The number of organizations in heterogeneous societies*

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Abstract

I consider a society with heterogeneous individuals who can form organizations for the production of a differentiated service. An arrangement of organizations is said to be split up stable when there is no majority to split any of the organizations. Unlike other equilibrium concepts in the literature, the largest number of organizations that is split up stable corresponds to the socially optimal number of organizations, with a possibility of over provision of one organization. The analysis is extended to a case with endogeneous membership, where it is shown that the results remain the same.

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1 Introduction

Organizations are everywhere in society, providing services ranging from those of the local football club to nation-wide unions and international organizations. For each of these services, there is a multitude of organizations providing differentiated but comparable tasks. This accommodates users with heterogeneous tastes and needs. A larger number of organizations guarantees each user access to services that fit his particular needs and desires closely. The flip side is higher total costs and hence higher costs per member.

Is a society able to get an appropriate number of such organizations when they are allowed to form freely? This is the topic of the current paper. This first requires assessing what the optimal number of organizations is, and how this depends on the level of heterogeneity and the cost structure of running organizations. Second, we need a concept of how organizations form. Then we can answer under which conditions we should expect this decentralized solution to achieve an optimal number of organizations. The answer to this question may also help answer whether we subsidization or taxation of running organizations is justified.

The analysis applies to many sorts of organizations. Economically, the most important may be the voluntary organizations providing welfare services complementing those provided by the public and private sectors (the so-called "third sector"). This sector organizes health care, education, any many other services, and accounts for as much as 20% of GDP in some developed countries (Evers and Laville, 2004). Hence efficiency here is important. Most other organizations providing excludable public goods are also relevant examples. Examples include churches (covering different denominations or religions), newspapers (with differentiated focus, political basis, etc.), sports and recreation facilities (for different sports, with different equipment, and at different location in space), housing cooperatives (different types of housing and location), and rotating savings groups (different income groups and risk profiles)[1] Other applications are to the number of political parties (several parties may be required to get a properly functioning democracy)[2] and unions (to accommodate differences in interests among sectors and level of employees). In all of these cases, there are good reasons to entrust provision to private actors. Still, the usual welfare theorems do not apply, so we are not guaranteed an optimal variation in service provision.

In this paper I consider a heterogeneous population, which can join organizations that provide certain services. The utility an agent derives from membership depends on how well the services provided by the organization match his needs and the cost of joining the organization. In more heterogeneous societies, a larger number of organizations is required to provide suitable services for everybody. The number of organizations is determined in a decentralized way, whereby an organizational structure is unstable if there is a majority

[1]See Gennicot and Ray (2003) and Bloch, Genicot, and Ray (2008) for further details on heterogeneity among ROSCAs and their members.

[2]See McGann (2002) for an extended discussion of this topic.

within any of the organizations for splitting the organization in two. When the organizational structure is such that split ups are only favored by minorities of the members, I label the structure *split up stable*. I consider equilibria where the number of organizations is found as the smallest number of organization that is split up stable.

As an initial benchmark, I use the situation where all individuals are required to join one organization, so their only decision is which organization to join. It is shown that the smallest split up stable number of organizations corresponds to the socially optimal one, although the integer nature of organizations may induce over provision of one organization.

As membership in an organization is voluntary in most cases of interest, however, assuming that everybody belongs to an organization is unsatisfactory. To model the membership decision, I equip all agents with an outside opportunity with heterogeneous value. Individuals with good outside opportunities remain outside the organizations, whereas individuals with bad outside opportunities join. This introduces a number of new features: First, the fraction of agents joining an organization depends on the agent types; agents who can find an organization close to what they prefer are more attracted to the organization, and hence join even when they have relatively good outside opportunities. Hence a higher fraction of these join compared to agents who are less satisfied with the organization. Also, as it is now optimal that some agents, those with good outside options, stay outside the organizations, the optimal number of organizations is lower than with full membership.

The main result is that the smallest split up stable number of organizations still corresponds to the socially optimal number of organizations. There are three factors behind this. First, when agents have outside opportunities, agents whose preferred type of services are close to those provided by the organization tend to be over-represented within the organization. These are less inclined to favor a split of the organization, reducing the pressure for splitting it. Second, as the cost of running the organization is split between the actual members, this also tend to limit the incentives to form new organizations. These two factors tend to give to few organizations. However, organizations are composed of agents with relatively bad outside opportunities, and when deciding on whether to spilt the organization or not, they do not take the preferences of agents with good outside opportunities into account. This third factor tends to give too many organizations. Under the conditions studied here, these factors perfectly balance, yielding the appropriate number of organizations.

There are two major novelties in this paper: First, the split up stability criterion has not been used before. This solution concept is interesting both because it provides a socially optimal number of organizations in decentralized solutions and hence provides a useful benchmark, and because it is a reasonable criterion for the study of organization formation. Second, earlier studies have not considered the issue of endogeneous membership, which is highly relevant for most organization, in contrast to e.g. jurisdictions and countries.

The paper is related to several strands of literature. First, the formation organizations may be seen as specific case of coalition formation, studied at length in cooperative game theory and related literature. The paper is also closely related to Cremer et al.'s (1985) model of the location of facilities in space and the extensions and improvements thereupon by Fujita and Thisse (2002). However, their approach is based on a society-wise decision mechanism so no group can choose to form a new facility, and membership is compulsory.[3] There is also a large related literature on local public goods provision starting with Tiebout (1956). Parts of this literature, such as Westhoff (1977) and Jehiel and Scotchmer (1997), ask similar questions to the present paper. The focus is largely on the effect of mobility on equilibria in different jurisdictions whereas I use a type of heterogeneity where sorting is not possible. In political science, there is also a small literature studying multi party systems in a similar fashion (McGann, 2002), but this literature pays little attention to the determination of the number of parties. The paper is also related to the literature on the size and number of nations (Bolton and Roland, 1997; Alesina and Spolaore, 1997, 2003; le Breton and Weber, 2003) and formation of international unions (Ruta, 2005), but as secessions in countries are different from secessions in organizations, the natural criteria for stability differ. As membership in countries is not voluntary, so there is no question of endogeneous membership in this literature, although le Breton, Makarov, Savvateev, and Weber (2007) consider the possibility of belongin to several nations. Also, Jehiel and Scotchmer (2001) consider the converse of my problem, the formation of jurisdictions where entrants can be denied. There is also a literature on the location and size of cities(Krugman, 1993; Tabuchi, Thisse, and Zeng, 2005), but this literature, although conceptually close, is quite different in the way the economy and the set of possible locations is modelled. Finally, it is related to the literature on club goods (Buchanan 1965; Ellickson, Grodal, Scotchmer, and Zame 1999 and a lot of others; see Scotchmer 2002 for an overview), the literature on group formation (e.g. Milchtaich and Winter, 2002), and the literature on private provision of public goods (Bergstrom, Blume, and Varian, 1986). This literature, however, is more centered on the problem of crowding, which I disregard. Also, stability is not considered, as they do not consider the threat of secession.

2 Basic setup

Consider a continuum of agents with type x uniformly[4] distributed on $[0, 1]$. An agent's location on the unit interval describes her preferences for the services provided by the organizations. There are N organizations that provide services to its members. Initially, I

[3] There are a large number of further contributions in this literature, including Casella and Feinstein (2002), Haimanko, le Breton, and Weber (2004), and Bogomolnaia, le Breton, Savvateev, and Weber (2007, 2008b,a).

