Paradoxical consumption behavior when economic activity has environmental effects

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Abstract

In a model where enhanced economic activity (accumulation of produced capital) leads to environmental effects (depletion of natural capital), competitive steady states corresponding to different discount rates are compared. For positive discount rates, the steady state stock of produced capital may exceed the size maximizing sustainable consumption. This implies *paradoxical consumption behavior*; that is, a lower discount rate may be associated with lower steady state consumption. The theoretical significance of this phenomenon for intergenerational equity is discussed, and examples indicating the empirical relevance of the underlying assumptions are presented.

Keywords and Phrases: Optimal growth, capital theory, environmental effects, intergenerational equity

JEL Classification Numbers: D63, 041, Q01

1 Introduction

This paper presents a model of an economy interacting with the natural environment. The term *natural environment* is meant to comprise non-produced resources,

the availability of which is measured by a one-dimensional indicator called *natural capital*. The economy is assumed to produce a composite good that can be either consumed or stored; the quantity being stored is referred to as *produced capital*. The dichotomy between natural and produced capital seems fundamental; by definition, saved production can only be turned into produced capital.

In the model analyzed here, enhanced economic activity, characterized by a larger stock of produced capital and a higher production rate, leads to increased environmental effects, resulting in a smaller stock of natural capital and a reduced production rate. Unlike the neoclassical one-sector growth model, an increased stock of produced capital does not without continuous reinvestment lead to a constant and perpetual stream of positive benefits. Because economic activity has environmental effects, the stream of net benefits is tilted towards the present and may eventually become negative. This points to the possibility that a lower discount rate may increase economic activity to the extent that the resulting environmental effects reduce steady state consumption, leading to *paradoxical steady states*. Such a positive relationship between the discount rate and steady state consumption may indeed occur in the model and is referred to as *paradoxical consumption behavior* (PCB).

A consequence of the occurrence of PCB and the related phenomenon of *reswitching* is that the economy's *rate of return on lending to the future* may not equal the *cost of borrowing from the future*. In particular, the real rate of interest, measuring produced capital's rate of net marginal productivity, depends on the rate at which future utilities are discounted and does not necessarily reflect the economy's ability to transform saved production into a constant and perpetual stream of additional future consumption. This property is related to the observation that, in the model, the value of natural capital in terms of produced capital is a decreasing function of the discount rate. Hence, capital cannot be aggregated, and (outside a steady state) current income cannot be defined independently of weight assigned to future consumption.

Resemblance to the problems stated here is found elsewhere in the economic literature: in the Cambridge controversies in the theory of capital (cf. Cohen and Harcourt 2003) it was argued that heterogeneous capital cannot be aggregated and does not determine a rate of marginal productivity by technical properties alone. However, the theoretical significance is different. The literature of the Cambridge controversies emphasizes what consequences the indeterminacy of capital's productiveness has for the functional distribution of income between capital and labor.

Here we emphasize what significance the indeterminacy of the economy's current productiveness has for the intertemporal distribution of consumption between the present and the future.

The present model is structurally similar to partial equilibrium models of resource exploitation (see, e.g., Clark et al. 1979) There are also papers showing within the context of partial equilibrium models that a lower discount rate

- need *not* lead to greater conservation (Farzin 1984, Hannesson 1986),

- may increase the social profitability of resource development projects in spite of delayed environmental costs (Prince and Rosser 1985).

Hence, the main contribution of the present paper is to present a *general equilibrium model* that connects the paradoxical phenomena like PCB and reswitching (see, e.g., Ahmad 1991) with the issue of whether the present shortchanges the future through its management of produced and natural capital (as discussed in the literature on *Hartwick's rule*; see, e.g., Solow 1974, Hartwick 1977, Dixit et al. 1980, Asheim et al. 2003). In this respect the present investigation is similar to the analysis of the Brock (1977) model presented by Becker (1982), who however does not provide sufficient conditions for paradoxical steady states nor argue for their empirical relevance.

There are by now many contributions on economic growth with environmental externalities (see, e.g., Beltratti 1996 and Smulders 2000 and the references therein), some of which discuss the effects of discounting on environmental preservation (see, e.g., Smulders 1999). The possibility and significance of the paradoxical phenomena of the Cambridge controversies do not seem to be emphasized in this literature.

The model is introduced in Section 2 where the existence, uniqueness, and stability of a competitive steady state (for a given discount rate) is shown. In Section 3, the characteristics of competitive steady states corresponding to different discount rates are compared, and the occurrence of PCB is established. In Section 4, the theoretical implications of PCB are explored, and it is pointed out that the economy's rate of return on lending to the future may not equal the cost of borrowing from the future. Empirical evidence is evaluated in Section 5, where it is argued that the assumptions underlying the paradoxes might be of a pervasive rather than exceptional nature. Some proofs are contained in a mathematical appendix.

2 The model

The model combines the neoclassical one-sector growth model with features of a biodynamic model due to Schaefer (1954) and others. The *Schaefer* model (discussed in, e.g., Clark 1990) is based on the differential equation

$$\dot{x}(t) = g(x(t)) - y(t), \tag{1}$$

where x denotes natural capital and y the rate of production, and where g, the natural growth function, is assumed to satisfy the following.

Assumption 1 g is a continuous, twice differentiable function defined for $x \in [0, \bar{x}]$ such that $g(0) = g(\bar{x}) = 0$ and, $\forall x \in (0, \bar{x})$, $g'' < 0$.

Assumption 1 implies that $g(x)/x$ is a decreasing function of x (and thus, $g'(x) < g(x)/x$), and it is satisfied by a logistic growth function such as the one illustrated in Figure 1. The production rate is assumed to depend on the stock of natural capital and a variable e, *productive effort*, such that y is proportional to x:

$$y(t) = x(t)e(t). \tag{2}$$

One important characteristic of the neoclassical one-sector growth model, as developed by Solow (1956) and Swan (1956), is the aggregate production function. Here productive effort is a function of produced capital, k, and labor, ℓ. The population and the supply of labor are assumed to be identical to one. Hence, no further reference will be made to whether variables express total or per capita quantities. By allowing for partial utilization of labor, we obtain

$$e(t) = \tilde{f}(k(t), \ell(t)), \tag{3}$$

where $0 \leq \ell(t) \leq 1$ and $\tilde{f}$ is assumed to have the usual neoclassical properties, including non-negative marginal productivities of both factors.

Equations (1), (2), and (3) yield the first of three *feasibility* conditions:

Condition 1 $\dot{x} = g(x) - x\tilde{f}(k, \ell)$.

Another characteristic of the *Solow-Swan* model is that production is a composite good that can be split between consumption and gross investment. We assume that

$$y(t) = c(t) + \dot{k}(t) + \delta k(t), \tag{4}$$

where δ denotes the positive rate at which produced capital depreciates. The rate of consumption c is required to be non-negative, and since irreversibility of investment is not assumed (for mathematical convenience), c is allowed to exceed y provided that k is positive.

Hence, the two remaining feasibility conditions are

Condition 2 $\quad \dot{k} = x\tilde{f}(k, \ell) - c - \delta k$,

Condition 3 $\quad c \geq 0, \quad c = 0$ when $k = 0$, and $0 \leq \ell \leq 1$.

For the analysis of Sections 2 and 3 we assume full utilization of labor ($\ell(t) \equiv 1$) and then show in the Appendix that this is without loss of generality. Define f by $f(k) = \tilde{f}(k, 1)$ for all k, where f has the usual neoclassical properties.

Assumption 2 f is a continuous, twice differentiable function defined for $k \in [0, \infty)$ such that $f(0) = 0$, $f'(0) = \infty$, $f'(\infty) = 0$, and, $\forall x \in (0, \infty)$, $f'' < 0$.

Note that any absolutely continuous path $\{k(t)\}_{t=0}^{\infty}$ determines a continuously differentiable path $\{x(t)\}_{t=0}^{\infty}$ through Condition 1 for a given initial stock of natural capital, $x(0)$. Since $d(g(x)/x)/dx < 0$ for $x \in (0, \bar{x})$, and thus $\dot{x}/x$ is a decreasing function of x for given k, we may define a continuous function ξ by

$$\lim_{t \to \infty} x(t) = \xi(k^*) \quad \text{when} \quad x(0) > 0 \text{ and } k(t) = k^*.$$

Hence, $\xi(k^*)$ is the stable equilibrium stock of natural capital when the stock of produced capital is identical to k^*. It follows from Assumptions 1 and 2 that

(a) $\xi(0) = \bar{x}$,

(b) $\xi(k^*) \in (0, \bar{x})$ and $\xi'(k^*) < 0$ (since $\xi' = \xi f'/(g'(\xi) - f)$ and $g'(\xi) < g(\xi)/\xi = f$) when $0 < f(k^*) < g'(0)$, and

(c) $\xi(k^*) = 0$ when $f(k^*)$ when $f(k^*) \geq g'(0)$.

The result that the equilibrium stock of natural capital is negatively related to the stock of produced capital shows that (in the model) economic activity has environmental effects.

