Sustainable recursive social welfare functions*

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Abstract

Koopmans' axiomatization of discounted utilitarianism is based on seemingly compelling conditions, yet this criterion leads to hard-to-justify outcomes. The present analysis considers a class of sustainable recursive social welfare functions within Koopmans' general framework. This class is axiomatized by means of a weak new equity condition ("Hammond Equity for the Future") and general existence is established. Any member of the class satisfies the key axioms of Chichilnisky's "sustainable preferences". The analysis singles out one of Koopmans' original conditions as particularly questionable from an ethical perspective.

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1 Introduction

How should we treat future generations? From a normative point of view, what are the present generation's obligations towards the future? What ethical criterion for intergenerational justice should be adopted if one seeks to respect the interests of future generations?

These questions can be approached and answered in at least two ways:

1. Through an axiomatic analysis one can investigate on what ethical conditions various criteria for intergenerational justice are based, and then proceed to evaluate the normative appeal of these conditions.

2. By considering different technological environments, one can explore the consequences of various criteria for intergenerational justice, and compare the properties of the intergenerational well-being streams that are generated.

It is consistent with Rawls' (1971) *reflective equilibrium* to do both: criteria for intergenerational justice should be judged both by the ethical conditions on which they build and by their consequences in specific technological environments. In particular, we may question the appropriateness of a criterion for intergenerational justice if it produces unacceptable outcomes in relevant technological environments. This view has been supported by many scholars, including Koopmans (1967), Dasgupta and Heal (1979, p. 311), and Atkinson (2001, p. 206).

When evaluating long-term policies, economists usually suggest to maximize the sum of discounted utilities. On the one hand, such *discounted utilitarianism* has been given a solid axiomatic foundation by Koopmans (1960).[1] On the other hand, this criterion has ethically questionable implications when applied to economic models with resource constraints. This is demonstrated by Dasgupta and Heal (1974) in the so-called Dasgupta-Heal-Solow (DHS) model of capital accumulation and resource depletion (Dasgupta and Heal, 1974, 1979; Solow, 1974), where discounted

[1] For an alternative set of axioms leading to discounted utilitarianism, see Lauwers (1997).

utilitarianism for any positive discount rate undermines the well-being of generations in far future, even if sustainable streams with non-decreasing well-being are feasible.

In this paper we revisit Koopmans' framework, with numerical representability, stationarity and sensitivity as its key features. In Section 2 we consider conditions that are sufficient to numerically represent the social welfare relation by means of a recursive social welfare function satisfying stationarity and sensitivity, thereby echoing the analysis of Koopmans (1960, Sections 3–7). In this framework we introduce the equity condition ("Hammond Equity for the Future"), capturing the following ethical intuition: A sacrifice by the present generation leading to a uniform gain for all future generations cannot lead to a less desirable stream of well-being if the present remains better-off than the future even after the sacrifice.[2]

In Section 3 we point out that "Hammond Equity for the Future" is weak, as it is implied by all the standard consequentialist equity conditions suggested in the literature. We show that adding this condition leads to a class of sustainable recursive social welfare functions, where the well-being of the present generation is taken into account if and only if the future is better-off. Furthermore, we establish general existence by means of an algorithmic construction. Finally, we show that any member of this class of sustainable recursive social welfare functions satisfies the key axioms of Chichilnisky's (1996) "sustainable preferences", namely "No Dictatorship of the Present" and "No Dictatorship of the Future".

In Section 4 we offer results that identify which of the conditions used by Koopmans (1960) to axiomatize discounted utilitarianism is particularly questionable from an ethical perspective. The condition in question, referred to as "Independent Present" by us and listed as Postulate $3'$a by Koopmans (1960, Section 14), requires that the evaluation of two streams which differ during only the first two periods *not* depend on what the common continuation stream is. It is only by means of "Independent Present"—which in the words of Heal (2005) is "restrictive"

[2]Our condition is inspired from Hammond's (1976) Equity condition, but – as we will see – it is weaker and has not only an egalitarian justification.

and "surely not innocent"—that Koopmans (1960, Section 14) moves beyond the recursive form to arrive at discounted utilitarianism. We single out "Independent Present" as the culprit by showing that the addition of this condition contradicts both "Hammond Equity for the Future" and the Chichilnisky (1996) conditions.

In Section 5 we apply sustainable recursive social welfare functions for studying optimal harvesting of a renewable resource that yields amenities. In a companion paper (Asheim and Mitra, 2008) it is demonstrated how such functions can be used to solve the distributional conflicts in the DHS model. In both settings, our new criterion yields consequences that differ from those of discounted utilitarianism.

Koopmans (1960) has often been interpreted as presenting the definitive case for discounted utilitarianism. In Section 6 we discuss how our results contribute to a weakening of this impression, by exploring other avenues within the general setting of his approach. We also investigate the scope for our new equity condition "Hammond Equity for the Future" outside the Koopmans framework by *not* imposing that the social welfare relation is numerically representable.

All proofs are relegated to an appendix.

2 Formal setting and basic result

Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{Z}_+$ the set of non-negative integers. Denote by ${}_0\mathbf{x} = (x_0, x_1, \ldots, x_t, \ldots,)$ an infinite stream, where $x_t \in Y$ is a one-dimensional indicator of the well-being of generation t, and $Y \subseteq \mathbb{R}$ is a non-degenerate interval of admissible well-beings.[3] We will consider the set $\mathbf{X}$ of infinite streams bounded in well-being (see Koopmans, 1986b, p. 89); i.e., $\mathbf{X}$ is given by

$$\mathbf{X} = \{{}_0\mathbf{x} \in \mathbb{R}^{\mathbb{Z}_+} \mid [\inf_t x_t, \sup_t x_t] \subseteq Y\}.$$

[3]A more general framework is, as used by Koopmans (1960), to assume that the well-being of generation t depends on an n-dimensional vector $\mathbf{x}_t$ that takes on values in a connected set $\mathbf{Y}$. However, by representing the well-being of generation t by a scalar x_t, we can focus on intergenerational issues. In doing so, we follow, e.g., Diamond (1965), Svensson (1980), Chichilnisky (1996), Basu and Mitra (2003) and Bossert, Sprumont and Suzumura (2007).

By setting $Y = [0,1]$, this includes the important special case where $\mathbf{X} = [0,1]^{\mathbb{Z}_+}$. However, the formulation allows for cases where Y is not compact.

Denote by ${}_0\mathbf{x}_{T-1} = (x_0, x_1, \ldots, x_{T-1})$ and ${}_T\mathbf{x} = (x_T, x_{T+1}, \ldots, x_{T+t}, \ldots,)$ the T-head and the T-tail of ${}_0\mathbf{x}$. Write ${}_{\text{con}}z = (z, z, \ldots)$ for the stream of a constant level of well-being equal to $z \in Y$. Throughout this paper we assume that the indicator of well-being is at least ordinally measurable and level comparable across generations; Blackorby, Donaldson and Weymark (1984) call this "level-plus comparability".

For all ${}_0\mathbf{x}$, ${}_0\mathbf{y} \in \mathbf{X}$, we write ${}_0\mathbf{x} \geq {}_0\mathbf{y}$ if and only if $x_t \geq y_t$ for all $t \in \mathbb{Z}_+$, ${}_0\mathbf{x} > {}_0\mathbf{y}$ if and only if ${}_0\mathbf{x} \geq {}_0\mathbf{y}$ and ${}_0\mathbf{x} \neq {}_0\mathbf{y}$, and ${}_0\mathbf{x} \gg {}_0\mathbf{y}$ if and only if $x_t > y_t$ for all $t \in \mathbb{Z}_+$.

A *social welfare relation* (SWR) is a binary relation $\succsim$ on $\mathbf{X}$, where for all ${}_0\mathbf{x}$, ${}_0\mathbf{y} \in \mathbf{X}$, ${}_0\mathbf{x} \succsim {}_0\mathbf{y}$ entails that ${}_0\mathbf{x}$ is deemed socially at least as good as ${}_0\mathbf{y}$. Denote by $\sim$ and $\succ$ the symmetric and asymmetric parts of $\succsim$; i.e., ${}_0\mathbf{x} \sim {}_0\mathbf{y}$ is equivalent to ${}_0\mathbf{x} \succsim {}_0\mathbf{y}$ and ${}_0\mathbf{y} \succsim {}_0\mathbf{x}$ and entails that ${}_0\mathbf{x}$ is deemed socially indifferent to ${}_0\mathbf{y}$, while ${}_0\mathbf{x} \succ {}_0\mathbf{y}$ is equivalent to ${}_0\mathbf{x} \succsim {}_0\mathbf{y}$ and $\neg({}_0\mathbf{y} \succsim {}_0\mathbf{x})$ and entails that ${}_0\mathbf{x}$ is deemed socially preferred to ${}_0\mathbf{y}$.

All comparisons are made at time 0; hence, the notation ${}_T\mathbf{x} \succsim {}_{T'}\mathbf{y}$ where T, $T' \geq 0$ means ${}_0\mathbf{x}' \succsim {}_0\mathbf{y}'$ where, for all t, $x'_t = x_{T+t}$ and $y'_t = y_{T'+t}$.

A *social welfare function* (SWF) representing $\succsim$ is a mapping $W : \mathbf{X} \to \mathbb{R}$ with the property that for all ${}_0\mathbf{x}$, ${}_0\mathbf{y} \in \mathbf{X}$, $W({}_0\mathbf{x}) \geq W({}_0\mathbf{y})$ if and only if ${}_0\mathbf{x} \succsim {}_0\mathbf{y}$. A mapping $W : \mathbf{X} \to \mathbb{R}$ is *monotone* if ${}_0\mathbf{x} \geq {}_0\mathbf{y}$ implies $W({}_0\mathbf{x}) \geq W({}_0\mathbf{y})$.

In the present section we impose conditions on the SWR sufficient to obtain a numerical representation in terms of an SWF with a recursive structure (see Proposition 2 below), similar to but not identical to Koopmans' (1960, Sections 3–7).

To obtain a numerical representation, we impose two conditions.

Condition O (*Order*) $\succsim$ is complete and transitive.

Condition RC (*Restricted Continuity*) For all ${}_0\mathbf{x}$, ${}_0\mathbf{y} \in \mathbf{X}$, if ${}_0\mathbf{x}$ satisfies $x_t = z$ for all $t \geq 1$, and the sequence of streams $\langle {}_0\mathbf{x}^n \rangle_{n \in \mathbb{N}}$ satisfies $\lim_{n \to \infty} \sup_t |x_t^n - x_t| = 0$

with, for each $n \in \mathbb{N}$, ${}_0\mathbf{x}^n \in \mathbf{X}$ and $\neg({}_0\mathbf{x}^n \prec {}_0\mathbf{y})$ (resp. $\neg({}_0\mathbf{x}^n \succ {}_0\mathbf{y})$), then $\neg({}_0\mathbf{x} \prec {}_0\mathbf{y})$ (resp. $\neg({}_0\mathbf{x} \succ {}_0\mathbf{y})$).

Condition **RC** is weaker than ordinary supnorm continuity.

Condition C (*Continuity*) For all ${}_0\mathbf{x}, {}_0\mathbf{y} \in \mathbf{X}$, if the sequence of streams $\langle {}_0\mathbf{x}^n \rangle_{n \in \mathbb{N}}$ satisfies $\lim_{n\to\infty} \sup_t |x_t^n - x_t| = 0$ with, for each $n \in \mathbb{N}$, ${}_0\mathbf{x}^n \in \mathbf{X}$ and $\neg({}_0\mathbf{x}^n \prec {}_0\mathbf{y})$ (resp. $\neg({}_0\mathbf{x}^n \succ {}_0\mathbf{y})$), then $\neg({}_0\mathbf{x} \prec {}_0\mathbf{y})$ (resp. $\neg({}_0\mathbf{x} \succ {}_0\mathbf{y})$).

Condition **C** is entailed by Koopmans' (1960) Postulate 1. As the analysis of Section 3 shows, the weaker continuity condition **RC** enables us to show existence of sustainable recursive social welfare functions.

The central condition in Koopmans' (1960) analysis is the stationarity postulate (Postulate 4). Combined with Koopmans' Postulate 3b, the stationarity postulate is equivalent to the following independence condition (where we borrow the name that Fleurbaey and Michel, 2003, use for a slightly stronger version of this condition).

Condition IF (*Independent Future*) For all ${}_0\mathbf{x}, {}_0\mathbf{y} \in \mathbf{X}$ with $x_0 = y_0$, ${}_0\mathbf{x} \succsim {}_0\mathbf{y}$ if and only if ${}_1\mathbf{x} \succsim {}_1\mathbf{y}$.

Condition **IF** means that an evaluation concerning only generations from the next period on can be made as if the present time (time 0) was actually at time 1; i.e., as if generations $\{0, 1, \dots\}$ would have taken the place of generations $\{1, 2, \dots\}$. If we extended our framework to also include comparisons at future times, then **IF** would imply time consistency as long as the SWR is time invariant.

With the well-being of each generation t expressed by a one-dimensional indicator x_t, it is uncontroversial to ensure through the following condition that a higher value of x_t cannot lead to a socially less preferred stream.

Condition M (*Monotonicity*) For all ${}_0\mathbf{x}, {}_0\mathbf{y} \in \mathbf{X}$, if ${}_0\mathbf{x} > {}_0\mathbf{y}$, then $\neg({}_0\mathbf{y} \succ {}_0\mathbf{x})$.

Combined with the completeness part of condition **O**, it follows from condition **M** that, for all ${}_0\mathbf{x}, {}_0\mathbf{y} \in \mathbf{X}$, if ${}_0\mathbf{x} \geq {}_0\mathbf{y}$, then ${}_0\mathbf{x} \succsim {}_0\mathbf{y}$. Condition **M** is obviously implied by the "Strong Pareto" condition.

Condition SP (*Strong Pareto*) For all ${}_0\mathbf{x}$, ${}_0\mathbf{y} \in \mathbf{X}$, if ${}_0\mathbf{x} > {}_0\mathbf{y}$, then ${}_0\mathbf{x} \succ {}_0\mathbf{y}$.