[4] A common finding in this class of models is that with general distributions, hardly any results exist; see e.g. McGann (2002).

assume that each individual is a member of one and only one organization. This assumption is relaxed in Section 3. Organizations choose the variety of services they supply, also described by points $q_i \in [0,1]$ on the unit interval. This is referred to as the organization's location. The services supplied by the organization are best suited for agents located in its proximity. The difference $|x - q_i|$ describes the *dissatisfaction* an agent of type x has with organization i. How large the loss from dissatisfaction is depends on how heterogeneous society is, measured by a parameter $a \in [0,1]$. This parameter can also be interpreted as a transport cost.

A member located at $x \in [0,1]$, belonging to an organization i located at q_i, and paying a membership fee c derives utility

$$U(x, q_i) = (1 - a|x - q_i|) - c$$

There is a fixed cost C of running the organization, financed through the fee c on each member.

2.1 The social optimum

The social optimum is found as the optimal trade off between higher costs of more organizations and on average higher dissatisfaction. The social planner's objective is

$$\max_{N,\{q_i\}} \int_0^1 U[x, q(x)] \; dx - NC$$

where $q(x) = \arg \min_{\tilde{q} \in \{q_i\}} |x - \tilde{q}|$ is the optimal organization to join for an agent located at x. Notice first that with N organizations, it is optimal to position the organizations to minimize $\int_0^1 |x - q_i| \; dx$, which implies and equal spacing of organizations. Formally we have

Proposition 1. *In a social optimum with N organizations, the organizations are located at*

$$q_i = \frac{2i-1}{2N}, \; i = 1, \ldots, N$$

Proof. Define $q_0 = 0$ and $q_{N+1} = 1$. Now for any $i \in [1, \ldots, N]$, the average difference between individuals located in $[q_{i-1}, q_{i+1}]$ and the closest organization is $\frac{(q_i - q_{i-1})^2}{4} + \frac{(q_{i+1} - q_i)^2}{4}$. Conditional on q_{i-1} and q_{i+1}, the optimal q_i is $q_i = \frac{q_{i-1}+q_{i+1}}{2}$. This only holds for all i if the Proposition is satisfied. □

This means that dissatisfaction with the organizations varies between 0 and $\frac{1}{2N}$ with a mean of $\frac{1}{4N}$. Hence the objective for the number of organizations becomes to maximize $1 - \frac{a}{4N} - NC$. If we initially disregard the integer nature of organizations, the optimal number of organizations is easily determined by simply taking the first order condition:

Proposition 2a. *With infinitely divisible organizations, the socially optimal number of organizations is*

$$N^* = \sqrt{\frac{a}{4C}}$$

There are reasons for studying non-integer numbers of organizations. First is of course the simplicity of the analysis. Second, they say something about how the number of organizations would change if the population where to increase, say by a factor of two. Still, the analysis in incomplete unless we derive the exact number of organizations which is socially optimal:

Proposition 2b. *With an integer number of organizations, the socially optimal number of organizations is*

$$\begin{aligned} N &= \max\left\{n \in \mathbb{N} : n(n-1) < \frac{a}{4C}\right\} \\ &= \min\left\{n \in \mathbb{N} : n(n+1) > \frac{a}{4C}\right\} \end{aligned}$$

To see this, notice that when $V(N) = 1 - \frac{a}{4N} - NC$ is the utility of N organization, the optimal number of organizations is the integer N such that $V(N-1) < V(N)$ and $V(N) > V(N+1)$.

In what follows, it turns out to be useful to be able to characterize the set of parameters that give exactly N organizations. If we consider (a, C)-space,[5] the combination of parameters where exactly N organizations is optimal is the cone defined by

$$\mathcal{S}_N = \left\{(a, C) \in [0,1]^2 : \frac{a}{4N(N+1)} < C < \frac{a}{4N(N-1)}\right\} \tag{1}$$

2.2 The split up stable outcome

The next step is to see how organizations are created and located when they are controlled by their members. The location of an organization is determined by popular vote among the organization's members, but agents can choose themselves which organization to join. As the organization's location is a unidimensional decision and preferences for the location are single peaked, the median voter theorem applies.

Proposition 3. *For a given number of organizations N, the locations of organizations in the decentralized solution are socially optimal, i.e. as in Proposition 1.*

To see this, notice first that if all organizations have the same size and all agents belong to their best choice of organization, the median voter theorem assures the locations given

[5] To be precise, we should consider the subset of $\mathbb{R}^2$ where it is optimal to have at least one organization, which is the set $\left\{(a, C) \in \mathbb{R}^2 \,\middle|\, 0 \leq a \leq 1, 0 \leq C \leq \frac{a}{4}\right\}$

by (1). Next, assume that organization 1 is larger than an adjacent organization 2. Then a member at the border between the two will strictly prefer to go to organization 2, so this cannot be an equilibrium. Hence the location of organizations correspond to the socially optimal one.

There are different ways to model how the number of organizations is determined. Here, I introduce na equilibrium concept I label *split up stability* which seems reasonable to understand the formation of organizations. Its relationship to some other equilibrium concepts found in the literature is discussed further in Section 2.3.

It is easily seen that an organization splits up more easily the higher the heterogeneity a is, the lower the cost of running organizations C is, and the larger the organization is, i.e. the smaller N is.[6] One criterion for stability is that there is no incentive to split the organization, in the sense that the majority of members in the organization prefers to keep it intact. If there is such a majority, a split could be achieved by a vote within the organization. It could not always be organized by a faction leaving the organization, though. It is easily seen that for sufficiently many organizations, split up stability always holds. More interesting is to see how few organizations we can have and still maintain this type of stability:

Definition 1. *The split up stable equilibrium is the smallest number of organizations where no organization contains a majority for splitting it into two organizations.*

One way to think about this equilibrium is to start with one organization. If there is a majority to split it, it splits, otherwise $N = 1$ is stable. With two organizations, we next have to check whether each of these are stable. If they are not, one of them splits and members are redistributed evenly among the three new organizations. The process continues until we reach stability.