Note that any piecewise continuous path $\{c(t)\}_{t=0}^{\infty}$ determines an absolutely continuous path $\{k(t), x(t)\}_{t=0}^{\infty}$ through Conditions 1 and 2 for given initial stocks of produced and natural capital, $(k(0), x(0))$. Condition 3 ensures that k and x are both
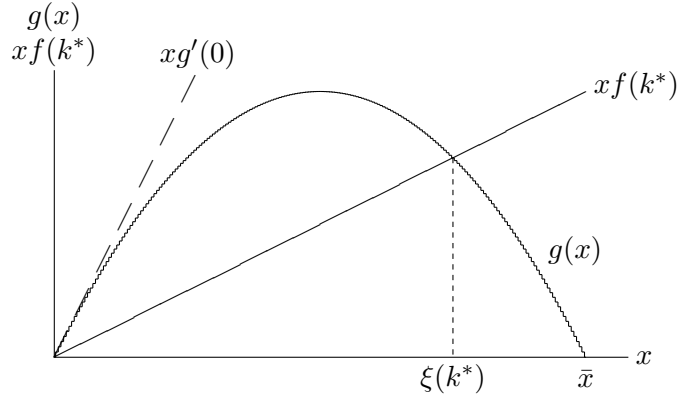
Figure 1: The natural growth function.

non-negative when Assumptions 1 and 2 are satisfied. A path $\{c(t), k(t), x(t)\}_{t=0}^{\infty}$ is *feasible from* $(k(0), x(0))$ if it satisfies Conditions 1–3 at all times and has initial stocks $(k(0), x(0))$, and $\{c(t)\}_{t=0}^{\infty}$ is piecewise continuous. A path $\{c(t), k(t), x(t)\}_{t=0}^{\infty}$ is *interior* if $(k(t), x(t)) \gg 0$ for all t. A *steady state* is a triple (c^*, k^*, x^*) with the property that the stationary path $\{c(t), k(t), x(t)\}_{t=0}^{\infty}$ defined by

$$(c(t), k(t), x(t)) = (c^*, k^*, x^*) \quad \text{for all } t \geq 0$$

is feasible from (k^*, x^*). (The set of steady states is the union of $\{(c^*, k^*, x^*) \mid x^* = \xi(k^*) \text{ and } c^* = x^* f(k^*) - \delta k^* \geq 0\}$ and $\{(0, 0, 0)\}$ since $x^* = 0$ is the only unstable equilibrium stock of x.)

The economy seeks to maximize the sum of discounted utilities, where utility is derived from consumption. In order words, a path $\{c_\rho(t), k_\rho(t), x_\rho(t)\}_{t=0}^{\infty}$ feasible from $(k(0), x(0))$ is *optimal from* $(k(0), x(0))$ if

$$\int_0^{\infty} \left[u(c(t)) - u(c_\rho(t)) \right] e^{-\rho t} dt \leq 0$$

for any path $(c(t), k(t), x(t))_{t=0}^{\infty}$ feasible from $(k(0), x(0))$. The rate of discount, ρ, is assumed to be positive, and the utility function satisfies the following assumption.

Assumption 3 *u is a continuous, twice differentiable, increasing and concave function defined for $c \in [0, \infty)$.*

The optimality of an interior path $\{c_\rho(t), k_\rho(t), x_\rho(t)\}_{t=0}^{\infty}$ implies the existence of continuously differentiable current value shadow prices of produced and natural capital, $p(t)$ and $q(t)$, such that the following conditions are satisfied at all times:

Condition 4 $\dot{q} = q\big(\rho - g'(x_\rho)\big) - (p-q)f(k_\rho)\,,$

Condition 5 $\dot{p} = p(\rho + \delta) - (p-q)x_\rho f'(k_\rho)\,,$

Condition 6 c_ρ maximizes $u(c) - pc\,.$

Conditions 4, 5, and 6 are *necessary* optimality conditions, obtained from Pontryagin's maximum principle by formulating the Hamiltonian:

$$H = \big[u(c) + p\big(xf(k) - c - \delta k\big) + q\big(g(x) - xf(k)\big)\big]e^{-\rho t}\,,$$

provided that the usual "normalization procedure" can be justified (see the technical note at the beginning of the Appendix). *Sufficient* optimality conditions are given in the Appendix.

A *competitive steady state* (CSS) is a triple $(c_\rho^*, k_\rho^*, x_\rho^*)$ with the properties that (i) the stationary path $\{c_\rho(t), k_\rho(t), x_\rho(t)\}_{t=0}^\infty$ defined by

$$(c_\rho(t), k_\rho(t), x_\rho(t)) = (c_\rho^*, k_\rho^*, x_\rho^*) \quad \text{for all } t \geq 0$$

is interior and feasible from (k_ρ^*, x_ρ^*), and (ii) Conditions 4–6 are satisfied from some pair of constant shadow prices:

$$(p(t), q(t)) = (p^*, q^*) \quad \text{for all } t \geq 0\,.$$

Assuming that $\{c_\rho(t), k_\rho(t), x_\rho(t)\}_{t=0}^\infty$ is interior excludes the steady state $(0, 0, 0)$, which is trivially feasible from $(0, 0)$, and thereby ensures that $x_\rho^* = \xi(k_\rho^*)$. The definition of a CSS combined with Assumptions 1 and 2 implies that

- x_ρ^* maximizes $(p^* - q^*)\,xf(k_\rho^*) - q^*\,(\rho x - g(x))$ over all x, and

- k_ρ^* maximizes $(p^* - q^*)\,x_\rho^*f(k) - p^*\,(\rho + \delta)\,k$ over all k.

In the remaining part of this section, we prove that there exists a unique CSS (for a given discount rate) and that the optimal path converges to this steady state. The former result (Lemma 1) forms the basis of the comparative dynamics analysis of Section 3. The latter (Lemma 2), which relies on the simplifying assumption of a linear utility function, has a two-fold purpose. *First*, it shows the stability of the CSS. This is of particular interest since Solow suggested that steady states at which PCB occurs might never be reached:

"The paradoxes themselves show that some simple conclusions deduced from models with one capital good may not hold for more general models, but it remains to be seen how significant this is. If the paradoxes matter at all, they are likely to matter for this ubiquitous question of convergence to steady states. The simpler question is whether such paradoxes can be observed in an optimizing economy, or whether if an optimal path comes upon such a situation, it will go around it, so to speak, so "paradoxical behavior" will never be observed along an optimal path" (Solow 1970, viii-ix).

This question of convergence to paradoxical steady states was subsequently analyzed by Burmeister and Hammond (1977), Burmeister and Long (1977), and Rosser (1983). *Second*, Lemma 2 describes the converging path, thereby giving insight into *why* paradoxical phenomena occur. The assumption of a linear utility function detracts from the generality of Lemma 2. Still, it shows that an optimal path can reach a paradoxical steady state and illustrates the properties of the converging path.

Provided that $p(t) > 0$, Condition 5 can be rewritten as follows:

$$\rho - \frac{\dot{p}}{p} = (1-v)x_\rho f'(k_\rho) - \delta = zf'(k_\rho) - \delta \,, \tag{5}$$

where $v(t) := q(t)/p(t)$ is the value of natural capital in terms of produced capital, and $z(t) := (1 - v(t))x_\rho(t)$ is the value of productive effort also in terms of k (since $1-v$ is the net value of production in terms of k, and x_ρ is the marginal product of e). Equation (5) shows that the *real interest rate* $\rho - \dot{p}/p$ (using the composite good as numeraire) measures *produced capital's rate of net marginal productivity* $zf'(k_\rho) - \delta$ (defined as the difference between its rates of gross productivity and depreciation).

The case where the shadow price $p(t)$ of produced capital is constant is of importance both in Lemma 1, which considers competitive steady states, and in the convergence phase of the optimal paths characterized in Lemma 2, which assumes a linear utility function. With a constant $p(t)$, equation (5) and the strict concavity of f imply that the optimal stock of produced capital is a continuous function of the value of productive effort: $k_\rho(t) = \kappa(z(t))$ when $p(t) \equiv p^* > 0$, where κ is defined by

$$\rho \equiv zf'(\kappa(z)) - \delta \,. \tag{6}$$

Equation (6) equalizes the discount rate and the rate of net marginal productivity of capital. It follows from Assumption 2 that κ is a one-to-one correspondence between positive values of k_ρ and z.

By these definitions, Conditions 1, 4, and 5 can be rewritten as

$$\dot{x}_\rho = g(x_\rho) - x_\rho f(\kappa(z)) \tag{7}$$

$$\dot{z} = \tfrac{z}{x_\rho} g(x_\rho) - (x_\rho - z)(\rho - g'(x_\rho)) \tag{8}$$

provided that $p(t) \equiv p^* > 0$. Note that the system of differential equations, (7) and (8), is determined given $(x_\rho(0), z(0))$. By the definition of κ, $x_\rho = \xi(\kappa(z))$ implies $\dot{x}_\rho = 0$. Similarly, one may define a continuous function ζ such that $z = \zeta(x_\rho)$ if and only if $\dot{z} = 0$. Through extending the domain of the function κ (so that $\kappa(z) = 0$ for $z \leq 0$), $\xi(\kappa(\zeta(\cdot)))$ becomes a continuous function from $[0, \bar{x}]$ into itself. Hence, by Brouwer's fixed point theorem there exists an $x_\rho^* \in [0, \bar{x}]$ such that $x_\rho^* = \xi(\kappa(\zeta(x_\rho^*)))$; in fact, $0 < z^* < x_\rho^* < \bar{x}$ (where $z^* = \zeta(x_\rho^*)$). Furthermore, the signs of the partial derivatives of $\dot{x}_\rho$ and $\dot{z}$ w.r.t. x_ρ and z imply that (x_ρ^*, z^*) is unique and has saddle point characteristics. (The details of the argument is given in the Appendix, and the results are illustrated in Figure 4.)