With condition **M** we need not impose Koopmans' (1960) extreme streams postulate (Postulate 5) and can consider the set of infinite streams bounded in well-being.

As the fifth and final condition of our basic representation result (Proposition 2), we impose the following efficiency condition.

Condition RD (*Restricted Dominance*) For all x, $z \in Y$, if $x < z$, then $(x, {}_{\mathrm{con}}z) \prec {}_{\mathrm{con}}z$.

To evaluate the implications of **RD**, consider the following three conditions.

Condition WS (*Weak Sensitivity*) There exist ${}_0\mathbf{x}$, ${}_0\mathbf{y}$, ${}_0\mathbf{z} \in \mathbf{X}$ such that $(x_0, {}_1\mathbf{z}) \succ (y_0, {}_1\mathbf{z})$.

Condition DF (*Dictatorship of the Future*) For all ${}_0\mathbf{x}$, ${}_0\mathbf{y} \in \mathbf{X}$ such that ${}_0\mathbf{x} \succ {}_0\mathbf{y}$, there exist $\underline{y}$, $\bar{y} \in Y$ with $\underline{y} \le x_t, y_t \le \bar{y}$ for all $t \in \mathbb{Z}_+$ and $T' \in \mathbb{Z}_+$ such that, for every ${}_0\mathbf{z}$, ${}_0\mathbf{v} \in [\underline{y}, \bar{y}]^{\mathbb{Z}_+}$, $({}_0\mathbf{z}_{T-1}, {}_T\mathbf{x}) \succ ({}_0\mathbf{v}_{T-1}, {}_T\mathbf{y})$ for all $T > T'$.

Condition NDF (*No Dictatorship of the Future*) Condition **DF** does not hold.

Condition **SP** implies condition **RD**, which in turn implies condition **WS**. Condition **WS** coincides with Koopmans' (1960) Postulate 2. Condition **NDF** generalizes one of Chichilnisky's (1996) two main axioms to our setting where we consider the set of infinite streams bounded in well-being.

Proposition 1 *Assume that the* SWR $\succsim$ *satisfies conditions* ***O*** *and* ***IF***. *Then* ***WS*** *is equivalent to* ***NDF***.

As already noted at the end of the introduction, the proof of this and later results are provided in an appendix.

Since **RD** strengthens **WS**, it follows from Proposition 1 that **RD** ensures "No Dictatorship of the Future", provided that the SWR satisfies conditions **O** and **IF**.

To appreciate why we cannot replace **RD** with an even stronger efficiency condition, we refer to the analysis of Section 3 and the impossibility result of Proposition 4.

To state Proposition 2, we introduce the following notation:

$$\mathcal{U} := \{U : Y \to \mathbb{R} \mid U \text{ is continuous and non-decreasing; } U(Y) \text{ is not a singleton}\}$$

$$\mathcal{U}_I := \{U : Y \to \mathbb{R} \mid U \text{ is continuous and increasing}\}$$

$$\mathcal{V}(U) := \{V : U(Y)^2 \to \mathbb{R} \mid V \text{ satisfies (V.0), (V.1), (V.2), and (V.3)}\},$$

where, for all $U \in \mathcal{U}$, $U(Y)$ denotes the range of U, and the properties of the *aggregator function* V, (V.0)–(V.3), are as follows:

(V.0) $V(u, w)$ is continuous in (u, w) on $U(Y)^2$.

(V.1) $V(u, w)$ is non-decreasing in u for given w.

(V.2) $V(u, w)$ is increasing in w for given u.

(V.3) $V(u, w) < w$ for $u < w$, and $V(u, w) = w$ for $u = w$.

Proposition 2 *The following two statements are equivalent.*

(1) The SWR $\succsim$ *satisfies conditions **O**, **RC**, **IF**, **M**, and **RD**.*

(2) There exists a monotone SWF $W : \mathbf{X} \to \mathbb{R}$ *representing* $\succsim$ *and satisfying, for some* $U \in \mathcal{U}_I$ *and* $V \in \mathcal{V}(U)$, $W({}_0\mathbf{x}) = V(U(x_0), W({}_1\mathbf{x}))$ *for all* ${}_0\mathbf{x} \in \mathbf{X}$ *and* $W({}_{\text{con}}z) = U(z)$ *for all* $z \in Y$.

For a given representation W (with associated utility function U) of an SWR satisfying conditions **O**, **RC**, **IF**, **M**, and **RD**, we refer to $U(x_t)$ as the *utility* of generation t and $W({}_0\mathbf{x})$ as the *welfare* derived from the infinite stream ${}_0\mathbf{x}$.

3 Hammond Equity for the Future

Discounted utilitarianism satisfies conditions **O**, **RC**, **IF**, **M**, and **RD**. Hence, these conditions do not by themselves prevent "Dictatorship of the Present", in the terminology of Chichilnisky (1996).

Condition DP (*Dictatorship of the Present*) For all ${}_0\mathbf{x}$, ${}_0\mathbf{y} \in \mathbf{X}$ such that ${}_0\mathbf{x} \succ {}_0\mathbf{y}$, there exist $\underline{y}$, $\bar{y} \in Y$ with $\underline{y} \leq x_t, y_t \leq \bar{y}$ for all $t \in \mathbb{Z}_+$ and $T' \in \mathbb{Z}_+$ such that, for any ${}_0\mathbf{z}$, ${}_0\mathbf{v} \in [\underline{y}, \bar{y}]^{\mathbb{Z}_+}$, $({}_0\mathbf{x}_{T-1}, {}_T\mathbf{z}) \succ ({}_0\mathbf{y}_{T-1}, {}_T\mathbf{v})$ for all $T > T'$.

Condition NDP (*No Dictatorship of the Present*) Condition **DP** does not hold.

Condition **NDP** generalizes the other of Chichilnisky's (1996) two main axioms to our setting where we consider the set of infinite streams bounded in well-being.

We impose a weak new equity condition that ensures **NDP**. Combined with **RC**, this condition entails that the interest of the present are taken into account only if the present is worse-off than the future. Consider a stream $(x, {}_{\text{con}}z)$ having the property that well-being is constant from the second period on. For such a stream we may unequivocally say that, if $x < z$, then the present is worse- off than the future. Likewise, if $x > z$, then the present is better-off than the future.

Condition HEF (*Hammond Equity for the Future*) For all x, y, z, $v \in Y$, if $x > y > v > z$, then $\neg\left((x, {}_{\text{con}}z) \succ (y, {}_{\text{con}}v)\right)$. [4]

For streams where well-being is constant from the second period on, condition **HEF** captures the idea of giving priority to an infinite number of future generations in the choice between alternatives where the future is worse-off compared to the present in both alternatives. If the present is better-off than the future and a sacrifice now leads to a uniform gain for all future generations, then such a transfer from the present to the future cannot lead to a less desirable stream, as long as the present remains better-off than the future.

To appreciate the weakness of condition **HEF**, consider first the standard "Hammond Equity" condition (Hammond, 1976) and a weak version of Lauwers' (1998) non-substitution condition.

Condition HE (*Hammond Equity*) For all ${}_0\mathbf{x}$, ${}_0\mathbf{y} \in \mathbf{X}$, if ${}_0\mathbf{x}$ and ${}_0\mathbf{y}$ satisfy that

[4] Condition **HEF** was introduced in a predecessor to this paper (Asheim and Tungodden, 2004b) and has been analyzed by Asheim, Mitra and Tungodden (2007) and Banerjee (2006).

there exists a pair τ', τ'' such that $x_{\tau'} > y_{\tau'} > y_{\tau''} > x_{\tau''}$ and $x_t = y_t$ for all $t \neq \tau'$, τ'', then $\neg({}_0\mathbf{x} \succ {}_0\mathbf{y})$.

Condition WNS (*Weak Non-Substitution*) For all x, y, z, $v \in Y$, if $v > z$, then $\neg\left((x, {}_{\text{con}}z) \succ (y, {}_{\text{con}}v)\right)$.

By assuming, in addition, that well-beings are at least cardinally measurable and fully comparable, we may also consider weak versions of the Lorenz Domination and Pigou-Dalton principles. Such equity conditions have been used in the setting of infinite streams by, e.g., Birchenhall and Grout (1979), Asheim (1991), Fleurbaey and Michel (2001), and Hari, Shinotsuka, Suzumura and Xu (2008).

Condition WLD (*Weak Lorenz Domination*) For all ${}_0\mathbf{x}$, ${}_0\mathbf{y} \in \mathbf{X}$, if ${}_0\mathbf{x}$ and ${}_0\mathbf{y}$ are such that ${}_0\mathbf{y}_{T-1}$ Lorenz dominates ${}_0\mathbf{x}_{T-1}$ and ${}_T\mathbf{x} = {}_T\mathbf{y}$ for some $T > 1$, then $\neg({}_0\mathbf{x} \succ {}_0\mathbf{y})$.

Condition WPD (*Weak Pigou-Dalton*) For all ${}_0\mathbf{x}$, ${}_0\mathbf{y} \in \mathbf{X}$, if ${}_0\mathbf{x}$ and ${}_0\mathbf{y}$ are such that there exist a positive number ϵ and a pair τ', τ'' satisfying $x_{\tau'} - \epsilon = y_{\tau'} \geq y_{\tau''} = x_{\tau''} + \epsilon$ and $x_t = y_t$ for all $t \neq \tau'$, τ'', then $\neg({}_0\mathbf{x} \succ {}_0\mathbf{y})$.

While it is clear that condition **HEF** is implied by **WNS**, it is perhaps less obvious that, under **O** and **M**, **HEF** is at least as weak as *each* of **HE**, **WPD**, and **WLD**.

Proposition 3 *Assume that the* SWR $\succsim$ *satisfies conditions* ***O*** *and* ***M****. Then each of* ***HE****,* ***WPD****, and* ***WLD*** *implies* ***HEF****.*

Note that condition **HEF** involves a comparison between a sacrifice by a single generation and a uniform gain for each member of an infinite set of generations that are worse-off. Hence, contrary to the standard "Hammond Equity" condition, if well-beings are made (at least) cardinally measurable and fully comparable, then the transfer from the better-off present to the worse-off future specified in condition **HEF** increases the sum of well-beings for a sufficiently large number T of generations. This entails that condition **HEF** is implied by both **WPD** and **WLD**, independently of what specific cardinal scale of well-beings is imposed (provided that conditions **O** and **M** are satisfied). Hence, "Hammond Equity for the Future" can be endorsed from both an egalitarian and utilitarian point of view. In particular, condition **HEF** is weaker and more compelling than the standard "Hammond Equity" condition.

However, in line with the Diamond-Yaari impossibility result (Diamond, 1965) on the inconsistency of equity and efficiency conditions under continuity,[5] the equity condition **HEF** is in conflict with the following weak efficiency condition under **RC**.

Condition RS (*Restricted Sensitivity*) There exist x, $z \in Y$ with $x > z$ such that $(x, {}_{\text{con}}z) \succ {}_{\text{con}}z$.

Condition **SP** implies condition **RS**, which in turn implies condition **WS**.

Proposition 4 *There is no* SWR $\succsim$ *satisfying conditions **RC**, **RS**, and **HEF**.*

Impossibility results arising from **HEF** are further explored in Asheim, Mitra and Tungodden (2007). Here we concentrate on SWRs that satisfy **HEF**. We note that it follows from Proposition 4 that **RD** is the strongest efficiency condition compatible with **HEF** under **RC**, when comparing streams $(x, {}_{\text{con}}z)$ where well-being is constant from the second period on with constant streams ${}_{\text{con}}z$.

The following result establishes that "Dictatorship of the Present" is indeed ruled out by adding condition **HEF** to conditions **O**, **RC**, **IF**, and **M**.

Proposition 5 *Assume that the* SWR $\succsim$ *satisfies conditions **O**, **RC**, **IF**, and **M**. Then **HEF** implies **NDP**.*

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[5]The Diamond-Yaari impossibility result states that the equity condition of "Weak Anonymity" (deeming two streams socially indifferent if one is obtained from the other through a finite permutation of well-beings) is inconsistent with the efficiency condition **SP** given **C**. See also Basu and Mitra (2003) and Fleurbaey and Michel (2003).

How does the basic representation result of Proposition 2 change if we also impose condition **HEF** on an SWR $\succsim$ satisfying conditions **O**, **RC**, **IF**, **M**, and **RD**? To investigate this question, introduce the following notation:

$$\mathcal{V}_S(U) := \{V : U(Y)^2 \to \mathbb{R} \mid V \text{ satisfies (V.0), (V.1), (V.2), and (V.3}')\},$$

where (V.3′) is given as follows:

(V.3′) $V(u,w) < w$ for $u < w$, and $V(u,w) = w$ for $u \geq w$.

Note that, for each $U \in \mathcal{U}$, $\mathcal{V}_S(U) \subseteq \mathcal{V}(U)$.

Proposition 6 *The following two statements are equivalent.*

(1) The SWR $\succsim$ *satisfies conditions* ***O****,* ***RC****,* ***IF****,* ***M****,* ***RD****, and* ***HEF****.*

(2) There exists a monotone SWF $W : \mathbf{X} \to \mathbb{R}$ *representing* $\succsim$ *and satisfying, for some* $U \in \mathcal{U}_I$ *and* $V \in \mathcal{V}_S(U)$, $W({}_0\mathbf{x}) = V(U(x_0), W({}_1\mathbf{x}))$ *for all* ${}_0\mathbf{x} \in \mathbf{X}$ *and* $W({}_{\text{con}}z) = U(z)$ *for all* $z \in Y$*.*

We refer to a mapping satisfying the property presented in statement (2) of Proposition 6 as a *sustainable recursive* SWF. Proposition 6 does not address the question whether there exists a sustainable recursive SWF for any $U \in \mathcal{U}_I$ and $V \in \mathcal{V}_S(U)$. This question of existence is resolved through the following proposition, which also characterizes the asymptotic properties of such social welfare functions.