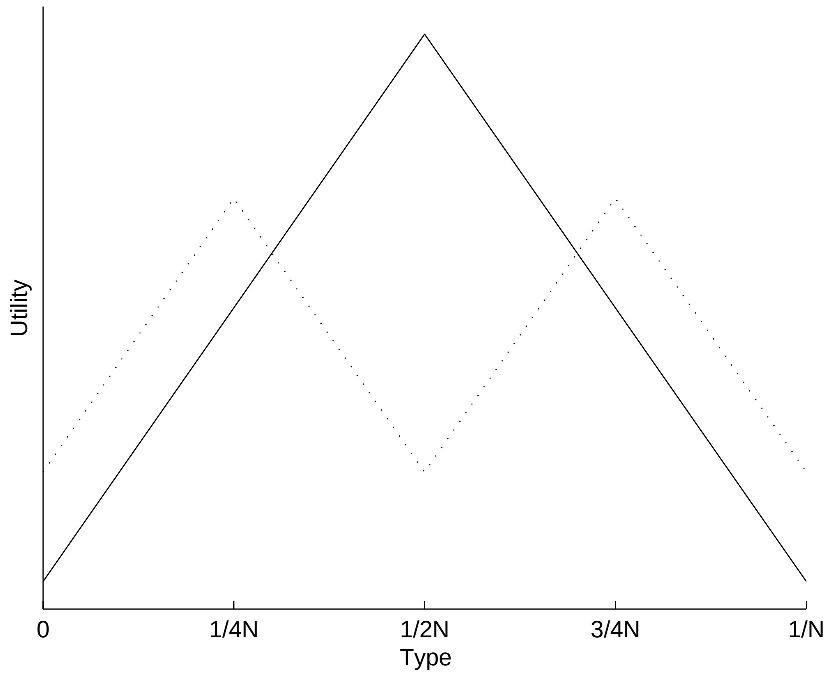
Under many equilibrium definitions in the literature (e.g. Alesina and Spolaore, 1997), there is a suboptimal number of organizations. As we see below, an equilibrium in which we have the smallest number of organizations that is split up stable has the remarkeable property of being socially optimal. Consequntly, this equilibrium concept also has a function as a benchmark – if there are too few organizations in the split up stable equilibrium, there will be too few organizations under other equilibrium concepts as well.

Consider without loss of generality the first organization, which covers the interval $\left[0, \frac{1}{N}\right]$. The membership fee is $c = CN$, and by the median voter theorem, the organization is located at $\frac{1}{2N}$. If the organization splits in two equal parts,the new membership fee becomes $2CN$, and the locations become $\frac{1}{4N}$ and $\frac{3}{4N}$. The distribution of preferences between splitting and not is shown in Figure 1.

If the members located at $\frac{1}{4N}$ and $\frac{3}{4N}$ favour a split, then so will the members at locations $x < \frac{1}{4N}$ and $x > \frac{3}{4N}$. These constitute a majority. If on the contrary the members located

[6] To see this, one can simply compare the utility with an without splitting an organization given in equation (2) below.

Figure 1: Utility from splitting up (dashed line) and not splitting up (solid line)



at $\frac{1}{4N}$ and $\frac{3}{4N}$ oppose a split, then so will the members located in $\left(\frac{1}{4N}, \frac{3}{4N}\right)$. These also constitute a majority. Hence the preferences of the members located at $\frac{1}{4N}$ and $\frac{3}{4N}$ (which coincide) is a Condorcet winner. Consequently, the smallest number of organizations which is split up stable is a number N such that these members are exactly indifferent between a split and no split. This is the case when

$$1 - a\frac{1}{4N} - CN = 1 - 2CN \quad (2)$$

which leads us to the following:

Proposition 4a. *With infinitely divisible organizations, the split up stable number of organizations is*

$$N = \sqrt{\frac{a}{4C}}$$

This corresponds exactly to the condition found in Proposition 2b, so with infinitely divisible organizations, the split up stable equilibrium corresponds to the social optimum. Notice that this equilibrium satisfies Alesina and Spolaraole's *A stability*-concept, so with a small perturbation of organization sizes, marginal members go towards the smaller equilibrium to re-establish equilibrium. To achieve an integer number of organizations,notice that

$N-1$ organizations is not split up stable if

$$1 - \frac{a}{4(N-1)} - C(N-1) < 1 - 2C(N-1)$$

whereas N organizations is stable when

$$1 - \frac{a}{4(N)} - C(N) \geq 1 - 2CN$$

hence we get the following result:

Proposition 4b. *The smallest split up stable number of organizations is N if the parameters are contained in the cone*

$$\mathcal{D}_N = \left\{ (a, C) \in 0, 1]^2 : \frac{a}{4N^2} < C < \frac{a}{4(N-1)^2} \right\} \tag{3}$$

As the optimal cone $\mathcal{S}_N$, defined by (1), does not perfectly overlap with $\mathcal{D}_N$, we are not guaranteed to get exactly the socially optimal number of organizations. The two sets are shown in Figure 2. The cones intersect, and it is easily seen that $\mathcal{S}_N \subset \mathcal{D}_N \cup \mathcal{D}_{N+1}$. This means that we get over-provision of one organization when $\frac{a}{4N(N+1)} < C < \frac{a}{4N^2}$, but the relative over-provision of organizations decreases as N grows.

