PARAMETRIC SOLUTION AND PROGRAMMING OF THE HICKSIAN MODEL

By RAGNAR FRISCH

Assisted by ASHOK K. PARIKH[1]

Institute of Economics, University of Oslo

THE HICKSIAN MODEL

In a nutshell the Hicksian model[2] can be described as follows, if time is denoted by t (say calender year):

Notation for five variables. C_t = private consumption ; I_t = net *induced* investment ; G_t = Government use of goods and services on current account ; H_t = net *autonomous* investment[3] ; Y_t = net national income[4].

Notation for four parameters : $\alpha, \beta, \kappa, \lambda$. Of these α and β are in the main structural parameters not subject to Government [illegible], while κ and λ may be thought of as being subject to Government decisio[illegible]st to some extent. The parameters κ and λ will define the Government s[illegible]ith respect to the total Government expenditure on current account.

Four equations:

Consumer behaviour,	$C_t = \alpha Y_{t-1}$,	... (1)
Behaviour of induced investors,	$I_t = \beta(Y_{t-1} - Y_{t-2})$	... (2)
Government strategy with respect to current account expenditure,	$G_t = \kappa Y_{t-1} - \lambda(Y_{t-1} - Y_{t-2})$	... (3)
Definition of net national income	$Y_t = C_t + I_t + G_t + H_t$.	... (4)

Degrees of freedom. As it stands the model has five variables and four equations, hence one degree of freedom. This degree of freedom may be thought of as being generated by the variable H_t, i.e. autonomous investment. This variable may be looked upon either as *deterministically* given (i.e. each value H_t is given for reasons outside of the model) or as *stochastically* given (i.e. the variables H_1, H_2 ...

---

[1] The Indian economist Mr. Ashok K. Parikh, M.Sc. (Econ.) (London School of Economics) stayed some months in the Oslo University Institute of Economics in the summer of 1962.

[2] J. R. Hicks (1950): *A Contribution to the Theory of the Trade Cycle*, Oxford. See also 7.3 in R. G. D. Allen (1957): *Mathematical Economics*, London.

[3] Hicks uses the symbol A_t for autonomous investment. I prefer to use H_t because it corresponds better to the Oslo notation.

[4] In standard national account terminology the Y_t would have to be called net national *product*. The designation "income" is the national account terminology connected with the concept of factor cost. I have on several occasions strongly criticized the concept as being neither logical nor fruitful. I am therefore quite happy to use the term "income" for Y_t, even though it may have nothing to do with factor cost.

have a given simultaneous probability distribution; possibly as a special case all the $H_1, H_2 \dots$ may be stochastically independent). In any case the values $H_1, H_2 \dots$ will have to be considered as *data* when we approach the solution of the model.[5]

This means that the solution should be sought in the form of four time-functions C_t, I_t G_t Y_t explicitly expressed in terms of the four parameters $\alpha, \beta, \kappa, \lambda$, the initial conditions and the values $H_1, H_2 \dots$.

Method of characteristic roots and method of recurrence. The solution may either be obtained by the *recurrence* method or by the method of *characteristic roots.*

In the recurrence method we start from a given initial situation of the variables and use the model (1)-(4) to compute the constellation of the system in next point of time and so on.

In the method of characteristic roots we start first by obtaining the solution of the associated *homogeneous* systems, i.e. the system that is identical with (1)-(4) except for the fact that the term H_t is *dropped.* In other words one puts $H_t = 0$ for all t. And this homogeneous system is solved in a particular way, namely, by computing the so-called characteristic roots. And by the help of the solution of the homogeneous system one finally derives the solution of the complete system (1)-(4).

It is very useful to have the solutions available in the both forms (cf. equations (35), (127), (130) and (133)).

The recurrence method has an advantage compared to that of the method of characteristic roots because the recurrence procedure yields immediately the forms which the functions in question must *necessarily* have in order that they form a solution. In the method of characteristic roots, we do not get immediately anything more than forms which are *sufficient* in order that the functions in question shall satisfy the system of difference equation. It follows, however, from classical mathematical theories (proved in a rather elaborate way) that under certain conditions the function forms obtained by means of the method of characteristic roots, are actually the forms which the functions must necessarily have in order that they shall satisfy (1)-(4).

In the sequel. I will use both these methods and compare the results.

The system (1)-(4) is a simple one because it can be *partitioned* in such a way that the whole problem is reduced to a consideration of a simple difference equation in one variable namely Y_t. Once the explicit expression for Y_t is obtained, the remaining follows from (1)-(3).

The equation in Y_t is derived by inserting (1)-(3) into (4). This gives

$$Y_t = \mu Y_{t-1} + \nu Y_{t-2} + H_t \quad \dots \quad (5)$$

(one equation in two time functions Y and H)

where

$$\mu = \alpha + \beta + \kappa - \lambda \quad \dots \quad (6)$$

$$\nu = \lambda - \beta. \quad \dots \quad (7)$$

[5] There is no particular reason to assume that H_t must be non-negative; it may in principle be positive, negative or zero, but of course in practice it will as a rule be positive.

A realistically weak point in the model. The equation (5) brings to light a realistically weak point in the model. The equation shows that whatever the past course of the economy has been, the net national income Y_t can in any given point of time t be rendered *arbitrary great* if H_t is chosen sufficiently great.

It is, of course, true that frequently the constellation of the economy, and its structure, may be such that an expansion will occur if *hidden resources* are released, either through determinate government action which will stimulate autonomous investment H_t, or through more or less accidental happenings (inventions, wars etc.) which will also stimulate autonomous investment. But I think that Western economic thinking on trade cycle regulation and on economic growth is all too one-sidedly concentrated on this point. The present model where H_t can *ad libitum* produce an expansion in Y_t is an example. H_t is by (5) so to speak an extra addition which we may, if we like, give to Y_t when the point of time t has been reached.

The analysis of the role of autonomous investment (or more precisely of autonomous investment *starting*) ought to proceed in a more thoroughgoing way by relating the possibility of autonomous investment to the existing production capacities and other *bounds* in the economy. In order that the additional term H_t in (4) and (5) shall have a meaning it must be looked upon as an erratic element (a "shock") which *itself* has taken care of the realistically existing bounds, or as an element given in some other way satisfying the bounds.

One very simple way in which we can explicitly take account of at least one important bound on the development of the economy, is by using the Harrod-Domar way of thinking where the need for *real capital* is introduced explicitly (and expressed through the capital-to-output ratio) and where this need for capital is connected with the *investment activity*. I do not think that the Harrod-Domar model gives an adequate analysis of the growth process in the economy.[6] In particular it is entirely inadequate for a discussion of the *decisions* connected with the growth problem in an underdeveloped country.[7] But the Harrod-Domar model takes at least account of one aspect of development which is completely neglected in the Hicksian model as described in (1)-(4). Another bound that may be considered is (148) below. The introduction of this latter bound will not restrict the number of degrees of freedom, so even with such a bound introduced, the model does not determine the actual solution of the system but will depend on what the autonomous investments H_t actually have been, subject to the constraint (148).

In the present paper I will, however, disregard this aspect of the Hicksian model and consider the nature of the solution as it follows from (1)-(5).

---

6 See for instance my paper in *Econometrica*, July 1961.

7 Some of my reasons for thinking so are given for instance in section III 1.a in my October 1960 paper in the journal *L'Egypt Contemporaine*, Cairo, and on pages 62–63 in my paper "Economic planning and the growth problem in developing countries", in the journal *Statsøkonomisk Tidsskrift*, Oslo, 1961.

THE SOLUTION BY RECURRENCE

It is essential to give the explicit solution of (5) *for any arbitrarily given function* H_t. We shall first approach this problem by the recurrence method.

Let the size of net national income in the points of time 0 and (-1), i.e. Y_0 Y_{-1}, be *given* as initial conditions. From (5) we then derive successively

$$
\begin{aligned}
Y_1 &= \mu Y_0 + \nu Y_{-1} + H_1,\\
Y_2 &= \mu^2(1+\omega)Y_0 + \mu\nu Y_{-1} + \mu H_1 + H_2,\\
Y_3 &= \mu^3(1+2\omega)Y_0 + \mu^2\nu(1+\omega)Y_{-1} + \mu^2(1+\omega)H_1 + \mu H_2 + H_3,\\
Y_4 &= \mu^4(1+3\omega+\omega^2)Y_0 + \mu^3\nu(1+2\omega)Y_{-1} + \mu^3(1+2\omega)H_1\\
&\quad + \mu^2(1+\omega)H_2 + \mu H_3 + H_4
\end{aligned}
\quad \ldots (8)
$$

(Parikh has continued this recurrence up to $t = 15$. i.e. up to Y_{15}).

where
$$\omega = \frac{\nu}{\mu^2}. \quad \ldots (9)$$

We can assume that
$$\mu \neq 0 \quad \ldots (10)$$
so that the parameter ω defined by (9) exists.

It is readily seen that in general the coefficient of Y_0 in the explicit expression for Y_t will be μ^t times a polynomial in ω of the order Ent $t/2$ where

Ent q denotes the largest integer contained in q. ... (11)

Denoting the coefficients of this polynomial by a_{ti} we can write this polynomial

$$A_t = a_{t0} + a_{t1}\omega + a_{t2}\omega^2 + \ldots \text{ up to the term with } \omega^{\text{Ent } t/2}. \quad \ldots (12)$$

By carrying out the recurrence procedure (8) for the first terms it is easily seen that in general the first two coefficients are

$$a_{t0} = 1 \qquad (t = 1, 2, \ldots, \infty) \quad \ldots (13)$$
$$a_{t1} = t-1 \qquad (t = 1, 2, \ldots, \infty). \quad \ldots (14)$$

The general expression for the coefficients a_{ti} is then easily obtained by noticing that the general expression for Y_t must be linear in $Y_0, Y_{-1}, H_1, H_2, \ldots, H_t$ and that the expression must hold good *identically* in these $t+2$ magnitudes.

Therefore if we put

$$Y_{-1} = H_1 = H_2 = \cdots = H_t = 0, \quad \ldots (15)$$

but
$$Y_0 \neq 0$$

and insert into (5) the expression for Y_t thus obtained, we get after having divided by Y_0 (assumed different from zero)

$$\mu^t \Sigma_i a_{ti}\omega^i = \mu.\mu^{t-1}\Sigma_i a_{t-1,\,i}\omega^i + \nu.\mu^{t-2}\Sigma_i a_{t-2,\,i}\omega^i. \quad \ldots (16)$$

The last expression in the right member of (16) can be written $\mu^t\Sigma_i a_{t-2,\,i}\,\omega^{i+1} = \mu^t\Sigma_i a_{t-2,\,i-1}\omega^i$, so that (16) can also be written

$$\mu^t\Sigma_i(a_{ti} - a_{t-1,\,i} - a_{t-2,\,i-1})\omega^i = 0 \qquad (t = 2, 3, \ldots, \infty). \quad \ldots (17)$$

By (10) this gives

$$\Sigma_i(a_{ti}-a_{t-1,\,i}-a_{t-2,\,i-1})\omega^i = 0 \qquad (t = 2, 3, \dots, \infty). \qquad \dots (18)$$

Since the polynomial (18) is to be zero for any value of ω, its coefficients must be zero separately. Hence

$$a_{ti}-a_{t-1,\,i} = a_{t-2,\,i-1} \qquad (t = 2, 3, \dots, \infty). \qquad \dots (19)$$

Adding the equations (19) for $t, t-1, \dots 3$, we get

$$a_{ti} = a_{2i}+\sum_{\tau=3}^{t} a_{\tau-2,\,i-1} \qquad (t = 3, 4, \dots, \infty). \qquad \dots (20)$$

In other words

$$a_{ti} = a_{2i}+\sum_{\tau=1}^{t-2} a_{\tau,\,i-1} \qquad (t = 3, 4, \dots, \infty). \qquad \dots (21)$$

We will only use this equation for $i = 2, 3, \dots \infty$. Since by the remark before (11)

$$a_{2i} = 0 \qquad \text{for } i \geqslant 2 \qquad \dots (22)$$

we get from (21)

$$a_{ti} = \sum_{\tau=1}^{t-2} a_{\tau,\,i-1} \qquad \begin{pmatrix} t = 3, 4, \dots, \infty \\ i \geqslant 2 \end{pmatrix}. \qquad \dots (23)$$

This together with (13), (14) and the general formula for binomial coefficients

$$\sum_{\tau=0}^{p}\binom{\tau-i}{i} = \binom{p-i+1}{i+1} \qquad \begin{pmatrix} p \geqslant 0 \\ i \geqslant 0 \end{pmatrix} \qquad \dots (24)$$

we get

$$a_{t2} = \sum_{\tau=1}^{t-2} a_{\tau 1} = \sum_{\tau=1}^{t-2}\binom{\tau-1}{1} = \binom{t-2}{2} \qquad (t = 3, 4, \dots, \infty). \qquad \dots (25)$$

From (8) we see that the last formula also holds good for $t = 1, 2$.