Proposition 7 *For all* $U \in \mathcal{U}_I$ *and* $V \in \mathcal{V}_S(U)$*, there exists a monotone mapping* $W : \mathbf{X} \to \mathbb{R}$ *satisfying* $W({}_0\mathbf{x}) = V(U(x_0), W({}_1\mathbf{x}))$ *for all* ${}_0\mathbf{x} \in \mathbf{X}$ *and* $W({}_{con}z) = U(z)$ *for all* $z \in Y$*. Any such mapping* W *satisfies, for each* ${}_0\mathbf{x} \in \mathbf{X}$*,*

$$\lim_{T\to\infty} W({}_T\mathbf{x}) = \liminf_{t\to\infty} U(x_t).$$

By combining Propositions 6 and 7 we obtain our first main result.

Theorem 1 *There exists a class of* SWRs $\succsim$ *satisfying conditions* ***O****,* ***RC****,* ***IF****,* ***M****,* ***RD****, and* ***HEF****.*

The proof of the existence part of Proposition 7 is based on an algorithmic construction. For any ${}_0\mathbf{x} \in \mathbf{X}$ and each $T \in \mathbb{Z}_+$, consider the following finite sequence:

$$\left.\begin{array}{l} w(T,T) = \liminf_{t\to\infty} U(x_t) \\ w(T-1,T) = V(U(x_{T-1}), w(T,T)) \\ \dots \\ w(0,T) = V(U(x_0), w(1,T)) \end{array}\right\} \tag{1}$$

Define the mapping $W_\sigma : \mathbf{X} \to \mathbb{R}$ by

$$W_\sigma({}_0\mathbf{x}) := \lim_{T\to\infty} w(0,T)\,. \tag{W}$$

In the proof of Proposition 7 we show that W_σ is a sustainable recursive SWF.

It is an open question whether W_σ is the *unique* sustainable recursive SWF given $U \in \mathcal{U}_I$ and $V \in \mathcal{V}_S(U)$. As reported in the following proposition, we can show uniqueness if the aggregator function satisfies a condition introduced by Koopmans, Diamond, and Williamson (1964, p. 88): $V \in \mathcal{V}(U)$ satisfies the property of *weak time perspective* if there exists a continuous increasing transformation $g : \mathbb{R} \to \mathbb{R}$ such that $g(w) - g(V(u,w))$ is a non-decreasing function of w for given u.

Proposition 8 *Let $U \in \mathcal{U}_I$ and $V \in \mathcal{V}_S(U)$. If V satisfies the property of weak time perspective, then there exists a unique monotone mapping $W : \mathbf{X} \to \mathbb{R}$ satisfying $W({}_0\mathbf{x}) = V(U(x_0), W({}_1\mathbf{x}))$ for all ${}_0\mathbf{x} \in \mathbf{X}$ and $W({}_{con}z) = U(z)$ for all $z \in Y$. This mapping, W_σ, is defined by (W).*

We have not been able to establish that the property of weak time perspective follows from the conditions we have imposed. However, it is satisfied in special cases; e.g., with V given by

$$V(u,w) = \begin{cases} (1-\delta)u + \delta w & \text{if } u < w \\ w & \text{if } u \geq w\,, \end{cases} \tag{2}$$

where $\delta \in (0,1)$.[6] We can also show that the set of supnorm continuous sustainable recursive SWFs contains at most W_σ. However, even though W_σ is continuous in the weak sense implied by condition **RC**, it need not be supnorm continuous.

Once we drop one of the conditions **RC**, **IF**, and **RD**, and combine the remaining two conditions with **O**, **M**, and **HEF**, new possibilities open up. It is clear that:

- The mapping $W : \mathbf{X} \to \mathbb{R}$ defined by $W({}_0\mathbf{x}) := \liminf_{t\to\infty} U(x_t)$ for some $U \in \mathcal{U}_I$ represents an SWR satisfying **O**, **RC**, **IF**, **M**, and **HEF**, but not **RD**.
- The maximin SWR satisfies **O**, **RC**, **M**, **RD**, and **HEF**, but not **IF**.
- Leximin and undiscounted utilitarian SWRs for infinite streams satisfy **O**, **IF**, **M**, **RD**, and **HEF**, but not **RC** (cf. Proposition 13).

It follows from Propositions 1, 5, and 6 that any sustainable recursive SWF represents an SWR satisfying **NDF** and **NDP**. Chichilnisky (1996, Definition 6) defines "sustainable preferences" by imposing **NDF** and **NDP** as well as numerical representability and **SP**. When showing existence in her Theorem 1, she considers SWRs violating condition **IF**. Hence, through showing general existence for our sustainable recursive SWF, we demonstrate that **NDF** and **NDP** can be combined with (a) numerical representability, (b) condition **IF** which implies stationarity, and (c) sensitivity to present well-being—and thus be imposed within the Koopmans framework—provided that **SP** is replaced by weaker dominance conditions.[7]

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[6]Sustainable recursive SWFs with aggregator function given by (2) are analyzed in the companion paper (Asheim and Mitra, 2008). Note that an SWR $\succsim$ represented by such a sustainable recursive SWF satisfies the following restricted form of the **IP** condition introduced in the next section:

For all ${}_0\mathbf{x}$, ${}_0\mathbf{y}$, ${}_0\mathbf{z}$, ${}_0\mathbf{v} \in \mathbf{X}$ such that $(x_0, x_1, {}_2\mathbf{z})$, $(y_0, y_1, {}_2\mathbf{z})$, $(x_0, x_1, {}_2\mathbf{v})$, $(y_0, y_1, {}_2\mathbf{v})$ are non-decreasing, $(x_0, x_1, {}_2\mathbf{z}) \succsim (y_0, y_1, {}_2\mathbf{z})$ if and only if $(x_0, x_1, {}_2\mathbf{v}) \succsim (y_0, y_1, {}_2\mathbf{v})$.

[7]Mitra (2008) shows by means of an example that "sustainable preferences" can be combined with **IF** in the case where $Y = [0,1]$ if we are willing to give up **RC**.

4 Independent Present

The following condition is invoked as Postulate 3′a in Koopmans' (1960) characterization of discounted utilitarianism.

Condition IP (*Independent Present*) For all ${}_0\mathbf{x}$, ${}_0\mathbf{y}$, ${}_0\mathbf{z}$, ${}_0\mathbf{v} \in \mathbf{X}$, $(x_0, x_1, {}_2\mathbf{z}) \succsim (y_0, y_1, {}_2\mathbf{z})$ if and only if $(x_0, x_1, {}_2\mathbf{v}) \succsim (y_0, y_1, {}_2\mathbf{v})$.

Condition **IP** requires that the evaluation of two streams differing only in the first two periods *not* depend on what the common continuation stream is. We suggest in this section that this condition may not be compelling, both through appeal to ethical intuition, and through formal results.

We suggest that it might be supported by ethical intuition to accept that the stream $(1, 4, 5, 5, 5, \dots)$ is socially better than $(2, 2, 5, 5, 5, \dots)$, while not accepting that $(1, 4, 2, 2, 2, \dots)$ is socially better than $(2, 2, 2, 2, 2, \dots)$. It is not obvious that we should treat the conflict between the worst-off and the second worst-off generation presented by the first comparison in the same manner as we treat the conflict between the worst-off and the best-off generation put forward by the second comparison.

Turn now to the formal results. Koopmans (1960) characterizes discounted utilitarianism by means of conditions **IF**, **WS**, and **IP**. However, it turns out that conditions **IF**, **WS**, and **IP** contradict **HEF** under **RC** and **M**. Furthermore, this conclusion is tight, in the sense that an SWR exists if any one of these conditions is dropped. This is our second main result.

Theorem 2 *There is no* SWR $\succsim$ *satisfying conditions* ***RC, IF, M, WS, HEF,*** *and* ***IP****. If one of the conditions* ***RC, IF, M, WS, HEF,*** *and* ***IP*** *is dropped, then there exists an* SWR $\succsim$ *satisfying the remaining five conditions as well as condition* ***O****.*

In the following proposition, we reproduce Koopmans' (1960) characterization of discounted utilitarianism within the formal setting of this paper.[8]

[8] See Bleichrodt, Rohde and Wakker (2008) for a simplified characterization of discounted utilitarianism on an extended domain, as well as an overview of related literature.

Proposition 9 *The following two statements are equivalent.*

(1) The SWR $\succsim$ *satisfies conditions* ***O****,* ***RC****,* ***IF****,* ***M****,* ***WS****, and* ***IP****.*

(2) There exists a monotone SWF $W : \mathbf{X} \to \mathbb{R}$ *representing* $\succsim$ *and satisfying, for some* $U \in \mathcal{U}$ *and* $\delta \in (0, 1)$*,* $W({}_0\mathbf{x}) = (1 - \delta)U(x_0) + \delta W({}_1\mathbf{x})$ *for all* ${}_0\mathbf{x} \in \mathbf{X}$*.*

Strengthening ***WS*** *to* ***RD*** *in statement (1) is equivalent to replacing* $\mathcal{U}$ *by* $\mathcal{U}_I$ *in statement (2).*

Furthermore, we note that the discounted utilitarian SWF exists and is unique.

Proposition 10 *For all* $U \in \mathcal{U}$ *and* $\delta \in (0, 1)$*, there exists a unique monotone mapping* $W : \mathbf{X} \to \mathbb{R}$ *satisfying* $W({}_0\mathbf{x}) = (1 - \delta)U(x_0) + \delta W({}_1\mathbf{x})$ *for all* ${}_0\mathbf{x} \in \mathbf{X}$*. This mapping,* W_δ*, is defined by, for each* ${}_0\mathbf{x} \in \mathbf{X}$*,*

$$W_\delta({}_0\mathbf{x}) = (1 - \delta)\sum\nolimits_{t=0}^{\infty} \delta^t U(x_t) \,.$$

Propositions 9 and 10 have the following implication.

Proposition 11 *There is no* SWR $\succsim$ *satisfying conditions* ***O****,* ***RC****,* ***IF****,* ***M****,* ***IP****,* ***NDP****, and* ***NDF****.*

To summarize, it follows from Theorem 2 and Propositions 1 and 11 that, within a Koopmans framework where **O**, **RC**, **IF**, **M**, and **WS** are imposed, condition **IP** contradicts both **HEF** and **NDP**. Hence, in such a framework, **IP** is in conflict with consequentialist equity conditions that respect the interests of future generations.

5 Applying sustainable recursive SWFs

We apply sustainable recursive SWFs for studying optimal harvesting of a renewable resource where, following Krautkraemer (1985), well-being may be derived directly from the resource stock. Using discounted utilitarianism in this setting reduces the resource stock below the green golden-rule (defined below) and leads to resource deterioration for sufficiently high discounting (Heal, 1998).

Maximizing sustainable recursive SWFs leads to very different conclusions, as reported in Proposition 12. Before stating this result, we introduce the model.

The law of motion governing the bio-mass of the renewable resource, k, is given by a standard increasing, concave stock-recruitment function, f, and therefore the production framework is formally the same as the standard neoclassical aggregate model of economic growth. The function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is assumed to satisfy:

(i) $f(0) = 0\,,$

(ii) f is continuous, increasing and strictly concave on $\mathbb{R}_+\,,$

(iii) $\lim_{k\to 0} \frac{f(k)}{k} > 1$ and $\lim_{k\to\infty} \frac{f(k)}{k} < 1\,.$

It can be shown that there exists a unique number $\bar{k} > 0$ such that $f(\bar{k}) = \bar{k}$ and $f(k) > k$ for $k \in (0, \bar{k})$.

A feasible path from $k \in [0, \bar{k}]$ is a sequence of *resource stocks* ${}_0\mathbf{k}$ satisfying:

$$k_0 = k\,, \quad 0 \le k_{t+1} \le f(k_t) \ \text{ for } t > 0\,.$$

It follows from the definition of $\bar{k}$ that $k_t \in [0, \bar{k}]$ for $t > 0$. Hence, $\bar{k}$ is the maximal attainable resource stock if one starts from an initial stock in $[0, \bar{k}]$. Associated with a feasible path ${}_0\mathbf{k}$ from $k \in [0, \bar{k}]$ is a *consumption* stream ${}_0\mathbf{c}$, defined by

$$c_t = f(k_t) - k_{t+1} \ \text{ for } t \ge 0\,.$$

Well-being, x, depends on consumption and resource amenities through a function $x : [0, \bar{k}]^2 \to \mathbb{R}$, which is assumed to satisfy:

$$\begin{aligned}&\text{(i) } x \text{ is continuous and quasi-concave on } [0,\bar{k}]^2,\\ &\text{(ii) } x \text{ is non-decreasing in } (c,k), \text{ and increasing in } c \text{ (when } k>0).\end{aligned} \tag{3}$$

The set of of admissible well-beings is given by $Y := [x(0,0), x(\bar{k},\bar{k})]$. Associated with a feasible path ${}_0\mathbf{k}$ from $k \in [0, \bar{k}]$ is a well-being stream ${}_0\mathbf{x}$, defined by

$$x_t = x(f(k_t) - k_{t+1}, k_t) \ \text{ for } t \ge 0\,.$$

For any $k \in [0, \bar{k}]$, the set of well-being streams associated with feasible resource paths from k is contained in $\mathbf{X} = Y^{\mathbb{Z}_+}$.

It follows from the continuity and strict concavity of f and the continuity and quasi-concavity of x, combined with property (3)(ii), that there exists a unique number $k^* \in [0, \bar{k}]$ such that $x(f(k^*) - k^*, k^*) \geq x(f(k) - k, k)$ for all $k \in [0, \bar{k}]$. Since, for any $k \in (0, \bar{k})$, $x(f(k) - k, k) > x(f(0) - 0, 0) = x(0, 0)$, we have that $k^* > 0$. Clearly, an additional assumption can be imposed to ensure the existence of $k \in (0, \bar{k})$ such that $x(f(k) - k, k) > x(f(\bar{k}) - \bar{k}, \bar{k}) = x(0, \bar{k})$, so that $k^* < \bar{k}$. The subsequent analysis holds with (and without) any such assumption.