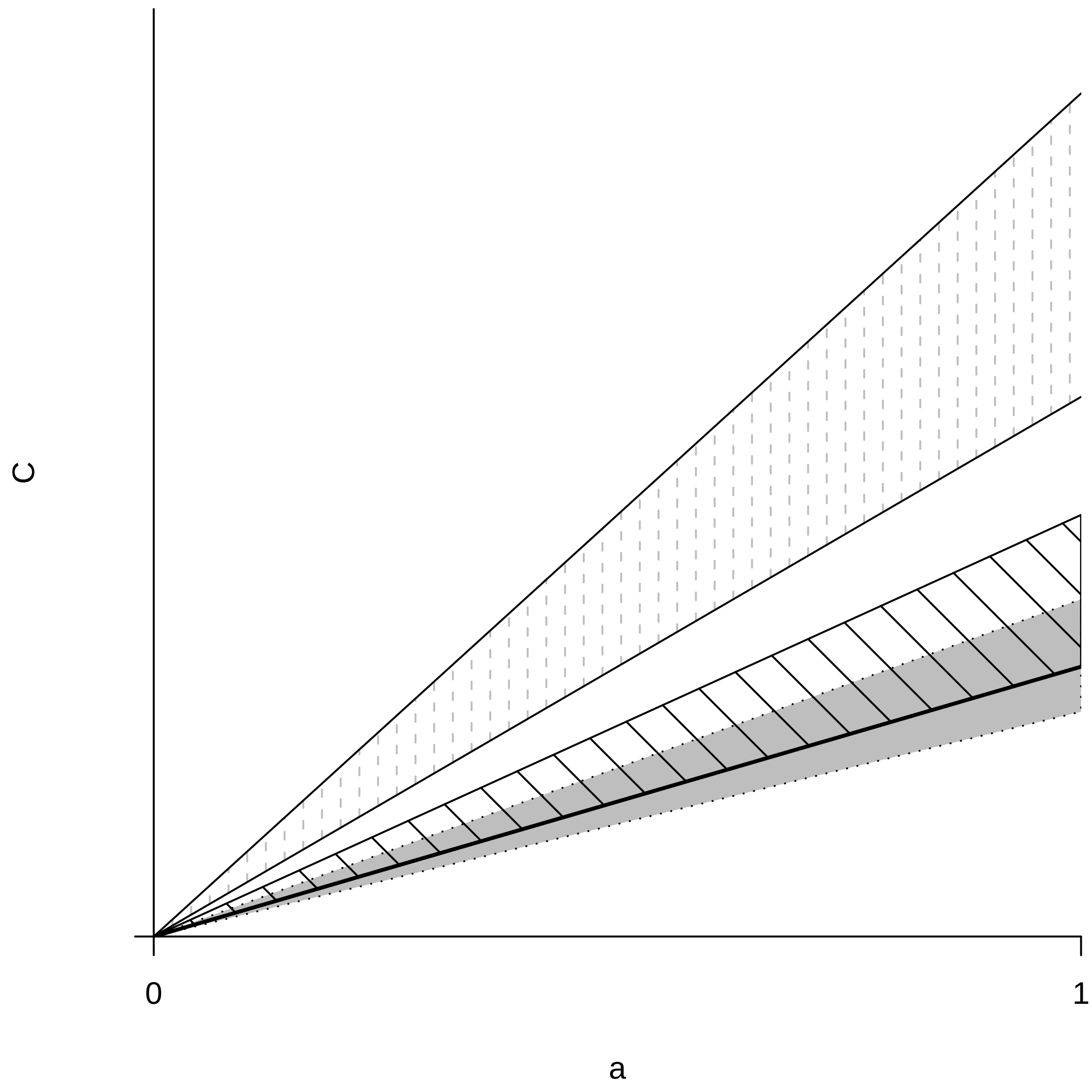
2.3 Split up stability

As the split up stability concept is novel, it deserves some discussion. One way to think about this concept is a society that starts with one large organization, which subsequently splits. When an equilibrium is reach, we obtain the split up stable equilibrium, and (almost) social optimality. We could also envisage the reverse process, starting with a large number of organization (or individuals) that merge until an equilibrium is reached. Then mergers stop at a too early stage, from a social point of view, yielding too many organizations. To see this, notice that a majority for two organizations to merge requires that the members located at the centre of each organization is indifferent, which occurs when $NC = \frac{a}{2N} + \frac{C}{2}$. Hence a merger is prevented whenever $N \leq \sqrt{\frac{a}{2C}} = \sqrt{2}N^*$, yielding a set of parameters where N orgnaizations are stable as

$$\mathcal{M}_N = \left\{ (a, C) \in 0, 1]^2 : \frac{a}{4N^2} < C < \frac{a}{4(N-1)^2} \right\} \tag{4}$$

This cone is also shown in Figure 2. It is seen that we generally get a larger number of organizations than the social optimum. This equilibrium condition corresponds to Alesina and Spolaore's(1997) definition of B-equilibrium. Notice that there is no incentive to merge organizations under a split up stable equilibrium.

Figure 2: Combinations of a and C that yield N organizations



Notes: Exactly N organizations is optimal along the solid line. The gray area shows the area $\mathcal{C}_N$ where N organizations are socially optimal, and the hatched area $\mathcal{D}_N$ where N is the smallest split up stable number of organizations. The area where N organizations is stable relative to mergers, $\mathcal{M}_N$ is shown in gray dashes.

When studying the number of countries, it seems reasonable to initially have a large number of extended households, who turn into villages, then chiefdoms, and through a long merging process into countries. Hence the B-equilibrium is the relevant equilibrium concept. For the types of organizations we study here, however, it seems more reasonable that we initially have one or a few organizations who gradually split up to suit the needs of the differentiated mass of members. Hence split up stability is a more appropriate equilibrium concept. These two processes are in some ways similar to the contrast between the von Neuman-Morgenstern stable set, where a member is allowed into a coalition only under unanimity, and Ray and Vohra's (1997) "equilibrium binding agreements", which may be seen as splitting from the grand coalition.

One could object that split up stability's requirement of a majority vote to split is conservative. Consider first other threats of unilateral secessions: first, if there is no majority to split the organization in two organization, there is no majority to split it into $q > 2$ organization.[7] One can also easily verify that no single group would unilaterally want to quit the organization if (2) holds. The reason is that the member located at $\frac{1}{2N}$ is the most reluctant to a split, so he will not support any split up. Hence the new organization would have a size below $\frac{1}{2N}$, which would give it larger costs than the majority vote of splitting in two. Hence no such group could be formed.

The equilibrium is not robust against a coalition of members from two adjacent organization. But this would be a very strong requirement as such criticisms could be raised against almost any equilibrium concept in game theory, including the Nash equilibrium. Hence this is of less concern.

The concept of split up stability may also be seen as a variety of "secession proofness" considered by le Breton and Weber (2003) and le Breton, Weber, and Drèze (2006). Secession proofness requires that there is no coalition within one organization that unilaterally wants to form a new organization, and is hence somewhat weaker than split up stability. Secession proof equilibria may exhibit efficiency properties similar to those of split up stability under the assumptions studied by le Breton and Weber (2003) and le Breton, Weber, and Drèze (2006). However, under the conditions studied here, there are situations with too few organizations from a social point of view that still are secession proof. The reason is that it requires a larger under-provision of organizations for there to be a coalition to break off – obtaining a majority for splitting within the organization requires less under-provision.

[7] This only achieves a majority if there are less than $\sqrt{2/q}N^*$ organizations.

3 Endogeneous membership

3.1 Social optimum

The model where everybody belongs to an organization is unrealistic unless the service provided is extremely valuable or there are legal obligations to belong to an organization. To analyse endogeneous membership, we need to introduce an outside option to capture the value of not joining any organization, for instance by abstaining from consuming the good provided by the organization or providing a substitute privately. We can then extend the types of agents to a two-dimensional space (x, ε) where as before x is the optimal location with a uniform distribution on $[0, 1]$ and ε is a variable, assumed to be uniformly distributed on the unit interval, representing the utility of not belonging to an organization. An individual will join an organization if, for the optimal choice of organization i, $U(x, q(x)) - c > \varepsilon$ where c is the membership fee (which is identical in all in organizations). Hence the fraction of agents of type x who belong to an organization is $U(x, q(x)) - c$ whenever this falls in the unit interval.

As before, the social planner's problem is to choose the number of organizations and their locations to maximize

$$\max_{N,\{q_i\}} \int_0^1 \int_0^1 \max\left(1 - a\left|x - q(x)\right|, \varepsilon\right) - NC \; d\varepsilon \; dx. \tag{5}$$

It is seen that in this problem, the optimal location of organizations for given N is still given by Proposition 1. The total benefit of all individuals with N organizations is then

$$\begin{aligned} W(N) &= 2N \int_{x=0}^{1/2N} \int_{u=0}^{1} \max(1 - ax, u) \;\; du \; dx \\ &= 1 - \frac{a}{4N} + \frac{a^2}{24N^2} \end{aligned}$$

Hence the optimal number of organizations maximizes $W(N) - NC$, yielding the first order condition

$$\frac{a}{4N^2} - \frac{a^2}{12N^3} = C. \tag{6}$$

This solution to this equation is the largest root of a cubic equation[8] Although it is solvable using Cardano's formulae, this yields no simple expression for the root. The equation can trivially be solved numerically, but I do not attempt to provide an analytic solution to the social planner's problem with infinitely divisible organizations. However, it is straightforward

[8] The equation is $N^3 - \frac{a}{4C}N + \frac{a^2}{12C} = 0$. To see why it is the largest root, notice first that the discriminant is $\Delta = \frac{a^3}{16C^3}(1 - 3aC)$. As $a \leq 1$ by assumption and the analysis is only interesting if $N^* = \sqrt{\frac{a}{4C}} \geq 1$, we have $aC < 1/4$. Hence $\Delta > 0$, so the polynomial has three real roots. One is negative, and the intermediate root violates the second order condition.

to find the subset of (a, C)-space where N organizations is optimal. In integer N is optimal if it satisfies both $W(N) - W(N-1) > C$ and $W(N+1) - W(N) \leq C$. Hence we have:

Proposition 5. *With endogeneous membership, N organizations is socially optimal when the parameters $(a, C) \in \mathcal{S}_N^e$ with*

$$\mathcal{S}_N^e = \left\{ a, C \in [0,1]^2 : \frac{a}{4N(N+1)} - \frac{a^2(2N+1)}{24N^2(N+1)^2} < C \leq \frac{a}{4N(N-1)} - \frac{a^2(2N+1)}{24N^2(N-1)^2} \right\},$$

This set is the area between two parabolae. Comparing it to $\mathcal{S}_N$, the one found for exogneous membership in (1), it is seen that the we require a lower cost to obtain the same number of organizations when membership is endogeneous. This means that the optimal number of organizations is (weakly) lower under endogeneous membership.