Hence we have quite generally

$$a_{t2} = \binom{t-2}{2} \qquad (t = 1, 2, \dots \infty). \qquad \dots (26)$$

Further we get by (22), (23) and (25)

$$a_{t3} = \sum_{\tau=1}^{t-2} a_{\tau 2} = \sum_{\tau=1}^{t-2}\binom{\tau-2}{2} = \binom{t-3}{3} \qquad (t = 3, 4, \dots \infty). \qquad \dots (27)$$

Since by (8) this formula also holds good for $t = 1, 2$, we have quite generally

$$a_{t3} = \binom{t-3}{3} \qquad (t = 1, 2, \dots, \infty). \qquad \dots (28)$$

Continuing in this way and taking account of (13) and (14) we get

$$a_{ti} = \binom{t-i}{i} \qquad \begin{pmatrix} t = 1, 2, \dots, \infty \\ i = 0, 1, \dots, \infty \end{pmatrix}. \qquad \dots (29)$$

We can check this formula by inserting into (5) the expression obtained for Y_t, Y_{t-1}, Y_{t-2} by using (29) and the assumptions (15). This gives for the left member, after dividing by Y_0

$$\mu^t \Sigma_i \binom{t-i}{i} \omega^i \qquad \dots \quad (30)$$

and for the right member we get

$$\mu \cdot \mu^{t-1} \Sigma_i \binom{t-1-i}{i} \omega^i + \nu \cdot \mu^{t-2} \Sigma_i \binom{t-2-i}{i} \omega^i = \mu^t \Sigma_i \binom{t-1-i}{i} \omega^i + \mu^t \Sigma_i \binom{t-2-i}{i} \omega^{i+1}.$$

The last expression here is equal to $\mu^t \Sigma_i \binom{t-1-i}{i-1} \omega^i$. The total right member is consequently

$$\mu^t \Sigma_i \left[\binom{t-1-i}{i} + \binom{t-1-i}{i-1} \right] \omega^i. \qquad \dots \quad (31)$$

By the classical formula for binomial coefficients

$$\binom{p}{i-1} + \binom{p}{i} = \binom{p+1}{i} \binom{p \geqslant 0}{i \geqslant 0} \qquad \dots \quad (32)$$

the bracket in (31) reduces to $\binom{t-i}{i}$ so that (31) is seen to be identical with (30).

From (8) we see heuristically that:

the coefficient of Y_{-1} *in* Y_t *is equal to* ν *times the coefficient of* Y_0 *in* Y_{t-1} ... (33)

and *the coefficient of* H_τ *in* Y_t *is equal to the coefficient of* Y_0 *in* $Y_{t-\tau}$. ... (34)

The two rules (33)-(34) can easily be verified by continuing the recurrence.

The general explicit expression for Y_t will consequently be

$$Y_t = \left[\mu^t \Sigma_i \binom{t-i}{i} \omega^i \right] Y_0 + \left[\mu^{t-1} \nu \Sigma_i \binom{t-1-i}{i} \omega^i \right] Y_{-1}$$

$$+ \sum_{\tau=1}^{t} \left[\mu^{t-\tau} \Sigma_i \binom{t-\tau-i}{i} \omega^i \right] H_\tau \quad (t = 1, 2, \dots, \infty). \qquad \dots \quad (35)$$

The brackets in (35) indicate the coefficients of Y_0, Y_{-1}, H_1, H_2, ..., H_T.

The first line in (35) can also be written

$$\mu^t \Sigma_i \frac{1}{i} \binom{t-i-1}{i-1} \left[\mu(t-i) Y_0 + \nu(t-2i) Y_{-1} \right] \omega^i$$

but if it is desired to bring out explicitly the separate effect of each initial condition, (35) is the proper form to use.

By inserting the general expression (35) into (5) we can check that the equation holds good whatever values we attribute to μ, ω, Y_0, Y_{-1}, H_1, H_2, ..., H_t.

Indeed, inserting (35) into (5) and bringing all the coefficients of Y_0, Y_{-1}, $H_1, \ldots, H_t$ together on one side, we get

$$\left[\mu^t\left\{\Sigma_i\binom{t-i}{i}\omega^i-\Sigma_i\binom{t-i-1}{i}\omega^i-\Sigma_i\binom{t-i-2}{i}\omega^{i+1}\right\}\right]Y_0$$

$$+\left[\mu^{t-1}\nu\left\{\Sigma_i\binom{t-i-1}{i}\omega^i-\Sigma_i\binom{t-i-2}{i}\omega^i-\Sigma_i\binom{t-i-3}{i}\omega^{i+1}\right\}\right]Y_{-1}$$

$$+\sum_{\tau=1}^{t-2}\left[\mu^{t-\tau}\left\{\Sigma_i\binom{t-\tau-i}{i}\omega^i-\Sigma_i\binom{t-\tau-i-1}{i}\omega^i-\Sigma_i\binom{t-\tau-i-2}{i}\omega^{i+1}\right\}\right]H_\tau$$

$$+\left[\mu\left\{\Sigma_i\binom{1-i}{i}\omega^i-\Sigma_i\binom{0-i}{i}\omega^i\right\}\right]H_{t-1}+[1-1]H_t.$$

In the last summation in first bracket above we replace i by $(i-1)$. By (32) the bracket is then seen to be zero. And the same applies to the following two brackets. The last two brackets are obviously zero.

The solution through characteristic roots

Another way to approach the solution of (5) is through characteristic roots. I repeat that it is essential to obtain the explicit solution of (5) for *any arbitrarily given function* H_t, not only for special analytical forms of H_t.

There are many text-books treating difference equations. But my experience is that in order to find in the text-books the results one needs, in a given problem, one has nearly always to wade through an enormous amount of symbolism. And the meaning of this symbolism can frequently be deciphered only by reading a great number of chapters different from the one where the results one is looking for is actually to be found. And even if one succeeds in deciphering the symbols the results are often not in *simple* form as one needs.

In view of the great importance for economic analysis of linear difference equations with constant coefficients and with an *arbitrary* additional term (such as H_t in (5)) I developed many years ago in my Oslo lecture (in Norwegian) an approach which is at the same time rigorous and, I believe, very easy to follow.[8] I shall reproduce here the "prescription" for the solution which this approach leads to. And I shall do it for the most general type of linear difference equation (with real constant coefficients), even though this most general case is more than needed in order to handle the simple equation (5).

The complete and the truncated difference equation. Let the difference equation in the unknown time function Y_t be

$$a_0Y_t+a_1Y_{t+1}+\ldots+a_nY_{t+n}=H_{t+n} \qquad \ldots \quad (36)$$

where $a_0, a_1, \ldots, a_n$ are given real coefficients and H_{t+n} an arbitrarily given real function of time. The integer n is called the *order* of the equation.[9]

[8] Appendix 7 in "Notater til ökonomisk teori" (mimeographed) 4. Edition, Oslo, 1947. Also, 'Ordinære lineære differentiallikninger", October, 1961.

[9] Several formulae pertaining to the solution of (36) will turn out in a much more handy form if we let a_ν denote the coefficient of $Y_{t+\nu}$ instead of the coefficient of $Y_{t-\nu}$.

It does not restrict generality if we assume

$$a_n = 1. \qquad \dots \quad (37)$$

Indeed if $a_n \neq 0$ we can simply divide the equation by a_n and if $a_n = 0$ *the equation* is simply an equation of the same kind but of lower order than n. In the sequel we will therefore assume (37).

The equation (36) with the additional term H_{t+n} will be called the *complete*, or non-homogeneous, equation.

To this complete, or non-homogeneous, equation we associate the corresponding *truncated*, or homogeneous, equation which is obtained by simply leaving out the additional term H_{t+n}, i.e. by putting $H_{t+n} = 0$ for all t. This truncated equation is consequently

$$a_0 Y_t + a_1 Y_{t+1} + \dots + a_n Y_{t+n} = 0. \qquad \dots \quad (38)$$

The solution of the truncated difference equation by means of characteristic roots. We will find first the general solution of the truncated equation (38). The basic fact in this connection (proved in an elementary way in my Oslo lecture) is the following: If we can only find in some way or other n *particular* solutions that are *linearly independent* over t, the problem is solved, because a linear combination of these n linearly independent particular solutions taken with arbitrary constant coefficients (constant in the sense of being independent of t) will give the general solution of (38). That is to say any function of t which satisfies (38) can be produced by attributing appropriate values to the n arbitrary constants.

Writing out in full: If $Y_{t1}, Y_{t2}, \dots, Y_{tn}$ are n particular solutions which are *linearly independent* over t, then

$$Y_t = C_1 Y_{t1} + C_2 Y_{t2} + \dots + C_n Y_{tn}, \qquad \dots \quad (39)$$

where $C_1, C_2, \dots, C_n$ are n arbitrary constants, will be the general solution of (38) in the sense that *any* function of t which is to satisfy (38) *must* be obtainable by attributing appropriate values to $C_1, C_2, \dots, C_n$ in (39).

The fact that n function $Y_{t1}, Y_{t2}, \dots, Y_{tn}$ are linearly independent over t means that it is *not* possible to find an *effective* set of constants $S_1, S_2, \dots, S_n$ such that

$$S_1 Y_{t1} + S_2 Y_{t2} + \dots + S_n Y_{tn} = 0, \quad \text{identically in } t. \qquad \dots \quad (40)$$

The magnitudes $S_1, S_2, \dots, S_n$ being *constant* means they are independent of t, and the set $S_1, S_2, \dots, S_n$ being *effective* means at least one of these constants is different from zero.

The fact that the linear combination (39) with $C_1, C_2, \dots, C_n$ as arbitrary coefficients will satisfy (38) if all the individual functions $Y_{t1}, Y_{t2}, \dots, Y_{tn}$ satisfy (38), is obvious. This is immediately seen by inserting (39) into (38). And this conclusion obviously holds good whether the individual functions are linearly independent or not. The basic proposition in the theory of linear homogeneous difference equation is the *inverse* of this, namely, that *any* solution of (38) must be of the form (39) provided the particular solution $Y_{t1}, Y_{t2}, \dots, Y_{tn}$ are linearly independent.

The problem of finding the general solution of (38) is therefore only to find *i n some way or other n* linearly independent particular solutions of (38).

The general procedure for finding particular solutions of (38) is to try if an *exponential function*

$$Y^{p}_{t} = \lambda^{t} \; (p = \text{"particular"}) \qquad \dots \quad (41)$$

should be a particular solution, if the constant λ is chosen in an appropriate way.

Inserting (41) into (38) we see that if (41) is to be a solution of (38) we must have

$$a_0\lambda^{t}+a_1\lambda^{t+1}+\dots+a_n\lambda^{t+n} = 0. \qquad \dots \quad (42)$$

This equation is to hold good for any value of t. Furthermore, since λ^t is different from zero if $\lambda \neq 0$ and t finite, we can divide the equation by λ^t. This shows that *if* (41) is to be a solution we *must* have

$$a_0+a_1\lambda^{1}+\dots+a_n\lambda^{n} = 0. \qquad \dots \quad (43)$$

In other words *if* (41) is to be a solution, it is *necessary* that λ be a root of the n-th degree polynomial written in the middle member of (43).

The equation defined by (43) is called the *characteristic equation* for (38). Further, the polynomial $f(\lambda) = a_0+a_1\lambda^1+\dots+a_n\lambda^n$ is called the *characteristic polynomial* for (38).

The characteristic equation (43) is simply an *algebraic equation* in λ with given coefficients. *This equation does not depend on t at all.* This is the essence of our tentative procedure for finding particular time functions that will satisfy (38).

The above reasoning shows that *if* (41) is to be a particular solution of (39) then λ *must* be a root of the characteristic equation (43). But the inverse also holds good. Obviously *any* root of the characteristic equation will yield a particular solution (41) which satisfies (38). Therefore if by any method whatsoever we have found a root of (43), we have by this fact also found a particular solution of (38).

It is a classical algebraic fact that a polynomial of degree n has exactly n zeros. This means that the characteristic equation (43) has exactly n roots (when each root is counted as many times as is indicated by its multiplicity). We may denote the roots $\lambda_1, \lambda_2, \dots, \lambda_n$. Hence if the n roots of the characteristic equation (43) are all *different* (i.e. all of them of multiplicity 1), we have immediately solved the problem of finding the general solution of (38). Two exponential functions

$$Y^{p}_{t} = \lambda^{t}_{p} \quad \text{and} \quad Y^{q}_{t} = \lambda^{t}_{q} \qquad \dots \quad (44)$$

where $\lambda_p \neq \lambda_q$ are indeed always linearly independent over t.

The roots of (43) may be real or complex, but whatever they are, the above remark about linear independence always applies provided the roots are different.

If (43) has complex roots it is a classical fact that they must always occur in *conjugate pairs*, i.e. if

$$\lambda = \theta + i\delta \qquad i = \sqrt{-1} \qquad \dots (45)$$

is a root with θ and δ real and $\delta \neq 0$ then also

$$\lambda_{\text{conj}} = \theta - i\delta \qquad \dots (46)$$

must be a solution, and if $\delta \neq 0$ these two roots are different.

We are only interested in *real* solutions of (38). The only two problems that remain in order to find the general solution of (38) in the most general case where the coefficient $a_n, a_{n-1}, \dots, a_0$ of (38) are *any* given real numbers (with $a_n = 1$), are therefore, first to see if and in what way the two particular solutions that correspond to a complex root of (43) can be *combined* so as to give a *real* component of the general solution (a component that would absorb two different roots of (35)), and second to discuss the case of multiple roots of the equation.

