i = \frac{\Sigma X_i}{\Sigma 1}$$

where the summations are extended over the field of observation. Let

$$(10.6)\qquad m_{ij} = \Sigma (X_i - \overline{X}_i)(X_j - \overline{X}_j) \qquad (i = 1, 2 \ldots n;\ j = 1, 2 \ldots n)$$

be the moments about the means and let

$$(10.7)\qquad \hat{m}_{ij} = \text{the adjoint matrix of (10.6)}$$

The diagonal mean regression can then be written

$$(10.8)\qquad \sum_{i=1}^{n} \varepsilon_i \sqrt{\hat{m}_{ii}}\; x_i = 0$$

where the square root is taken positive and ε_i is the *sign* one wishes to attribute to the term with x_i. If the signs in the various rows of (10.7)

(1) The diagonal method is discussed in connection with (4.22) in my paper in the Nordic Statistical Journal. Vol. 8, 1928. Compare Section 14.

are compatible (which means that all the *elementary* regressions are sign compatible) this sequence of signs may be accepted. Or the signs may be selected on a priori considerations (they ought to check with the compatibility rule stated if the fit is reasonably good). The regression equation for the variables measured from the origin is obtained by inserting (10.4) into (10.8).

11. Moving determination of the coefficients Q_j.

Another, and perhaps more practical, way to introduce non linearity in the marginal preferences, is to perform a *moving* determination of the coefficients Q_j. This can be done for instance in the way that we split the summations in (5.10) - (5.11) into two parts, one over an S range around a value S' and the other over an S range around a value S''. This gives us by (5.9) two values of Q_j°, namely $Q_j^\circ(S')$ and $Q_j^\circ(S'')$. Assuming Q_j° to be a linear function [1] of ϕ_j^S

$$Q_j^\circ = A_j + B_j \phi_j^S \tag{11.1}$$

where A_j and B_j are constants independent of S, we get for these constants

$$A_j = \frac{Q_j^\circ(S'')\,\phi_j^{S'} - Q_j^\circ(S')\,\phi_j^{S''}}{\phi_j^{S'} - \phi_j^{S''}} \tag{11.2}$$

$$B_j = \frac{Q_j^\circ(S') - Q_j^\circ(S'')}{\phi_j^{S'} - \phi_j^{S''}} \tag{11.3}$$

Inserting (11.1) into the term $Q_j^\circ \phi_j^S$, we get this term replaced by

$$Q_j^\circ \phi_j^S = A_j \phi_j^S + B_j (\phi_j^S)^2 \tag{11.4}$$

A similar treatment of $P_j^{\circ\circ}$ would replace this constant by a linear expression of ϕ_j^S so that we would end up with an expression of the form

$$F_g^C = C + P_g^{\circ\circ} + Q_g^\circ x_g + \quad R_g^\circ x_g^2 \qquad (g \text{ reg}) \tag{11.5}$$

to replace (5.19), $P_g^{\circ\circ}$, Q_g° and R_g° being constants determined from interview data on distributions.

[1] Note that S in (5.9) - (5.10) is only a dummy affix while in (11.1) it is not.

12. Interview variables and model variables.

Suppose that the interviewed person wishes to express his preferences for certain variables that are not *explicity* contained amongst the model variables, i.e. those specified in (2.1). This situation can be handled relatively easily if each of the extra variables which the interviewed person is thinking of, can be expressed as a (linear or non linear) function of some or all of the variables that are in the model.

Quite generally let $y_1, y_2 \ldots y_M$ be the variables which the interviewed person wants to consider, and let each of them be expressible in terms of the variables $x_1, x_2 \ldots x_N$ of the model. Let these expressions be

$$y_j = G_j(x_1 \ldots x_N) \qquad (j = 1,2 \ldots M) \tag{12.1}$$

where the functions G_j are known.

Further let

$$E(y_1 \ldots y_M) \tag{12.2}$$

be the interview preference function, i.e. the function by which we assume that the interviewed individual is guided when he gives his answers. When the expressions (12.1) are inserted in (12.2), we get a function of the model variables which plays the rôle of our previously considered preference function. Let the function be

$$F(x_1 \ldots x_N) = E(y_1 \ldots y_M) \tag{12.3}$$

The last relation holds by definition identically in all the variables of the model $x_1 \ldots x_N$, when we insert for $y_1 \ldots y_M$ in the right member of (12.3) the expressions given by (12.1).

The interview preference function E can be studied from the viewpoint that all its variables $y_1 \ldots y_M$ are free variables, and the numerical form of the function can be constructed by the methods of the preceding sections. When this is done, the substitution (12.2) carries us over into the model preference function $F(x_1 \ldots x_N)$. In particular we have

$$F_g = \sum_{j=1}^{M} E_j G_{jg} \qquad (g = 1, 2 \ldots N) \tag{12.4}$$

where

$$E_j = \frac{\partial E\,(y_1 \ldots y_M)}{\partial y_j} \qquad (j = 1, 2 \ldots M) \tag{12.5}$$

and

$$G_{jg} = \frac{\partial G_j\,(x_1 \ldots x_N)}{\partial x_g} \quad (j = 1,2 \ldots M;\ g = 1,2 \ldots N) \tag{12.6}$$

In many practical cases the functions G_j for many of the variables will simply be x_j so that for these values of j we have

$$y_j = x_j \tag{17.7}$$

while for some additional y variables we may have more complicated functions of $x_1 \ldots x_N$. It is an advantage if we can use linear approximations to (12.1).