3.2 Membership in the decentralized solution

Before we can discuss split up stability in the decentralized solution, we need to study the pattern of membership for a given number N of organizations. Without loss of generality, consider the organization located at $\left[0, \frac{1}{N}\right]$, where the good produced is of type $\frac{1}{2N}$. As membership is now endogeneous and members located around the center of the organization derive more utility from joining than members closer to the boundaries, a larger fraction of agents with types close to $\frac{1}{2N}$ are members. Particularly, the fraction of agents of type x who are members is implicitly defined by

$$\psi_N(x) = \max\left\{1 - a\left|x - \frac{1}{2N}\right| - \frac{C}{\int_0^{1/N} \psi_N(x)\ dx}, 0\right\}.$$

As ψ_N is symmetric around $\frac{1}{2N}$ on $\left[0, \frac{1}{N}\right]$, the median member of the organization is located at $\frac{1}{2N}$. The median voter theorem still applies, so the variety chosen by the location is the variety in the center, $\frac{1}{2N}$. To study membership in equilibrium, notice that total membership in an organization $\Psi_N = \int_0^{1/N} \psi_N(x)\ dx$, solves

$$\Psi_N = 2\int_0^{1/2N} \max\left\{1 - ax - \frac{C}{\Psi_N}, 0\right\} dx.$$

If Ψ_N exists, the existence of ψ_N follows trivially. The fraction Ψ_N is a fixed points of the mapping

$$\Lambda : \Psi_N \mapsto 2\int_0^{1/2N} \max\left\{1 - ax - \frac{C}{\Psi_N}, 0\right\} dx.$$

This mapping can be rewritten

$$\Lambda(\Psi) = \begin{cases} 0 & \text{if } \Psi < C \\ \frac{1}{a}\left(1 - \frac{C}{\Psi}\right)^2 & \text{if } C \leq \Psi < \frac{C}{1-\frac{a}{2N}} \\ \frac{1}{N}\left(1 - \frac{a}{4N} - \frac{C}{\Psi}\right) & \text{if } \frac{C}{1-\frac{a}{2N}} \leq \Psi \end{cases} \tag{7}$$

The proof is provided in Appendix A.1

The function Λ is strictly increasing on $[C, 1]$, continuous, and continuously differentiable for all $\Psi_N \neq C$. The equation has a trivial solution at $\Psi_N = 0$, and for some parameter configurations there is also another equilibrium.

Proposition 6. *If the equation $N\Psi_N^2 - \left(1 - \frac{a}{4N}\right)\Psi_N + C = 0$ has real roots, the equation $\Psi_N = \Lambda(\Psi_N)$ has a unique stable interior equilibrium given by*

$$\Psi_N^* = \frac{\left(1 - \frac{a}{4N}\right) + \sqrt{\left(1 - \frac{a}{4N}\right)^2 - 4NC}}{2N}$$

The proof is provided in Appendix A.2

3.3 The split up stable outcome

The concept of split up stability is still applicable and useful, but a new complication arises when considering endogeneous membership. Consider again without loss of generality the first organization covering $\left[0, \frac{1}{N}\right]$. There are no longer equally many members of each type as ψ has a peak at $\frac{1}{2N}$. Hence the majority for a split up decision is not determined by the members located at $\frac{1}{4N}$ and $\frac{3}{4N}$, but a set of voters closer to the centre of the organization. This tends to reduce the pressure for splitting the organization, and hence reduce the equilibrium number of organizations. To ease the exposition, I postpone a proper dicsussion of this to Section 3.4, and for the time being condition of pivotal voters located at $\frac{1}{4N}$ and $\frac{3}{4N}$.

The members located at $\frac{1}{4N}$ and $\frac{3}{4N}$ are indifferent between splitting and not splitting when[9]

$$\frac{a}{4N} + \frac{C}{\Psi_N} = \frac{C}{\Psi_{2N}} \tag{8}$$

From this expression, we can for any set of parameters find the smallest the smallest split up stable number of organizations. As for the social optimum, is it not trivial to find a closed

[9]One could imagine that instead of the smallest split up stable number of organizations determining the number of organizations, it could be that the maximal number of organizations, i.e. the largest N such that $\Psi_N = \Lambda(\Psi_N)$ has a solution would limit the number of organizations. Bu using the ensuing number of organizations, it is easily verified that this is not the case.

form solution to this equation, but we can find combinations of a and C which yields a given number of organizations in equilibrium:

Proposition 7. *With fixed pivotal voters, N organizations is split up stable when*

$$C \geq C_N^{\mathcal{D}} := \frac{8Na - 15a^2 + 3a\sqrt{64N^2 - 48Na + 25a^2}}{128N^3} \tag{9}$$

The proof is provided in Appendix A.3

From this expression we can also find the part of (a, C)-space where N is the smallest split up stable number of organizations. This is the set

$$\mathcal{D}_N^f = \left\{(a, C) \in [0,1]^2 : C_N^{\mathcal{D}} \leq C < C_{N-1}^{\mathcal{D}}\right\}$$

We are now ready to analyse the provison of organizations in the decentralized case. The main finding is:

Proposition 8. *With fixed pivotal members, for any number of organizations N, the limiting cost for split up stability $C_N^{\mathcal{D}}$ is below the the socially optimal iso-organization cost $C_N^{\mathcal{S}}$ but still $\mathcal{D}_N^f \in \mathcal{S}_N^e \cup \mathcal{S}_{N+1}^e$.*