The first of those two problems is a simple one, but the second demands a little closer analysis. It will turn out that in the case of multiple roots also we can find the necessary number n of *linearly independent* particular solutions. Therefore the complete discussion of all the n roots of the algebraic equation (43) (each root counted a number of times equal to its multiplicity) will yield the general solution of (38) in the general case where the coefficients $a_0, a_1, \dots, a_n$ are *any* given real numbers.

The problem of actually finding all the roots of (43) is a simple one for small values of n, say for $n = 2$. In this case it can even be done explicitly in terms of the coefficients. But for high values of n the problem cannot be handled except in the case where the coefficients are *numerically* given and effective numerical methods of computation are used.

If the roots are close together, the numerical task is always a difficult one. And if the roots are multiple a special numerical procedure must be applied. In the sequel I shall make a few remarks on the numerical aspect of finding the roots of the characteristic equation, but a full discussion of this numerical problem is beyond the scope of this paper.

For numerical reasons, both theoretical and practical, it is most convenient to concentrate first on the *real* roots. For each root λ_p which we succeed in determining, we can—by dividing the characteristic polynomial by $(\lambda - \lambda_p)$—*lower* the degree of the characteristic polynomial and thus reduce the difficulty of the search for further root.

If the problem is not so simple that the roots can be expressed explicitly in terms of the coefficients, then the problem must be approached numerically. The first thing is to draw a *graph* of $f(\lambda)$ as a function of λ. A study of this curve, including a study of the behaviour of the curve as λ tends towards $+\infty$ or $-\infty$, will indicate whether real roots are present, and if so, indicate approximately *where* they are. In other words we know already approximate values for the real roots. These values can subsequently be improved upon by known numerical methods. As a rule it is advantageous first to determine the root where the graph indicates that the curve cuts through the λ-axis in the *steepest* way. Here a numerical approximation method will work at its best. When this root, say λ_p, is determined with a high degree of accuracy, the polynomial $f(\lambda)$ is divided by $(\lambda - \lambda_p)$ and the work is continued on the resulting polynomial.

Already when considering the real roots we must be prepared for the possibility that any of the roots may be of higher multiplicity than one.

Since the polynomial $f(\lambda)$ is always continuous and with a continuous derivative, a graphical consideration is sufficient to indicate the existence of a root of multiplicity 2. The case of two roots coinciding means that the polynomial $f(\lambda)$ does not *cut through* the λ axis but only *touches* the λ axis in the point considered. This means that such a point λ_p must not only be a solution of $f(\lambda) = 0$, i.e. of (43) but must also be a solution of $f'(\lambda) = 0$, where $f'(\lambda) = df/d\lambda$ is the derivative of $f(\lambda)$ with respect to λ. This condition is not only necessary but also sufficient for the point λ_p to be of multiplicity 2, or higher. In general the necessary and sufficient condition for a point λ_p to be a root of multiplicity μ_p is that it satisfies at the same time all the μ_p equations

$$\begin{aligned} f(\lambda) &= 0 \\ f'(\lambda) &= 0 \\ f''(\lambda) &= 0 \\ &\cdots \\ f^{(\mu_p - 1)}(\lambda) &= 0. \end{aligned} \qquad \ldots \quad (47)$$

In this case the polynomial $f(\lambda)$ is said to have contact of the order μ_p with the λ axis on the point considered.

The curve that depicts the shape of $f(\lambda)$ will suggest if and where a multiple root is located. For instance, if it looks as if the curve has a minimum or a maximum on the λ-axis, this suggests the existence of a double root or more generally the existence of a root of even multiplicity i.e. $\mu_p = 2$ or 4 or 6. etc. If the curve has an inflexional point with the λ-axis itself as the inflexion tangent, this suggests the existence of a root of odd multiplicity $\mu_p = 3$ or 5 or 7 etc.

The equations (47) offer a grip on the problem of determining numerically and in an exact way the multiplicity of a root. We can analytically derive the successive polynomials $f'(\lambda), f''(\lambda)$... etc. and then for each of these polynomials draw a curve that depicts its course in the vicinity of the point considered. The first of them that has an ordinate different from zero in the point considered indicates the multiplicity of the root in question.

If the shapes of the curves are such that the situation in the vicinity of the point considered is not discernible in a sufficiently convincing way, one may have to draw successive sets of new curves on larger and larger scales but over a more and more restricted λ-range. This graphical procedure combined with one of the several available methods of numerical determination of the roots of non-linear equations will in practically all cases permit to determine all the real roots with any desired accuracy, as well as, the exact multiplicity of each root.

Any such real root, say λ_p, will yield a particular solution to (38) of the form (41). And if we have found such a particular solution, it is obvious that the function obtained by multiplying (41) by an arbitrary constant will also be a solution.

The case where simply a constant (independent of t) is a solution of (38) is only a special case of what has been said above. It means that the characteristic equation has the (simple or multiple) root $\lambda = 1$. Necessary and sufficient for this case is obviously that the sum of coefficients $a_0, a_1, \ldots, a_n$ is zero.

We know that if a certain number of roots $\lambda_p, \lambda_q, \ldots, \lambda_r$ are *different*, then the corresponding time-functions of the form (41) are not only solutions of (38) but they are also linearly independent. We have thus advanced at least part of the way towards a complete set of n linearly independent particular solutions. And if we actually have found n *different* roots of (38) the job is completed. It suffices then to form a linear aggregate with arbitrary constant coefficients of the n functions of the form (41). But if one or more of the roots are multiple we cannot complete the job merely by functions of the form (41). Indeed the number of roots of (42), when each root is counted a number of times according to its multiplicity, is exactly n. Hence we cannot get a sufficient number of linearly independent solutions merely by considering functions of the form (41). Therefore any multiple root raises a new problem of how to construct more linearly independent solutions.

To complete our discussion of real roots let λ_p be a real root of multiplicity μ_p where $\mu_p > 1$ (obviously we must have $\mu_p \leqslant n$). In this case any time-function of the form

$$Y_{tp\mu} = t^{\mu}\lambda_p^t \qquad \ldots \quad (48)$$

(μ equal to any of the numbers $0, 1, \ldots, \mu_p - 1$), must also satisfy the difference equation (38).

Indeed, by inserting (48) into (38) the left member becomes

$$\sum_{\nu=0}^{n} a_\nu (t+\nu)^{\mu} \lambda_p^{t+\nu} = \lambda_p^t \left[\sum_{\nu=0}^{n} a_\nu (t+\nu)^{\mu} \lambda_p^{\nu} \right]. \qquad \ldots \quad (49)$$

In the bracket to the right we develop $(t+\nu)^{\mu}$ by the binomial formula. This brings (49) into the form

$$\lambda_p^t \sum_{\nu=0}^{n} \sum_{i=0}^{\mu} a_\nu \binom{\mu}{i} \nu^i t^{\mu-i} \lambda_p^{\nu}. \qquad \ldots \quad (50)$$

The power z^i of any number z can be developed in terms of the factorials $z^{[j]}$ by the formula

$$z^i = \sum_{j=0}^{i} \binom{i}{j} B_{i-j}^{(-j)} z^{[j]} \qquad \dots \quad (51)$$

where the factorial $z^{[j]}$ is defined by

$$z^{[j]} = \begin{cases} 1 & \text{if } j = 0 \\ z(z-1) \dots (z-j+1) & \text{if } j \geqslant 1. \end{cases} \qquad \dots \quad (52)$$

and $B_{i-j}^{(-j)}$ are the Bernoullian numbers. They are pure numerical coefficients.[10]

Putting $z = \nu$ the expression (50) can therefore also be written

$$\lambda_p^{t-n} \sum_{\nu=0}^{n} \sum_{i=0}^{\mu} \sum_{j=0}^{i} a_\nu \binom{\mu}{i} \binom{i}{j} B_{i-j}^{(-j)} \nu^{[j]} t^{\mu-i} \lambda_p^\nu. \qquad \dots \quad (53)$$

In this expression the limits 0 and n of the first sum are *fixed* numbers. We can therefore move this summation sign to the right so that it becomes the last of the three summation operations to be performed. At the same time we replace λ_p^ν by $\lambda_p^{\nu-j}$. λ^j. This being done the expression (53) becomes

$$\lambda_p^{t-n} \sum_{i=0}^{\mu} \sum_{j=0}^{i} \binom{\mu}{i} \binom{i}{j} B_{i-j}^{(-j)} t^{\mu-i+j} \left[\sum_{\nu=0}^{n} a_\nu \, \nu^{[j]} \lambda_p^{\nu-j} \right]. \qquad \dots \quad (54)$$

The bracket to the right in this expression is the value in the point $\lambda = \lambda_p$ of the j-th derivative $f^{(j)}(\lambda)$ of the characteristic polynomial. Since we have assumed that μ is *less* than the multiplicity μ_p, and by the first two summation signs in (54) $i \leqslant \mu$ and $j \leqslant i$, and consequently $j \leqslant \mu$ we must have j less than the multiplicity μ_p. Consequently by (47) all the brackets in (54) must vanish. Hence the expression (54) must vanish for any value of t. Consequently, the left member of (49)—which is the same as (54)—must vanish identically in t. That is to say the function (48) must satisfy the difference equation (38).

This shows that if λ_p is a root of multiplicity μ_p of the characteristic equation (43), all the μ_p time-functions

$$\lambda_p^t, \; t\lambda_p^t, \; t^2\lambda_p^t, \; \dots \; t^{\mu_p - 1}\lambda_p^t \qquad \dots \quad (55)$$

must satisfy (38). *All the μ_p time-functions are linearly independent* (and, of course, also linearly independent of corresponding time-functions corresponding to any other root λ_q).

Therefore to each root λ_p in (38), whether simple or multiple, we have determined as many linearly independent particular solutions of (38) as is indicated by the multiplicity of the root λ_p.

[10] See for instance formula (9b) and table 1, p. 12, in my memoire "Sur les semi-invariants et moments employes dans l'etude des distributions statistique." Det Norske Videnskap-Akademi i Oslo, 1926.

In the above argument we reasoned as if λ_p should be a *real* root. But the whole argument *is equally valid in the case of a complex root.* Therefore whatever the nature of the characteristic polynomial we have actually determined exactly n linearly independent solutions of the difference equation (38). Hence we have obtained the *general* solution of this equation.

This general solution will be of the form (39), i.e. it will contain n arbitrary constants.

Furthermore, these constants appear in the formula in such a way that if we consider the equation (39) for n arbitrarily given points of time where the values of Y_t are *prescribed*, and if these points of time are such that the $n \times n$ matrix of values Y_{ti} in the right member of (39) is non-singular, we can determine—and in a unique way—the values which it is necessary and sufficient to attribute to the constants in order that the solution shall assume the prescribed n values of Y_t.

Therefore the only thing which now remains is to show how the complex solutions can be combined into pairs in such a way that we obtain *real* time-functions and we retain *the same number n* of arbitrary constants and these occur in such a manner that by a suitable choice of them we can make the solution to assume n prescribed values of Y_t.

To show how this combination of complex roots can be achieved let us consider a complex root (45). The fact that (45)—and hence (46)—satisfies the characteristic equation means that

$$\sum_{\nu=0}^{n} a_\nu \,(\theta+i\delta)^\nu = 0. \qquad \dots (56)$$

Developing the parenthesis in (56) by the binomial formula we get

$$\sum_{\nu=0}^{n} a_\nu \sum_{\mu=0}^{\nu} \binom{\nu}{\mu} \theta^{\nu-\mu} \,(i\delta)^\mu. \qquad \dots (57)$$

The summation over μ can be split up in the following four sums

$$\left[\sum_{\mu=0,4,8..} \binom{\nu}{\mu} \theta^{\nu-\mu}\,\delta^\mu - \sum_{\mu=2,6,10\dots} \binom{\nu}{\mu} \theta^{\nu-\mu}\,\delta^\mu \right]$$

$$-i \left[\sum_{\mu=1,5,9\dots} \binom{\nu}{\mu} \theta^{\nu-\mu}\,\delta^\mu - \sum_{\mu=3,7,11\dots} \binom{\nu}{\mu} \theta^{\nu-\mu}\,\delta^\mu \right] \qquad \dots (58)$$

where all the summations over μ are to be continued to the last not vanishing term.

The first bracket in (58) is the real part and the second bracket the imaginary part. Since these parts must be zero separately in order that the whole expression be zero, we get

$$\sum_{\nu=0}^{n} a_\nu \left[\sum_{\mu=0,4,8\dots} \binom{\nu}{\mu} \theta^{\nu-\mu}\,\delta^\mu - \sum_{\mu=2,6,10\dots} \binom{\nu}{\mu} \theta^{\nu-\mu}\delta^\mu \right] = 0 \qquad \dots (59)$$

$$\sum_{\nu=0}^{n} a_\nu \left[\sum_{\mu=1,5,9\dots} \binom{\nu}{\mu} \theta^{\nu-\mu}\,\delta^\mu - \sum_{\mu=3,7,11\dots} \binom{\nu}{\mu} \theta^{\nu-\mu}\,\delta^\mu \right] = 0. \qquad \dots (60)$$

These two equations contain only real numbers and are therefore amendable to numerical computation. The values of δ and θ that satisfy these two simultaneous equations define a complex root of the form (45).

It is easily seen that if the set of two numbers (δ, θ) satisfy (59)-(60), the conjugate set $(-\delta, \theta)$ must also satisfy. These two roots $(\theta+i\delta)$ and its conjugate $(\theta-i\delta)$ are precisely the roots we want to combine.