The existence and uniqueness of (x_ρ^*, z^*) is the key to the following lemma.

Lemma 1 *Let the functions g, f, and u satisfy Assumptions 1, 2, and 3 respectively. For each $\rho > 0$, there exists a unique* CSS *$(c_\rho^*, k_\rho^*, x_\rho^*)$. Furthermore, the stationary path $\{c_\rho(t), k_\rho(t), x_\rho(t)\}_{t=0}^\infty$ defined by $(c_\rho(t), k_\rho(t), x_\rho(t)) = (c_\rho^*, k_\rho^*, x_\rho^*)$ for all $t \geq 0$ is optimal from (k_ρ^*, x_ρ^*).*

Proof. *Existence.* The triple $(c_\rho^*, k_\rho^*, x_\rho^*)$, where $k_\rho^* = \kappa(z^*)$ and $c_\rho^* = x_\rho^* f(k_\rho^*) - \delta k_\rho^*$, is a steady state. It follows from (6) and the concavity of f that $c_\rho^* > 0$. Moreover, (6), (7), and (8) imply that Conditions 4, 5, and 6 are satisfied by letting $p(t) \equiv p^*$ and $q(t) \equiv p^* v^*$ (where $p^* := u'(c_\rho^*) > 0$ and $v^* := 1 - z^*/x_\rho^*$).

Uniqueness. Let $(c_\rho^*, k_\rho^*, x_\rho^*)$ be any CSS. Since $c_\rho^* > 0$, it follows from the definition of ξ that $x_\rho^* = \xi(k_\rho^*)$ and by Condition 6 that $p(t) \equiv p^* = u'(c_\rho^*) > 0$. By Condition 5, there exists z^* such that $q(t) \equiv p^*(1 - z^*/x_\rho^*)$ and $k_\rho^* = \kappa(z^*)$. Furthermore, Condition 4 implies that $z^* = \zeta(x_\rho^*)$. Thus, any CSS corresponds to (x_ρ^*, z^*).

Optimality of $\{c_\rho(t), k_\rho(t), x_\rho(t)\}_{t=0}^\infty$ is shown in the Appendix. ∎

The unique CSS can be implemented in a competitive market economy, provided that the discount rate ρ reflects the intertemporal preferences of the consumers (including their intergenerational altruism) and the cost of using the services of the natural environment is internalized. In such an intertemporal competitive equilibrium, ρ equals the real interest rate measuring produced capital's rate

of net marginal productivity, and consumption c_ρ^* is split into the functional shares $z^* \left(f'(k_\rho^*) - \delta \right) k_\rho^*$ for the owners of produced capital, $z^* \left(f(k_\rho^*) - f'(k_\rho^*)k_\rho^* \right)$ for the workers, and $(x_\rho^* - z^*)f(k_\rho^*) = v^* x_\rho^* f(k_\rho^*) = v^* g(x_\rho^*)$ for the owners of natural capital.

Assume now for mathematical convenience that the utility function is linear, and let the initial stocks of produced/natural capital be larger/smaller than the equilibrium stocks, corresponding to what might be the empirical relevant case. Then the saddle point characteristics of (x_ρ^*, z^*) entail that the CSS is stable, as stated by the following lemma.

Lemma 2 *Let the functions g and f satisfy Assumptions 1 and 2 respectively, and assume that the function u is of the following form:*

$$u(c) = p^* c \quad \text{where } p^* > 0 \text{ and } c \in [0, \infty).$$

If $\rho > 0$ and $x(0) = \xi(k(0)) \in (x_\rho^*, \bar{x})$, then there exists a unique optimal path $\{c_\rho(t), k_\rho(t), x_\rho(t)\}_{t=0}^{\infty}$ from $(k(0), x(0))$ converging to the CSS; that is,

$$\lim_{t \to \infty} \left(c_\rho(t), k_\rho(t), x_\rho(t) \right) = (c_\rho^*, k_\rho^*, x_\rho^*).$$

The path is characterized by an initial investment phase, in which $c_\rho(t) = 0$, followed by a convergence phase, in which $c_\rho(t) > c_\rho^*$.

Proof. *The convergence phase.* Let $x_\rho(\tau) \in (x_\rho^*, \bar{x})$. Since there is a stable manifold leading to the saddle point (x_ρ^*, z^*) (see Figure 4), there exists $z(\tau) \in (0, x_\rho^*)$ such that the path $\{x_\rho(t), z(t)\}_{t=\tau}^{\infty}$ satisfies (7) and (8) and converges to (x_ρ^*, z^*). The path $\{c_\rho(t), k_\rho(t), x_\rho(t)\}_{t=\tau}^{\infty}$, where $k_\rho = \kappa(z)$ and $c_\rho = x_\rho f(k_\rho) - \delta k_\rho - \dot{k}_\rho$, is feasible from $(k_\rho(\tau), x_\rho(\tau))$. It follows from (6), the concavity of f, and the properties of the path that $c_\rho(t) > c_\rho^*$ for all $t > \tau$.

The investment phase. Since the initial stock of produced capital is smaller than the size corresponding to the converging interior path, the latter is not feasible from $(k(0), x(0))$. Consider the path $\{c_\rho(t), k_\rho(t), x_\rho(t)\}_{t=0}^{\infty}$ feasible from $(k(0), x(0))$, where $c_\rho(t) = 0$ during the investment phase. Throughout the investment phase, $\dot{k}_\rho > 0$ and $\dot{x}_\rho \leq 0$ by Conditions 1 and 2, since $\xi(k(0)) < \bar{x}$ (and thus $k(0) > 0$) and, for each (x_ρ, k_ρ), $k_\rho < \kappa(z)$ where (x_ρ, z) is on the stable manifold (and thus $x_\rho f(x_\rho) - \delta k_\rho > 0$ by (6) and the concavity of f). Hence, the path reaches the converging interior path at some $t = \tau$.

By the Appendix, $\{c_\rho(t), k_\rho(t), x_\rho(t)\}_{t=0}^{\infty}$ is the unique optimal path from $(k(0), x(0))$. ∎
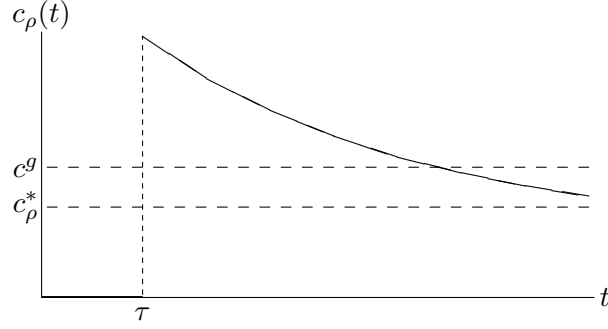
Figure 2: The optimal path of consumption.

It follows from the analysis of the Appendix that

$$p > p^* \quad \text{and} \quad p > q + p \frac{\rho + \delta}{x_\rho f'(k_\rho)}$$

in the investment phase $(0 < t < \tau)$. Hence, the marginal benefit of k accumulation exceeds both (a) the marginal utility of consumption and (b) the marginal cost of x depletion plus the marginal steady state cost of production. This leads to zero consumption and maximizes accumulation of produced capital.

In the convergence phase (all $t > \tau$), the rates of consumption and k accumulation are adjusted such that

$$p^* = p = q + p \frac{\rho + \delta}{x_\rho f'(k_\rho)} \ .$$

The resulting consumption path is illustrated in Figure 2.

The unique optimal path can be implemented in a competitive market economy, provided that the consumers lack intertemporal inequality aversion (as entailed by the linear utility function), their intertemporal preferences (including their intergenerational altruism) is reflected by the discount rate ρ, and the cost of using the services of the natural environment is internalized. The former of these assumption (that consumers have no preferences for consumption smoothing) is troublesome. Hence, the purpose of Lemma 2 is not to describe the smooth path that an actual economy would implement, but to demonstrate the *possibility* of convergence and highlight the *qualitative* properties of the converging path.

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3 Comparative dynamics

The set of steady states is one-dimensional. Since economic activity has environmental effects, a larger stock of produced capital leads to a smaller stock of natural capital. This negative relationship is expressed by the function ξ:

Condition 1* $\quad x^* = \xi(k^*)$.

Hence, steady state consumption is a function solely of k^*:

Condition 2* $\quad c^* = \xi(k^*)f(k^*) - \delta k^* \equiv g(\xi(k^*)) - \delta k^*$.