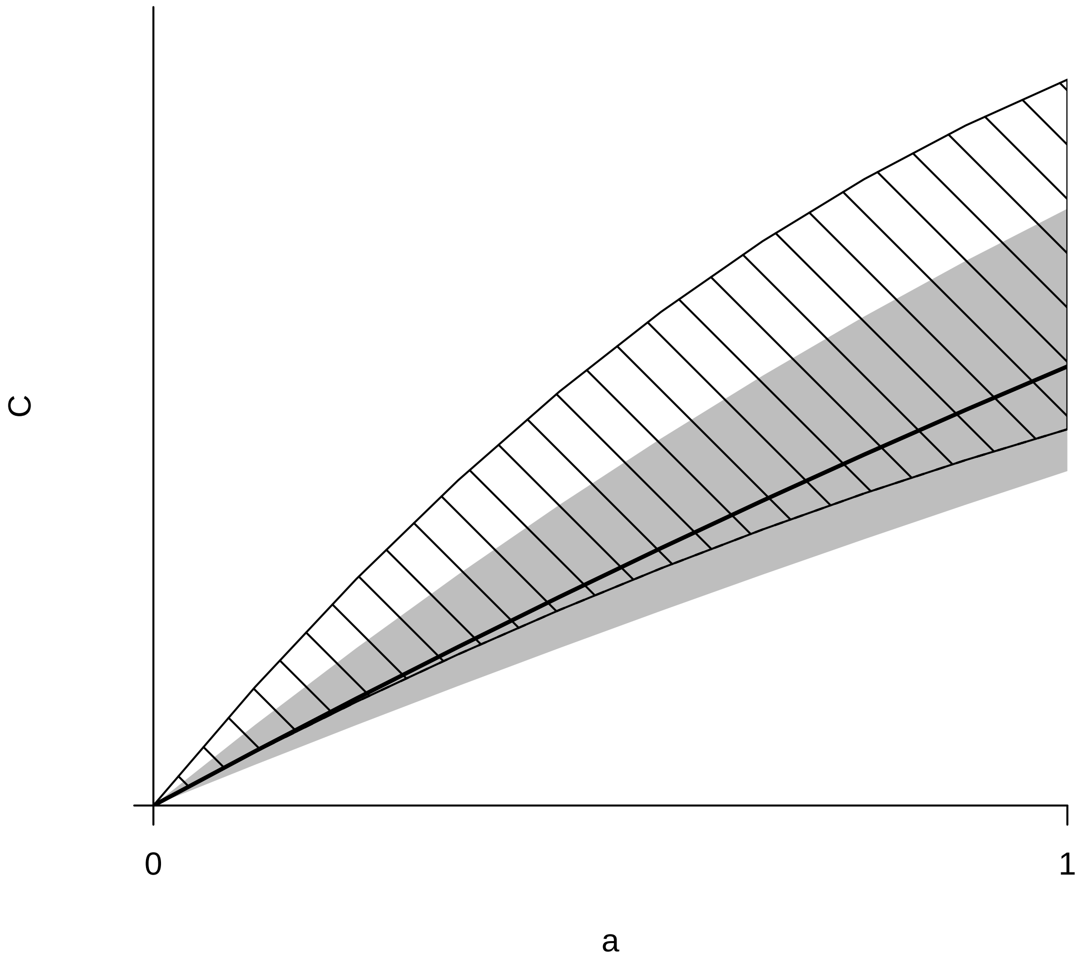
For a proof, see Appendix A.4

This shows that the smallest split up stable equilibrium either corresponds to the social optimum or under provision of one organization as was the case with full membership. There are two opposing forces giving a different outcome than in the case of exogeneous membership: First, members of organizations tend to have relatively bad outside opportunities and hence over value the number of organizations. The social planner acknowledges that there are outside opportunities, and, as seen above, reduces the number of organizations relative to the number with exogeneous membership. This is not acknowledged by the actual members of the organizations, and tend to give to many organizations. Second, as some choose to not join organizations, so $\Psi_N < 1/N$, there is an increased cost per member for a given number of organizations, tending to give too few organizations. The two effects almost cancel out, but the latter is somewhat stronger. Hence $C_N^{\mathcal{D}} < C_N^{\mathcal{S}}$, so we get underprovision for a smaller combinations of a and C so we can say that we are more likely to achieve the social optimum. The situation is depicted in Figure 3.

3.4 The pivotal members

The analysis so far has been conditional on the members pivotal in the split up decision being fixed at the positions they had with exogeneous membership. However, as membership varies with each individual's evaluation of the organization's appropriateness, a larger share

Figure 3: Combinations of a and C that yield N organizations with endogeneous organization membership



Notes: Exactly N organizations is optimal along the solid line depicting $C_N^{\mathcal{S}}$. The gray area shows $\mathcal{S}_N^e$, the area where N organizations are socially optimal, and the hatched area $\mathcal{D}_N^f$, the set where N is the smallest split up stable number of organizations.

of individuals are members close to the organization centre. The position of the pivotal members are given by the following lemma:

Lemma 1. *The pivotal members are located at* $\frac{1}{2N} \pm m$ *where*

$$m = \frac{\left(1 - \frac{C}{\Psi_N}\right) - \sqrt{\left(1 - \frac{C}{\Psi_N}\right)^2 - \frac{a}{2N}\left(1 - \frac{a}{4N} - \frac{C}{\Psi_N}\right)}}{a} \tag{10}$$

Proof. As ψ is symmetric around $\frac{1}{2N}$ on $\left[0, \frac{1}{N}\right]$, there is some m such that the pivotal members are located at $\frac{1}{2N} \pm m$. As we need an equal mass of members on $[0, m]$ as on $\left[m, \frac{1}{2N}\right]$, we have

$$\int_0^m \max\left\{1 - ax - \frac{C}{\Psi_N}, 0\right\}\, dx = \int_m^{1/2N} \max\left\{1 - ax - \frac{C}{\Psi_N}, 0\right\}\, dx$$

From Proposition 7, $\psi(x) > 0$ for all x, so the expression reduces to $\int_0^m 1 - ax - \frac{C}{\Psi_N}\, dx = \int_m^{1/2N} 1 - ax - \frac{C}{\Psi_N}\, dx$. Integrating and solving, this reduces to the quadratic equation

$$am^2 - 2m\left(1 - \frac{C}{\Psi_N}\right) + \frac{1}{2N}\left(1 - \frac{C}{\Psi_N} - \frac{a}{4N}\right) = 0.$$

Only the smaller root guarantees $m \in \left[0, \frac{1}{4N}\right]$. □

If the organization decides to split, it is as before split in two equal organizations, yielding new organization centra at $\frac{1}{4N}$ and $\frac{3}{4N}$. Hence the pivotal members prefer no split whenever

$$am + \frac{C}{\Psi_N} \leq a\left(\frac{1}{4N} - m\right) + \frac{C}{\Psi_{2N}} \tag{11}$$

This expression no longer permit a closed form solutions for the cut off cost. However, we can show that the conclusions from Proposition 8 still holds:

Proposition 9. *With pivotal members determined as in Lemma 1, we have* $\mathcal{D}_N^e \subset \mathcal{S}_N^e \cup \mathcal{S}_{N+1}^e$.

The proof is provided in Appendix A.5

In addition to the two changes from the exogneous case mentioned in Section 3.3, there is now a third factor, the change in the identity of the pivotal voters, which also tend to limit the number of organizations. However, this effect is not strong enough to change the above conclusions. However, a this reduces the demand for organizations, it increases the likelihood that the exact number of organizations is formed (in the sense that the critical level of C is reduced). But we may still get overprovision of one organization.

4 Concluding remarks

We have seen that in a heterogeneous society where new organizations are created if and only if there is a majority within an organization to split, labeled a split up stable equilibrium, we obtain either the socially optimal number of organizations or an over provison of a single organization du to an integer problem. Hence when there are a large number of organizations, the relative mis-allocating is small and there is little need for public interventions. In cases with few organizations, there may be a scope for taxing the formation of organizations as there may be over provision.