The numerical solution of the system consisting of the two simultaneous equations (59)-(60) in the two real variables δ and θ will, of course, as a rule be more laborious than the solution of the single equation (43) in one real variable λ. But I shall not go into details about this. I shall only show in a subsequent note that the numerical work can also, if we like, be performed in terms of real trigonometric functions.

In order to show that a complex root of the characteristic equation (43) and its conjugate root will together yield two linearly independent *real* time-functions satisfying (38), we make the transformation[11]

$$\lambda = e^{\gamma} \qquad \ldots \quad (61)$$

$e = 2.71828\ldots$ (basis of natural logarithms)

where[12] $$\gamma = \beta+i\alpha \quad i = \sqrt{-1}, \beta \text{ and } \alpha \text{ real numbers} \qquad \ldots \quad (62)$$

and the complex form of λ is given by (45). Then by classical formulae

$$\lambda = \theta+i\delta = r(\cos\alpha+i\sin\alpha) \qquad \ldots \quad (63)$$

where $$r = \sqrt{\theta^2+\delta^2} \quad \theta = r\cos\alpha; \quad \delta = r\sin\alpha. \qquad \ldots \quad (64)$$

All the magnitudes in (64) are real, and $r = 0$. Obviously it does not restrict generality if we assume

$$0 \leqslant \alpha < 2\pi. \qquad \ldots \quad (65)$$

Raising the *left* member of (61) to the power t is the same as to raise the last member of (63) to the power t. By classical formulae this gives

$$\lambda^t = r^t\,(\cos\alpha t+i\sin\alpha t). \qquad \ldots \quad (66)$$

On the other hand by raising directly the *right* member of (61) to the power t we get

$$e^{\gamma t} = e^{(\beta+i\alpha)t} \qquad \ldots \quad (67)$$

which by classical formulae reduces to

$$e^{\gamma t} = e^{\beta t(\cos\alpha t+i\sin\alpha t)}. \qquad \ldots \quad (68)$$

Comparing (68) with (66) we see that

$$r = e^{\beta}. \qquad \ldots \quad (69)$$

Therefore (67) can also be written

$$e^{\gamma t} = r^t e^{i\alpha t}. \qquad \ldots \quad (70)$$

---

[11] When handling differential equations as distinct from difference equations we are led immediately to a characteristic equation in λ not to one in γ and hence need not consider a subsequent transformation of the kind (61).

[12] The letters α and β in (62) should not be confused with the structural constants α and β in (1)-(7).

The conjugate root is obtained by replacing δ by $(-\delta)$, which by the first and last equation in (64) is the same as to replace α by $(-\alpha)$ but retaining r. In other words the two time-functions

$$r^t e^{i\alpha t} \quad \text{and} \quad r^t e^{-i\alpha t} \qquad \dots \ (71)$$

are both solutions of (38). They are linearly independent when $\alpha \neq 0$, which simply means that the root in question is actually complex.

To combine the two time-functions in (71) in such a way as to obtain two *real* and linearly independent time-functions we multiply the first of them by an arbitrary complex constant and the second of them by a complex constant that is the conjugate of the first.

Let the two conjugate constants be

$$(A-iB) \quad \text{and} \quad (A+iB) \quad i = \sqrt{-1} \qquad \dots \ (72)$$

where A and B are arbitrary *real* constants. Using the classical formula

$$e^{iz} = \cos z + i \sin z \quad i = \sqrt{-1} \qquad \dots \ (73)$$

we get $$(A-iB)r^t e^{i\alpha t} + (A+iB)r^t e^{-i\alpha t} = 2Ar^t \cos \alpha t + 2Br^t \sin \alpha t. \qquad \dots \ (74)$$

The two time-functions written to the right in (74) are real and linearly independent. The aggregate satisfies the difference equation for any values we may choose for A and B. Choosing first $A = \frac{1}{2}$, $B = 0$ and next $A = 0$, $B = \frac{1}{2}$ we see that each of the two functions

$$r^t \cos \alpha t \quad \text{and} \quad r^t \sin \alpha t \qquad \dots \ (75)$$

is a solution of (38). They are linearly independent if $r \neq 0$ and $\alpha \neq 0$. These two conditions are obviously fulfilled if we actually have a complex root.

If we like we can combine the two time-functions (75)—each taken with an arbitrary real coefficient—in such a way that the aggregate appears in the form of *a wave component* satisfying (38). The wave will be damped, explosive or non-changing according to the value of r in the following way

$$r < 1 \text{ damped} \quad r > 1 \text{ explosive}, \quad r = 1 \text{ non-changing}. \qquad \dots \ (76)$$

We have indeed[13] $$Ar^t \cos \alpha t + Br^t \sin \alpha t = Kr^t \sin(v + \alpha t) \qquad \dots \ (77)$$

where $$K = |\sqrt{A^2+B^2}| \ ; \ A = K \sin v \ ; \ B = K \cos v. \qquad \dots \ (78)$$

Obviously it does not restrict generality if we assume

$$0 \leqslant v \leqslant 2\pi \qquad \dots \ (79)$$

K is the *scale factor* and v *the phase* of the wave component. Between the two sets of constants (A, B) in the left member of (77) and (K, v) in the right member there is a one-to-one correspondence. Any given set (A, B) will by (78)-(79) lead to a uniquely determined set (K, v). And inversly any given set (K, v) will by the last two equations in (78) lead to a uniquely determined set (A, B). Thus, whether we look at the left or the right expression in (77) we have two arbitrary constants at disposal.

[13] In (77) A and B are used in the sense of one half the values A and B used in (73). This, of course is only a typographical convenience.

/
74

For computational reasons it is as a rule simplest to use the left hand expression in (77). Here the nature of the aggregate as the sum of two linearly independent time-functions is also brought clear to light. On the other hand if we want to plot a graph of the aggregate as a time-function and study its nature as a *wave component* satisfying (38), the right hand expression in (77) is useful.

The treatment of *multiple* complex roots of (43) can be handled in a way which is quite analogous to the way in which we handled multiple *real* roots in (47)-(55).

Note on the solution of the characteristic equation by means of trigonometric functions. Inserting (61)-(63) into the left member of the characteristic equation (43) we get by (73)

$$\sum_{\nu=0}^{n} a_\nu \lambda^\nu = \sum_{\nu=0}^{n} a_\nu r^\nu (\cos \alpha\nu + i \sin \alpha\nu) \tag{80}$$

separating here the real and imaginary parts we see that the single equation (43) in the complex variable λ is equivalent with the following two equations in the real magnitudes r and α

$$\sum_{\nu=0}^{n} a_\nu r^\nu \sin \alpha\nu = 0 \quad \text{and} \quad \sum_{\nu=0}^{n} a_\nu r^\nu \cos \alpha\nu = 0 \qquad \dots \tag{81}$$

where it does not restrict generality if we impose (65).

If we want to handle the two simultaneous characteristic equations numerically in the trigonometric form (81) the following formula may be found useful. Putting for brevity

$$s = \sin \alpha \qquad c = \cos \alpha \qquad \dots \tag{82}$$

we have

$$\begin{aligned} \sin 2\alpha &= 2cs & \cos 2\alpha &= 1-2s^2 \\ \sin 3\alpha &= s(3-4s^2) & \cos 3\alpha &= c(1-4s^2) \\ \sin 4\alpha &= cs(4-8s^2) & \cos 4\alpha &= 1-8s^2+8s^4. \end{aligned} \qquad \dots \tag{83}$$

These formulae can easily be extended to subsequent values of ν by the recurrence formula

$$\sin(\nu+1)\alpha = c \,.\, \sin \nu\alpha + s.\cos \nu\alpha \quad \text{and} \quad \cos(\nu+1)\alpha = c.\cos \nu\alpha - s.\sin \nu\alpha \quad \dots \tag{84}$$

The general solution of the complete (non-truncated equation). Let Y_t^P and Y_t^Q be any two particular solutions of the complete (non-homogeneous) difference equation (36). I have used capital letters for superscripts to indicate that the two time-functions are solutions of the complete equation.

If Y_t^P and Y_t^Q are any solution of the complete equation, then their difference $(Y_t^P - Y_t^Q)$ must be a solution of the corresponding *truncated* (homogeneous) equation. This is immediately seen by inserting first Y_t^P and next Y_t^Q in the complete equation and taking the difference between the two equations thus obtained.

This being so, let Y_t^Q stand for any particular solution (of the complete equation) which we *know* in its whole course over t, and let Y^P stand for a particular solution (of the complete equation) which is so far not known in its whole course over t but is only fixed through its specialized initial conditions. Then there must, by the argument in the beginning of this section, exist *some specific* solution y_t^p of the *truncated* equation which represents the difference $(Y_t^P - Y_t^Q)$. In other words, for any t we must have

$$Y_t^P = y_t^p + Y_t^Q \qquad \dots \quad (85)$$

where y_t^p is a particular solution of the truncated equation, its particularization depending on what particular solutions Y_t^P and Y_t^Q we are considering.

Inversely if in the right member of (85) we insert any particular solution y_t^p (of the truncated equation) which we may happen to think of, we will through (85) have generated some particular solution Y_t^P of the complete equation. Furthermore, if we let y_t^p vary in all possible ways, we will by (85) generate a whole lot of different time-functions y_t^p, all of which are solutions of the complete equation. In fact we will generate *all possible* solutions of the complete equation. This follows from the fact that "all possible" solutions y_t^p in the right member of (85) means a function with n arbitrary constants (they occur as we know linearly). And n arbitrary constants in the right member of (85) means that the function Y_t^P defined by (85) will contain n arbitrary constants (appearing linearly).

This shows that if we insert in (85) for y_t^p the *general* solution $Y_t^{\text{gen.trun}}$ of the truncated equation (this general solution will contain n arbitrary constants) we will have generated the *general* solution of the complete (non-homogeneous) equation. In other words

$$Y_t^{\text{gen.comp}} = Y_t^{\text{gen.trun}} + Y_t^Q \qquad \dots \quad (86)$$

where $Y_t^{\text{gen.comp}}$ denotes the general solution of the complete (non-homogeneous) difference equation, and Y_t^Q denotes *any* particular solution of the complete equation which we have been able to obtain in some way or the other.

The formula (86) may give rise to many *different forms* for presenting the general solution of the complete equation. If we have found one particular solution Y_t^Q of the complete equation we can add to it any particular solution Y_t^q of the truncated equation. The result $Y_t^R = Y_t^Q + Y_t^q$ will also be a particular solution of the complete equation and can be used as the second term in the right member of (86).

The problem of finding the general solution of the complete equation is by (86) reduced to that of first finding the *general* solution of the *truncated* equation—a problem which was entirely solved in previous sections—and second finding *any* particular solution of the complete equation.

Sometimes we may be able to find a particular solution of the complete equation by a happy guess. But in general we cannot, of course, rely on this. We must therefore have a systematic way of finding one such particular solution.

The determination of a particular solution of the complete equation. The following is an elementary but rigorous procedure for obtaining a particular solution of the complete equation. The procedure is built on the knowledge of the *general* solution of the truncated equation.

Let the general solution of the truncated equation be known and let η_t be the particular solution of the truncated equation which is obtained by specifying the following initial conditions

$$\begin{aligned} a_n\eta_0 &= 1 \\ a_n\eta_1+a_{n-1}\eta_0 &= 0 \\ a_n\eta_2+a_{n-1}\eta_1+a_{n-2}\eta_0 &= 0 \\ \cdots \quad \cdots \quad \cdots \\ a_n\eta_{n-1}+a_{n-1}\eta_{n-2}\cdots+a_1\eta_0 &= 0\,. \end{aligned} \qquad \ldots \quad (87)$$

The arguments 0, 1, 2, ... etc. on η in (87) are not obtained by a conventional choice of origin for t in η_t, but have the *absolute* meaning of 0, 1, 2, ... etc. They represent number of intervals of the same sort as occur in the difference equation itself.

More briefly (87) can be written

$$a_n\eta_0 = 1$$

$$\sum_{\nu=\tau}^{n} a_\nu\eta_{\nu-\tau} = 0, \text{ for } \quad \tau = 1, 2, \ldots (n-1). \qquad \ldots \quad (88)$$

This particular solution of the truncated equation will be called the cumulator.

Since by (37) $a_n \neq 0$, the first equation above determines η_0, similarly η_1 is determined by the second equation (if a_{n-1} should happen to be zero we get $\eta_1 = 0$). Similarly η_2 is determined by the third equation etc., and finally η_{n-1} by the n-th equation. These initial conditions $\eta_0, \eta_1, \ldots, \eta_{n-1}$ determine the particular solution in a unique way. The particular solution η_t obtained in this way will by definition satisfy the difference equation

$$\sum_{\nu=0}^{n} a_\nu\eta_{t+\nu} = 0 \qquad \text{for any value of } t. \qquad \ldots \quad (89)$$

Consider the time-function

$$Y_t^Q = \sum_{\tau=0}^{t-s} \eta_t H_{t-\tau} \; (t = s, s+1, \ldots, \infty) \qquad \ldots \quad (90)$$

where s is the earliest point of time such that H_T is known for $T = s, s+1, \ldots, \infty$. This means that $t = s$ is the earliest point of time for which the expression (90) is applicable. For precision of thinking it is essential to specify the point of time s. Since we may, of course, if we like *disregard* our knowledge of H_t over certain time ranges, the parameter s in (90) may be interpreted as *any* point of time which is such that in this point of time and in all *subsequent* points of time H_t is known. In this sense the upper limit of summation in (90) contains an *arbitrary* element.