Figure 2 illustrates the consumption path during the transition between steady states where the new equilibrium has a larger stock of produced capital, but lower steady state consumption. Such a transition may be undertaken at a positive discount rate even when leading to a lower steady state state consumption because the intermediate gain may more than cancel out initial and eventual losses. Impatience clearly prevents the transition at higher values of ρ (since then the initial loss dominates the intermediate gain), suggesting that steady state consumption may be positively related to the discount rate (i.e., PCB). The transition is also suboptimal at lower discount rates (since then the eventual loss is weighted too heavily compared to the intermediate gain), indicating that steady state consumption approaches the maximum sustainable level as ρ approaches zero.

This section is organized as follows. The stock of produced capital maximizing steady state consumption (i.e., corresponding to the *Golden Rule*) is identified in Lemma 3. Steady states in which k^* exceeds the Golden Rule stock are referred to as *paradoxical*. It is demonstrated in Lemma 4 that for some positive discount rates, the optimal path may converge to a paradoxical steady state. The reason is that by increasing the stock of produced capital above the Golden Rule size, the economy can for a finite period of time enjoy a consumption rate exceeding the maximum sustainable level. The main result (that the optimality of paradoxical steady states implies PCB) is pointed out in the Theorem. The propositions are illustrated by a numerical example.

By Conditions 4 and 5, the CSS corresponding to ρ satisfies

Condition 4* $\quad \rho = \left(\frac{1}{v^*} - 1\right) f(k_\rho^*) + g'(\xi(k_\rho^*))\,,$

Condition 5* $\quad \rho = (1 - v^*)\xi(k_\rho^*)f'(k_\rho^*) - \delta\,.$

Since $\xi(0)f'(0) = \infty$, $\xi(\infty)f'(\infty) = 0$, and $d(\xi f')/dk^* < 0$ when $\xi(k^*)f'(k^*) > 0$, there exists a unique stock $\bar{k}$ of produced capital satisfying $0 = \xi(\bar{k})f'(\bar{k}) - \delta$. We have that $\bar{k}$ is an upper bound for the stock of produced capital at dynamically efficient steady states. It follows from the proof of Lemma 1 that (a) $v^* \in (0,1)$ for all $\rho > 0$ (since $z^* \equiv (1 - v^*)x_\rho^* \in (0, x_\rho^*)$), and (b) $k_\rho^* \in (0, \bar{k})$ for all $\rho > 0$ (since $x_\rho^* = \xi(k_\rho^*) \in (0, \bar{x})$, and $\xi(k_\rho^*)f'(k_\rho^*) - \delta > 0$ by Conditions 5*).

Define v^g and k^g as the limits of v^* and k_ρ^* as ρ approaches zero. Conditions 4* and 5* imply that $v^g \in (0,1)$ and $k^g \in (0, \bar{k})$. The following result establishes that k^g corresponds to the Golden Rule in the sense that steady state consumption is maximized when the stock of produced capital equals k^g.

Lemma 3 *Let c^* be steady state consumption when the stock of produced capital equals k^*. Then*

$$\frac{dc^*}{dk^*} > 0 \quad for \quad k^* \in (0, k^g),$$
$$\frac{dc^*}{dk^*} < 0 \quad for \quad k^* \in (k^g, \bar{k}).$$

Proof. Conditions 4* and 5* imply $g'(\xi(k^g)) + \delta f(k^*)/(\xi(k^g)f'(k^g) - \delta) = 0$. Furthermore, $d(g'(\xi) + \delta f/(\xi f' - \delta))/dk^* > 0$ for $k^* \in (0, \bar{k})$. Finally,

$$\frac{dc^*}{dk^*} = g'(\xi)\xi' - \delta = \left(g'(\xi) + \frac{\delta f}{\xi f' - \delta}\right) \frac{\xi f' - \delta}{g'(\xi) - f}$$

by Condition 2* since $\xi' = \xi f'/(g'(\xi) - f)$ for $k^* \in (0, \bar{k})$. Hence,

$$\text{sgn}\left(\frac{dc^*}{dk^*}\right) = -\text{sgn}\left(g'(\xi) + \frac{\delta f}{\xi f' - \delta}\right) = \text{sgn}(k^g - k^*)$$

since $\xi f' - \delta > 0$ and $g'(\xi) - f < 0$ for $k^* \in (0, \bar{k})$. ∎

By rewriting Conditions 2* as follows

$$c^* = (1 - v^*)\xi(k^*)f(k^*) - \delta k^* + v^* g(\xi(k^*))$$

and applying Conditions 4* and 5* we obtain an alternative result for dc^*/dk^*:

$$\begin{aligned}
\frac{dc^*}{dk^*} &= (1 - v^*)\xi(k^*)f'(k^*) - \delta + \left((1 - v^*)f(k^*) + v^* g'(\xi(k^*))\right)\xi'(k^*) \\
&= \rho(1 + v^*\xi'(k^*)).
\end{aligned} \tag{9}$$

Since steady state consumption must decrease when moving away from the Golden Rule stocks, it follows from (9) that k_ρ^* must increase beyond k^g for small discount

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rates if and and only if $1 + v^g \xi'(k^g) < 0$. Moreover, since Conditions 4^* and 5^* and the properties of the functions f, g, and ξ imply that the steady state value of natural capital in terms of produced capital decreases with the discount rate,

$$\frac{dv^*}{d\rho} < 0 \quad \text{for all} \quad \rho > 0,$$

and $v^* \to 0$ as $\rho \to \infty$, it follows that k_ρ^* must eventually backtrack and decrease with steady state consumption for $k_\rho^* < k^g$. This behavior of the steady state stock of produced capital as a function of ρ is shown in the following result.

Lemma 4 *Let $(c_\rho^*, k_\rho^*, x_\rho^*)$ be the CSS corresponding to the discount rate ρ. Then there exists $\rho_1 \geq 0$, where $\rho_1 > 0$ if and only if $1 + v^g \xi'(k^g) < 0$, such that*

$$\frac{dk_\rho^*}{d\rho} < 0 \quad \text{for all} \quad \rho > \rho_1 \,.$$

If $\rho_1 > 0$, then

$$\frac{dk_\rho^*}{d\rho} > 0 \quad \text{for} \quad 0 < \rho < \rho_1 \,.$$

Furthermore, if $\rho_1 > 0$, then there exists ρ_2, where $\rho_2 > \rho_1$, such that $k_{\rho_2}^* = k^g$.

Proof. Conditions 4^* and 5^* imply that k_ρ^* is a continuous and differentiable function of ρ satisfying that $\text{sgn}\left(dk_\rho^*/d\rho\right) = \text{sgn}\left(n_\rho\right)$, where n_ρ is a continuous and differentiable function of ρ defined by

$$n_\rho := v^* - \varpi(k_\rho^*) \quad \text{and} \quad \varpi(k^*) := \sqrt{\frac{f(k^*)}{\xi(k^*) f'(k^*)}} \,.$$

Since $dv^*/d\rho < 0$, it follows that $dn_\rho/d\rho < 0$ if $n_\rho = 0$. Hence, there are two cases.

Case 1: $v^g \leq \varpi(k^g)$. $n_\rho < 0$ and $dk_\rho^*/d\rho < 0$ for all $\rho > 0$.

Case 2: $v^g > \varpi(k^g)$. Suppose $n_\rho > 0$ for all $\rho > 0$. Then $k_\rho^* > k^g$ and $\xi(k_\rho^*) f'(k_\rho^*) < \xi(k^g) f'(k^g)$ for all $\rho > 0$. This is impossible since, by Condition 5^*,

$$\xi(k^g) f'(k^g) = (1 - v^*) \xi(k_{\bar\rho}^*) f'(k_{\bar\rho}^*) < \xi(k_{\bar\rho}^*) f'(k_{\bar\rho}^*) \,,$$

where $\bar\rho$ is determined by $\bar\rho = \xi(k^g) f'(k^g) - \delta$. Hence, let $\rho_1 > 0$ be defined by $n_{\rho_1} = 0$. Then $n_\rho > 0$ and $dk_\rho^* > 0$ for $0 < \rho < \rho_1$, and $n_\rho < 0$ and $dk_\rho^*/d\rho < 0$ for all $\rho > \rho_1$. Furthermore, since k_ρ^* is a continuous function of ρ and $k_{\rho_1}^* < k^g < k_{\bar\rho}^*$, there exists ρ_2, where $\rho_1 < \rho_2 < \bar\rho$, such that $k_{\rho_2}^* = k^g$. We have that

$$\frac{f(k^g)}{v^g \xi(k^g) f'(k^g)} = -\frac{g'(\xi(k^g))}{\delta} = -\frac{1}{\xi'(k^g)} \,,$$

where the left equality follows from Conditions 4^* and 5^* and the right inequality is implied by Lemma 3 since $dc^*/dk^* = g'(\xi)\xi' - \delta$. Hence, $v^g > \varpi(k^g)$ if and only if $v^g > -1/\xi'(k^g)$ (or equivalently, $1 + v^g\xi'(k^g) < 0$). ∎

Lemma 4 proves that the CSS is paradoxical for positive discount rates smaller than ρ_2, where ρ_2 is positive if produced capital is sufficiently productive; see remark at the end of this section. Hence, it may be optimal (and therefore, dynamically efficient) to maintain the stock of produced capital above the Golden Rule size. In the Solow-Swan model, all such paths are dynamically inefficient (see Burmeister and Dobell 1970, pp. 52–53). Paradoxical steady states are optimal in the present model because production both requires prior accumulation of produced capital, since $y = xf(k)$, and leads to posterior depletion of natural capital, since $\dot{x} = g(x) - y$. Thus, the costs of production are not only borne by the past, but also by the future through environmental effects. A positive discount rate may (by discounting future environmental effects) lead to "too much" economic activity. In the Solow-Swan model, a positive discount rate leads to "too little" economic activity since production only requires prior accumulation of produced capital.