We write $c^* := f(k^*) - k^*$ and $x^* := x(c^*, k^*)$. By keeping the resource stock constant at k^*, a maximum sustainable well-being equal to x^* is attained; this corresponds to the *green golden-rule* (Chichilnisky et al., 1995). The following result shows that if $k \in [k^*, \bar{k}]$ and a sustainable recursive SWF is maximized, then welfare corresponds to the green golden-rule, and the resource stock never falls below the green golden-rule level.

Proposition 12 *Assume that an economy maximizes a sustainable recursive* SWF $W : \mathbf{X} \to \mathbb{R}$ *on the set of well-being streams associated with feasible resource paths from* $k \in [k^*, \bar{k}]$*. Then an optimum exists, and for any optimal resource path* ${}_0\hat{\mathbf{k}}$*, with associated well-being stream* ${}_0\hat{\mathbf{x}}$*,*

$$W({}_t\hat{\mathbf{x}}) = W({}_{\text{con}}x^*), \quad \hat{x}_t \geq x^* \,, \quad \textit{and } \hat{k}_t \geq k^* \textit{ for } t \geq 0 \,.$$

Hence, in contrast to the existence problem encountered when Chichilnisky's (1996) "sustainable preferences" are applied to such a setting (see Figuieres and Tidball, 2009, where this problem motivates an interesting analysis), optima exist when sustainable recursive SWFs are used to evaluate streams (at least, for $k \in [k^*, \bar{k}]$). Moreover, in contrast to the outcome under discounted utilitarianism, sustainable recursive SWFs sustain well-being at or above its maximum sustainable level, by sustaining the resource stock at or above the green golden-rule level.

In a companion paper (Asheim and Mitra, 2008) it is demonstrated how sustainable recursive SWFs can be used to resolve in an appealing way the interesting distributional conflicts that arise in the DHS model of capital accumulation and resource depletion. In particular, applying sustainable recursive SWFs in this setting leads to growth and development at first when capital is productive, while protecting the generations in the distant future from the grave consequences of discounting when the vanishing resource stock undermines capital productivity.

6 Concluding remarks

Koopmans (1960) has often been interpreted as presenting the definitive case for discounted utilitarianism. In Sections 2 and 3 we have sought to weaken this impression by exploring other avenues within the general setting of his approach. In particular, by not imposing condition **IP**, used by Koopmans (1960) to characterize discounted utilitarianism, we have been able to combine our new equity condition **HEF** with the essential features of the Koopmans framework: (a) numerical representability, (b) condition **IF** which includes Koopmans' stationarity postulate, and (c) sensitivity to the interests of the present generation. This leads to a non-empty class of sustainable recursive social welfare functions. We have argued that condition **HEF** is weak, as it is implied by all the standard consequentialist equity conditions suggested in the literature, yet strong enough to ensure that the Chichilnisky (1996) conditions are satisfied. As we have discussed in Section 5, sustainable recursive social welfare functions are applicable and yield consequences that differ from those of discounted utilitarianism.

In this final section we note that even wider possibilities open up if we are willing to give up numerical representability by not imposing **RC**. In particular, we are then able to combine the equity condition **HEF** and the independence condition **IP** with our basic conditions **O** and **IF**, while strengthening our efficiency conditions **M** and **RD** to condition **SP**.

Proposition 13 *There exists an* SWR $\succsim$ *satisfying conditions* ***O, IF, SP, HEF,*** *and* ***IP****.*

The proof of this proposition employs the leximin and undiscounted utilitarian SWRs for infinite streams that have been axiomatized in recent contributions (see Asheim and Tungodden, 2004a; Basu and Mitra, 2007; Bossert, Sprumont and Suzumura, 2007).

We end by making the observation that continuity is not simply a "technical" condition without ethical content. In a setting where **RC** (or a stronger continuity condition like **C**) is combined with **RS** (or a stronger efficiency condition like **SP**), it follows from Proposition 4 that condition **HEF** is not satisfied. Hence, on this basis one may claim that, in combination with a sufficiently strong efficiency condition, continuity rules out SWFs that protect the interests of future generations by implying that the equity condition **HEF** does not hold. In the main analysis of this paper we have avoided the trade-off between continuity and numerical representability on the one hand, and the ability to impose the equity condition **HEF** on the other hand, by weakening the efficiency condition in an appropriate way.

Appendix: Proofs

Proof of Proposition 1. *Part I: **WS** implies **NDF**.* Assume that the SWR $\succsim$ satisfies conditions **O** and **WS**. By **WS**, there exist ${}_0\mathbf{x}$, ${}_0\mathbf{y} \in \mathbf{X}$ with ${}_1\mathbf{x} = {}_1\mathbf{y}$ such that ${}_0\mathbf{x} \succ {}_0\mathbf{y}$. Let ${}_0\mathbf{z}$, ${}_0\mathbf{v} \in \mathbf{X}$ be given by ${}_0\mathbf{z} = {}_0\mathbf{v} = {}_0\mathbf{x}$. We have that, for any $\underline{y}$, $\bar{y} \in Y$ satisfying $\underline{y} \leq x_t, y_t \leq \bar{y}$ for all $t \in \mathbb{Z}_+$, ${}_0\mathbf{z}$, ${}_0\mathbf{v} \in [\underline{y}, \bar{y}]^{\mathbb{Z}_+}$. Still, for all $T > 0$, $({}_0\mathbf{z}_{T-1}, {}_T\mathbf{x}) = {}_0\mathbf{x} = ({}_0\mathbf{x}_{T-1}, {}_T\mathbf{y}) = ({}_0\mathbf{v}_{T-1}, {}_T\mathbf{y})$, implying by **O** that $({}_0\mathbf{z}_{T-1}, {}_T\mathbf{x}) \sim ({}_0\mathbf{v}_{T-1}, {}_T\mathbf{y})$. This contradicts **DF**.

*Part II: **NDF** implies **WS**.* Assume that the SWR $\succsim$ satisfies conditions **O** and **IF**. Suppose that **WS** does not hold, i.e., for all ${}_0\mathbf{x}'$, ${}_0\mathbf{y}' \in \mathbf{X}$ with ${}_1\mathbf{x}' = {}_1\mathbf{y}'$, we have that ${}_0\mathbf{x}' \sim {}_0\mathbf{y}'$. Suppose ${}_0\mathbf{x}, {}_0\mathbf{y} \in \mathbf{X}$ are such that ${}_0\mathbf{x} \succ {}_0\mathbf{y}$. Let ${}_0\mathbf{z}$, ${}_0\mathbf{v}$ be arbitrary streams in $\mathbf{X}$. We have that ${}_{T-1}\mathbf{x} \sim (z_{T-1}, {}_T\mathbf{x})$ for all $T > 0$ since **WS** does not hold. By **IF** and the above argument,

$${}_{T-2}\mathbf{x} = (x_{T-2}, {}_{T-1}\mathbf{x}) \sim (x_{T-2}, z_{T-1}, {}_T\mathbf{x}) \sim ({}_{T-2}\mathbf{z}_{T-1}, {}_T\mathbf{x}) .$$

By invoking **O** and applying **IF** and the above argument repeatedly, it follows that ${}_0\mathbf{x} \sim ({}_0\mathbf{z}_{T-1}, {}_T\mathbf{x})$ for all $T > 0$. Likewise, ${}_0\mathbf{y} \sim ({}_0\mathbf{v}_{T-1}, {}_T\mathbf{y})$ for all $T > 0$. By **O**, $({}_0\mathbf{z}_{T-1}, {}_T\mathbf{x}) \succ ({}_0\mathbf{v}_{T-1}, {}_T\mathbf{y})$ for all $T > 0$. This establishes **DF**, implying that **NDF** does not hold. ■

The following lemma is useful for proving Proposition 2 and subsequent results.

Lemma 1 *Assume that the* SWR $\succsim$ *satisfies conditions* ***O****,* ***RC****,* ***M****. Then, for all* ${}_0\mathbf{x} \in \mathbf{X}$*, there exists* $z \in Y$ *such that* ${}_{\text{con}}z \sim {}_0\mathbf{x}$*. If condition* ***RD*** *is added, then* z *is unique.*

Proof. Assume that the SWR $\succsim$ satisfies conditions **O**, **RC**, and **M**. By **O**, **M**, and the definition of **X**, there exists $z \in Y$ such that $\inf\{v \in Y \mid {}_{\text{con}}v \succsim {}_0\mathbf{x}\} \leq z \leq \sup\{v \in Y \mid {}_{\text{con}}v \precsim {}_0\mathbf{x}\}$. By **O** and **RC**, ${}_{\text{con}}z \sim {}_0\mathbf{x}$.

If condition **RD** is added, then by **O**, **M**, and **RD** we have that

$$ {}_{\text{con}}v = (v, {}_{\text{con}}v) \precsim (v, {}_{\text{con}}z) \prec {}_{\text{con}}z \quad \text{if } v < z \,, \tag{4} $$

so that $\inf\{v \in Y \mid {}_{\text{con}}v \succsim {}_0\mathbf{x}\} = \sup\{v \in Y \mid {}_{\text{con}}v \precsim {}_0\mathbf{x}\}$ and z is unique. ■

Proof of Proposition 2. *Part I: (1) implies (2).* Assume that the SWR $\succsim$ satisfies conditions **O**, **RC**, **IF**, **M**, and **RD**. In view of Lemma 1, determine $W : \mathbf{X} \to Y$ by, for all ${}_0\mathbf{x} \in \mathbf{X}$, $W({}_0\mathbf{x}) = z$ where ${}_{\text{con}}z \sim {}_0\mathbf{x}$. By **O** and (4), $W({}_0\mathbf{x}) \geq W({}_0\mathbf{y})$ if and only if ${}_0\mathbf{x} \succsim {}_0\mathbf{y}$. By **M**, W is monotone.

Let $U \in \mathcal{U}_I$ be given by $U(x) = x$ for all $x \in Y$, implying that $U(Y) = Y$. Hence, by construction of W, $W({}_{\text{con}}z) = z = U(z)$ for all $z \in Y$. It follows from **IF** that, for given $x_0 \in Y$, there exists an increasing transformation $V(U(x_0), \cdot) : Y \to Y$ such that, for all ${}_1\mathbf{x} \in \mathbf{X}$, $W(x_0, {}_1\mathbf{x}) = V(U(x_0), W({}_1\mathbf{x}))$. This determines $V : Y \times Y \to Y$, where $V(u, w)$ is increasing in w for given u, establishing that V satisfies (V.2). By **M**, $V(u, w)$ is non-decreasing in u for given w, establishing that V satisfies (V.1). Since $\neg\,((x, {}_{\text{con}}z) \prec {}_{\text{con}}v)$ (resp. $\neg\,((x, {}_{\text{con}}z) \succ {}_{\text{con}}v)$) if and only if

$$ V(x, z) = V(U(x), W({}_{\text{con}}z)) = W(x, {}_{\text{con}}z) \geq v \quad (\text{resp. } \leq v), $$

RC implies that V satisfies (V.0). Finally, since

$$ \begin{aligned} V(z, z) &= V(U(z), W({}_{\text{con}}z)) = W({}_{\text{con}}z) = z \\ V(x, z) &= V(U(x), W({}_{\text{con}}z)) = W(x, {}_{\text{con}}z) < W({}_{\text{con}}z) = z \text{ if } x < z \,, \end{aligned} $$

by invoking **RD**, it follows that V satisfies (V.3). Hence, $V \in \mathcal{V}(U)$.

Part II: (2) implies (1). Assume that the monotone mapping $W : \mathbf{X} \to \mathbb{R}$ is an SWF and satisfies, for some $U \in \mathcal{U}_I$ and $V \in \mathcal{V}(U)$, $W({}_0\mathbf{x}) = V(U(x_0), W({}_1\mathbf{x}))$ for all ${}_0\mathbf{x} \in \mathbf{X}$ and $W({}_{\text{con}}z) = U(z)$ for all $z \in Y$. Since the SWR $\succsim$ is represented by the SWF W, it follows that $\succsim$ satisfies **O**. Moreover, $\succsim$ satisfies **M** since W is monotone, $\succsim$ satisfies **IF** since V satisfies (V.2), and $\succsim$ satisfies **RD** since $U \in \mathcal{U}_I$ and V satisfies (V.3). The following argument shows that $\succsim$ satisfies **RC**.

Let ${}_0\mathbf{x}$, ${}_0\mathbf{y} \in \mathbf{X}$, and let $x_t = z$ for all $t \geq 1$. Let ${}_0\mathbf{x}^n \in \mathbf{X}$ for $n \in \mathbb{N}$, with the property that $\lim_{n\to\infty} \sup_t |x_t^n - x_t| = 0$ and, for each $n \in \mathbb{N}$, $\neg({}_0\mathbf{x}^n \prec {}_0\mathbf{y})$. We have to show that $\neg({}_0\mathbf{x} \prec {}_0\mathbf{y})$, or equivalently, $W({}_0\mathbf{x}) \geq W({}_0\mathbf{y})$. Define $\epsilon_0(n)$ and $\epsilon(n)$ for $n \in \mathbb{N}$ by, for each $n \in \mathbb{N}$, $\epsilon_0(n) := \max\{0, x_0^n - x_0\}$ and $\epsilon(n) := \max\{0, \sup_{t\geq 1}(x_t^n - x_t)\}$, so that $\lim_{n\to\infty} \epsilon_0(n) = 0$ and $\lim_{n\to\infty} \epsilon(n) = 0$. For each $n \in \mathbb{N}$,

$$\begin{aligned} V(U(x_0 + \epsilon_0(n)), U(z + \epsilon(n))) &= V(U(x_0 + \epsilon_0(n)), W({}_{\text{con}}(z + \epsilon(n)))) \\ &= W(x_0 + \epsilon_0(n), {}_{\text{con}}(z + \epsilon(n))) \geq W({}_0\mathbf{x}^n) \geq W({}_0\mathbf{y}) \end{aligned}$$

since W is monotone and represents $\succsim$, and $\neg({}_0\mathbf{x}^n \prec {}_0\mathbf{y})$. This implies that

$$W({}_0\mathbf{x}) = V(U(x_0), W({}_{\text{con}}z)) = V(U(x_0), U(z)) \geq W({}_0\mathbf{y})$$

since U and V are continuous and $\lim_{n\to\infty} \epsilon(n) = 0$. The same kind of argument can be used to show that $\neg({}_0\mathbf{x} \succ {}_0\mathbf{y})$ if, for each $n \in \mathbb{N}$, $\neg({}_0\mathbf{x}^n \succ {}_0\mathbf{y})$. ■

Proof of Proposition 3. Assume $x > y > v > z$. We must show under **O** and **M** that each of **HE**, **WLD**, and **WPD** implies $\neg\,((x, {}_{\text{con}}z) \succ (y, {}_{\text{con}}v))$.