The time-function (90) will satisfy the difference equation (36) for any point of time $t = s, s+1, s+2 \ldots$ ad. infin.

Indeed, if we insert (90) into the left member of (36) we get

$$\sum_{\nu=0}^{n} a_\nu\, Y_{t+\nu}^Q = \sum_{\nu=0}^{n} a_\nu \sum_{\tau=0}^{t+\nu-s} \eta_t H_{t+\nu-\tau} = \sum_{\nu=0}^{n} \sum_{\tau=-\nu}^{t-s} a_\nu\eta_{\tau+\nu}H_{t-\tau} \quad (t = s, s+1, \ldots, \infty) \ldots \quad (91)$$

In the expression to the right in (91) we interchange the order of the two summation operations. Since

$$\sum_{\nu=0}^{n} \sum_{\tau=-\nu}^{t-s} = \sum_{\tau=0}^{t-s} \sum_{\nu=0}^{n} + \sum_{\tau=-1}^{-n} \sum_{\nu=-\tau}^{n} \quad \dots \quad (92)$$

the expression to the right in (91) becomes

$$\sum_{\tau=0}^{t-s} H_{t-\tau} \left[\sum_{\nu=0}^{n} a_\nu \eta_{\tau+\nu} \right] + \sum_{\tau=-1}^{-n} H_{t-\tau} \left[\sum_{\nu=-\tau}^{n} a_\nu \eta_{\tau+\nu} \right]. \quad \dots \quad (93)$$

The bracket in the first term of (93) is by (89) zero for any value of τ. Therefore the first term of (93) vanishes.

If we introduce $(-\tau)$ instead of τ as a summation affix in the last term of (93), this last term becomes

$$\sum_{\tau=1}^{n} H_{t+\tau} \left[\sum_{\nu=\tau}^{n} a_\nu \eta_{\nu-\tau} \right]. \quad \dots \quad (94)$$

For $\tau = n$ the bracket in (94) becomes $\sum_{\nu=n}^{n} a_\nu \eta_{\nu-n} = a_n \eta_0$ which by the first equation in (88) is equal to 1. For $\tau = n$ (94) will therefore give H_{t+n}. For $\tau = 1, 2, \dots (n-1)$ the bracket in (94) will vanish by the last equation in (88).

All that is left of (93)—which is the same as the expression in the left member of (91)—is consequently H_{t+n}. In other words the time-function $Y^{0}_{t+\nu}$ satisfies the complete difference equation (36).

The *general* solution $Y_t^{\text{gen. comp}}$ of the *complete* difference equation is therefore by (68) and (90)

$$Y_t^{\text{gen. comp}} = Y_t^{\text{gen. trun}} + \sum_{\tau=0}^{t-s} \eta_\tau H_{t-\tau} \qquad (t = s, s+1, \dots, \infty) \quad \dots \quad (95)$$

$Y_t^{\text{gen. trun}}$ in (95) denoting the *general* solution of the truncated (homogenous) difference equation (38),—a solution that contains n arbitrary constants, which may, if we like, be chosen as real constants—, and η_τ in (95) denoting the *particular* solution of the truncated (homogeneous) difference equation (38) which satisfies the initial conditions (87).

Since the general solution of the truncated (homogeneous) difference equation (38) is given by the reasoning in (41)-(47), the right hand member of (95) gives the explicit expression for the general solution of the complete (non-homogeneous) difference equation (36). This general solution is applicable for $t = s, s+1 \dots, \infty$ in the sense that (36) for $t = s, s+1 \dots, \infty$ will be satisfied if we insert (95). This limitation on the applicability of (95) will, of course, not prevent us from assuming that the difference equation (36) *itself* is valid even for earlier points of time, but then the difference equation can only be used *by recurrence* for these earlier points.

Application to the Hicksian difference equation. In order to apply (95) to the equation (5) we replace in (5) t by $(t+2)$ and write the equation in the following form which is immediately comparable with (5)

$$-\nu Y_t - \mu Y_{t+1} + Y_{t+2} = H_{t+2}. \qquad \dots \quad (96)$$

That is we have $$a_0 = -\nu, \quad a_1 = -\mu, \quad a_2 = 1. \qquad \dots \quad (97)$$

The special case $\nu = 0$ means a first order equation. ... (98)

The special case $\mu = 0$ ... (99)

can be reduced to a first order equation by doubling of the interval considered. These first order cases will be discussed later. We begin by assuming

Both conditions $\nu \neq 0$ *and* $\mu \neq 0$ *fulfilled so that*
(96) *is a true second order difference equation.* ... (100)

In the case (100) we must distinghish between the three subcases,

I.	*two real and different roots of the characteristic equation;*	
II.	*two real and equal roots;*	... (101)
III.	*two complex (and then necessarily conjugate) roots: the wave-case;*	

We will discuss these cases one by one.

The roots of the characteristic equation corresponding to the truncated Hicksian difference equation. The characteristic equation is

$$\lambda^2 - \mu\lambda = \nu \qquad \dots \quad (102)$$

the roots of which are $$\lambda_1 = \frac{\mu}{2}(1+\sqrt{1+4\omega}) \qquad \dots \quad (103)$$

$$\lambda_2 = \frac{\mu}{2}(1-\sqrt{1+4\omega}). \qquad \dots \quad (104)$$

By convention we interpret the square root sign in (103)-(104) as the positive branch, i.e.

$$\sqrt{1+4\omega} = \begin{cases} \text{absolute value of } \sqrt{1+4\omega} & \text{if } 1+4\omega \geqslant 0 \\ \sqrt{-1} \text{ times absolute value of } \sqrt{(1+4\omega)} & \text{if } 1+4\omega < 0. \end{cases} \qquad \dots \quad (105)$$

The three cases (101) will be characterized by

I.	if $1+4\omega > 0$	(Two real and different roots)	
II.	if $1+4\omega = 0$	(Two real and equal roots)	... (106)
III.	if $1+4\omega < 0$	(Two complex roots).	

In the case I: Two real and different roots we have by the convention (105)

absolute value of $\lambda_1 >$ absolute value of λ_2. ... (107)

In terms of the original parameters α, β, κ, λ of the model (1)-(4)[14] the nature of the characteristic equation may be characterised by the set of two parameters (μ, ω) where μ is defined by (6) and ω by

$$\omega = \frac{\lambda-\beta}{(\alpha+\beta+\kappa-\lambda)^2}. \qquad \ldots \quad (108)$$

Or it may be characterised by the set of two coefficients (ν, π) where ν is defined by (7) and π by

$$\pi = \kappa+\alpha.$$

Between the two sets (μ, ω) and (ν, π) there is a one-to-one correspondence, because we have

$$\mu = \pi-\nu. \qquad \ldots \quad (110)$$

$$\omega = \frac{\nu}{(\pi-\nu)^2}. \qquad \ldots \quad (111)$$

and

$$\nu = \omega\mu^2 \qquad \ldots \quad (112)$$

$$\pi = \mu(1+\omega\mu). \qquad \ldots \quad (113)$$

From the viewpoint of the *government strategy* in matters of current account expenditure it might be most convenient to consider the set (ν, π) since by (7) ν differs from the strategy parameter λ only by the addition of a *structurally* defined constant, namely $(-\beta)$, and by (109) π differs from the strategy parameter κ only by the addition of a *structurally* defined constant, namely α : But from the viewpoint of the explicit expression (35), obtained by recurrence, and from the viewpoint of the expression (103)-(104) for the characteristic roots the set (μ, ω) is unquestionably the most convenient.

The condition for the non-wave cases I and II which should be aimed at when determining government strategy — is simply

$$1+4\omega \geqslant 0. \qquad \ldots \quad (114)$$

This can be looked upon as a (μ, ω) formulation of the non-wave condition. The corresponding (ν, π) formulation (in the cases of a true second order difference equation) is

$$(\pi-\nu)^2+4\nu \geqslant 0. \qquad \ldots \quad (115)$$

In the special case where the difference equation degenerates to a first order difference equation — for instance because government strategy has the features $\lambda = \beta$ — the question of a wave formed evolution over time will, of course, not arise.

[14]The parameter λ occurring in the model must, of course, not be confounded with the unknown λ occurring in the characteristic equation (102). In the sequel no confusion is possible since we use the notation λ_1 and λ_2 for the two roots of the characteristic equation. We might perhaps have used a different letter for the parameter λ occuring in (5). The structural constants α and β in (1)-(7) should not be confused with the letters α and β in (62).

In the study of the true second order case of (96) the following formulae, which easily follow from (103)-(104) may be useful

$$2r^2 = \mu^2(1+2\omega) \qquad \dots (116)$$

$$\lambda_1+\lambda_2 = \mu \qquad \dots (117)$$

$$\lambda_1\lambda_2 = -\nu \qquad \dots (118)$$

$$\lambda_1-\lambda_2 = \mu\sqrt{1+4\omega} \qquad \dots (119)$$

$$\frac{\lambda_1^2-\lambda_2^2}{\lambda_1-\lambda_2} = \mu \qquad \dots (120)$$

$$\frac{\lambda_1^3-\lambda_2^3}{\lambda_1-\lambda_2} = \lambda_1^2+\lambda_1\lambda_2+\lambda_2^2 = \mu^2+\nu = \mu^2(1+\omega). \qquad \dots (121)$$

In the sequel we will simply disregard the wave-case, i.e., case III in (101) and (106) because this case *can be prevented so easily*, simply by imposing on the government strategy parameters κ, λ the condition (114), which is equivalent to (115). But then, of course, this condition must not be forgotten in any possible programming formulation.

Case I : *The explicit expression for the general solution of the complete Hicksian difference equation in terms of the characteristic roots when the roots are real and different.*

The cumulator η_t—cfr. (87), (88) and (89)—is in the present case

$$\eta_t = c_1\lambda_1^t+c_2\lambda_2^t \qquad \dots (122)$$

where the constant c_1 and c_2 ought to be determined so that the cumulator satisfies the intial conditions

$$a_2\eta_0 = 1$$

$$a_2\eta_1+a_1\eta_0 = 0 \qquad \dots (123)$$

where now $\quad a_2 = 1 \quad a_1 = -\mu \qquad \dots (124)$

which gives $\quad \eta_0 = 1 \quad \eta_1 = \mu. \qquad \dots (125)$

This gives, in the case $\lambda_1 \neq \lambda_2$

$$c_1 = \frac{\mu-\lambda_2}{\lambda_1-\lambda_2} \quad c_2 = \frac{\lambda_1-\mu}{\lambda_1-\lambda_2}. \qquad \dots (126)$$

Inserting this into (122) we get after some reductions

$$\eta_t = \frac{\lambda_1^{t+1}-\lambda_2^{t+1}}{\lambda_1-\lambda_2} \quad (t = \text{any point of time}). \qquad \dots (127)$$

The general solution of the complete difference equation (96) will therefore, (cf. (95)) be of the form

$$Y_t = C_1\lambda_1^t+C_2\lambda_2^t+\sum_{\tau=0}^{t-s}\eta_\tau H_{t-\tau} \quad (t = s, s+1 \dots \infty) \qquad \dots (128)$$

s being a point of time such that H_t is known for $t = s$ and for all *subsequent* points of the time, and C_1 and C_2 being two arbitrary constants.

If we take—as we did in (35) —Y_0 and Y_{-1} as expressions for the initial conditions, and assume that H_t is known for $t = s, s+1 \ldots, \infty$ the procedure for expressing the coefficients C_1 and C_2 in terms of Y_0 and Y_{-1} will be, first to compute Y_1 and Y_2 by recurrence in terms of Y_0 and Y_{-1}, H_1 and H_2 (the result is given by the first two expressions in (8)) and then to put down the two equations which state that these expressions for Y_1 and Y_2 ought to be the same as the right member of (128).

Doing this, we get after some reduction, using (117)-(119) the following expression for the general solution of the complete equation (96)

$$Y_t = \eta_t Y_0 + \nu\eta_{t-1} Y_{-1} + \sum_{\tau=0}^{t-s} \eta_\tau H_{t-\tau} \qquad \ldots \quad (129)$$

$$(t = s, s+1 \ldots, \infty); \quad (s = 1, 2, \ldots, \infty)$$

By following the above argument literally we would have to put $s = 1$ in (129), but it is easily seen that (129) applies[15] for any $1 \leqslant s \leqslant t$. The parameter s is precisely the *arbitrary* element in the summation limit which we considered in connection with (90). By inserting (129) into the difference equation we see in fact that all values of H_T will disappear for $T \geqslant t$ where t is the earliest point of time that occurs for H_T when (129) is inserted in the difference equation. In other words if we only think of the use of (129) *in the difference equation*, we may attribute to H_T for $T < t$ any values we like (but it must be the *same* values of H_T for *all* the t in (129) that are used in the difference equation). On the other hand if we want to take (129) as an *independent definition* of a function Y_t regardless of the use we may want to make of this function, we must assume that H_T is known for $T \geqslant s$.