If $\rho_1 > 0$, then the CSS corresponding to a discount rate ρ between 0 and ρ_1 corresponds also to some discount rate $\bar{\rho}$ between ρ_1 and ρ_2. Since ρ equals the real interest rate in steady states, it follows that the same technique is competitive at two different interest rates, but not at the intermediate interest rates. In the terminology of the Cambridge controversies, this is *reswitching* (see, e.g., Starrett 1969, p. 673, and Robinson 1975, p. 35).

The behavior of steady state consumption as a function of the discount rate is determined by Lemmas 3 and 4. The consumption behavior is normal for $\rho > \rho_2$; lower discount rates lead to larger stocks of produced capital and higher steady state consumption. Consumption reaches its maximum sustainable level c^g at ρ_2. For $\rho_1 < \rho < \rho_2$, lower discount rates lead to increasingly excessive stocks of produced capital and lower steady state consumption; PCB *occurs in this interval*. Consumption reaches a local minimum at ρ_1. Since steady state consumption approaches the Golden Rule level as ρ approaches zero, the economy switches back to smaller stocks of produced capital and higher steady state consumption for $0 < \rho < \rho_1$. We have established the following main result.

Theorem 1 *Let $(c_\rho^*, k_\rho^*, x_\rho^*)$ be the CSS corresponding to the discount rate ρ. Then*

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Table 1: The behavior of k_ρ^* and c_ρ^* when $f(k) = 3\sqrt{k}$, $g(x) = x(2-x)$, and $\delta = 1$

ρ	k_ρ^*	c_ρ^*
0	$\frac{9}{100}$	$\frac{9}{10}$
0.5	$\frac{1}{9}$	$\frac{8}{9}$
2.0	$\frac{1}{9}$	$\frac{8}{9}$
$\rho_2 = 3.4$	$\frac{9}{100}$	$\frac{9}{10}$

there exists $\rho_2 \geq 0$, where $\rho_2 > 0$ if and only if $1 + v^g \xi'(k^g) < 0$, such that

$$\frac{dc_\rho^*}{d\rho} < 0 \quad for\ all \quad \rho > \rho_2\,.$$

If $\rho_2 > 0$, then there exists ρ_1, where $0 < \rho_1 < \rho_2$, such that

$$\frac{dc_\rho^*}{d\rho} < 0 \quad for \quad 0 < \rho < \rho_1\,,$$

$$\frac{dc_\rho^*}{d\rho} > 0 \quad for \quad \rho_1 < \rho < \rho_2\,.$$

Furthermore, $c_{\rho_2}^* = c^g$.

Note that the economy is *regular* in the terminology of Burmeister and Long (1977, Definition 2; i.e., $dk_\rho^*/d\rho + v^* dx_\rho^*/d\rho < 0$ for all $\rho > 0$) if and only if PCB does *not* occur since, by (9),

$$\frac{dc_\rho^*}{d\rho} = \frac{dc^*}{dk^*}\frac{dk_\rho^*}{d\rho} = \rho\big(1 + v^* \xi'(k^*)\big)\frac{dk_\rho^*}{d\rho} = \rho\left(\frac{dk_\rho^*}{d\rho} + v^*\frac{dx_\rho^*}{d\rho}\right)\,.$$

Table 1 illustrates functional forms and parameter values for which the economy is not regular and PCB occurs in some range for the discount rate ρ.

If the technical productiveness (as expressed through the function f) is negligible compared to the natural productiveness (as expressed through the function g), then an increase in k has a negligible effect on the equilibrium stock of x ($\xi(k^*) \simeq \bar{x}$ and $\xi'(k^*) \simeq 0$ for relevant sizes of k^*), natural capital is abundant ($v^g \simeq 0$ and $k^g \simeq \bar{k}$), and the present model reduces to a Solow-Swan model. But if the technical productiveness is high compared to the natural productiveness, due to a large population and/or a high level of technology, then an increase in k has a large effect on the equilibrium stock of x ($|\xi'(k^g)|$ is large), natural capital is relatively scarce (v^g is close to 1), and paradoxical phenomena occur ($1 + v^g \xi'(k^g) < 0$).

4 Discussion

Within the setting of the model, it is only by accumulating produced capital (and thereby enhancing economic activity) that current production can be transferred to the future. Since economic activity has environmental effects, such a transfer is costly in terms of natural capital. The occurrence of PCB signifies that optimally accumulated produced capital (at some positive discount rate) cannot give rise to a perpetual stream of additional consumption due to this kind of negative effects on the stock of natural capital. Hence, the produced capital's rate of net marginal productivity in a steady state does not necessarily reflect the economy's ability to transform saved production into a constant and perpetual stream of additional future consumption. This line of thought suggests that the economy's rate of return on lending to the future may not equal the cost of borrowing from the future.

To facilitate this discussion, consider paths being optimal according to the maximin criterion. A path $\{c_\mu(t), k_\mu(t), x_\mu(t)\}_{t=0}^\infty$ feasible from $(k(0), x(0))$ is *maximin from* $(k(0), x(0))$ if

$$\inf_{t\geq 0} c(t) - \inf_{t\geq 0} c_\mu(t) \leq 0$$

for any path $(c(t), k(t), x(t))_{t=0}^\infty$ feasible from $(k(0), x(0))$.

Lemma 3 can used to analyze the properties of maximin paths in this model. There exists no pair of initial stocks from which $\inf_{t\geq 0} u(c_\mu(t)) > c^g$. By letting $\{c_\mu(t)\}_{t=0}^\infty$ satisfy $0 < c_\mu(t) = c^* \leq c^g$ for all t and applying Lemma 3, one obtains a phase diagram in (x, k)-space for each c^*. The converging paths, which are illustrated in the lower panel of Figure 3, can be shown to be efficient maximin paths; except for the efficient path with $c_\mu(t) \equiv c^g$ leading to the Golden Rule stocks, the paths are *regular maximin paths* (Burmeister and Hammond 1977, Dixit et al. 1980). It is consistent with the analysis of Burmeister and Hammond that no regular maximin path converges to a paradoxical steady state.

The following observations conclude the analysis of maximin paths in the vicinity of CSS. If $x(0) > \xi(k^g)$ and $k(0)$ is greater than the stock of produced capital corresponding to the path converging to the Golden Rule stocks, then a maximin path entails consuming more than c^g initially so that convergence to the Golden Rule stocks is ensured. If $\xi(k(0)) < x(0) \in (\xi(\bar{k}), \xi(k^g)]$, then the maximin path entails partial utilization of labor initially, converging to some $(\tilde{x}, \tilde{k})$ satisfying $\xi(\tilde{k}) = \tilde{x} \geq x(0)$, and then staying put at $(\tilde{x}, \tilde{k})$ if reached in finite time.

Let the economy be in the CSS $(c_\rho^*, k_\rho^*, x_\rho^*)$ corresponding to ρ. Consider an

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increase in the stock of produced capital, $\Delta k > 0$, and let $\Delta c = \inf_{t \geq 0} c_\mu - c_\rho^*$ denote the maximal uniform increase in consumption that such a stock increase can lead to, where $\{c_\mu(t), k_\mu(t), x_\mu(t)\}_{t=0}^\infty$ is maximin from $(k_\rho^* + \Delta k, x_\rho^*)$. Let

$$\rho_\ell = \limsup_{\Delta k \downarrow 0^+} \frac{\Delta c}{\Delta k}$$

be *the rate of return on lending to the future* when current positive saving are transformed into a uniform increase in future consumption. Likewise, consider a decrease in the stock of produced capital, $\Delta k < 0$, and let $\Delta c = \inf_{t \geq 0} c_\mu - c_\rho^*$ denote the minimal uniform decrease in consumption that such a stock increase must lead to, where $\{c_\mu(t), k_\mu(t), x_\mu(t)\}_{t=0}^\infty$ is maximin from $(k_\rho^* + \Delta k, x_\rho^*)$. Let

$$\rho_b = \liminf_{\Delta k \uparrow 0^-} \frac{\Delta c}{\Delta k}$$

be *the cost of borrowing from the future* when current negative savings are transformed into a uniform decrease in future consumption. The optimality under discounted utilitarianism of the CSS (cf. Lemma 2) implies that

$$\rho_\ell \leq \rho \leq \rho_b, \tag{10}$$

because otherwise it would be possible to increase the sum of discounted utilities by either positive or negative initial savings followed by maximin behavior.

