The Dual Approach for Measuring Multidimensional Deprivation and Poverty

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Abstract. This paper is concerned with the problem of ranking and quantifying the extent of deprivation exhibited by multidimensional distributions, where the multiple attributes in which an individual can be deprived are represented by dichotomized variables. To this end we first aggregate deprivation for each individual into a distribution of deprivation count, representing the number of dimensions for which the population suffer from deprivation. Next, by drawing on the dual social evaluation framework that originates from Yaari (1987, 1988) social evaluation functions are used to construct summary measures of deprivation. Moreover, the measures of deprivation are proven to admit decomposition into the mean and the dispersion of the distribution of multiple deprivations. Two alternative criteria of second-degree count distribution dominance are shown to divide the family of dual measures of deprivation into two separate subfamilies, which differ with regard to whether concern is turned towards those people suffering from deprivation on all dimensions or those suffering from at least one dimension. To provide a normative justification of the dominance criteria we introduce alternative principles of association rearrangements, where the mean deprivation is assumed to be kept fixed. An empirical application based on data for 26 European countries illustrates the usefulness of the proposed framework and shows how different ethical views lead to different results.

KEY WORDS: Multidimensional deprivation, counting approach, partial orderings, dual measures of deprivation, principles of mean preserving association rearrangements.

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1. Introduction

Since the seminal papers of Sen (1976) and Foster-Greer-Thorbecke (1984), a flourishing literature has extended the normative approach of poverty measurement to the multidimensional case for continuous variables. In this paper we focus on situations where the multiple attributes in which an individual can be deprived are represented by dichotomized variables. The number of dimensions for which each individual suffers from deprivation may therefore be summarised in a "deprivation count" (see Atkinson, 2003)[1]. The purpose of this paper is *not* to discuss the justification for counting the deprivation indicators; we take it for granted by referring to the extensive practice of Statistical Agencies to publish such data; normally summarized by three summary measures: The proportion of people suffering from at least one deprivation indicator, the proportion of people suffering from all deprivation indicators and the average number of deprivations in the population. The importance of collecting such data has also been emphasized by the European Union and was adopted as part of the European 2020 Agenda measures. As a consequence EUROSTAT (the Statistical Agency of the EU) collects counting data on a regular basis, as part of the EU-SILC microdata on level of living. These facts form a motivating background for investigating deprivation and poverty in deprivation count distributions.

Being deprived on a single dimension could result from the combination of a threshold and a continuous or discrete variable (e.g. income, or number of healthy days for year). In what follows it is supposed that available data only contain information on whether an individual is deprived or not on each dimension. This simplification allows us to delve into the question of how to measure (overall) deprivation in a country. As for the analysis of poverty in multidimensional distributions of continuous variables the order of aggregation is of crucial importance for the measurement of deprivation in count distributions. Data limitations might in some cases only allow to first aggregate across individuals for each dimension and next aggregate the dimension-specific proportions into an overall measure of deprivation (or poverty). The Human Poverty Index (HPI) is a prominent example of this approach.[2] However, when data provide information on all dimensions for the same individuals then it is more relevant to employ the opposite order of aggregation. Otherwise, essential information about the association between deprivation indicators would have been lost[3]. First, by aggregating across the single dimensions for each individual a "deprivation count" is identified, representing the number of dimensions for which the individual suffers from deprivation. Second, by aggregating across individuals we obtain a count distribution, which will form the basis of the development of the methods introduced in this paper.

Atkinson's (2003) illuminating discussion of the relationship between social welfare, measurement of deprivation and association between different attributes has formed the motivation and inspiration for this paper.[4] However, as opposed to the approach discussed by Atkinson (2003), which can be justified by the "primal independence axiom" of the expected utility theory[5], the methods proposed in this paper rely on an alternative independence axiom called the "dual independence axiom" by Yaari (1986). The deprivation measures justified

[1] Bossert et al. (2007) use the counting approach to analyse social exclusion in a dynamic context, whilst Bossert et al. (2012) and Lasso de La Vega and Urrutia (2011) provide alternative axiomatic foundations of deprivation measures based on the counting approach

[2] See also Anand and Sen (1997).

[3] The importance of accounting for the association between dimensions in analyses of multidimensional inequality and poverty has been underlined by e.g. Atkinson and Bourguignon (1982), Tsui (1999), Atkinson (2003), Bourguignon and Chakravarty (2003) and Alkire and Foster (2011).

[4] See also Duclos et al. (2006).