Since $x > y > v > z$, there exist an integer T and utilities $x', z' \in Y$ satisfying $y > x' \geq v > z' > z$ and $x - x' = T(z' - z)$.

By **O** (completeness) and **HE**, $(x', z', {}_{\text{con}}z) \succsim (x, {}_{\text{con}}z)$, and by **M**, $(y, {}_{\text{con}}v) \succsim (x', z', {}_{\text{con}}z)$. By **O** (transitivity), $(y, {}_{\text{con}}v) \succsim (x, {}_{\text{con}}z)$.

Consider next **WLD** and **WPD**. Let ${}_0\mathbf{x}^0 = (x, {}_{\text{con}}z)$, and define ${}_0\mathbf{x}^n$ for $n \in \{1, \ldots, T\}$ inductively as follows:

$$\begin{aligned} x_t^n &= x_t^{n-1} - (z' - z) && \text{for } t = 0 \\ x_t^n &= z' && \text{for } t = n \\ x_t^n &= x_t^{n-1} && \text{for } t \neq 0, n\,. \end{aligned}$$

By **O** (completeness) and **WLD**, ${}_0\mathbf{x}^T \succsim {}_0\mathbf{x}^0$, and by **M**, $(y, {}_{\text{con}}v) \succsim {}_0\mathbf{x}^T$. By **O** (transitivity), $(y, {}_{\text{con}}v) \succsim (x, {}_{\text{con}}z)$ since ${}_0\mathbf{x}^0 = (x, {}_{\text{con}}z)$.

By **O** (completeness) and **WPD**, ${}_0\mathbf{x}^n \succsim {}_0\,\mathbf{x}^{n-1}$ for $n \in \{1, \ldots, T\}$, and by **M**, $(y,\, {}_{\text{con}}v) \succsim {}_0\mathbf{x}^T$. By **O** (transitivity), $(y,\, {}_{\text{con}}v) \succsim (x,\, {}_{\text{con}}z)$ since ${}_0\mathbf{x}^0 = (x,\, {}_{\text{con}}z)$. ■

Proof of Proposition 4. This follows from Asheim, Mitra and Tungodden (2007, Proposition 2). ■

Proof of Proposition 5. Assume that the SWR $\succsim$ satisfies conditions **O**, **RC**, **M**, **IF**, and **HEF**. Let ${}_0\mathbf{x}$, ${}_0\mathbf{y} \in \mathbf{X}$ satisfy ${}_0\mathbf{x} \succ {}_0\mathbf{y}$, and let $\underline{y}$, $\bar{y} \in Y$ satisfy $\underline{y} \leq x_t, y_t \leq \bar{y}$ for all $t \in \mathbb{Z}_+$. For any $T \in \mathbb{Z}_+$ with $x_{T-1} > \underline{y}$, Proposition 4 implies that $(x_{T-1},\, {}_{\text{con}}\underline{y}) \succ {}_{\text{con}}\underline{y}$ would contradict **RC** and **HEF**. Since $x_{T-1} \geq \underline{y}$, it follows from **O** and **M** that $(x_{T-1},\, {}_{\text{con}}\underline{y}) \sim {}_{\text{con}}\underline{y}$ for all $T > 0$. By **IF** and the above argument,

$$({}_{T-2}\mathbf{x}_{T-1},\, {}_{\text{con}}\underline{y}) = (x_{T-2},\, x_{T-1},\, {}_{\text{con}}\underline{y}) \sim (x_{T-2},\, {}_{\text{con}}\underline{y}) \sim {}_{\text{con}}\underline{y}\,.$$

By invoking **O** and applying **IF** and the above argument repeatedly, $({}_0\mathbf{x}_{T-1},\, {}_{\text{con}}\underline{y}) \sim {}_{\text{con}}\underline{y}$ for all $T > 0$. Likewise, $({}_0\mathbf{y}_{T-1},\, {}_{\text{con}}\underline{y}) \sim {}_{\text{con}}\underline{y}$ for all $T > 0$.

Let ${}_0\mathbf{z}$, ${}_0\mathbf{v} \in [\underline{y}, \bar{y}]^{\mathbb{Z}_+}$ be given by ${}_0\mathbf{z} = {}_0\mathbf{v} = {}_{\text{con}}\underline{y}$. Since $({}_0\mathbf{x}_{T-1},\, {}_{\text{con}}\underline{y}) \sim {}_{\text{con}}\underline{y} \sim ({}_0\mathbf{y}_{T-1},\, {}_{\text{con}}\underline{y})$ for all $T > 0$, we have by **O** that $({}_0\mathbf{x}_{T-1},\, {}_T\mathbf{z}) \sim ({}_0\mathbf{y}_{T-1},\, {}_T\mathbf{v})$ for all $T > 0$. This contradicts **DP**. ■

The following result is useful for the proof of Proposition 6.

Lemma 2 *Assume that the* SWR $\succsim$ *satisfies conditions* ***O**, **RC**, **IF**, **M**, **RD**, and **HEF**. Then, for all* ${}_0\mathbf{x} \in \mathbf{X}$ *and* $T \in \mathbb{Z}_+$, ${}_T\mathbf{x} \precsim {}_{T+1}\mathbf{x}$.

Proof. Assume that the SWR $\succsim$ satisfies conditions **O**, **RC**, **IF**, **M**, **RD**, and **HEF**. By the interpretation of ${}_T\mathbf{x}$, it is sufficient to show that ${}_0\mathbf{x} \precsim {}_1\mathbf{x}$. Suppose on the contrary that ${}_0\mathbf{x} \succ {}_1\mathbf{x}$. By Lemma 1, there exist z^0, $z^1 \in Y$ such that ${}_{\text{con}}z^0 \sim {}_0\mathbf{x}$ and ${}_{\text{con}}z^1 \sim {}_1\mathbf{x}$, where, by **O**, (4), and ${}_0\mathbf{x} \succ {}_1\mathbf{x}$, it follows that $z^0 > z^1$. Furthermore, since ${}_1\mathbf{x} \sim {}_{\text{con}}z^1$, it follows by **IF** that $(x_0,\, {}_1\mathbf{x}) \sim (x_0,\, {}_{\text{con}}z^1)$. Hence, ${}_0\mathbf{x} \sim (x_0,\, {}_{\text{con}}z^1)$.

If $x_0 \leq z^0$, then,

$$\begin{aligned}
{}_0\mathbf{x} \sim (x_0,\, {}_{\text{con}}z^1) \prec (x_0,\, {}_{\text{con}}z^0) \quad & \text{by (4) and condition } \mathbf{IF} \text{ since } z^1 < z^0 \\
\precsim (z^0,\, {}_{\text{con}}z^0) = {}_{\text{con}}z^0 \sim {}_0\mathbf{x} \quad & \text{by conditions } \mathbf{O} \text{ and } \mathbf{M} \text{ since } x_0 \leq z^0.
\end{aligned}$$

This contradicts condition **O**, ruling out this case.

If $x_0 > z^0$, then, by selecting some $v \in (z^1, z^0)$,

$$
\begin{aligned}
{}_0\mathbf{x} \sim (x_0, {}_{\text{con}}z^1) \precsim (z^0, {}_{\text{con}}v) &\quad \text{by conditions } \mathbf{O} \text{ and } \mathbf{HEF} \text{ since } x_0 > z^0 > v > z^1 \\
\prec (z^0, {}_{\text{con}}z^0) \sim {}_0\mathbf{x} &\quad \text{by (4) and condition } \mathbf{IF} \text{ since } v < z^0 .
\end{aligned}
$$

This contradicts condition **O**, ruling out also this case. ■

Proof of Proposition 6. *Part I: (1) implies (2).* Assume that the SWR $\succsim$ satisfies conditions **O**, **RC**, **IF**, **M**, **RD**, and **HEF**. By Proposition 2, the SWR $\succsim$ is represented by a monotone SWF $W : \mathbf{X} \to \mathbb{R}$ satisfying, for some $U \in \mathcal{U}_I$ and $V \in \mathcal{V}(U)$, $W({}_0\mathbf{x}) = V(U(x_0), W({}_1\mathbf{x}))$ for all ${}_0\mathbf{x} \in \mathbf{X}$ and $W({}_{\text{con}}z) = U(z)$ for all $z \in Y$. It remains to be shown that $V(u, w) = w$ for $u > w$, implying that V satisfies (V.3′) and, thus, $V \in \mathcal{V}_S(U)$.

Since $V(u, w)$ is non-decreasing in u for given $w \in U(Y)$ and $V(u, w) = w$ for $u = w$, suppose that $V(u, w) > w$ for some u, $w \in U(Y)$ with $u > w$. Since $U \in \mathcal{U}_I$, the properties of W imply that there exist x, $z \in Y$ with $x > z$ such that

$$
\begin{aligned}
W(x, {}_{\text{con}}z) &= V(U(x), W({}_{\text{con}}z)) = V(U(x), U(z)) \\
&= V(u, w) > w = U(z) = W({}_{\text{con}}z) .
\end{aligned}
$$

Since the SWR $\succsim$ is represented by the SWF W, it follows that $(x, {}_{\text{con}}z) \succ {}_{\text{con}}z$. This contradicts Lemma 2.

Part II: (2) implies (1). Assume that the monotone mapping $W : \mathbf{X} \to \mathbb{R}$ is an SWF and satisfies, for some $U \in \mathcal{U}_I$ and $V \in \mathcal{V}_S(U)$, $W({}_0\mathbf{x}) = V(U(x_0), W({}_1\mathbf{x}))$ for all ${}_0\mathbf{x} \in \mathbf{X}$ and $W({}_{\text{con}}z) = U(z)$ for all $z \in Y$. By Proposition 2, it remains to be shown that the SWR $\succsim$, represented by the SWF W, satisfies **HEF**. We now provide this argument.

Let x, y, z, $v \in Y$ satisfy $x > y > v > z$. We have to show that $\neg\left((x, {}_{\text{con}}z) \succ (y, {}_{\text{con}}v)\right)$, or equivalently, $W(x, {}_{\text{con}}z) \le W(y, {}_{\text{con}}v)$. By the properties of W,

$$
\begin{aligned}
W(x, {}_{\text{con}}z) &= V(U(x), W({}_{\text{con}}z)) = V(U(x), U(z)) = U(z) \\
&< U(v) = V(U(y), U(v)) = V(U(y), W({}_{\text{con}}v)) = W(y, {}_{\text{con}}v) ,
\end{aligned}
$$

since $x > y > v > z$, $U \in \mathcal{U}_I$, and $V \in \mathcal{V}_S(U)$ ■

Proof of Proposition 7. Fix $U \in \mathcal{U}_I$ and $V \in \mathcal{V}_S(U)$. The proof has two parts.

Part I: $\lim_{T\to\infty} W({}_T\mathbf{x}) = \liminf_{t\to\infty} U(x_t)$. Assume that the monotone mapping $W : \mathbf{X} \to \mathbb{R}$ satisfies $W({}_0\mathbf{x}) = V(U(x_0), W({}_1\mathbf{x}))$ for all ${}_0\mathbf{x} \in \mathbf{X}$ and $W({}_{\text{con}}z) = U(z)$ for all $z \in Y$. Hence, by Proposition 6, the SWF W represents an SWR $\succsim$ satisfying **O**, **RC**, **M**,

RD, **IF**, and **HEF**. By Lemma 1, for all ${}_0\mathbf{x} \in \mathbf{X}$, there exists $z \in Y$ such that ${}_{\text{con}}z \sim {}_0\mathbf{x}$. By Lemma 2, $W({}_t\mathbf{x})$ is non-decreasing in t.

Step 1: $\lim_{t\to\infty} W({}_t\mathbf{x})$ *exists.* Suppose $W({}_\tau\mathbf{x}) > \limsup_{t\to\infty} U(x_t)$ for some $\tau \in \mathbb{Z}_+$. By the premise and the fact that $U \in \mathcal{U}_I$, there exists $z \in Y$ satisfying

$$W({}_\tau\mathbf{x}) \;\geq\; U(z) \;>\; \limsup{}_{t\to\infty} U(x_t)$$

and $T \geq \tau$ such that $z > v := \sup_{t\geq T} x_t$. By **RD**, **O**, and **M**, ${}_{\text{con}}z \succ (v, {}_{\text{con}}z) \succsim {}_T\mathbf{x}$, and hence, by **O**, ${}_{\text{con}}z \succ {}_T\mathbf{x}$. However, since $W({}_t\mathbf{x})$ is non-decreasing in t, $W({}_T\mathbf{x}) \geq W({}_\tau\mathbf{x}) \geq U(z)$. This contradicts that W is an SWF. Hence, $W({}_t\mathbf{x})$ is bounded above by $\limsup_{t\to\infty} U(x_t)$, and the result follows since $W({}_t\mathbf{x})$ is non-decreasing in t.