Extensions of the model could overturn this finding, though. More general cost structures seem to have little effect on the properties of the decentralized solution[10] However, the distribution of the outside opportunities ma potentially have major impacts. With non-uniform distributions, closed form solutions are generally not available, so general conclusions are difficult to draw.

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[10]Some calculations along these lines are provided upon request.

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A Proofs

A.1 Proof of equation (12)

Organizational membership Ψ_N solves

$$\Psi_N = \Lambda(\Psi_N) \text{ with } \Lambda(\Psi) = \begin{cases} 0 & \text{if } \Psi < C \\ \frac{1}{a}\left(1 - \frac{C}{\Psi}\right)^2 & \text{if } C \leq \Psi < \frac{C}{1-\frac{a}{2N}} \\ \frac{1}{N}\left(1 - \frac{a}{4N} - \frac{C}{\Psi}\right) & \text{if } \frac{C}{1-\frac{a}{2N}} \leq \Psi \end{cases} \tag{12}$$

Proof. Consider first the case $\frac{a}{2N} + \frac{C}{\Psi_N} < 1$, so there are participants at all values of x. Then

$$\begin{aligned} \Psi_N &= 2\int_0^{1/2N} 1 - ax - \frac{C}{\Psi_N}\, dx \\ &= \frac{1}{N} - \frac{C}{N\Psi_N} - \frac{a}{4N^2} \end{aligned}$$

When $\frac{a}{2N} + \frac{C}{\Psi_N} > 1$ and $\frac{C}{\Psi_N} < 1$, there are some member types x where no-one choose to join the organization and some member types x where at least some agents join. We now get total member ship as

$$\begin{aligned}\Psi_N &= 2\int_0^{\frac{1}{a}\left(1-\frac{C}{\Psi_n}\right)} 1 - ax - \frac{C}{\Psi_N}\,dx \\ &= \frac{1}{a}\left(1-\frac{C}{\Psi_N}\right)^2\end{aligned}$$

Finally, when $\frac{C}{\Psi_N} > 1$, no-one wants to join so $\Psi_N = 0$. □

A.2 Proof of Proposition 6

Proposition 6. *If the equation* $N\Psi_N^2 - \left(1-\frac{a}{4N}\right)\Psi_N + C = 0$ *has real roots, equation (12) has a unique stable interior equilibrium given by*

$$\Psi_N^* = \frac{\left(1-\frac{a}{4N}\right) + \sqrt{\left(1-\frac{a}{4N}\right)^2 - 4NC}}{2N}$$

Proof. The roots of $N\Psi_N^2 - \left(1-\frac{a}{4N}\right)\Psi_N + C = 0$ are given by

$$\Psi_N = \frac{\left(1-\frac{a}{4N}\right) \pm \sqrt{\left(1-\frac{a}{4N}\right)^2 - 4NC}}{2N},$$

where only the larger root satisfies $\frac{C}{1-\frac{a}{2N}} \le \Psi_N$. We now need to show that (i) this root always satisfies $\frac{C}{1-\frac{a}{2N}} \le \Psi_N$, and (ii) that it is the unique interior solution of the equation.

To show (i), we know that $\left(1-\frac{a}{4N}\right)^2 > 4NC$ as the roots by assumption are real, so $C < \frac{1}{4N} - \frac{a}{8N} + \frac{a^2}{16N^2}$. As $\Psi_N \le \frac{1-\frac{a}{4N}}{2N}$, it suffices to show that $\frac{C}{1-\frac{a}{2N}} < \frac{1-\frac{a}{4N}}{2N}$ which holds when $2NC < \left(1-\frac{a}{4N}\right)\left(1-\frac{a}{2N}\right)$i.e. when $C < \frac{1}{2N} - \frac{3a}{8N^2} + \frac{a^2}{16N^2}$. Then (i) holds when $\frac{1}{4N} - \frac{a}{8N} + \frac{a^2}{16N^2} < \frac{1}{2N} - \frac{3a}{8N^2} + \frac{a^2}{16N^2}$, which holds when $0 < \frac{1}{4N}\left(1 - \frac{a}{N} + \frac{3a^2}{16N^2}\right)$, which again holds when $a < \frac{4}{3}$. As $a \le 1$ by assumption, we have $\frac{C}{1-\frac{a}{2N}} \le \Psi_N^*$.

To show (ii), it is easily seen that Λ is convex for $\Psi_N < \frac{3C}{2}$ and concave for $\Psi_N > \frac{3C}{2}$. As Λ is continuously differentiable at $\Psi_N = \frac{C}{1-\frac{a}{2N}}$, concavity also hold in this point. The as Ψ_N^* is real, (12) has two interior solutions, one stable and one unstable. □

A.3 Proof of Proposition 7

Proposition 7. *With fixed pivotal voters, N organizations is split up stable when*

$$C \ge C_N^{\mathcal{D}} := \frac{8Na - 15a^2 + 3a\sqrt{64N^2 - 48Na + 25a^2}}{128N^3} \tag{13}$$

Proof. Whenever there are members of organization when there are $2N$ organizations, the minimum sustainable number is determined by (8). Define

$$\theta_N = \frac{1}{\Psi_N} = \frac{1 - \frac{a}{4N} - \sqrt{\left(1 - \frac{a}{4N}\right)^2 - 4NC}}{2C}, \tag{14}$$

so the split up stability condition becomes

$$\begin{aligned} \frac{a}{4N} &= C\theta_{2N} - C\theta_N \\ &= \frac{1}{2}\left[\left(1 - \frac{a}{8N} - \sqrt{\left(1 - \frac{a}{8N}\right)^2 - 8NC}\right) - \left(1 - \frac{a}{4N} - \sqrt{\left(1 - \frac{a}{4N}\right)^2 - 4NC}\right)\right] \end{aligned}$$

Taking squares of the expression and solving, we get

$$\frac{192CN^3 - 32N^2 + 12Na + a^2}{16N^2} = -2\sqrt{\left(\left(1 - \frac{a}{4N}\right)^2 - 4NC\right)\left(\left(1 - \frac{a}{8N}\right)^2 - 8NC\right)}.$$

Taking squares again and solving yields the equation

$$256C^2N^5 - 32CN^3a + 60CN^2a^2 - 8Na^2 + 3a^3 = 0.$$

This is a quadratic equation in C where one the largest root gives a positive C, and hence yielding the solution (7).

To see that $2N$ organizations is sustainablewhenever this condition holds, notice that this holds iff $\left(1 - \frac{a}{8N}\right)^2 - 8NC \geq 0$. Hence a sufficient condition is that $C^{DE} \leq \frac{a^2}{512N^3} - \frac{a}{32N^2} + \frac{1}{8N}$. Using the derived expression for the iso-organization line, this expression is satisfied when $12a\sqrt{-48Na + 64N^2 + 25a^2} \leq 64N^2 - 48Na + 61a^2$. Taking squares and simplifying, this condition reduces to $0 \leq (64N^2 - 48Na - 11a^2)^2$ which is trivially satisfied for all real a and N. □

A.4 Proof of Proposition 8

Proposition 8. *With fixed pivotal members, for any number of organizations N, the limiting cost for split up stability $C_N^{\mathcal{D}}$ is below the the socially optimal iso-organization cost $C_N^{\mathcal{S}}$ but still $\mathcal{D}_N^f \in \mathcal{S}_N^e \cup \mathcal{S}_{N+1}^e$.*

Proof. To show $C_N^{\mathcal{D}} < C_N^{\mathcal{S}}$, we need to show that

$$\frac{8Na - 15a^2 \pm 3a\sqrt{-48Na + 64N^2 + 25a^2}}{128N^3} < \frac{3Na - a^2}{12N^3}$$

which holds whenever $24Na - 45a^2 + 9a\sqrt{-48Na + 64N^2 + 25a^2} < 96Na - 32a^2$ or $81\left(-48Na + 64N^2 + 25a^2\right) < (72N + 13a)^2$. For $a > 0$, this condition reduces to $5760N - 1856a > 0$, which always hold when $N \geq 1$ and $a \leq 1$.