[5] This approach has been considered by Chakravarty and D'Ambrosio (2006), Bossert, D'Ambrosio and Peragine (2007), Alkire and Foster (2011) and Aaberge and Brandolini (2014, 2015).

by the dual independence axiom (and some additional standard axioms) are obtained by aggregating a transformation of the count distribution over the range of counts, and moreover prove to admit a linear decomposition with regard to the mean and the dispersion of deprivation counts. The transformation function can be considered as the preference function of a social planner. The shape of the preference function reveals whether the concern of the social planner is turned towards those people suffering from deprivation on all dimensions (convex preference function) or those suffering from at least one dimension (concave preference function). This distinction is demonstrated also to be captured by two alternative partial orders; second-degree upward and downward count distribution dominance, which refines the trivial ranking of deprivation count distributions provided by Pareto dominance (or first-degree stochastic dominance).

In line with the proposal of Alkire and Foster (2011) we also make a distinction between the notions of deprivation and poverty by introducing a cut-off z in the deprivation count distribution and define people that suffer from at least z deprivations as poor. It is demonstrated how the framework for measuring multidimensional deprivation can be extended to measure multidimensional poverty, when the available information is restricted to deprivation indicators.

A normative justification of the dominance criteria is provided through alternative principles of association rearrangements, where the mean deprivation is assumed to be kept fixed. Our approach departs from the correlation-based rearrangement principles discussed in the multidimensional inequality and poverty literature (se e.g. Atkinson and Bourguignon, 1982, Bourguignon and Chakravarty, 2003 and Atkinson, 2003) and rests on a less restrictive mean preserving association rearrangement principle. Moreover, as opposed to the previous literature, we stress the importance of making a distinction between whether an association rearrangement comes from a distribution characterized by positive or negative association between two or several deprivation indicators, in the spirit of the statistical literature on measurement of association in multidimensional contingency tables (formed by two or several dichotomous variables).The introduced mean preserving association increasing/decreasing rearrangement principles will be proved to support second-degree downward/upward dominance and to divide the family of dual deprivation measures into two subfamilies, determined by whether the preference function is convex or concave.

The paper is organized as follows. Section 2 first presents second-degree upward and downward dominance criteria as suitable refinements of first-order stochastic dominance. These criteria are able to capture alternative ethical views of a social planner mainly interested in individuals suffering from few or many deprivations, respectively. Next, we introduce a family of deprivation measures that is analogous to the family of dual (rank-dependent) measures of social welfare. These measures are shown to admit a useful decomposition with regard to the extent and the spread of deprivation counts. Moreover, the proposed deprivation measures prove to form a useful basis for defining measures of poverty for count distributions. A generalization to the case where the attributes under exam are differently weighted concludes the section. Section 3 introduces various mean preserving association rearrangement principles which are shown to justify the employment of second-degree upward and downward dominance criteria and two subfamilies of dual deprivation measures. The main results of the paper are collected and presented in two theorems. Section 4 provides an application of the framework based on material deprivation indicators (Eurostat, 2014) for 26 European countries. Section 5 provides a brief summary of the paper and discuss possible further developments. Proofs are gathered in the Appendix.

2. Ranking distributions of deprivation counts

We consider a situation where individuals might suffer from r different dimensions of deprivation. Let X_i be equal to 1 if an individual suffers from deprivation in the dimension i and 0 otherwise. Moreover, let

$$X = \sum_{i=1}^{r} X_i$$

be a random variable with cumulative distribution function F and mean μ, and let F^{-1} denote the left inverse of F, that is $F^{-1}(t) = \inf\{k : F(k) \geq t\}$. Thus, $X = 1$ means that the individual suffers from one deprivation, $X = 2$ means that the individual suffers from two deprivations, etc. We call X the deprivation count. Furthermore, let $q_k = \Pr\left(X = k\right)$ which yields

(2.1) $$F(k) = \sum_{j=0}^{k} q_j,\ k = 0,1,2...,r$$

and

(2.2) $$\begin{aligned}&\mu = \sum_{k=1}^{r} kq_k = r - \sum_{k=0}^{r}(r-k)q_k = r - \left[rq_0 + (r-1)q_1 + \cdots + q_{r-1}\right] = \\ &r - \sum_{k=0}^{r-1}\sum_{j=0}^{k} q_j = r - \sum_{k=0}^{r-1} F(k)\end{aligned}.$$

Section 2.4 considers comparison of distributions of weighted deprivation indicators.