Step 2: $\lim_{t\to\infty} W({}_t\mathbf{x}) \geq \liminf_{t\to\infty} U(x_t)$. Suppose $\lim_{t\to\infty} W({}_t\mathbf{x}) < \liminf_{t\to\infty} U(x_t)$. By the premise and the fact that $U \in \mathcal{U}_I$, there exists $z \in Y$ satisfying

$$\lim{}_{t\to\infty} W({}_t\mathbf{x}) \;\leq\; U(z) \;<\; \liminf{}_{t\to\infty} U(x_t)$$

and $T \geq 0$ such that $z < v := \inf_{t\geq T} x_t$. By **O**, **M**, and **RD**, ${}_{\text{con}}z \precsim (z, {}_{\text{con}}v) \prec {}_{\text{con}}v \precsim {}_T\mathbf{x}$, and hence, by **O**, ${}_{\text{con}}z \prec {}_T\mathbf{x}$. However, since $W({}_t\mathbf{x})$ is non-decreasing in t, $W({}_T\mathbf{x}) \leq \lim_{t\to\infty} W({}_t\mathbf{x}) \leq U(z)$. This contradicts that W is an SWF.

Step 3: $\lim_{t\to\infty} W({}_t\mathbf{x}) \leq \liminf_{t\to\infty} U(x_t)$. Suppose $\lim_{t\to\infty} W({}_t\mathbf{x}) > \liminf_{t\to\infty} U(x_t)$. By Lemma 1, there exists, for all $t \in \mathbb{Z}_+$, $z^t \in Y$ such that ${}_{\text{con}}z^t \sim {}_t\mathbf{x}$. Since $U \in \mathcal{U}_I$, $z \in Y$ defined by $z := \lim_{t\to\infty} z^t$ satisfies $U(z) = \lim_{t\to\infty} W({}_t\mathbf{x})$. By the premise and the fact that $U \in \mathcal{U}_I$, there exists $x \in Y$ satisfying

$$\liminf{}_{t\to\infty} U(x_t) \;<\; U(x) \;<\; U(z)$$

and a subsequence $(x_{t_\tau}, z^{t_\tau})_{\tau\in\mathbb{Z}_+}$ such that, for all $\tau \in \mathbb{Z}_+$, $x_{t_\tau} \leq x < z^{t_\tau}$. Then

$${}_{\text{con}}z^{t_\tau} \;\sim\; {}_{t_\tau}\mathbf{x} = (x_{t_\tau},\, {}_{t_\tau+1}\mathbf{x}) \precsim (x,\, {}_{\text{con}}z^{t_{\tau+1}}) \precsim (x,\, {}_{\text{con}}z)\,,$$

since z^t is non-decreasing in t. By **O**, **RC**, and the definition of z, ${}_{\text{con}}z \precsim (x, {}_{\text{con}}z)$. Since $x < z$, this contradicts **RD**.

Part II: Existence. Let ${}_0\mathbf{x} \in \mathbf{X}$. This implies that there exist $\underline{y}, \bar{y} \in Y$ such that, for all $t \in \mathbb{Z}_+$, $\underline{y} \leq x_t \leq \bar{y}$. For each $T \in \mathbb{Z}_+$, consider $\{w(t,T)\}_{t=0}^T$ determined by (1).

Step 1: $w(t,T)$ is non-increasing in T for given $t \leq T$. Given $T \in \mathbb{Z}_+$,

$$w(T, T+1) = V(U(x_T), w(T+1, T+1)) \leq w(T+1, T+1) = \liminf_{t\to\infty} U(x_t) = w(T,T)$$

by (1) and (V.3′). Thus, applying (V.2), we have

$$w(T-1, T+1) = V(U(x_{T-1}), w(T, T+1)) \le V(U(x_{T-1}), w(T, T)) = w(T-1, T)\,.$$

Using (V.2) repeatedly, we obtain

$$w(t, T+1) \le w(t, T) \;\text{ for all } t \in \{0, ..., T-1\}\,,$$

which establishes that $w(t,T)$ is non-increasing in T for given $t \le T$.

Step 2: $w(t,T)$ is bounded below by $U(\underline{y})$. By (1), (V.1), (V.2), and (V.3′), $w(T,T) = \liminf_{t\to\infty} U(x_t) \ge U(\underline{y})$, and for all $t \in \{0, ..., T-1\}$,

$$w(t+1, T) \ge U(\underline{y}) \text{ implies } w(t,T) = V(U(x_t), w(t+1, T)) \ge V(U(\underline{y}), U(\underline{y})) = U(\underline{y})\,.$$

Hence, it follows by induction that $w(t,T)$ is bounded below by $U(\underline{y})$.

Step 3: Definition and properties of W_σ. By steps 1 and 2, $\lim_{T\to\infty} w(t,T)$ exists for all $t \in \mathbb{Z}_+$. Define the mapping $W_\sigma : \mathbf{X} \to \mathbb{R}$ by (W). We have that W_σ is monotone by (1), (V.1), and (V.2). As $w(0,T) = V(U(x_0),\, w(1,T))$ and V satisfies (V.0), we have that $W_\sigma({}_0\mathbf{x}) = V(U(x_0), W_\sigma({}_1\mathbf{x}))$. Finally, if ${}_0\mathbf{x} = {}_{\text{con}}z$ for some $z \in Y$, then it follows from (1) and (V.3′) that $w(t,T) = U(z)$ for all $T \in \mathbb{Z}_+$ and $t \in \{0, ..., T\}$, implying that $W_\sigma({}_0\mathbf{x}) = U(z)$. ■

Proof of Proposition 8. Suppose there exists a monotone mapping $W : \mathbf{X} \to \mathbb{R}$ satisfying $W({}_0\mathbf{y}) = V(U(y_0), W({}_1\mathbf{y}))$ for all ${}_0\mathbf{y} \in \mathbf{X}$ and $W({}_{\text{con}}z) = U(z)$ for all $z \in Y$ such that $W({}_0\mathbf{x}) \ne W_\sigma({}_0\mathbf{x})$. Since V satisfies the property of weak time perspective, there is a continuous increasing transformation $g : \mathbb{R} \to \mathbb{R}$ such that $|g(W({}_0\mathbf{x})) - g(W_\sigma({}_0\mathbf{x}))| = \epsilon > 0$, and furthermore, $|g(W({}_t\mathbf{x})) - g(W_\sigma({}_t\mathbf{x}))| = |g(V(U(x_t), W({}_{t+1}\mathbf{x}))) - g(V(U(x_t), W_\sigma({}_{t+1}\mathbf{x})))| \le |g(W({}_{t+1}\mathbf{x})) - g(W_\sigma({}_{t+1}\mathbf{x}))|$ for all $t \in \mathbb{Z}_+$. It now follows, by induction, that

$$|g(W({}_T\mathbf{x})) - g(W_\sigma({}_T\mathbf{x}))| \ge \epsilon > 0$$

for all $T \in \mathbb{Z}_+$. However this contradicts that, for all $T \in \mathbb{Z}_+$,

$$\lim_{T\to\infty} W({}_T\mathbf{x}) = \liminf_{t\to\infty} U(x_t) = \lim_{T\to\infty} W_\sigma(\mathbf{x})$$

by Proposition 7, since g is a continuous increasing transformation. ■

For the proofs of the results of Section 4, the following notation is useful, where ${}_0\mathbf{z} =$

$(z_0,\, {}_1\mathbf{z}) = (z_0,\, z_1,\, {}_2\mathbf{z}) \in \mathbf{X}$ is a fixed but arbitrary reference stream:

$x_0 \succsim_0^{\mathbf{z}} y_0$	means	$(x_0,\, {}_1\mathbf{z}) \succsim (y_0,\, {}_1\mathbf{z})$
${}_1\mathbf{x} \;{}_1\!\succsim^{\mathbf{z}} {}_1\mathbf{y}$	means	$(z_0,\, {}_1\mathbf{x}) \succsim (z_0,\, {}_1\mathbf{y})$
$(x_0,\, x_1) \;{}_0\!\succsim_1^{\mathbf{z}} (y_0,\, y_1)$	means	$(x_0,\, x_1,\, {}_2\mathbf{z}) \succsim (y_0,\, y_1,\, {}_2\mathbf{z})$
${}_2\mathbf{x} \;{}_2\!\succsim^{\mathbf{z}} {}_2\mathbf{y}$	means	$(z_0,\, z_1,\, {}_2\mathbf{x}) \succsim (z_0,\, z_1,\, {}_2\mathbf{y})$
$x_1 \succsim_1^{\mathbf{z}} y_1$	means	$(z_0,\, x_1,\, {}_2\mathbf{z}) \succsim (z_0,\, y_1,\, {}_2\mathbf{z})\,.$

Say that $\succsim_0^{\mathbf{z}}$ is *independent of* ${}_0\mathbf{z}$ if, for all ${}_0\mathbf{x}$, ${}_0\mathbf{y}$, ${}_0\mathbf{z}$, ${}_0\mathbf{v} \in \mathbf{X}$, $x_0 \succsim_0^{\mathbf{z}} y_0$ if and only if $x_0 \succsim_0^{\mathbf{v}} y_0$, and likewise for ${}_1\!\succsim^{\mathbf{z}}$, ${}_0\!\succsim_1^{\mathbf{z}}$, ${}_2\!\succsim^{\mathbf{z}}$, and $\succsim_1^{\mathbf{z}}$. In this notation and terminology, condition **IF** implies that ${}_1\!\succsim^{\mathbf{z}}$ is independent of ${}_0\mathbf{z}$, while condition **IP** states that ${}_0\!\succsim_1^{\mathbf{z}}$ is independent of ${}_0\mathbf{z}$. The following result due to Gorman (1968b) indicates that imposing condition **IP** is consequential.

Lemma 3 *Assume that the* SWR $\succsim$ *satisfies conditions* ***IF*** *and* ***IP****. Then* $\succsim_0^{\mathbf{z}}$*,* ${}_1\!\succsim^{\mathbf{z}}$*,* ${}_0\!\succsim_1^{\mathbf{z}}$*,* ${}_2\!\succsim^{\mathbf{z}}$*, and* $\succsim_1^{\mathbf{z}}$ *are independent of* ${}_0\mathbf{z}$*.*

Proof. Assume that the SWR $\succsim$ satisfies conditions **IF** and **IP**. By repeated application of **IF**, ${}_1\!\succsim^{\mathbf{z}}$ and ${}_2\!\succsim^{\mathbf{z}}$ are independent of ${}_0\mathbf{z}$, while **IP** states that ${}_0\!\succsim_1^{\mathbf{z}}$ is independent of ${}_0\mathbf{z}$. By **IF**, $(x_1,\, {}_2\mathbf{z}) \succsim (y_1,\, {}_2\mathbf{z})$ is equivalent to $(z_0,\, x_1,\, {}_2\mathbf{z}) \succsim (z_0,\, y_1,\, {}_2\mathbf{z})$, which, by **IP**, is equivalent to $(z_0,\, x_1,\, {}_2\mathbf{v}) \succsim (z_0,\, y_1,\, {}_2\mathbf{v})$, which in turn, by **IF**, is equivalent to $(x_1,\, {}_2\mathbf{v}) \succsim (y_1,\, {}_2\mathbf{v})$, which finally, by **IF**, is equivalent to $(v_0,\, x_1,\, {}_2\mathbf{v}) \succsim (v_0,\, y_1,\, {}_2\mathbf{v})$, where ${}_0\mathbf{v} \in \mathbf{X}$ is some arbitrary stream. Hence, $\succsim_0^{z}$ and $\succsim_1^{z}$ are independent of ${}_0\mathbf{z}$. ∎

Proof of Theorem 2. *Part I:* This part is proved in three steps.

Step 1: By Lemma 3, **IF** and **IP** imply that $\succsim_0^{\mathbf{z}}$ is independent of ${}_0\mathbf{z}$.

Step 2: By condition **WS**, there exist ${}_0\mathbf{x}$, ${}_0\mathbf{y}$, ${}_0\mathbf{z} \in \mathbf{X}$ such that $x_0 \succ_0^{\mathbf{z}} y_0$. This rules out that $x_0 = y_0$, and by **M**, $x_0 < y_0$ would lead to a contradiction. Hence, $x_0 > y_0$. Since $\succsim_0^{\mathbf{z}}$ is independent of ${}_0\mathbf{z}$, this implies **RS**.

Step 3: By Proposition 4, there is no SWR $\succsim$ satisfying **RC**, **RS**, and **HEF**.

Part II: To establish this part, consider dropping a single condition.

Dropping ***IP****.* Existence follows from Theorem 1 since **RD** implies **WS**.

Dropping ***HEF****.* Existence follows from Propositions 9 and 10.

Dropping ***WS****.* All the remaining conditions are satisfied by the SWF $\succsim$ being represented by the mapping $W : \mathbf{X} \to \mathbb{R}$ defined by $W({}_0\mathbf{x}) := \liminf_{t\to\infty} x_t$.

*Dropping **M**.* All the remaining conditions are satisfied by the SWF $\succsim$ being represented by the mapping $W : \mathbf{X} \to \mathbb{R}$ defined by $W({}_0\mathbf{x}) := -x_0 + \liminf_{t\to\infty} x_t$.

*Dropping **IF**.* All the remaining conditions are satisfied by the SWF $\succsim$ being represented by the mapping $W : \mathbf{X} \to \mathbb{R}$ defined by $W({}_0\mathbf{x}) := \min\{x_0,\, x_1\}$.