The formula (129) shows the basic role played by the cumulator η_t, defined as the particular solution of the truncated equation which satisfies the initial condition (123), or in general (87). *The whole problem is really solved as soon as the cumulator is obtained and the H_T are known for all the T we need.*

In (35) we assume that H_1 was the earliest known value of H_T. The same assumption may also be made in (129), i.e. we may put

$$s = 1. \qquad \ldots \quad (130)$$

This will reduce (129) to (35) identically in t and identically in the parameters μ and ω.

To permit an easier comparison with (35) we can transform the expression for the cumulator as follows, (cf. (119))

$$\eta_t = \frac{\lambda_1^{t+1} - \lambda_2^{t+1}}{\lambda_1 - \lambda_2} = \frac{1}{\varepsilon\mu}\left(\frac{\mu}{2}\right)^{t+1}[(1+\varepsilon)^{t+1} - (1-\varepsilon)^{t+1}] \qquad \ldots \quad (131)$$

where we have put for brevity

$$\varepsilon = \sqrt{1+4\omega}.$$

---

15 Since (129) only applies for $t=s$ the value of y must be computed by recurrence from $t=-1$ to $t=s-1$.

Developing $(1+\varepsilon)$ and $(1-\varepsilon)$ by the binomial formula the bracket in (131) becomes

$$\sum_{i=0}^{t+1}\binom{t+1}{i}(\varepsilon^i-(-\varepsilon)^i)=\sum_{i=0}^{t+1}\binom{t+1}{i}\varepsilon^i(1-(-1)^i)$$

which reduces to $$\eta_t=\left(\frac{\mu}{2}\right)^t\sum_{i=1,3,5\ldots}\binom{t+1}{i}(1+4\omega)^{(i-1)/2}. \qquad \ldots (133)$$

This expression is identically equal to the coefficient of Y_0 in (35). For $t=3$ we get for instance by (133)

$$\left(\frac{\mu}{2}\right)^3\left[\binom{4}{1}+\binom{4}{3}(1+4\omega)\right]=\mu^3(1+2\omega) \qquad \ldots (134)$$

which checks with the coefficient of Y_0 on the third line in (8). Similarly the coefficient of Y_{-1} in (35) is what is obtained by replacing t by $(t-1)$ in (133) and multiplying by ν. (cf. the second term in the right member of (130)).

A warning note in connection with the general solution of the complete Hicksian difference equation. One sometimes encounters analytical work where the reasoning runs as if the characteristic roots were sufficient to describe the time shape of the solution of a complete, i.e. non-homogeneous, difference equation. This is not so. The characteristic roots described the time shape of the *cumulator*, not that of the time-function Y_t which is a solution of the complete. i.e. non-truncated, difference equation. The solution Y_t is indeed *not yet determined* if only the roots are known. The time shape of Y_t may be anything depending on the time shape of the additional term H_t in the difference equation (5). This is only another aspect of the fact that (5) is one equation in two variables Y and H, hence one degree of freedom.

Case II. *A real double root of the characteristic equation corresponding to the truncated Hicksian difference equation.* In this case the argument in (131) does not hold good because $(\lambda_1-\lambda_2)$ vanishes. But the expression (133) can be used also in this case because the denominator $(\lambda_1-\lambda_2)$ which is equal to $\mu\sqrt{1+4\omega}$, has been *removed* when we pass from (131) to (133). Compare also (149).

The case $\lambda_1=\lambda_2$ is obtained from (133) simply by putting $1+4\omega=0$, (cf. (103)-(104)). Only one term will then be left in (133) namely the term corresponding to $i=1$. This gives

$$\eta_t=(t+1)\left(\frac{\mu}{2}\right)^t \quad (\text{when } \lambda_1=\lambda_2). \qquad \ldots (135)$$

This expression inserted into (129) will give the general solution of the complete difference equation in the case of two equal roots (the roots are necessarily real when they are equal).

The same expression (135) for the cumulator is obtained by following the general rule (56). This leads to

$$\eta_t=c_1\left(\frac{\mu}{2}\right)^t+c_2t\left(\frac{\mu}{2}\right)^t. \qquad \ldots (136)$$

Determining here the constants c_1 and c_2 in such a way that (136) satisfies the initial conditions (125), we get back to (135).

Degeneration to a first order difference equation. The general solution in the case where the difference equation degenerates into a first order equation because of

$$\nu = 0 \qquad \dots \quad (137)$$

follows immediately from (35) by putting $\omega = 0$. This leaves in the first and third brackets of (35) only one term, namely, the one for $i = 0$, while the second bracket vanishes. This gives

$$Y_t = \mu^t Y_0 + \sum_{\tau=1}^{t} \mu^{t-\tau} H_\tau \qquad \dots \quad (138)$$

which can also be written

$$Y_t = \mu^t Y_0 + \sum_{\tau=0}^{t-1} \mu^\tau H_{t-\tau} \quad (t = 1, 2, \dots, \infty, \text{ when } \nu = 0). \qquad \dots \quad (139)$$

The latter expression corresponds to (129) for $\nu = 0$, $s = 1$ and $\eta_t = \mu^t$.

The above simple remarks which immediately lead to (135) and (139) show the usefulness of having different expressions for the solution. The whole situation is under more complete control when we approach the problem both by recurrence and by means of characteristic roots.

The programming approach

Smoothness and rapidity. The programming approach within a given time horizon $t = 1, 2, \dots, T$ may be formulated in different ways.

A common feature of all these formulations is that we want to produce a development of national income Y_t which is in some sense "smooth" and at the same time "rapid". In practice other desiderata may also be implied but for the time being let us assume that the "smoothness" and "rapidity" of the development Y_t are our desiderata.

The "smoothness" of Y_t will depend on two things : the nature of the cumulator η_t as defined by (131) and (125) and on the possible vicissitudes of H_t over time. The way in which "smoothness" depends on the nature of the cumulator can be disposed of simply by saying that the cumulator η_t should not be wave shaped[16]. This is obtained simply by imposing (114), which is equivalent to (115). This is only a single *bound* on the government strategy parameters. In some of the programming formulations the effect of vicissitudes in H_t may be taken account of in special ways exemplified in the sequel.

The "rapidity" of the development of a time-function offers no definitional difficulty in the case where the function is simply an exponential in t, because in this case its relative rate of growth is constant. But in our problem the relative rate will as a rule not be constant, (cf. (127) and 129)). We therefore face the question whether we want to attribute most importance to a rapid growth in the near future or most importance to the final achievements in the long run. This is a matter to

[16] Cf. the remarks after (148) as well as (155).

be decided by the policy maker (whether it is a democratic parliament or some dictator). It does not restrict generality if we let $t = 0$ denote the *decision time point* (the plan making point of time if a true planning procedure is envisaged, otherwise it would simply be a point of time when discussion takes place).

The policy maker's decision on the relative preference between the present and the future may be expressed by a preference function of the form

$$F = \sum_{t=1} P_t Y_t = \text{max.} \qquad \text{... (140)}$$

where the P_t are the policy maker's preference coefficients, and T a time horizon adopted for the repercussions to be taken account of. T is the *repercussion period.* To put $T = \infty$ is sheer abstractism based on the assumption of constant structural coefficients and strategy coefficients over an indefinite future. In practice a much shorter repercussion period will have to be considered.

A particularly simple form of (140) would be to put all P_t equal to zero except P_T. In this case the preference function is simply

$$F = P_T Y_T. \qquad \text{... (141)}$$

Another particularly simple form of (140) is

$$P_t = d^t \qquad \text{... (142)}$$

where d (generally positive and less than unity) is a discount factor. This discount factor would (in an economy which is not completely ruled by the time honoured unenlightened financialism) have nothing to do with any "market rate of interest" but would simply be an expression for the policy maker's desires.

More complicated forms of the preference function than the linear form (140) may be considered for special purposes.

So much for the desiderata to be achieved. Differences in the choice of the *means to be used* (and to be analyzed in terms of the model) will result in different programming formulations.

Formulation 1 : Let the initial conditions Y_0, Y_{-1} as well as the parameters μ and ω be *given* (the latter being given through known constant values of the structural parameter α, β and through such decided upon values of the strategy coefficients κ, λ as will satisfy (114)).

In such a situation at time 0 how can the autonomous investments H_1, H_2 ..., H_T be chosen so as to obtain a "smooth" and at the same time "rapid" development of Y_t within the horizon T ?

In this formulation the application of (35) or, if we like (129) must be forward looking, i.e. the point of time 0 must be today and $t = 1, 2, \ldots$ must be points of time in the future.

This formulation of the programming problem has an obvious interest in the case where government is at least to some degree able to *influence* the course of H

in the future. And even in the case where government has left autonomous investment largely in the hands of private business, it might be interesting for government to know what the nationally desirable evolution of autonomous investment is, and to measure how far the statistically observed action of private business is from achieving the nationally desirable optimum. So in any case the present formulation of the programming problem might be interesting.

If the preference function is of the linear form (140) we see, by inserting from (129), that F will also be linear in the T variables $H_1, H_2 \ldots H_T$. Hence so far as the preference function is concerned the programming problem is linear.

So far as the bounds are concerned we would in a realistic analysis have to impose the T conditions.

$$\underline{C}_t \leqslant C_t \quad (t = 1, 2 \ldots, T) \qquad \ldots \quad (143)$$

where $\underline{C}_t$ $(t = 1, 2 \ldots, T)$ are lower bounds for consumption.

These lower consumption bounds may be taken as decisionally ***given***, i.e. as magnitudes not to be determined through the programming analysis. If so, the policy maker would have to decide on how stringent these bounds should be (perhaps near to the physical minimum of existence in the beginning of the repercussion period). If the bounds $\underline{C}_t$ are given, and if we insert from (114) and make use of the known initial condition Y_0, the bounds (143) will also emerge as linear in the decisional variables $H_1, H_2 \ldots, H_T$.

If the lower bounds $\underline{C}_t$ are not taken as given in an absolute sense but determined as certain given *fractions* of Y_t, the bounds (143) would again become linear in terms of the decisional variables $H_1, H_2 \ldots, H_T$. The same is even true if each $\underline{C}_t$ is determined as a linear function of all the $H_1, H_2 \ldots, H_T$ with given constants.

In all these cases the bounds (143) will be linear.

But (143) are not the only bounds that must realistically be taken account of. One must *at least* consider the bounds springing from the need to consider *production capacities*, (cf. my introductory remarks on the realism of the model).

One simple way in which to take account of this type of bounds would be to use the Harrod-Domar idea in the following way.

Let $\overline{Y}_t$ be the production capacity as determined by the size of real capital in the point of time t. This means that we must have

$$Y_t \leqslant \overline{Y}_t \qquad (t = 1, 2 \ldots, \infty). \qquad \ldots \quad (144)$$

By Harrod-Domar reasoning $\overline{Y}_t$ would be of the form

$$\overline{Y}_t = \sigma K_t \qquad (t = 1, 2 \ldots, \infty) \qquad \ldots \quad (145)$$

where K_t is real capital at time t, and σ is the *output-to-capital ratio*, supposed structurally given and constant over time (which means that we refrain from considering the specific technological effects of new investments on σ).

Since by definition

$$K_t = K_0 + \sum_{\tau=1}^{t} (I_\tau + H_\tau) \qquad (t = 1, 2 \ldots, \infty) \qquad \ldots \quad (146)$$

where K_0 is the initial size of real capital, we get from (144)—(145)

$$Y_t \leqslant \sigma\left[K_0 + \sum_{\tau=1}^{t} (I_\tau + H_\tau)\right]. \qquad \ldots \quad (147)$$

This again is, by (2) and (129) a bound which is linear in the decisional parameters $H_1, H_2 \ldots, H_T$.

The introduction of the two new variables $\bar{Y}_t$ and K_t does not change the number of degrees of freedom since we also have two new equations, namely (145) and (146).

If we want to introduce labour requirement we may do it in the form of the bound

$$Y_t \leqslant \phi \bar{N}_t \quad (t = 1, 2 \ldots, \infty) \qquad \ldots \quad (148)$$

where ϕ is a structurally given labour coefficient and $\bar{N}_t$ a given forecast of the working population (so far without taking account of the problem of specialized skilled labour, which is so conspicuously important, not least in developing countries). In this case too the bound (148) is linear. And the inclusion of unemployment $(\bar{N}_t - Y_t/\phi)$ as part of the preference function would not change the linear character of this function.

So, in all we get a programming model which is completely linear both with regard to preference function and bounds. It may therefore be solved by one of the known methods.[17]

If we want to consider a smooth course of the H_t as an additional desideratum besides the main desideratum expressed in (140) we may add for instance $\left[-P_x \sum_{t=1}^{T} (H_t - H_x)^2\right]$ to (140), where P_x is a positive preference coefficient and $H_x = \frac{1}{T} \sum_{t=1}^{T} H_t$ the average of H_t over the repercussion period. The preference function F thus obtained is concave (since the matrix of its second order derivatives will be negative definite). The bounds remain linear. The ensuing programming problem is therefore not a particularly difficult one.[18] Any other additional term which retains the concave character of the preference function could be considered.

[17] Since we will in this case have a moderate number of degrees of freedom, namely T, the number of decisional variables $H_1, H_2 \ldots, H_T$, but a much larger number of dependent variables, the problem is one where my multiplex method will work to great advantage. It is coded by Mr. Ole-Johan Dahl for use on the electronic computer of the Norwegian Defence organization, and has been very successfully used here for rather large size problems.