If $\rho_1 > 0$, so that PCB occur, then the CSS corresponding to a discount rate ϱ between 0 and ρ_1 corresponds also to some discount rate $\bar{\rho}$ between ρ_1 and ρ_2. Hence, it follows from (10) that, in paradoxical steady states, the rate of return on lending to the future is smaller than the cost of borrowing from the future:

$$\rho_\ell \leq \varrho < \bar{\rho} \leq \rho_b.$$

Schedules for ρ_ℓ and ρ_b satisfying the above condition are sketched in the upper panel of Figure 3. This implies that there is a wedge between ρ_ℓ and ρ_b for CSS corresponding to $\rho \in [0, \rho_2]$. On the other hand, it follows from the properties of the regular maximin paths (see lower panel of Figure 3) and Assumptions 1–3 that the rate of return on lending to the future equals the cost of borrowing from the future for CSS corresponding to $\rho > \rho_2$.

The discrepancy between the rate of return on lending to the future and the cost of borrowing from the future for $0 \leq \rho \leq \rho_2$ represents a cost of intertemporal transfers of income. Since ρ_ℓ and ρ_b are not equal for $0 \leq \rho \leq \rho_2$, the technological
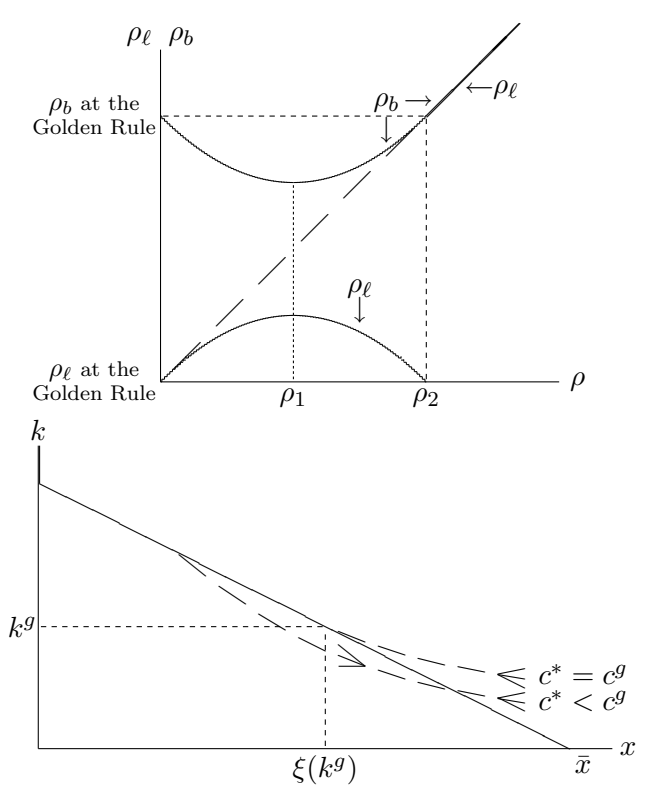
Figure 3: Rate of return on lending and cost of borrowing.

trade-off between well-being now and in the future is not measured by the rate of marginal productivity of "aggregated capital", as in the Solow-Swan model.

Because production requires prior accumulation of produced capital and leads to posterior depletion of natural capital, the value of natural capital in terms of produced capital, v, is positively related to the relative cost of future environmental effects in terms of past economic activity. Since there is no well-defined trade-off between the present and the future, this intertemporal (or intergenerational) comparison of costs cannot be based solely on a dynamic efficiency criterion. As v is a decreasing function of the rate at which future utilities are discounted, the comparison depends on the weight assigned to future consumption and, thus, must also involve normative judgements concerning intertemporal (or intergenerational) equity. This inference, which is supported by the result in Section 3 concerning the behavior of the relative value at the CSS ($dv^*/d\rho < 0$), has two consequences:

(a) Natural and produced capital cannot be aggregated.

(b) Along an optimal path that is not in a steady state, income (defined as $c_\rho + \dot{k}_\rho + v\dot{x}_\rho$, the sum of consumption and the value of net investments) depends on normative judgements concerning intergenerational equity. In particular, if natural capital is turned into produced capital, then the economy's income is an increasing function of ρ.

The two consequences are interrelated. The essence of the aggregation problem is the ambiguity as to whether the economy is saving or dissaving when its structure of capital is altered. To what extent can accumulated produced capital make up for depleted natural capital when their relative value is dependent on the utility discount rate? Since the contributions of Solow (1974) and Hartwick (1977), this question has been related to the concept of Hartwick's rule and a discussion of the significance of this rule (see, e.g., Asheim et al. 2003). An investigation into how, in the present model, a change in consumption affects income in the steady state satisfying the Golden Rule sheds light on this question. The following result is useful for this purpose.

Lemma 5 *Along an optimal path $\{c_\rho(t), k_\rho(t), x_\rho(t)\}_{t=0}^\infty$, the time derivative of the economy's income, $c_\rho + \dot{k}_\rho + v\dot{x}_\rho$, equals the real interest rate, $\rho - \frac{\dot{p}}{p}$, times the value of net investments, $\dot{k}_\rho + v\dot{x}_\rho$:*

$$\frac{d}{dt}\left(c_\rho + \dot{k}_\rho + v\dot{x}_\rho\right) = \left(\rho - \frac{\dot{p}}{p}\right)\left(\dot{k}_\rho + v\dot{x}_\rho\right). \tag{11}$$

Proof. The lemma follows from Asheim and Weitzman (2001, Proposition 3) (see also Dixit et al. 1980, proof of Theorem 1), and can alternatively be shown in the context of this model by using Conditions 1, 2, 4, and 5. ∎

Starting from a steady state (e.g., with initial stocks corresponding to the Golden Rule), the consumption rate c cannot instantaneously influence the stock of natural capital ($\dot{x} = 0$). By letting c fall below the steady state consumption level, only produced capital is accumulated ($\dot{k} > 0$), so that (11) simplifies to

$$\frac{d}{dt}\left(c_\rho + \dot{k}_\rho + v\dot{x}_\rho\right) = \left(\rho - \frac{\dot{p}}{p}\right)\dot{k}_\rho$$

at the moment the path leaves the steady state.

Under a zero discount criterion, there is a *zero* growth potential with initial steady state stocks corresponding to the Golden Rule:

$$\rho - \frac{\dot{p}}{p} = (1 - v^g)\xi(k^g)f'(k^g) - \delta = 0$$

by equation (5) since Condition 5* is satisfied for $\rho = 0$, $k_\rho^* = k^g$, and $v^* = v^g$. Accumulated produced capital cannot offset the depletion of natural capital because k^g maximizes sustainable consumption.

When future utilities are discounted at a positive rate, there is a *positive* growth potential even with initial stocks corresponding to the Golden Rule. If paradoxical steady states are optimal for $0 < \rho < \rho_2$, then equation (5) and the analysis of the Appendix imply that

$$\rho - \frac{\dot{p}}{p} = (1 - v)\xi(k^g)f'(k^g) - \delta > \rho > 0$$

for $0 < \rho < \rho_2$ (keeping in mind that we are at the beginning of the investment phase of the path considered in Lemma 2). Implementing such growth through the accumulation of produced capital would lead to a potential Pareto improvement for all future generations if the economy's rate of return on lending to the future ρ_ℓ were equal to the produced capital's rate of net marginal productivity $(1-v)\xi(k^g)f'(k^g) - \delta$. However, no *actual* Pareto improvement is feasible since the economy is not able to transform saved production into a constant and perpetual stream of additional future consumption, recalling that ρ_ℓ equals zero for initial stocks corresponding to the Golden Rule. The accumulated produced capital offsets the depletion of natural capital only if the economy is willing to accept present gains of a finite duration as an adequate compensation for infinitely lasting future losses. Present benefits cannot

be redistributed to the future due to the cost of intertemporal (or intergenerational) transfers of income: no reinvestment possibilities exist.

Page (1977) has suggested that discounting additive utility is not an appropriate criterion of intergenerational equity in such cases. Instead the economy should adopt an 'almost anywhere dominant' standpoint, which would rule out consumption paths (like the one illustrated in Figure 2) favoring a finite number of generations at the expense of an infinite number, and thus, prevent accumulation of produced capital above the Golden Rule size. Asheim et al. (2006) have recently shown how such a normative position can be incorporated in an axiomatic analysis in the tradition of Koopmans (1960), entailing that sustainability is imposed as a side constraint in a discounting procedure.

5   Concluding remarks

By analyzing a generalized Solow-Swan model, we have shown that in an economy interacting with the natural environment, intertemporal (or intergenerational) transfers of consumption are costly. Due to this cost, substituting produced capital for depleted natural capital have consequences for intergenerational equity: well-being of future generations depends on maintaining natural capital. In this respect, the model formalizes the opinion of many conservationists, namely that a high level of present economic activity may not benefit the future due to environmental effects. Here such a problem of intergenerational equity has been related to the observation that the economy's rate of return on lending to the future may not equal the cost of borrowing from the future.