*Dropping **RC**.* Existence follows from Proposition 13 since **SP** implies **M** and **WS**. ■

Proof of Proposition 9. The proof is based on standard results for additively separable representations (Debreu, 1960; Gorman, 1968a; Koopmans, 1986a), and is available at http://folk.uio.no/gasheim/srswfs2.pdf . ■

Proof of Proposition 10. Available at http://folk.uio.no/gasheim/srswfs2.pdf . ■

Proof of Proposition 11. Assume that the SWR $\succsim$ satisfies conditions **O**, **RC**, **IF**, **M**, **IP**, and **NDF**. By Proposition 1, **O**, **IF**, and **NDF** imply **WS**. Hence, by Propositions 9 and 10, the SWR $\succsim$ is represented by $W_\delta : \mathbf{X} \to \mathbb{R}$ defined by, for each ${}_0\mathbf{x} \in \mathbf{X}$,

$$W_\delta({}_0\mathbf{x}) = (1-\delta)\sum\nolimits_{t=0}^{\infty} \delta^t U(x_t)\,,$$

for some $U \in \mathcal{U}$ and $\delta \in (0,1)$. This implies **DP**, thus contradicting **NDP**. ■

The proof of Proposition 12 needs some preliminaries. A sustainable recursive SWF $W : \mathbf{X} \to \mathbb{R}$ is given, with $W({}_0x) = V(U(x_0), W({}_1x))$ for all ${}_0x \in \mathbf{X}$ and $W({}_{con}z) = U(z)$ for all $z \in Y$. A utility stream ${}_0\mathbf{u}$ is associated with a feasible path ${}_0\mathbf{k}$ from $k \in [0, \bar{k}]$ if $u_t = U(x(f(k_t) - k_{t+1}, k_t))$ for $t \geq 0$. Write $u^* \equiv U(x^*) = U(x(c^*, k^*)) = U(x(f(k^*) - k^*, k^*))$.

Write $S := \{(c,k) \in [0, \bar{k}] \mid x(c,k) = x^*\}$. Since $S \neq \emptyset$, we can define:

$$I = \{k \in (0, \bar{k}] \mid \text{there is some } c \geq 0 \text{ satisfying } (c,k) \in S\}$$

Note that $k^* \in I$. Let $k \in I$; then there is $c \geq 0$ such that $x(c,k) = x^*$. Now, let $k' \in I$ satisfy $k' > k$. Then $x(c,k') \geq x(c,k) = x^*$, while $x(0,k') \leq x(0,\bar{k}) = x(f(\bar{k}) - \bar{k}, \bar{k}) \leq x^*$. Thus, by continuity of x, there is some $c' \geq 0$, such that $x(c',k') = x^*$. This shows that I is a sub-interval of $(0, \bar{k}]$, containing $[k^*, \bar{k}]$.

Define, for each $k \in I$, the set $\phi(k) = \{c \geq 0 \mid (c,k) \in S\}$. By definition of I, $\phi(k)$ is non-empty for each $k \in I$. Since $k \in I$ implies $k > 0$, $\phi(k)$ is a singleton by property (3)(ii) of the function x. Thus, ϕ is a function from I to $\mathbb{R}_+$, and by definition, $x(\phi(k), k) = x^*$ for all $k \in I$, so $c^* = \phi(k^*)$. By property (3)(ii), ϕ is non-increasing on I.

Lemma 4 *For every $k \in [k^*, \bar{k}]$, there exists a feasible resource path, ${}_0\hat{\mathbf{k}}$, from k where the associated well-being stream, ${}_0\hat{\mathbf{x}}$, satisfies $W({}_t\hat{\mathbf{x}}) = W({}_{\text{con}}x^*)$ for $t \geq 0$.*

Proof. Let $k \in [k^*, \bar{k}]$, and consider the resource path ${}_0\hat{\mathbf{k}}$ defined by

$$k_0 = k\,, \quad k_{t+1} = f(k_t) - \phi(k_t) \text{ for } t > 0\,.$$

Note that, if $k_t \in [k^*, \bar{k}]$, then

$$\bar{k} \geq f(\bar{k}) \geq f(k_t) - \phi(k_t) \geq f(k_t) - \phi(k^*) = f(k_t) - [f(k^*) - k^*] \geq k^*\,.$$

Hence, $k_{t+1} \in [k^*, \bar{k}]$ and, by induction, $k_t \in [k^*, \bar{k}]$ for $t \geq 0$. This shows that ${}_0\hat{\mathbf{k}}$ is feasible from $k \in [k^*, \bar{k}]$. By the definition of $\phi(\cdot)$, $x_t = x(\phi(k_t), k_t) = x^*$ for $t \geq 0$. ■

Lemma 5 *Let ${}_0\hat{\mathbf{k}}$ be a feasible resource path from $k \in [0, \bar{k}]$ with associated utility stream, ${}_0\mathbf{u}$. Given any $\varepsilon > 0$, there is some $T \geq 0$ such that $u_T < u^* + \varepsilon$.*

Proof. Suppose, on the contrary, there is some $\varepsilon > 0$, such that $u_t \geq u^* + \varepsilon$ for all $t \geq 0$. By continuity of U, there is $\delta > 0$, such that whenever $x \in Y$ and $|x - x^*| < \delta$, we have $|U(x) - U(x^*)| < \varepsilon$. Thus, we must have $|x_t - x^*| \geq \delta$ for all $t \geq 0$. Further, since U is an increasing function, we must have $x_t \geq x^* + \delta$ for all $t \geq 0$. This implies:

$$x(f(k_t) - k_{t+1}, k_t) = x(c_t, k_t) > x^* \text{ for all } t \geq 0\,.$$

Since $x(f(k_t) - k_t, k_t) \leq x^*$, property (3)(ii) implies that $k_{t+1} < k_t$ for all $t \geq 0$. Thus, ${}_0\mathbf{k}$ must converge to some $\kappa \in [0, \bar{k}]$. The continuity of f and x then imply that $x(f(\kappa) - \kappa, \kappa) \geq x^* + \delta$, and this contradicts the definition of x^*. ■

Lemma 6 *Let ${}_0\hat{\mathbf{k}}$ be a feasible resource path from $k \in [0, \bar{k}]$ with associated well-being stream, ${}_0\hat{\mathbf{x}}$. Then, we have $W({}_0\hat{\mathbf{x}}) \leq W({}_{\text{con}}x^*)$.*

Proof. Suppose, by way of contradiction, that there exist $k \in [0, \bar{k}]$ and a feasible resource path, ${}_0\hat{\mathbf{k}}$, from k where the associated well-being stream, ${}_0\hat{\mathbf{x}}$, satisfies $W({}_0\hat{\mathbf{x}}) > W({}_{\text{con}}x^*) = U(x^*) = u^*$. Denote by ${}_0\hat{\mathbf{u}}$ the associated utility stream (i.e., $\hat{u}_t = U(\hat{x}_t)$ for $t \geq 0$). Since $W({}_t\hat{\mathbf{x}})$ is non-decreasing in t, and is bounded above by $U(x(\bar{k}, \bar{k}))$ (by the properties of a sustainable recursive SWF), it converges to some $\omega \leq U(x(\bar{k}, \bar{k}))$. Hence, $\omega \geq W({}_0\hat{\mathbf{x}}) > u^*$. Since the aggregator function V satisfies (V.3′), we must have $V(u^*, \omega) < V(\omega, \omega) = \omega$. Using the continuity of V, we can find $\varepsilon > 0$ such that

$$V(u^* + \varepsilon, \omega) < V(\omega, \omega) = \omega\,. \tag{5}$$

Write $\theta := \omega - V(u^* + \varepsilon, \omega)$. By (5), $\theta > 0$.

Choose $T \in Z_+$ large enough so that for all $t \geq T$, $W({}_t\hat{\mathbf{x}}) \geq \omega - (\theta/2)$. By Lemma 5, $\hat{u}_t < u^* + \varepsilon$ for some $t \geq T$. Let τ be the first period ($\geq T$) for which $\hat{u}_t < u^* + \varepsilon$. Then:

$$\begin{aligned} \omega - \tfrac{\theta}{2} \;\leq\; W({}_\tau\hat{\mathbf{x}}) \;&=\; V(u_\tau, W({}_{\tau+1}\hat{\mathbf{x}})) \\ &\leq\; V(u^* + \varepsilon, W({}_{\tau+1}\hat{\mathbf{x}})) \;\leq\; V(u^* + \varepsilon, \omega) \;=\; \omega - \theta \;<\; \omega - \tfrac{\theta}{2}\,, \end{aligned}$$

which is a contradiction. ■

Lemma 7 *If a feasible resource path, ${}_0\hat{\mathbf{k}}$, from $k \in [0, \bar{k}]$ has an associated well-being stream, ${}_0\hat{\mathbf{x}}$, which satisfies $W({}_0\hat{\mathbf{x}}) = W({}_{\text{con}}x^*)$, then (i) $\hat{x}_t \geq x^*$ for all $t \geq 0$; and (ii) $\hat{k}_t \geq k^*$ for all $t \geq 0$.*

Proof. Assume that a feasible resource path, ${}_0\hat{\mathbf{k}}$, from $k \in [0, \bar{k}]$ has an associated well-being stream, ${}_0\hat{\mathbf{x}}$, which satisfies $W({}_0\hat{\mathbf{x}}) = W({}_{\text{con}}x^*)$. Since $W({}_t\hat{\mathbf{x}})$ is non-decreasing in t, it follows from Lemma 6 that $W({}_t\hat{\mathbf{x}}) = W({}_{\text{con}}x^*) = u^*$ for all $t \geq 0$.

To establish (i), suppose, by way of contradiction, that $\hat{x}_t < x^*$ for some $t \geq 0$. Then, since $U(\hat{x}_t) < U(x^*) = u^*$, (V.3′) implies:

$$u^* \;=\; W({}_t\hat{\mathbf{x}}) \;=\; V(U(\hat{x}_t), u^*) \;<\; V(u^*, u^*) \;=\; u^*,$$

which is a contradiction.

To establish (ii), suppose, on the contrary, that $\hat{k}_\tau < k^*$ for some $\tau \geq 0$. Then, by the fact that $x(f(k) - k, k) < x^*$ if $k \neq k^*$, we have $x(f(\hat{k}_\tau) - \hat{k}_\tau, \hat{k}_\tau) < x^*$, while:

$$x^* \leq \hat{x}_\tau = x(\hat{c}_\tau, \hat{k}_\tau) = x(f(\hat{k}_\tau) - \hat{k}_{\tau+1}, \hat{k}_\tau)\,.$$

So, $\hat{k}_{\tau+1} < \hat{k}_\tau < k^*$, and repeating this step, $\hat{k}_{t+1} < \hat{k}_t$ for all $t \geq \tau$. Thus, ${}_0\hat{k}$ must converge to some $\kappa \in [0, \bar{k}]$, with $\kappa \leq \hat{k}_\tau < k^*$. The continuity of f and x then imply that:

$$x(f(\kappa) - \kappa, \kappa) \geq x^*$$

using (i). But, this contradicts that $x(f(k) - k, k) < x^*$ if $k \neq k^*$. ■

Proof of Proposition 12. Lemmas 4 and 6 establish existence of an optimum and that any optimal well-being stream satisfies $W({}_t\hat{\mathbf{x}}) = W({}_{\text{con}}x^*)$ for $t \geq 0$. Lemma 7 shows that any optimal resource path ${}_0\hat{\mathbf{k}}$, with associated well-being stream ${}_0\hat{\mathbf{x}}$, satisfies $\hat{x}_t \geq x^*$ and $\hat{k}_t \geq k^*$ for $t \geq 0$. ■

Proof of Proposition 13. Asheim and Tungodden (2004a), Basu and Mitra (2007), and Bossert, Sprumont and Suzumura (2007) define different incomplete leximin and undiscounted utilitarian SWRs, each of which is given an axiomatic characterization. Denote by $\succsim$ one such incomplete SWR. It can be verified that $\succsim$ is reflexive, transitive and satisfies **IF**, **SP**, **HEF** (with $(x, {}_{\text{con}}z) \precsim (y, {}_{\text{con}}v)$ if $x > y > v > z$), and **IP**. Completeness (and thereby condition **O**) can be satisfied by invoking Arrow's (1951) version of Szpilrajn's (1930) extension theorem (see also Svensson, 1980).

Since $\succsim$ satisfies conditions **SP** and **HEF** (with $(x, {}_{\text{con}}z) \precsim (y, {}_{\text{con}}v)$ if $x > y > v > z$), so will any completion. Since, for all ${}_0\mathbf{x}$, ${}_0\mathbf{y}$, ${}_0\mathbf{z} \in \mathbf{X}$, $(x_0, x_1)\ {}_0\succsim_1^{\mathbf{z}}\ (y_0, y_1)$ or $(x_0, x_1)\ {}_0\precsim_1^{\mathbf{z}}\ (y_0, y_1)$, and $\succsim$ satisfies **IP**, so will any completion. However, special care must be taken to ensure that the completion satisfies **IF**.

Consider $\mathbf{X}_0^2 = \{({}_0\mathbf{x}, {}_0\mathbf{y}) \in \mathbf{X}^2 \mid x_0 \neq y_0\}$, and invoke Arrow's (1951) version of Szpilrajn's (1930) extension theorem to complete $\succsim$ on this subset of $\mathbf{X}^2$. For any $({}_0\mathbf{x}, {}_0\mathbf{y}) \in \mathbf{X}$ with ${}_0\mathbf{x} \neq {}_0\mathbf{y}$, let ${}_0\mathbf{x}$ be at least as good as ${}_0\mathbf{y}$ if and only if ${}_T\mathbf{x}$ is at least as good as ${}_T\mathbf{y}$ according to the completion of $\succsim$ on $\mathbf{X}_0^2$, where $T := \min\{t \mid x_t \neq y_t\}$. Since $\succsim$ satisfies **IF**, this construction constitutes a complete SWR satisfying **IF**. ■

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Proofs of Propositions 9 and 10, not to be included in the paper.

Proof of Proposition 9. *Part I: (1) implies (2).* Assume that the SWR $\succsim$ satisfies conditions **O**, **RC**, **IF**, **M**, **WS**, and **IP**.