To show $\mathcal{D}_N^f \in \mathcal{S}_N^e \cup \mathcal{S}_{N+1}^e$, it suffices to show that $C_N^{\mathcal{D}} \in \mathcal{S}_N^e$ as this also implies $C_{N+1}^{\mathcal{D}} \in \mathcal{S}_{N+1}^e$. To do so, we need to show

$$\frac{8Na - 15a^2 + 3a\sqrt{64N^2 - 48Na + 25a^2}}{128N^3} < \frac{a}{4N(N+1)} - \frac{a^2(2N+1)}{24N^2(N+1)^2} \tag{15}$$

which holds whenever

$$16N\left[6N\left(N-1\right) - a(2N+1)\right] < 3\left(N+1\right)^2\left(8N - 15a + 3\sqrt{64N^2 - 48Na + 25a^2}\right).$$

Simplyfying and squaring, we see that this expression holds when

$$\left(45a - 24N + 13N^2a + 74Na - 144N^2 + 72N^3\right)^2 < 81\left(N+1\right)^4\left(64N^2 - 48Na + 25a^2\right)$$

If we define

$$\begin{aligned} P\left(a,N\right) =& 180N^4a - 1296N^4 - 58N^3a^2 + 702N^3a - 432N^3 - 193N^2a^2 + 246N^2a \\ & - 432N^2 - 172Na^2 - 30Na - 144N - 45a^2 + 54a \end{aligned}$$

expression (15) holds whenever $32NP\left(a,N\right) < 0$, so we want to show that $P\left(a,N\right) < 0$ for all $a \in [0,1]$ and all $N \geq 1$. We have

$$P_a'\left(a,N\right) = 2\left[123N^2 + 351N^3 + 90N^4 - 15N + 27 - \left(45 + 193N^2 + 58N^3 + 172N\right)a\right],$$

and it is easily verified that for all $N \geq 1$, $P_a'\left(0,N\right) > 0$ and $P_a'\left(1,N\right) > 0$. By the linearity of P_a' in a, it follows that for all $N \geq 1$ and all $a \in [0,1]$, $P\left(a,N\right) \leq P\left(1,N\right)$. The polynomial $P\left(1,N\right) = -1116N^4 + 212N^3 - 379N^2 - 346N + 9$ is clearly decreasing in N for $N \geq 1$ as $P_N'\left(1,N\right) = -4464N^3 + 636N^2 - 758N - 346$, and as $P\left(1,1\right) = -1620$ it follows that $P\left(a,N\right) < 0$ for all $N \geq 1$ and all $a \in [0,1]$. □

A.5 Proof of Proposition 9

Proposition 9. *With pivotal members determined as in Lemma 1, we have* $\mathcal{D}_N^e \subset \mathcal{S}_N^e \cup \mathcal{S}_{N+1}^e$.

Proof. As we have lower pressure for splitting organizations (details?), it follows that $\mathcal{D}_N^e \subsetneq \mathcal{S}_{N+k}^e$ for $k > 1$. To see that $\mathcal{D}_N^e \subsetneq \mathcal{S}_{N-k}^e$ for $k > 0$, it sufficies to show that at $C_{low} = \frac{a}{4N(N+1)} - \frac{a^2(2N+1)}{24N^2(N+1)^2}$, the lower boundary of $\mathcal{S}_N^e$, we have $am + \frac{C_{low}}{\Psi_N} \leq a\left(m - \frac{1}{4N}\right) + \frac{C_{low}}{\Psi_{2N}}$.

To see this, it follows from (14) that

$$\frac{C}{\Psi_N} = \frac{1}{2}\left(1 - \frac{a}{4N}\right) + \sqrt{\left[\frac{1}{2}\left(1 - \frac{a}{4N}\right)\right]^2 - NC} \tag{16}$$

Combining this with Lemma 1, we get

$$am = \left(\frac{1}{2} + \frac{a}{8N} - \sqrt{\left(\frac{1}{2} - \frac{a}{8N}\right)^2 - NC}\right)$$
$$- \sqrt{\left(\frac{1}{2} + \frac{a}{8N} - \sqrt{\left(\frac{1}{2} - \frac{a}{8N}\right)^2 - NC}\right)^2 - \frac{a}{2N}\left(\frac{1}{2} + \frac{a}{8N} - \sqrt{\left(\frac{1}{2} - \frac{a}{8N}\right)^2 - NC} - \frac{a}{4N}\right)}$$

so

$$\begin{aligned} 2am =& 1 + \frac{a}{4N} - 2\sqrt{\left(\frac{1}{2} - \frac{a}{8N}\right)^2 - NC} \\ & - 2\sqrt{\frac{1}{2} + \frac{3a^2}{32N^2} - 2\left(\frac{1}{2} - \frac{a}{8N}\right)\sqrt{\left(\frac{1}{2} - \frac{a}{8N}\right)^2 - NC} - NC} \end{aligned}$$

Using (16) in (11) and inserting $C_{low} = \frac{a}{4N(N+1)} - \frac{a^2(2N+1)}{24N^2(N+1)^2}$, we get that the expression holds whenever

$$\begin{aligned} Q(a,N) =& \frac{\sqrt{A(a,N)}}{\sqrt{768}N(N+1)} + \frac{\sqrt{B(a,N)}}{\sqrt{192}N(N+1)} \\ & + \sqrt{\frac{D(a,N)}{24N^2(N+1)^2} - \left(1 - \frac{a}{4N}\right)\frac{\sqrt{B(a,N)}}{\sqrt{12}N(N+1)}} + \frac{a}{16N} - 1 \geq 0 \end{aligned}$$

with

$$A(a,N) = 192N^4 - 432N^3a + 384N^3 + 131N^2a^2 - 480N^2a + 192N^2 + 70Na^2 - 48Na + 3a^2$$
$$B(a,N) = 48N^4 - 72N^3a + 96N^3 + 19N^2a^2 - 96N^2a + 48N^2 + 14Na^2 - 24Na + 3a^2$$
$$D(a,N) = 48N^4 - 24N^3a + 96N^3 + 17N^2a^2 - 24N^2a + 48N^2 + 22Na^2 + 9a^2$$

It is seen that $Q(0,N) = 0$. Also for all $N > 1$, Q is increasing in a, so for all $a \geq 0$ and $N > 1$ we have $Q(a,N) \geq 0$. Finally, one can verify that $Q(a,1)$ is minimized at $a = 0$ yielding $Q(0,1) = 0$. □