The model is clearly restrictive in the sense that sustainable consumption is effectively bounded by the natural productiveness. Empirical evidence does however support its postulated complementarity between economic activity and environmental effects: important classes of production techniques that require prior accumulation of produced capital do in fact lead to posterior depletion of natural capital. Several such examples are the following:

(a) Fossil fuel consumption, being the result of prior economic development, increases the stock of greenhouse gases in the atmosphere, having potentially catastrophic long term environmental effects;

(b) Nuclear power generation, being the result of prior investment, gives rise to

long lasting reduction of environmental quality due to accumulation of radio-active waste products;

(c) Resource exploitation (e.g., strip mining; see Prince and Rosser 1985), requiring prior development costs, leads to serious long term environmental effects;

(d) Intensive agricultural practices using fertilizers and pesticides, having been developed through prior research in order to enhance current production, seem to have negative long term effects on the soil's productiveness;

(e) Medical treatment by means of antibiotics, having been developed through prior research in order to improve health conditions, leads to the emergence of resistant strains of bacteria and thus increased future health problems.

In general, economic activity affects the natural environment by extracting and emitting materials. Empirical observation indicates that by capital intensifying production techniques, the material throughput is increased or new materials are introduced into the economic process, such that the availability of non-produced resources tends to be reduced. Moreover, theoretical considerations suggest that production techniques not having future environmental effects are prone to be exceptional if future utilities are discounted: at any point in time, the economy will seek to engage in activity that converts natural capita into sufficiently large short run benefits. Such behavior is reinforced by a large population and a high technological level, and it may well be prevalent even if natural capital is relatively scarce.

Appendix: Sufficient conditions for optimality

Conditions 4–6 are necessary for optimality if one can justify the usual "normalization procedure" (i.e. setting equal to unity the constant multiplier, say λ_0, associated with the integrand of the objective function) when formulating the Hamiltonian. One can set $\lambda_0 = 1$ if certain growth conditions are satisfied (cf. Sydsæter et al., 2005, Theorem 9.11.2, and its more general version in Seierstad and Sydsæter, 1987, Theorem 16). This result cannot be applied here without modifying the optimization problem slightly.

In this appendix we show that paths satisfying Conditions 4–6 as well as the additional properties (12)–(14) are indeed optimal, and that these conditions and

properties are satisfied for the paths considered in Lemmas 1 and Lemma 2. None of the results of this paper depends on Conditions 4–6 being necessary.

Proposition 1 *Let $\{c_\rho(t), k_\rho(t), x_\rho(t)\}_{t=0}^{\infty}$ be an interior path feasible from $(k(0), x(0))$, which satisfies $\ell(t) = 1$ (i.e., full utilization of labor) for all t. If there exist continuously differentiable functions $p(t)$ and $q(t)$ satisfying Conditions 4–6 and the following additional properties,*

$$p(t) > 0 \quad and \quad q(t) > 0 \quad for \ all \ t, \tag{12}$$

$$\lim_{t\to\infty} p(t)e^{-\rho t} = 0 \quad and \quad \lim_{t\to\infty} q(t)e^{-\rho t} = 0 \tag{13}$$

$$\frac{d\left((p(t) - q(t))\, x_\rho(t) e^{-\rho t}\right)}{dt} \leq 0 \quad for \ all \ t, \tag{14}$$

then $\{c_\rho(t), k_\rho(t), x_\rho(t)\}_{t=0}^{\infty}$ is the unique optimal path from $(k(0), x(0))$.

Proof. Showing sufficiency is complex because $xf(k)$ is not a concave function. Note that (13) and (14) imply $(p - q)x_\rho \geq 0$ and $p - q \geq 0$ for all $t \geq 0$ since the path is interior. Let $\{c(t), k(t), x(t)\}_{t=0}^{\infty}$ be any path feasible from $(k(0), x(0))$. This alternative path may have partial utilization of labor; hence, we only require that $\{\ell(t)\}_{t=0}^{\infty}$ is a piecewise continuous path satisfying $0 \leq \ell(t) \leq 1$ for all t. Then

$$\int_0^\infty \left[u(c) - u(c_\rho)\right] e^{-\rho t} dt$$

$$= \int_0^\infty \left[u(c) - u(c_\rho) + p\left(\left(x\tilde{f}(k, \ell) - c - \delta k - \dot{k}\right) - \left(x_\rho f(k_\rho) - c_\rho - \delta k_\rho - \dot{k}_\rho\right)\right)\right.$$

$$\left. + q\left(\left(g(x) - x\tilde{f}(k, \ell) - \dot{x}\right) - \left(g(x_\rho) - x_\rho f(k_\rho) - \dot{x}_\rho\right)\right)\right] e^{-\rho t} dt$$

(by Conditions 1 and 2)

$$\leq \int_0^\infty \left[(p - q)\left(x - x_\rho\right)\left(\tilde{f}(k, \ell) - f(k_\rho)\right)\right.$$

$$- p\left((\dot{k} + \delta k) - (\dot{k}_\rho + \delta k_\rho)\right) + (p - q)x_\rho\left(\tilde{f}(k, \ell) - f(k_\rho)\right)$$

$$\left. - q\left((\dot{x} - g(x)) - (\dot{x}_\rho - g(x_\rho))\right) + (p - q)\left(x - x_\rho\right)f(k_\rho)\right] e^{-\rho t} dt$$

(by Condition 6 and by rearranging terms). Conditions 4 and 5 imply that the four last terms are non-positive since

(a) $(p - q)x_\rho\left(\tilde{f}(k, \ell) - f(k_\rho)\right) \leq \left(k - k_\rho\right)(p - q)x_\rho f'(k_\rho)$

$$\text{by } f'' < 0 \text{ and } (p - q)x_\rho \geq 0 \text{ since } \tilde{f}(k, l) \leq f(k)$$

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(b) $q\big(g(x) - g(x_\rho)\big) \leq (x - x_\rho)qg'(x_\rho)$ by $g'' < 0$ and $q > 0$,

(c) $-\int_0^\infty p(\dot{k} - \dot{k}_\rho)e^{-\rho t}dt = \int_0^\infty (k - k_\rho)(\dot{p} - p\rho)e^{-\rho t}dt$,

(d) $-\int_0^\infty q(\dot{x} - \dot{x}_\rho)e^{-\rho t}dt = \int_0^\infty (x - x_\rho)(\dot{q} - q\rho)e^{-\rho t}dt$,

where (c) and (d) follows from (13) using integration by parts. Concerning the first term, Assumption 1 and Condition 1 imply that

$$(x - x_\rho)\big(\tilde{f}(k, \ell) - f(k_\rho)\big) \leq (x - x_\rho)\left(\frac{\dot{x}_\rho}{x_\rho} - \frac{\dot{x}}{x}\right),$$

because $\mathrm{sgn}(x - x_\rho) = \mathrm{sgn}(g(k_\rho)/x_\rho - g(x)/x) = \mathrm{sgn}((\dot{x}_\rho + x_\rho f(k_\rho))/x_\rho - (\dot{x} + \tilde{f}(k, \ell))/x) = \mathrm{sgn}((\dot{x}_\rho/x_\rho - \dot{x}/x) - (\tilde{f}(k, \ell) - f(k_\rho)))$. Since $p - q \geq 0$, it therefore follows that

$$\int_0^\infty \big[u(c) - u(c_\rho)\big]e^{-\rho t}dt \leq \int_0^\infty (p - q)(x - x_\rho)\left(\frac{\dot{x}_\rho}{x_\rho} - \frac{\dot{x}}{x}\right)e^{-\rho t}dt$$

$$= \int_0^\infty (p - q)x_\rho\left(\frac{x}{x_\rho} - 1\right)\tfrac{d}{dt}\big(\ln x_\rho - \ln x\big)e^{-\rho t}dt = \int_0^\infty (p - q)x_\rho e^{-\rho t}\big(e^{-m} - 1\big)\dot{m}dt,$$

where $m(t)$ is defined by $m(t) := \ln x_\rho - \ln x$. Note that $m(0) = 0$. Let T be an arbitrary positive number. Then, using integration by parts,

$$\int_0^T (p - q)x_\rho e^{-\rho t}\big(e^{-m} - 1\big)\dot{m}dt$$

$$= (p(0) - q(0))\,x_\rho(0) - (p(T) - q(T))\,x_\rho(T)e^{-\rho T}\big(e^{-m(T)} + m(t)\big)$$

$$+ \int_0^T \tfrac{d}{dt}\big((p - q)x_\rho e^{-\rho t}\big)\big(e^{-m} + m\big)dt$$

$$\leq (p(0) - q(0))\,x_\rho(0) - (p(T) - q(T))\,x_\rho(T)e^{-\rho T} + \int_0^T \tfrac{d}{dt}\big((p - q)x_\rho e^{-\rho t}\big)dt,$$

since $(p(T) - q(T))\,x_\rho(T) \geq 0$, $d\big((p-q)x_\rho e^{-\rho t}\big)/dt \leq 0$ for all $0 < t < T$ by (14), and $e^{-m} + m \geq 1$ for all m. The last expression is equal to zero, and the optimality of $\{c_\rho(t), k_\rho(t), x_\rho(t)\}_{t=0}^\infty$ follows by letting T approach infinity. Uniqueness is implied by $g'' < 0$ and $q > 0$. ∎

The existence, uniqueness, and saddle point characteristics of (x_ρ^*, z^*).