[18] For problems with linear (upper and/or lower) bounds and a concave preference function the multiplex method works very well, as I reported on at the 1960 Tokyo meeting of the International Statistical Institute. The paper is printed in the report of the meeting.

Formulation 2 : This is similar to Formulation 1 but $H_1, H_2 \ldots, H_T$ are now not deterministically defined but defined through the simultaneous probability distribution of $H_1, H_2 \ldots, H_T$. The technical aspect of this programming solution in this case will in principle be somewhat similar to that under Formulation 1, except for the fact that the decisional parameters will now be the parameters that determine the simultaneous distribution of $H_1, H_2 \ldots, H_T$. It goes, however, without saying that this stochastic twist of the problem will in practice increase the computational difficulties very much.

Formulation 3 : This formulation is similar to the Formulation 1, in so far as the magnitudes $H_1, H_2 \ldots, H_T$ are also now considered as deterministic parameters to be determined by the programming procedure but in addition the government strategy parameters κ and λ are also considered as decisional, i.e. as variables to be decided upon through the programming analysis, subject however to the condition (114).

Since the parameters κ and λ—through μ and ω—enter in an extremely non-linear manner in (35) and (129), and hence by (1) in the corresponding expression for consumption $C_1, C_2 \ldots, C_T$, real capital $K_1, K_2 \ldots, K_T$ etc., the computational difficulties will now be great, but possibly not insurmountable for a moderately large horizon T. (cf. the remarks on iteration at the end of this paper).

Formulation 4 : Same as Formulation 3 except for the fact that $H_1, H_2 \ldots, H_T$ are now considered not as deterministically defined but as stochastically defined through the simultaneous distribution of $H_1, H_2 \ldots, H_T$.

This problem will be computationally very hard to handle. It is very doubtful whether an attempt in this direction is worth the trouble, particularly in view of the fact that it will probably not lead to any essentially new and practically more useful results than those that could be obtained through a solution of the Formulation 3.

Formulation 5 : *Solution when autonomous investment is not under government control.* If autonomous investment is not under government control, the only thing government can do, within the framework of the model (1)-(4), is to handle the strategy parameters (κ, λ) in some way which is as "optimal" as it can be made without knowing the autonomous investment.

A plausible thing to do under these circumstances is to consider a (κ, λ) strategy which is optimal on the assumption that there will be *no* autonomous investments in the future i.e. for $t = 1, 2 \ldots \infty$ ($t = 0$ being "today" i.e. the day when the decision about (κ, λ) is made). Any actually occurring autonomous investment will then simply come as an *addition* to the national income which is theoretically produced under the strategy considered[19].

We denote the solution of the difference equation (5) under this assumption by the letter y_t. By (131) we have

$$y_t = \eta_t Y_0 + \nu \eta_{t-1} Y_{-1} \quad (t = 1, 2 \ldots, \infty) \qquad \ldots (149)$$

where Y_0 and Y_{-1} are the *actually realized* magnitudes of the national income in years 0 and (−1) respectively regardless how those magnitudes have been achieved.

[19] As an alternative we may assume that the H_t are given through some sort of *estimate* made *independently* of the model. This would make the computations more complicated but would not cause insurmountable difficulties.

As before, we assume provisionally the non-wave case for the cumulator. This is equivalent to imposing the condition (114) which is the same as (115). It will turn out that this is not a restrictive assumption since the non-wave case is actually the optimal solution.

If the two characteristic roots (provisionally real by assumption) are different, the expression for the cumulator is (127). This expression can be transformed in such a way that it becomes applicable even in the case $\lambda_1 = \lambda_2$. We simply divide by $(\lambda_1 - \lambda_2)$ which gives

$$\eta_t = \sum_{v=0}^{t} \lambda_1^v \lambda_2^{t-v} = \sum_{v=0}^{t} \lambda_1^{t-v} \lambda_2^v \qquad (\lambda_1 = \lambda_2 \text{ or } \lambda_1 \neq \lambda_2). \quad \dots \quad (150)$$

The two expressions in (150) for η_t are equivalent.

Since by (118) the expression (149) can be written

$$y_t = \eta_t Y_0 - \lambda_1 \lambda_2 \eta_{t-1} Y_{-1} \quad (t = 1, 2 \dots, \infty) \quad \dots \quad (151)$$

we can by (150) write the expression for y_t in either of the two following forms

$$y_t = Y_0 \sum_{v=0}^{t} \lambda_1^v \lambda_2^{t-v} - Y_{-1} \sum_{v=0}^{t-1} \lambda_1^{v+1} \lambda_2^{t-v} \quad (\lambda_1 = \lambda_2 \text{ or } \lambda_1 \neq \lambda_2) \quad \dots \quad (152)$$

$$y_t = Y_0 \sum_{v=0}^{t} \lambda_1^{t-v} \lambda_2^v - Y_{-1} \sum_{v=0}^{t-1} \lambda_1^{t-v} \lambda_2^{v+1} \quad (\lambda_1 = \lambda_2 \text{ or } \lambda_1 \neq \lambda_2) \quad \dots \quad (153)$$

We may, if we like, consider the programming formulation as one in (λ_1, λ_2). Indeed, to any set (λ_1, λ_2) corresponds by (117)-(118) one and only one set (μ, ν), and inversely to any set (μ, ν) corresponds a set (λ_1, λ_2) which is uniquely determined, apart from an interchange of the numbering of the two roots λ_1 and λ_2 (we may, if we like, conventionally choose the numbering so that $\lambda_1 \geqslant \lambda_2$).

Considering the programming problem as a problem in (λ_1, λ_2) we take the partial derivatives of y_t with respect to λ_1 and λ_2 respectively. When determining the former of these two derivatives it is most convenient to use (152) and when determining the latter derivative it is most convenient to use (153).

After some transformations we get

$$\frac{\partial y_t}{\partial \lambda_1} = \left[\sum_{v=1}^{t} v \lambda_1^{v-1} \lambda_2^{t-v} \right] (Y_0 - \lambda_2 Y_{-1}) \quad \dots \quad (154)$$

$$\frac{\partial y_t}{\partial \lambda_2} = \left[\sum_{v=1}^{t} v \lambda_1^{t-v} \lambda_2^{v-1} \right] (Y_0 - \lambda_1 Y_{-1}). \quad \dots \quad (155)$$

This shows that we can determine values of λ_1 and λ_2 that are independent of t and such that they *annihilate* both derivatives simultaneously. These values are simply obtained by putting the last parenthesis in (154) and (155) equal to zero. This gives

$$\lambda_1 = \lambda_2 = \frac{Y_0}{Y_{-1}}. \quad \dots \quad (156)$$

Without going into a detailed algebraic proof we see from numerical examples, (cf. tables (180) and (181)), that (156) actually yields a maximum.

The single parameter strategy. We are thus led to the case of two equal roots. Since the two roots cannot be equal unless they are real and since the above argument is valid *regardless* of whether λ_1 and λ_2 are looked upon as real or complex, we have actually proved that the programming in λ_1 and λ_2 leads to the case of two equal and real roots. That is to say we must have the Case II in (106).

The programming problem *is thus reduced to a single-parameter problem.*

Let $$r = \frac{\mu}{2} \quad \text{(the general case of two equal roots)} \qquad \dots \quad (157)$$

be the common value of the two roots. The parameter r in (157) has the same meaning as in (64) and (69). The problem is to determine the parameter r in such a way that y_t becomes as large as possible when Y_0 and Y_{-1} are given. The value of r which achieves this, follows directly from (156). We denote this value by the subsscript 1. That is, we have

$$r_1 = \frac{Y_0}{Y_{-1}} \text{(the optimal value of the two equal roots).} \qquad \dots \quad (158)$$

The symbol r_1 can be looked upon as designating both the optimum value of the strategy parameter r and the initially observed *actual growth factor*[20].

The cumulator in the case of two equal roots is easily derived from (135) and (157). We get

$$\eta_t = (t+1)r^t \quad (t = 1, 2, \dots, \infty) \quad (r \text{ arbitrary, not necessarily equal to } r_1). \qquad \dots \quad (159)$$

This expression can also be obtained by the general method of (56), (89)-(91).

From (117)-(118) and (157) we now get

$$\mu = 2r \qquad \dots \quad (160)$$

$$\nu = -r^2. \qquad \dots \quad (161)$$

So that by (6)-(7)

$$\kappa = -\alpha + 2r - r^2 \quad \text{(for any } r) \qquad \dots \quad (162)$$

$$\lambda = \beta - r^2 \quad \text{(for any } r) \qquad \dots \quad (163)$$

The expression (162) shows how much larger the strategy parameter κ must be chosen than the structural constant α in the Case II in (108) when $(\beta - \lambda)$ has been chosen.

Inserting (158) into (162)-(163) we get the optimal strategy values

$$\kappa_0 = -\alpha + 2r_1 - r_1^2 \quad \text{(optimal)} \qquad \dots \quad (164)$$

$$\lambda_0 = \beta - r_1^2 \quad \text{(optimal)} \qquad \dots \quad (165)$$

where r_1 is given by (158).

20When I speak here of the "growth factor" (which is unity plus the growth rate) I take the word factor simply in the sense of "a number by which to multiply", not in the sense of "something which can explain". In the latter sense the word factor is used for instance in biology when one speaks of a "growth factor" or "factor for growth."

In any *realistic case in economics* r_1 determined by (158) will be *positive*, so that we do not get the zig-zag case (the case where η_t changes from plus to minus in every other point of time).

Government current account expenditure is given by (3) which we can also write

$$G_t = (\kappa - \lambda) Y_{t-1} + \lambda Y_{t-2} \quad (t = 1, 2, \ldots, \infty). \qquad \ldots (166)$$

Inserting (162)-(163) in (166) we get

$$G_t = [2r - (\alpha + \beta)] Y_{t-1} + [\beta - r^2] Y_{t-2} \qquad \ldots (167)$$

(for any r in the case of two equal roots)

If we insert the optimal strategy value (158) for r in (167) we get the *optimal* government current account expenditure in year t, when Y_{t-1} and Y_{t-2} are the *actual* national incomes in year $(t-1)$ and $(t-2)$ regardless of the way in which these magnitudes happen to have emerged through previous autonomous investments. This is the meaning of our determining (by a decision at time $t = 0$) the optimal strategy parameters κ_0 and λ_0 *as if* all the $H_1, H_2 \ldots, H_{t-1}$ are going to be zero.

In the general case of a double root—i.e. with an arbitrary value of r we have

$$\eta_{t+1} - \eta_t = r^t[r(t+2) - (t+1)] = \eta_t \left[r \frac{t+2}{t+1} - 1 \right] \qquad \ldots (168)$$

(for any r in the case of two equal roots)

and by (129) $$Y_{t+1} - Y_t = H_{t+1} + \left[r^t[r(t+2) - (t+1)] Y_0 - r^{t+1}[r(t+1) - t] Y_{-1} \right]$$

$$+ \sum_{\tau=0}^{t-s} r^\tau \left[r(\tau+2) - (\tau+1) \right] H_{t-\tau} \qquad \ldots (169)$$

$$(t = s, s+1, \ldots, \infty); \quad (s = 1, 2, \ldots, \infty).$$

The middle term in (169), i.e. the big bracket represents *the after-effect of the initial conditions*, the last term, i.e. the summation over τ represents the *after-effect of the autonomous investments* up to t inclusive and the first term in (169) represents *the effect of the new autonomous investments* at time $t+1$.

As t tends toward infinity we get from (168)

$$\frac{\eta_{t+1} - \eta_t}{\eta_t} = (r-1) \qquad \text{(as } t \to \infty \text{ for any value of } r). \qquad \ldots (170)$$

By the passage to $t \to \infty$ and using the formulae (129), (168), (170) and (159) become

$$\frac{Y_{t+1} - Y_t}{Y_t} = (r-1) + \frac{H_{t+1}}{Y_t} + \frac{r \sum_{\tau=0}^{t-s} r^\tau H_{t-\tau}}{Y_t} \quad \text{(as } t \to \infty \text{ for any value of } r) \qquad \ldots (171)$$

(170) and (171) show that *asymptotically* the rate of growth of the cumulator will be $(r-1)$—as it would be for *any* t if the cumulator were a simple exponential—, and

asymptotically the rate of growth of national income itself will be $(r-1)$ plus two correctional terms. The first is due to H_{t+1}, which we can in principle choose arbitrarily inside the model (1)-(4). And the second correctional term represents the after-effect of previous autonomous investments. All we can say about the latter of the correctional terms is that they will be positive if the autonomous investment are on the average" positive. Even if $H_{t+1} = 0$ it will therefore be safe to assume that asymptotically the growth rate of Y_t will be *at least* $(r-1)$.

More important than to study the passage $t \to \infty$ is to see what happens in the first few years after the point of time 0. I will illustrate this below by some numerical examples. They will also show the effect of changing r. These examples will verify the above theoretical results. We consider the case where all the autonomous investment are zero.

For any value of r (i.e. assuming two equal roots) we have by (160)-(161)

$$y_t = 2ry_{t-1} - r^2 y_{t-2} \qquad \ldots \quad (172)$$

(the recurrent formula in the case of two equal roots),

$$y_t = (t+1)r^t Y_0 - tr^{t+1} Y_{-1} \qquad \ldots \quad (173)$$

(the ab-initio formula in the case of two equal roots).