Say that $\succsim_0^{\mathbf{z}}$ is *sensitive* if there exist ${}_0\mathbf{x}$, ${}_0\mathbf{y}$, ${}_0\mathbf{z} \in \mathbf{X}$ such that $x_0 \succ_0^{\mathbf{z}} y_0$, and likewise for ${}_1\succsim^{\mathbf{z}}$, ${}_0\succsim_1^{\mathbf{z}}$, ${}_2\succsim^{\mathbf{z}}$, and $\succsim_1^{\mathbf{z}}$. By **WS**, $\succsim_0^{\mathbf{z}}$ is sensitive. By **IF**, $(x_1, {}_2\mathbf{z}) \succ (y_1, {}_2\mathbf{z})$ implies $(z_0, x_1, {}_2\mathbf{z}) \succ (z_0, y_1, {}_2\mathbf{z})$. Since $\succsim_0^{\mathbf{z}}$ is sensitive, there exist ${}_0\mathbf{x}$, ${}_0\mathbf{y}$, ${}_0\mathbf{z} \in \mathbf{X}$ such that $x_1 \succ_1^{\mathbf{z}} y_1$, meaning that $\succsim_1^{\mathbf{z}}$ is sensitive. Since Y is not a singleton, it follows from (4) and **IF** that ${}_1\succsim^{\mathbf{z}}$ is sensitive and, by an additional application of **IF**, that ${}_2\succsim^{\mathbf{z}}$ is sensitive. By Lemma 3, $\succsim_0^{\mathbf{z}}$, ${}_1\succsim^{\mathbf{z}}$, ${}_0\succsim_1^{\mathbf{z}}$, ${}_2\succsim^{\mathbf{z}}$, and $\succsim_1^{\mathbf{z}}$ are independent of ${}_0\mathbf{z}$.

By **O** and **M**, there exists a continuous function $\tilde{U} : Y \to \mathbb{R}$ satisfying $\tilde{U}(z) \geq \tilde{U}(v)$ if and only if ${}_{\text{con}}z \succsim {}_{\text{con}}v$. In view of Lemma 1, determine $\tilde{W} : \mathbf{X} \to \mathbb{R}$ by, for all ${}_0\mathbf{x} \in \mathbf{X}$, $\tilde{W}({}_0\mathbf{x}) = \tilde{U}(y)$ where ${}_{\text{con}}y \sim {}_0\mathbf{x}$. By **O**, $\tilde{W}({}_0\mathbf{x}) \geq \tilde{W}({}_0\mathbf{y})$ if and only if ${}_0\mathbf{x} \succsim {}_0\mathbf{y}$. By construction of $\tilde{W}$, $\tilde{W}({}_{\text{con}}z) = \tilde{U}(z)$ for all $z \in Y$. By **IF**, for given $x_0 \in Y$, there exists an increasing transformation $\tilde{V}(\tilde{U}(x_0), \cdot) : \mathbb{R} \to \mathbb{R}$ such that, for all ${}_1\mathbf{x} \in \mathbf{X}$, $\tilde{W}(x_0, {}_1\mathbf{x}) = \tilde{V}(\tilde{U}(x_0), \tilde{W}({}_1\mathbf{x}))$. This determines $\tilde{V} : \tilde{U}(Y)^2 \to \mathbb{R}$. Since $\neg\,((x, {}_{\text{con}}z) \prec {}_{\text{con}}v)$ (resp. $\neg\,((x, {}_{\text{con}}z) \succ {}_{\text{con}}v)$) if and only if

$$\tilde{V}(\tilde{U}(x), \tilde{U}(z)) = \tilde{V}(\tilde{U}(x), \tilde{W}({}_{\text{con}}z)) = \tilde{W}(x, {}_{\text{con}}z) \geq \tilde{U}(v) \quad (\text{resp.} \leq \tilde{U}(v)),$$

RC implies that $\tilde{V}$ is continuous in (u, w) on $\tilde{U}(Y)^2$.

Hence, on the set of streams in $\mathbf{X}$ of the form $(x_0, x_1, {}_{\text{con}}v)$, $\succsim$ is represented by $\tilde{W}(x_0, x_1, {}_{\text{con}}v) = \tilde{V}(x_0, \tilde{W}(x_1, {}_{\text{con}}v)) = \tilde{V}(x_0, \tilde{V}(x_1, \tilde{U}(v)))$, which is continuous in (x_0, x_1, v) on Y^3. Since $\succsim_0^{\mathbf{z}}$, $\succsim_1^{\mathbf{z}}$, and ${}_2\succsim^{\mathbf{z}}$ are sensitive (in the case of ${}_2\succsim^{\mathbf{z}}$ also within the set of constant streams, by **O**, **M**, and **WS**), and $\succsim_0^{\mathbf{z}}$, ${}_1\succsim^{\mathbf{z}}$, ${}_0\succsim_1^{\mathbf{z}}$, ${}_2\succsim^{\mathbf{z}}$, and $\succsim_1^{\mathbf{z}}$ are all independent of ${}_0\mathbf{z}$, it now follows from standard results for additively separable representations (Debreu, 1960; Gorman, 1968a; Koopmans, 1986a) that there exist continuous functions $U_0 : Y \to \mathbb{R}$, $U_1 : Y \to \mathbb{R}$, and $U : Y \to \mathbb{R}$, such that $W_0 : \{{}_0\mathbf{x} \in \mathbf{X} \mid x_t = v \text{ for all } t \geq 2\} \to \mathbb{R}$ defined by

$$W_0(x_0, x_1, {}_{\text{con}}v) = U_0(x_0) + U_1(x_1) + U(v) \tag{6}$$

is an SWF. By repeated applications of **IF**, it follows from Lemma 1 that W_0 can be extended to all ${}_0\mathbf{x} \in \mathbf{X}$:

$$W_0({}_0\mathbf{x}) = U_0(x_0) + U_1(x_1) + U(W^*({}_2\mathbf{x}))\,,$$

where $W^* : \mathbf{X} \to Y$ maps any ${}_0\mathbf{y} \in \mathbf{X}$ into *some* $z \in Y$ satisfying ${}_{\text{con}}z \sim {}_0\mathbf{y}$. It follows from **IF** that $W_1 : \mathbf{X} \to \mathbb{R}$ defined by

$$W_1({}_0\mathbf{x}) = U_1(x_0) + U(W^*({}_1\mathbf{x}))\,,$$

is also an SWF. The additively separable structure between time 0 and times $1, 2, \dots$ means that, for all ${}_0\mathbf{x} \in \mathbf{X}$, $W_1({}_0\mathbf{x}) = \delta W_0({}_0\mathbf{x}) + \epsilon$, $U_1(x_0) = \delta U_0(x_0) + \epsilon$, and

$$U(W^*({}_1\mathbf{x})) = \delta\big(U_1(x_1) + U(W^*({}_2\mathbf{x}))\big) + \epsilon\,. \tag{7}$$

Furthermore, by inserting ${}_{\text{con}}z$ in (7) and keeping in mind that $U(W^*({}_{\text{con}}z)) = U(z)$, we obtain $U(z) = \delta\big(U_1(z) + U(z)\big) + \epsilon$, or equivalently,

$$U_1(z) = \tfrac{1-\delta}{\delta}U(z) - \tfrac{\epsilon}{\delta} \tag{8}$$

for all $z \in Y$. By defining $W : \mathbf{X} \to \mathbb{R}$ by, for all ${}_0\mathbf{x} \in \mathbf{X}$, $W({}_0\mathbf{x}) = U(W^*({}_0\mathbf{x}))$, it follows from (7) and (8) that the SWF W satisfies $W({}_0\mathbf{x}) = (1-\delta)U(x_0) + \delta W({}_1\mathbf{x})$ for all ${}_0\mathbf{x} \in \mathbf{X}$, where $\delta \in (0,1)$ since both $\succsim_0^{\mathbf{z}}$ and ${}_1\!\succsim^{\mathbf{z}}$ are sensitive. By **M**, W is monotone and U is non-decreasing. By **WS**, $U(Y)$ is not a singleton; hence, $U \in \mathcal{U}$.

If **WS** is strengthened to **RD**, then it follows from (4), (6), and repeated applications of **IF** that $U(Y)$ is increasing; hence, $U \in \mathcal{U}_I$.

Part II: (2) implies (1). Assume that the monotone mapping $W : \mathbf{X} \to \mathbb{R}$ is an SWF and satisfies, for some $U \in \mathcal{U}$ and $\delta \in (0,1)$, $W({}_0\mathbf{x}) = (1-\delta)U(x_0) + \delta W({}_1\mathbf{x})$ for all ${}_0\mathbf{x} \in \mathbf{X}$. Note that, for each $U \in \mathcal{U}$ and each $\delta \in (0,1)$, $V : U(Y)^2 \to \mathbb{R}$ defined by $V(u,w) = (1-\delta)u + \delta w$ is an element of $\mathcal{V}(U)$; hence,

$$\{V : U(Y)^2 \to \mathbb{R} \mid V(u,w) = (1-\delta)u + \delta w \text{ for some } \delta \in (0,1)\} \;\subseteq\; \mathcal{V}(U)\,.$$

Also, $W({}_{\text{con}}z) = (1-\delta)U(z) + \delta W({}_{\text{con}}z)$ implies $W({}_{\text{con}}z) = U(z)$. Hence, by Proposition 2, if $U \in \mathcal{U}_I$, it remains to be shown that the SWR $\succsim$, represented by the SWF W, satisfies **IP**. The following argument shows that $\succsim$ satisfies **IP**.

Let ${}_0\mathbf{x}$, ${}_0\mathbf{y}$, ${}_0\mathbf{z}$, ${}_0\mathbf{v} \in \mathbf{X}$, and let $(x_0, x_1) \;{}_0\!\succsim_1^{\mathbf{z}}\, (y_0, y_1)$, or equivalently, $W(x_0, x_1, {}_2\mathbf{z}) \geq W(y_0, y_1, {}_2\mathbf{z})$. We have to show that $(x_0, x_1) \;{}_0\!\succsim_1^{\mathbf{v}}\, (y_0, y_1)$, or equivalently, $W(x_0, x_1, {}_2\mathbf{v}) \geq W(y_0, y_1, {}_2\mathbf{v})$. By the properties of W,

$$\begin{aligned} W(x_0, x_1, {}_2\mathbf{z}) - W(y_0, y_1, {}_2\mathbf{z}) &= (1-\delta)\big[\big(U(x_0) - U(y_0)\big) + \delta\big(U(x_1) - U(y_1)\big)\big] \\ &= W(x_0, x_1, {}_2\mathbf{v}) - W(y_0, y_1, {}_2\mathbf{v})\,, \end{aligned}$$

since $W({}_0\mathbf{x}') = (1-\delta)\big(U(x_0') + \delta U(x_1')\big) + \delta^2 W({}_2\mathbf{x}')$ for all ${}_0\mathbf{x}' \in \mathbf{X}$.

If $U \in \mathcal{U} \backslash \mathcal{U}_I$, then above analysis goes through, except that it does not follow that the SWR $\succsim$ satisfies **RD**. Instead, the property that $U(Y)$ is not a singleton implies that SWR $\succsim$ satisfies **WS**. ■

Proof of Proposition 10. Fix $U \in \mathcal{U}$ and $\delta \in (0,1)$, and let ${}_0\mathbf{x} \in \mathbf{X}$, implying that there exist $\underline{y}$, $\bar{y} \in Y$ such that, for all $t \in \mathbb{Z}_+$, $\underline{y} \leq x_t \leq \bar{y}$.

Part I: Existence. For each $T \in \mathbb{Z}_+$, consider the following finite sequence:

$$
\begin{aligned}
&w(T,T) = U(\bar{y}) \\
&w(T-1,T) = (1-\delta)U(x_{T-1}) + \delta w(T,T) = (1-\delta)U(x_{T-1}) + \delta U(\bar{y}) \\
&\dots \\
&w(0,T) = (1-\delta)U(x_0) + \delta w(1,T) = (1-\delta)\sum\nolimits_{t=0}^{T-1} \delta^t U(x_t) + \delta^T U(\bar{y})
\end{aligned}
$$

Since $w(t,T)$ is non-increasing in T for given $t \leq T$ and bounded below by $U(\underline{y})$, $\lim_{T\to\infty} w(t,T)$ exists for all $t \in \mathbb{Z}_+$. Define the monotone mapping $W_\delta : \mathbf{X} \to \mathbb{R}$ by

$$W_\delta({}_0\mathbf{x}) := \lim\nolimits_{T\to\infty} w(0,T) = (1-\delta)\sum\nolimits_{t=0}^{\infty} \delta^t U(x_t)\,.$$

As $w(0,T) = (1-\delta)U(x_0) + \delta w(1,T)$, we have that $W_\delta({}_0\mathbf{x}) = (1-\delta)U(x_0) + \delta W_\delta({}_1\mathbf{x})$.

Part II: Uniqueness. Suppose there exists a monotone mapping $W : \mathbf{X} \to \mathbb{R}$ satisfying $W({}_0\mathbf{y}) = (1-\delta)U(y_0) + \delta W({}_1\mathbf{y}))$ for all ${}_0\mathbf{y} \in \mathbf{X}$ such that $W({}_0\mathbf{x}) \neq W_\delta({}_0\mathbf{x})$. Since $W({}_t\mathbf{x}) - W_\delta({}_t\mathbf{x}) = \delta\big(W({}_{t+1}\mathbf{x}) - W_\delta({}_{t+1}\mathbf{x})\big)$ for all $t \in \mathbb{Z}_+$,

$$|W({}_T\mathbf{x}) - W_\delta({}_T\mathbf{x})| = \tfrac{1}{\delta^T}|W({}_0\mathbf{x}) - W_\delta({}_0\mathbf{x})| > U(\bar{y}) - U(\underline{y})$$

for some $T \in \mathbb{Z}_+$. However this contradicts that, for all $T \in \mathbb{Z}_+$,

$$U(\underline{y}) = W({}_{\text{con}}\underline{y}) \leq W({}_T\mathbf{x}) \leq W({}_{\text{con}}\bar{y}) = U(\bar{y})$$

(and likewise for $W_\delta({}_T\mathbf{x})$) by the facts that W is monotone and $W({}_{\text{con}}z) = (1-\delta)U(z) + \delta W({}_{\text{con}}z)$ implies $W({}_{\text{con}}z) = U(z)$. ■