13. Example of the actual determination of a macroeconomic preference function.

Following the principles developed above, and using the data given in tab. (3.1) - (3.2) as well as other material, a macroeconomic preference function has actually been constructed for the Oslo Median Model (1). This was made possible through the wholehearted collaboration of a prominent Norwegian politician who gave much of his valuable time to experiments in working out our interview technique.

The preference function as it now appears is only a device for use in illustrating methods of linear and quadratic programming on the Oslo Median Model. The results of this experimental work have, however, been so promising that I have no doubt that we have here the outlines of a technique which is going to play a great rôle in macroeconomic programming in the future.

The quadratic preference function constructed contains 14 variables all of which are assumed to be measured in millions of Kroner. The P_j will — if they are determined through (5.8) or (9.1) — be independent of the unit of measurement, but the Q_j as determined by (5.9) or (9.2) will depend on the unit.

(1) The model is described in the memorandum of 10 October 1956 « Main features of the Oslo Median Model ».

The coefficients determined are given in tab. (13.1). The ordinal numbers for the variables, as well as their terminological description in tab. (13.1), are the same as in the memorandum of 10 October 1956 on the Oslo Median Model.

TAB. (13.1) - Coefficients P_j and Q_j for a quadratic preference function for programming in the Oslo Median Model

Variable No.	Description	P_j	Q_j (To be applied to figures expressed in millions of kroner)
331'	Export surplus	4.76401	—0.00313
332	Disposable income in households whose main income consists of wages	29.08563	—0.00246
333	Disposable income in households of entrepreneurs	21.13082	—0.00526
334	Disposable income in households of peasants and fishermen	20.05801	—0.00701
335	Disposable income for persons living on pensions and persons living in institutions	17.15249	—0.00732
346	Government investment	11.70987	—0.00587
347	Investments in extractive industries excluding mining	24.96073	—0.02414
348	Investment in sea transport and whaling	24.05431	—0.00805
349	Investment in manufacturing industries producing for the export market	12.81269	—0.01261
350	Investment in land and air transport, post and telecommunications	17.10417	—0.01261
351	Investment in mining and in manufacturing industries producing capital goods for the home market (capital goods used by Norwegian sectors)	17.39012	—0.01261
352	Investment in manufacturing industries producing consumers goods for the home market	11.81650	—0.02414
353	Investment in housing, power plants and gas works	21.09005	—0.00960
355	Government purchases of goods and services on current account	16.49737	—0.00368

14. Note on the diagonal or other forms of mean regressions.

The term « regression » in the strictly stochastic sense is derived from the concept of a simultaneous distribution of several variables. If in such a distribution we consider all the variables as *given* except one of them, i.e. if we consider that particular part of the total distribution where these variables have preassigned values (or are lying in infinitesimal intervals around such values), we get the conditional distribution of the remaining variable, and we may compute the mathematical expectation in this distribution. It will, of course, in general depend on the levels that were prescribed for the other variables. The function that expresses how the conditional mathematical expectation considered depends on the prescribed levels of the other variables is the (elementary) regression of that variable on the others.

When we use the term « regression » in the mean or symmetric sense, we may or may not have a strictly stochastic model in mind. In either case the fundamental assumption is now that there exists nothing in the problem which permits us to single out one of the variables from the others and treat this one differently from the rest. We simply have a more or less vague idea that from the viewpoint underlying the problem — such as for instance the equality of two marginal preferences in a substitumal point — the variables are « approximately connected » by an equation, perhaps a linear equation, i.e. the observations should be looked upon as lying approximately on a surface, but the *polarization* of the equation, i.e. the direction of the « cause », is completely lacking. It may well be that we perform a smaller distortion of the reality by using some sort of empirical curve fitting that actually treats the variables in a non polarized way, than by using a highly refined stochastic machinery with assumptions that do introduce polarization. Example: The rough diagonal regression will probably give a better result than any of the elementary regressions applied to (5.5).

The use of such devices as to introduce « errors in the equation » instead of introducing « errors in the variables » is no answer at all to the difficulty discussed, because the result obtained from an « error in the equation »-approach will depend fundamentally on how we choose to write the equation (how we choose to arrange its terms) *before* we inject the errors. As a rule there will be nothing a priori (except convenience of calculation) that can permit us to prefer one form of the equation for another. The « error in the equation »-approach is therefore only a concealed and unfounded form of polarization.

It can only be through some sort of classification of the individual variables according to their individual « error variability » (for instance a model with an additive error term on each variable) that a polarization can be introduced. The extreme case here is, of course, that of the elementary regressions, where all the variability *a priori* is thrown into one single variable and all the others are left without variability.