Consider equations (7) and (8). Extend the domain of $\kappa(\cdot)$ to non-positive values of z as follows: $\kappa(z) = 0$ for $z \in (-\infty, 0]$. Then $\xi(\kappa(\cdot))$ is a continuous function from $\mathbb{R}$ into $[0, \bar{x}]$ with the following properties:

$$x_\rho = \xi(\kappa(z)) \quad \text{implies} \quad \dot{x}_\rho = 0$$

$$\xi(\kappa(z)) = \bar{x} \ \text{ for } z \in (-\infty, 0] \quad \text{and} \quad \xi(\kappa(z)) \in [0, \bar{x}] \ \text{ for } z \in (0, \infty).$$
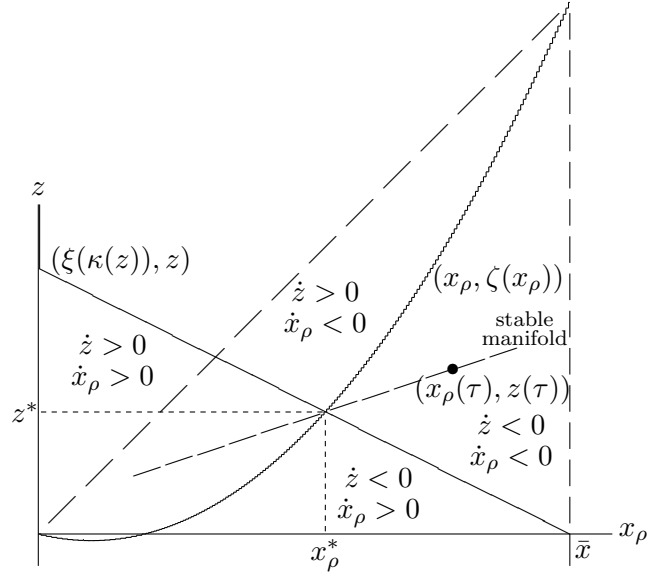
Figure 4: A phase diagram.

Since $\partial \dot{z}/\partial z = g(x_\rho)/x_\rho - g'(x_\rho) + \rho \geq \rho > 0$ for all z and x_ρ and $\dot{z} = g(x_\rho)$ if $z = x_\rho$, there exists a continuous function from $\zeta(\cdot)$ from $[0, \bar{x}]$ into $\mathbb{R}$ with the following properties

$$z = \zeta(x_\rho) \quad \text{if and only if} \quad \dot{z} = 0$$

$$\zeta(0) = 0, \quad \zeta(x_\rho) < x_\rho \text{ for } x_\rho \in (0, \bar{x}), \quad \text{and} \quad \zeta(\bar{x}) = \bar{x}.$$

By Brouwer's fixed point theorem, there exists $x_\rho^* \in [0, \bar{x}]$ such that $x_\rho^* = \xi(\kappa(\zeta(x_\rho^*)))$. It follows that $0 < x_\rho^* < \bar{x}$ and $z^* = \zeta(x_\rho^*) > 0$ since

$$x_\rho^* = 0 \quad \Rightarrow \quad z^* = \zeta(x_\rho^*) \leq 0 \quad \Rightarrow \quad x_\rho^* = \xi(\kappa(z^*)) = \bar{x}$$

$$\Rightarrow \quad z^* = \zeta(x_\rho^*) > 0 \quad \Rightarrow \quad x_\rho^* = \xi(\kappa(z)) < \bar{x}.$$

Furthermore, $z^* < x_\rho^*$, since $z^* = \zeta(x_\rho^*) \geq x_\rho^*$ implies $x_\rho^* = 0$ or $x_\rho^* = \bar{x}$. Finally, (x_ρ^*, z^*) is unique and has saddle point characteristics since

$$\frac{\partial \dot{x}_\rho}{\partial x_\rho} < 0 \quad \text{and} \quad \frac{\partial \dot{x}_\rho}{\partial z} < 0 \quad \text{when} \quad x_\rho = \xi(\kappa(z)) \in (0, \bar{x})$$

$$\frac{\partial \dot{z}}{\partial x_\rho} < 0 \quad \text{and} \quad \frac{\partial \dot{z}}{\partial z} > 0 \quad \text{when} \quad z = \zeta(x_\rho) \in (0, x_\rho).$$

These results are illustrated by a phase diagram in Figure 4.

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Lemma 1 and sufficient optimality conditions.

Properties (12) and (13) are satisfied since both p and q are positive and constant. Property (14) is satisfied since $(p - q)x_\rho \equiv p^*(1 - v^*)x_\rho^* \equiv p^*z^*$ is positive and constant.

Lemma 2 and sufficient optimality conditions.

If $p > 0$, then it follows from Conditions 1, 4, and 5 that

$$\dot{z} = \tfrac{z}{x_\rho}g(x_\rho) - (x_\rho - z)\big(zf'(k_\rho) - \delta - g'(x_\rho)\big),\qquad(15)$$

which becomes (8) under the additional assumption that $\dot{p} = 0$. Since $\{k_\rho\}_{t=0}^\infty$ and $\{x_\rho\}_{t=0}^\infty$ are absolutely continuous, it follows that z is continuously differentiable.

The convergence phase. By the linearity of u, Condition 6 is satisfied by letting $p = p^* > 0$ for all $t \geq \tau$. Since $p > 0$ and $\dot{p} = 0$ for $t \geq \tau$, $\{x_\rho(t), z(t)\}_{t=\tau}^\infty$ moves along the stable manifold of Figure 4 leading to the saddle point (x_ρ^*, z^*). The properties of the converging path from $(x_\rho(\tau), z(\tau))$ with $x_\rho(\tau) > x_\rho^*$ imply that $z > 0$ and $\dot{z} < 0$ for all $t \geq \tau$. It follows from (6), (7), and (8) that Conditions 4 and 5 are satisfied by letting $q = p^*v \equiv p^*\big(1 - z/x_\rho\big)$ for all $t \geq \tau$.

The investment phase. Turn now to the properties of p and q for $0 < t < \tau$ when Conditions 4 and 5 are imposed. Since $p(\tau) = p^* > 0$ and p is continuous, we must have $p > 0$ for all $t > \tau - \epsilon$, for some $\epsilon > 0$. Hence, (15) holds for $\tau - \epsilon < t < \tau$. Furthermore, for $\tau - \epsilon < t < \tau$,

(a) $z = x_\rho$ implies $\dot{z} > 0$ by (15) since $0 < x_\rho < 1$,

(b) $z \in (0, x_\rho)$ and $\dot{z} = 0$ imply that $d\dot{z}/dt$ exists and is positive since (15) holds, and $\dot{k}_\rho > 0$ and $\dot{x}_\rho \leq 0$ (by the proof of Lemma 2).

Hence, $z(\tau) > 0$ and $\dot{z}(\tau) < 0$ imply that $z < x$ for $\tau - \epsilon < t < \tau$ by (a) and $z > 0$ and $\dot{z} < 0$ for $\tau - \epsilon < t < \tau$ by (b). Therefore, $0 < z < x_\rho$ and $\dot{z} < 0$ for $\tau - \epsilon < t < \tau$.

It follows from the previous paragraph that $k_\rho < k_\rho(\tau) = \kappa(z(\tau)) < \kappa(z)$ for $\tau - \epsilon < t < \tau$, since $\dot{k}_\rho > 0$ and $\dot{z} < 0$ for $\tau - \epsilon < t < \tau$ and κ is strictly increasing. By (5) (which follows from Condition 5) and (6) this means that

$$-\tfrac{\dot{p}}{p} = zf'(k) - (\rho + \delta) > zf'(\kappa(z)) - (\rho + \delta) \equiv 0$$

for $\tau - \epsilon < t < \tau$ since $z > 0$, implying that $\dot{p} < 0$ for $\tau - \epsilon < t < \tau$. This implies that we can set $\epsilon = \tau$, so that $p > p^*$ and $\dot{p} < 0$ for $0 < t < \tau$. Therefore, $0 < z < x_\rho$ and $\dot{z} < 0$ for $0 < t < \tau$.

It follows from the above analysis that, for $0 < t < \tau$, Conditions 4 and 5 are satisfied by construction, while Condition 6 is satisfied since $p > p^*$ and $c_\rho = 0$.

The analysis of the convergence and investment phases entails that $p > 0$, $\dot{p} \leq 0$, $0 < z < x_\rho$, and $\dot{z} < 0$ for all $t > 0$. Since thus $p > 0$ and $q = pv = p(1 - z/x_\rho) > 0$ for all $t > 0$, property (12) is satisfied. Furthermore, both p and q converge to the positive constants p^* and $q^* := p^*(1 - z^*/x_\rho^*)$, implying that property (13) is satisfied. Finally, $(p - q)x_\rho \equiv pz$ is positive and decreasing, implying that property (14) is satisfied.

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