2.1. *Partial orders*

As for distributions of continuous variables (like income) comparisons of count distributions can be achieved by employment of appropriate dominance criteria. The condition of first-degree dominance, i.e. $F_1(k) \geq F_2(k)$ for all $k = 0,1,2,...,r-1$ and the inequality holds strictly for some k, justifies the claim that F_1 exhibits less deprivation than F_2.

To deal with situations where deprivation count distributions intersect, weaker dominance criteria than first-degree dominance are called for. As will be demonstrated below it will be useful to make a distinction between aggregating across count distributions from below and from above. We first introduce the "second-degree downward dominance" criterion.

DEFINITION 2.1A. *A deprivation count distribution F_1 is said to second-degree downward dominate a deprivation count distribution F_2 if*

$$\sum_{k=s}^{r-1} F_1(k) \geq \sum_{k=s}^{r-1} F_2(k) \ \textit{for}\ s = 0,1,...,r-1$$

and the inequality holds strictly for some k.

A social planner who implements second-degree downward count distribution dominance is especially concerned about those people who suffer from deprivation over many dimensions. However, an alternative ranking criterion that focuses attention on those who suffer deprivation from few dimensions can be obtained by aggregating the deprivation count distribution from below.

DEFINITION 2.1B. *A deprivation count distribution* F_1 *is said to second-degree upward dominate a deprivation count distribution* F_2 *if*

$$\sum_{k=0}^{s} F_1(k) \geq \sum_{k=0}^{s} F_2(k) \;\; \textit{for} \;\; s = 0,1,...,r-1,$$

and the inequality holds strictly for some s.

Note that second-degree downward as well as upward count distribution dominance preserves first-degree dominance since first-degree dominance implies second-degree downward and upward dominance.

The following example illustrates the difference between the two principles: Consider two counting distributions F_1 and F_2. In distribution F_1 individual *i* suffers from *h* deprivations and individual *j* from *l* (*l*<*h*) deprivations. In distribution F_2 individual *i* suffers from *h+1* deprivations and individual *j* from *l-1* deprivations. The remaining individuals of the population have identical status in F_1 and F_2. A social planner who supports the condition of second-degree downward count distribution dominance will consider F_1 to be preferable to F_2. By contrast, a social planner who supports the condition of second-degree upward count distribution dominance will prefer F_2 to F_1. Thus, for a fixed number of deprivations second-degree downward dominance will rank the distribution with the lowest proportion suffering from all dimensions as more favourable then the distribution with the lowest proportion suffering from at least one dimension, whereas second-degree upward dominance provides a reverse ranking. As will be demonstrated in Section 2.3 these properties are associated with the union and intersection approaches for measurement of multidimensional poverty, when people are defined as poor if they suffer from at least one deprivation dimension.

2.2. Complete orderings – the dual approach

Since both second-degree downward and second-degree upward dominance in many cases will fail to provide rankings of deprivation count distributions, it will be helpful to introduce and employ summary measures of deprivation. Moreover, summary measures of deprivation do not only rank distributions with regard to deprivation, but do also provide an estimate of the extent of deprivation exhibited by a multidimensional distribution of level of living indicators.

Next, let ***F*** denotes the family of deprivation count distributions. A social planner's ranking over ***F*** can be represented by a preference relation $\succeq$, which will be assumed to be continuous, transitive and complete. It is well known that a preference ordering that satisfies these three conditions can be represented by an increasing and continuous preference functional (see Debreu 1964). To make the preference ordering $\succeq$ empirically relevant, it is required to impose further conditions on $\succeq$. To this end, we introduce the following independence condition:

Axiom *(Dual Independence). Let F_1, F_2 and F_3 be members of* ***F*** *and let* $\alpha \in [0,1]$. *Then* $F_1 \succeq F_2$ *implies* $\left(\alpha F_1^{-1} + (1-\alpha) F_3^{-1}\right)^{-1} \succeq \left(\alpha F_2^{-1} + (1-\alpha) F_3^{-1}\right)^{-1}$.

The dual independence axiom was introduced by Yaari (1987) as an alternative to the independence axiom of the expected utility theory for choice under uncertainty[6]. This axiom requires that the ordering of distributions is invariant with respect to certain changes in the distributions being compared. If F_1 is weakly preferred to F_2, then the dual independence axiom states that any mixture on F_1^{-1} is weakly preferred to the corresponding mixture on F_2^{-1}. The intuition is that identical mixing interventions on the inverse distribution functions being compared do not affect the ranking of distributions.