Whether such an assumption about polarization shall be permissible or not, will depend on knowledge we have *before the statistical fitting process*. We must know something about the way in which the observed phenomenon is actually produced.

Take for instance the case of interview data from distribution questions discussed in the previous sections. Here it may be a reasonable model to assume that the total S is given and the individual distribution figure x_j is a stochastic variable whose expectation depends on S according to a regression equation which may, say, be assumed of the form

$$P_j + Q_j x_j = A + BS \tag{14.1}$$

where now x_j stands for the conditional expectation of x_j, and P_j, Q_j, A and B are constants. Obviously all these constants cannot be determined from the substitumal data. Some degrees of freedom are left. But if we assume A and B to be given, the elementary regression would give

$$P_j = (A + B\overline{S}) - Q_j \overline{\phi}_j \tag{14.2}$$

$$Q_j = \frac{m_{oo}}{m_{j\bullet}} B \tag{14.3}$$

where

$$m_{oo} = \Sigma (S - \overline{S})^2 \tag{14.4}$$

$$m_{oj} = \Sigma (S - \overline{S})(\phi_j^s - \overline{\phi}_j) = m_{j\bullet} \tag{14.5}$$

$$\overline{S} = \frac{\Sigma S}{\Sigma 1} \tag{14.6}$$

$$\overline{\phi}_j = \frac{\Sigma \phi_j^s}{\Sigma 1} \tag{14.7}$$

ϕ_j^s being the observed distribution numbers.

Intuitively we may conclude that if the data are rough, the determination of Q_j will be less stable by (14.3) than by (5.9). And in this connection it is not only a question of determining stability by the usual standard errors on regression coefficients and similar devices, but judgement of how far individual and fairly big « shocks » may come into the picture and effect results. Looking at the matter from a purely heuristic viewpoint, we see that the denominator of (5.9) can only be zero if ϕ_j^s is rigorously independent of S — and then it will simply be pushed over into the constant term of the preference function as explained above, so that no trouble arises, while in (14.3) the denominator may become zero if positive and negative terms in the product under the summation sign cancel out.

But, at least, (14.2) - (14.3) would be very much better than the result that would emerge if we calculated in (5.5) the elementary regression of ϕ_β^s on ϕ_3^s or inversely. Even though these elementary regression coefficients turned out to be very « significant » from the viewpoint of the stochastic model that treats the two variables in a polarized way, we could say beforehand that in reality such coefficients are not significant in the deeper meaning of the word because the polarized model does not apply.

It is my conviction that much of the current practice of equation fitting — individual equations or systems of simultaneous equations — is off the mark because too much emphasis is put on an exact derivation of stochastic conclusions from given assumptions and too little is put on the nature of the mechanism that has produced the observed phenomenon. The latter discussion must necessarily be on an a priori basis (a priori in relation to the data which are to be analyzed by the stochastic model). When we recognize this and time does not permit us to await the construction of a stochastic super-model that could analyze also the genesis of the phenomenon for which data are now at hand (a super-model which would again have to be based on what is a priori in relation to this model), two alternatives may be open: either to use a refined stochastic model which is provided with tests of significance that are very satisfactory when the model is granted, but is based on certain assumptions that even by a « horse sense » judgement must appear as in flagrant contradiction to the little we know about the genesis of the phenomenon (drastically exemplified if we would determine the coefficients of (5.5) by one of the two elementary regression coefficients supplemented by its standard error) — *or* to use heuristically some cruder empirical fitting procedure about which we know at least that it does not contradict what we know about the genesis of the phenomenon (exemplified by using the diagonal regression to determine the coefficients of (5.5)).

In many cases one would do better to follow the latter procedure. And in fact much statistical analysis does follow — and correctly so — this « horse sense » procedure. But, of course, this should not be an excuse for always choosing the crudest and simplest solution. Empirical fitting procedures are permissible in a new field where we must make a dash forward. But the need for improvements should never be forgotten.

The further we can go in the direction of developing stochastic models that are *really adequate* to the specific problem, the better. If a model is to be really adequate, it must take account of the genesis of the phenomenon. This means that we should perhaps be more careful than we have been hitherto when adapting methods from one field to another. And in particular we should perhaps be a little more sceptical about methods that from a formal point of view seem to be so general as to be applicable to practically any problem.

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Additional Note 1. Regarding (5.6).

The first and second elementary regression equation respectively is the equation obtained from (5.6) by inserting in the right member the factor $r_{\alpha\beta}$ and $\frac{1}{r}$ respectively, where $r_{\alpha\beta}$ is the correlation coefficient between ϕ_α and $\phi_\beta^{\alpha\beta}$. The geometric average between these two equations is (5.6).

Additional Note 2. Regarding the reasoning before (5.14).

There may be different choice indicators and even different indifference maps that lead to the same substitumal. For programming purposes it is *not sufficient* to know the class of all the choice indicators that lead to the same substitumal. We must know the more restricted class that leads to the actually existing (and in principle observable) indifference surfaces. This means that the level term C must be determined. This can (in principle) be done if we have information about two indifferent points, at least one of which is not substitumal.