By (167) the government expenditure on current account will, on the assumption of no autonomous investments, be

$$g_t = [2r-(\alpha+\beta)]y_{t-1} + [\beta - r^2]y_{t-2} \quad (t = 1, 2, \ldots, \infty \text{ for any } r). \qquad \ldots \quad (174)$$

For $t = 1$ we will in (174) have to put y_0 and y_{-1} equal to respectively Y_0 and Y_{-1} where Y_0 and Y_{-1} are the magnitudes actually realized in these two points of time.

From (173) we get

$$\frac{dy_t}{dr} = t(t+1)r^{t-1}(Y_0 - rY_{-1}) \qquad \ldots \quad (175)$$

and using this in (174) we get

$$\frac{dg_t}{dr} = \left[[2r-(\alpha+\beta)]r(t-1)t + [\beta - r^2](t-2)(t-1) + 2r^2 \right] r^{t-3}(Y_0 - rY_{-1}). \qquad \ldots \quad (176)$$

Comparing (175) and (176) we see that during the variation of r, g_t and y_t must have extremum for the same value of r namely the value r_1 defined by (158).

By inserting the ab-initio formulae (173) for y_{t-1} and y_{t-2} we can derive the corresponding ab-initio formula for g_t, but this is unnecessary for our purpose.

Numerical examples. In our numerical examples we use

$$\alpha = 0.8 \qquad \beta = 2. \qquad \ldots \quad (177)$$

And we consider two sets of initial conditions, namely

Example A : $Y_{-1} = 100$ and $Y_0 = 103$ ... (178)

Example B : $Y_{-1} = 100$ and $Y_0 = 107$. ... (179)

The result for different values of r are given in (180) and (181) respectively.

Example A: $Y_{-1} = 100$ and $Y_0 = 103$

	$t=1$			$t=2$			$t=3$			
alternative r	y_t	g_t	$\frac{100g_t}{y_t}$	y_t	g_t	$\frac{100g_t}{y_t}$	y_t	g_t	$\frac{100g_t}{y_t}$	
1.00	106.00	17.60	16.60	109.00	18.20	16.70	112.00	18.80	16.79	
1.01	106.05	17.65	16.64	109.15	18.21	16.68	112.30	18.78	16.72	
1.02	106.08	17.68	16.66	109.24	18.22	16.67	112.49	18.77	16.69	
1.03	106.09	17.69	16.67	109.27	18.22	16.67	112.55	18.77	16.67	
1.04	106.08	17.68	16.66	109.24	18.21	16.67	112.49	18.77	16.69	
1.05	106.05	17.65	16.64	109.15	18.21	16.68	112.29	18.77	16.72	
1.06	106.00	17.60	16.60	108.99	18.19	16.68	111.96	18.79	16.78	... (180)
1.07	105.93	17.53	16.55	108.77	18.16	16.69	111.48	18.80	16.86	
1.08	105.84	17.44	16.48	108.48	18.12	16.70	110.85	18.80	16.96	
1.09	105.73	17.33	16.39	108.12	18.07	16.71	110.08	18.81	17.09	
1.10	105.60	17.20	16.29	107.69	18.01	16.72	109.14	18.81	17.23	
1.20	103.20	14.80	14.34	99.36	16.40	16.51	89.86	18.05	20.09	
1.30	98.80	10.40	10.53	82.81	12.18	14.71	48.33	14.07	29.10	
1.40	92.40	4.00	4.33	56.84	4.12	7.25	−21.95	3.70	−16.84	
1.50	84.00	−4.40	−5.24	20.25	−8.95	−44.20	−128.25	−16.95	13.22	

Example B: $Y_{-1} = 100$ and $Y_0 = 107$

1.00	114.00	14.40	12.63	121.00	15.80	13.06	128.00	17.20	13.44	
1.01	114.13	14.53	12.73	121.39	15.83	13.04	128.79	17.15	13.32	
1.02	114.24	14.64	12.82	121.73	15.85	13.02	129.47	17.11	13.22	
1.03	114.33	14.73	12.88	122.00	15.88	13.02	130.03	17.08	13.14	
1.04	114.40	14.80	12.94	122.22	15.90	13.01	130.48	17.07	13.08	
1.05	114.45	14.85	12.98	122.38	15.92	13.01	130.81	17.05	13.04	... (181)
1.06	114.48	14.88	13.00	122.47	15.93	13.01	131.01	17.05	13.01	
1.07	114.49	14.89	13.01	122.50	15.93	13.01	131.08	17.05	13.01	
1.08	114.48	14.88	13.00	122.47	15.93	13.01	131.01	17.05	13.01	
1.09	114.45	14.85	12.98	122.37	15.91	13.00	130.80	17.05	13.04	
1.10	114.40	14.80	12.94	122.21	15.89	13.00	130.44	17.05	13.07	
1.20	112.80	13.20	11.70	116.64	14.80	12.69	117.50	16.51	14.05	
1.30	109.20	9.60	8.79	103.09	11.33	10.99	83.49	13.23	15.85	
1.40	103.60	4.00	3.86	80.36	4.28	5.33	21.95	4.14	18.88	
1.50	96.00	−3.60	−3.75	47.25	−7.55	−15.98	−74.25	−14.55	19.60	

These tables show clearly that $r_1 = \frac{Y_0}{Y_{-1}} = 1.03$ is the optimum strategy in the first example, and $r_1 = \frac{Y_0}{Y_{-1}} = 1.07$ the optimum strategy in the second example.

The optimum relative government expenditure on current account. In the case of the optimal strategy we get

$$y_{t.0} = r_1^t Y_0 \qquad \dots (182)$$

$$g_{t.0} = r_1^{t-2} Y_0[r_1^2-(\alpha+\beta)r_1+\beta] \qquad \dots (183)$$

where the subscript 0 on $y_{t.0}$ and $g_{t.0}$ indicates the result of the optimum choice (158).

From (182)-(183) we get immediately

$$\frac{g_{t.0}}{y_{t.0}} = 1-\frac{\alpha+\beta}{r_1}+\frac{\beta}{r_1^2} \quad \dots \quad (184)$$

and hence

$$\frac{d\left(\frac{g_{t.0}}{y_{t.0}}\right)}{dr_1} = \frac{\alpha+\beta}{r_1^3}\left(r_1-\frac{2\beta}{\alpha+\beta}\right). \quad \dots \quad (185)$$

The formula (184) shows that the optimum *relative* government expenditure—under the conditions we have assumed for the optimum—*is independent of t*. But it will change with the initial conditions (which determine r_1), and the way this change takes place is given in (185).

It is an interesting fact that the above optimalization process is *independent of t*. This means that in *any* point of time $(t-1)$ we can determine the optimum value of the two strategy parameters $(\kappa-\lambda)$ and λ, as if it were only a question of achieving the best result in the *next year t*, assuming that no autonomous investment is to take place in the next year. Whatever autonomous investment that is *actually going* to be achieved in the next year t will only produce so much additional increase in the national income Y_t. The development will therefore *gain momentum* from year to year by any autonomous investments which the nation's real resources will permit and the private investors choose to make. The actual growth process will therefore be more rapid than that illustrated in (180) and (181). Each year will so to speak represent a change of initial conditions, and hence a new optimal value for the parameter r.

Government current account budget strategy under non-controlled investments

The above consideration leads to the following rule for a *moving optimalization*.

Moving optimalization. Each year t the *actually realized* national income in year $t-1$ and $t-2$ will be known. From these data Y_{t-1} and Y_{t-2} one should compute the optimum factor r_t, determined by

$$r_t = \frac{Y_{t-1}}{Y_{t-2}} \quad \text{(The optimum factor computed at } t) \quad (t = 1, 2, \dots, \infty). \quad \dots \quad (186)$$

This value of r_t is inserted for r in (167). This gives the optimum government current acount expenditure in year t.

That is, in year t the government current account expenditure ought to be put equal to the optimum value

$$G_{t.0} = [2r_t-(\alpha+\beta)]Y_{t-1}+[\beta-r_t^2]Y_{t-2} = Y_{t-1}[r_t+\frac{\beta}{r_t}-(\alpha+\beta)] \quad \dots \quad (187)$$

where r_t is given by (186).

This figure will, of course, in practice only indicate a *guiding signal* for government expenditures. It has been computed only by considerations on the general growth rate of the economy. Other considerations (social justice, defence etc.) may motivate an action whereby one sacrifices part of the overall national growth rate.

But this being said, I believe that in the absence of a real control over the autonomous investments in the nation, and of a complete programming solution involving also the magnitudes $H_1, H_2 \ldots$, the above expenditure rule is as good a rule as can reasonably be deviced. But the rule only leads to protection against decline, not to real *progress* as will be seen from the subsequent argument.

By the right hand expression in (187) the optimal expenditure ratio $G_{t.0}/Y_{t-1}$ *depends only on the recent growth rate* r_t (and on the two structural coefficients α and β). Furthermore, the way in which the optimal expenditure ratio depends on r_t transpires immediately from (187) and from the expression for the derivative

$$\frac{d\left(\frac{G_{t.0}}{Y_{t-1}}\right)}{dr_t} = 1 - \frac{\beta}{r_t^2} = \text{negative in realistic cases.} \quad \ldots \quad (188)$$

Since we may assume α and β positive—and in realistic case α smaller than unity and β larger than unity—the optimal expenditure ratio is by (187) very large for very small positive values of r_t. With increasing r_t it decreases, passing the value $(1-\alpha)$ for $r_t = 1$, continuing to decrease and reaches its minimum value $(1-\alpha)-(\sqrt{\beta}-1)^2$ for $r_t = \sqrt{\beta}$. From this point on it increases without interruptions as r_t continues to increase.

The most realistic range for r_t is between 1 and $\sqrt{\beta}$ and in this range we can draw the following conclusion : The lower the growth rate for national income has been in recent years the *larger* should the optimum government expenditure on current account be in relation to national income. This conclusion is rather at variance with the favourite conservative argument about the advantage of low government expenditure. This need to *increase* government expenditure (in order to obtain optimality) is all the smaller the larger β, i.e. (188) *more* negative when β is large.

Protection against decline, not a driving force for progress. If no autonomous investments accrue, the growth rate of national income (under optimal government current account expenditure at all times) is *constant* and determined by some distant initial conditions (cf. 182). In the model (1)-(4) the optimalization of government current account expenditure is therefore only a *protection against decline*. It protects what we had, namely r_1 (or r_t in moving optimalization), but is no driving force for progress. Only through autonomous investments can the growth rate be increased. And when it is increased, the optimal expenditure ratio should (in order to protect the new growth rate) be adapted in the way we have just discussed in connection with (187) and (188). If the model should have been able to display a driving force for progress, it would have been necessary to add a relation that indicates an effect of government current expenditure on autonomous investment.

Attempts at reaching higher growth rates by influencing autonomous investment. If account is taken of bounds such as (143), (144) and (148), the more complete programming Formulations 1-4 may be realistic enough to make it possible to reach much higher growth rates. But then autonomous investments must be controlled.

One may simplify the computation by an iterative procedure which consists in first determining the optimal strategy parameter r_t by (186) and then proceeding to the solution of the Formulation 1 problem, which is simply a linear programming problem. Having solved this problem one may reshape Formulation 5 so as to take account of the autonomous investments now determined. Thus we reach an improved value for r_t. This improved value may again be taken as the basis for a programming problem of type 1. And so on. This might lead to an approximate solution to the Formulation 3 problem and would give *coordination* of current account and investment policies.

The need for a more complete planning model. But this whole approach is as yet too simple to be taken as a complete planning model, because it does not treat explicitly such important aspects as *import-export* effects (balance of payments), *sectorial breakdown, substitution possibilities* among sector inputs (i.e. not all the input coefficients being constants), the *infra effect* of investments in changing the coefficients of a model, the problem of *education* and *skilled labour* (which formally can be handled very much in the same way as the problem of capital formation and with bound-effects similar to that produced by material capital). All these problems are basically important for planning in an underdeveloped country. Nor can they be neglected in a more advanced country which will introduce a certain amount of planning in its economy.

In this connection I shall not discuss the various attempts I have made of developing a complete planning model.[21]

Summary

An explicit general solution of the Hicksian Model, holding true for any period of time t and with any arbitrarily given right member (the term which makes the difference equation non-homogeneous) is given. The solution has one degree of freedom (one arbitrary function of time) when the parameters of the model are given.

Programming techniques are indicated either to determine desirable values of the parameters in the Hicksian Model or to choose one of the variables as a decisional function of time. In both cases, we may haze a linear or a more general preference function having the aim of minimising fluctuations in incomes from their long run growth path and/or assuring rapid growth in the long run.

There are five formulations, making different assumptions about the time shape of autonomous investment. One formulation applies to the case where the autonomous investment is not under government control.

The recurrence method as well as the method of characteristic roots for solving a linear difference equation of any order with constant co-efficients and an arbitrary right member (the non-homogeneous case) are stated in a precise and easily understandable form with special application to the Hicksian Model.

---

[21] See for instance, "An implementation system for optimal national economic planning without detailed quantity fixation from a central authority". Memorandum, 3 January 1963 from the Oslo University Institute of Economics. Soon to appear in print.