To illustrate this averaging operation, let us consider the problem of evaluating the average deprivation within couples obtained by matching men and women with the same rank in the male and female deprivation count distributions (i.e. the most deprived man is matched with the most deprived woman, the second deprived man with the second deprived woman, and so on). Dual independence means that, given any initial distribution F_3 of deprivation over the female population, if within the male population, distribution F_1 is deemed to contain less deprivation than distribution F_2, this judgement is preserved after the matching with the women. The dual independence axiom requires this property regardless of the initial patterns of deprivation and of the weights associated to male and female deprivation counts computing the average deprivation at the household level.

Provided that the preference relation $\succeq$ is continuous, transitive and complete and satisfies first-degree stochastic dominance, the dual independence axiom justifies the following family of social evaluation functions:

(2.3) $$W_{\Gamma}(F) = \sum_{k=0}^{r-1} \Gamma(F(k)),$$

where Γ, with $\Gamma(0)=0$ and $\Gamma(1)=1$, is a non-negative and non-decreasing function that represents the preferences of the social planner[7]. The social planner will consider the distribution that produces the largest $W_{\Gamma}(F)$ as the most favourable distribution. Thus, the social evaluation function $W_{\Gamma}(F)$ provides a normative justification of the following family of deprivation measures:

(2.4) $$D_{\Gamma}(F) = r - \sum_{k=0}^{r-1} \Gamma(F(k)).$$

Since F denotes the distribution of the deprivation count, $D_{\Gamma}(F)$ can be considered as a summary measure of deprivation exhibited by the distribution F. The social planner considers the distribution F that minimizes $D_{\Gamma}(F)$ to be the most favorable among those being compared. Contrasting (2.4) and (2.2), it is easy to observe that if $\Gamma(t)=t$, then $D_{\Gamma}(F)=\mu$.

Atkinson et al. (2002) and Atkinson (2003) call attention to the distinction between the union and intersection approaches for measuring deprivation. A social planner who supports the union approach is concerned with the proportion of people who suffers from at least one dimension of deprivation $(1-q_0)$, whereas a social planner in favour of the intersection

[6] Weymark (1981) denoted this axiom Weak Independence of Income Source and used it to justify rank-dependent measures of inequality.

[7] Since the ordering relation defined on the set of inverse distribution functions is equivalent to the ordering relation defined on $\boldsymbol{F}$, the proof of the axiomatic characterization of W_{Γ} defined by (2.3) can be derived from the proof of the expected utility theory for choice under uncertainty. For alternative proofs see Weymark (1981) and Yaari (1987).

approach will focus attention on the proportion of people deprived on all dimensions (q_r) . By choosing the following function for Γ ,

(2.5) $$\Gamma(t)=\begin{cases} t & if\ t=q_0 \\ 1 & if\ q_0<t\leq 1 \end{cases}$$

we get $D_\Gamma(F)=1-q_0$, which means that the proportion that suffer from at least one dimension can be considered as a limiting case of the D_Γ -family of deprivation measures for concave Γ . The following alternative specification of the preference function,

(2.6) $$\Gamma(t)=\begin{cases} 0 & if\ q_0\leq t<1-q_r \\ t & if\ t=1-q_r \end{cases}$$

yields $D_\Gamma(F)=r-1+q_r$, which means that the proportion that suffer from all dimensions represents a limiting case of the D_Γ -family of deprivation measures for convex Γ . Although the proportions suffering from at least one dimension and all dimensions do not belong to the D_Γ -family (which is generated by continuous Γ functions) these deprivation measures can be approximated within this class (see Le Breton and Peluso 2010 for general approximation results).

Decomposition of deprivation measures

As is well-known social welfare measures derived from the expected and rank-dependent utility theories, called primal and dual approaches below, allow multiplicative decompositions with regard to the mean and the inequality of income distributions (see Atkinson, 1970 and Yaari, 1987). An extension to measurement of multidimensional inequality has been considered by Weymark (2006). In this section we show that the deprivation measures introduced above admit a decomposition with regard to the mean and the dispersion of the deprivation count distributions. Moreover, it is demonstrated that the structure of this decomposition depends on whether the preferences of the social planner are associated with the union or with the intersection approach.

The following example motivates the methods introduced in this section:

Example 1. Two alternative policies produce the following distributions of two-dimensional deprivation: F_1, where 50 per cent of the population suffers from one dimension and the remaining 50 per cent suffers from the other dimension; F_2 where 50 per cent of the population does not suffer from any deprivation and the remaining 50 per cent suffers from both dimensions. Thus, the mean number of deprivation is 1 for both distributions, but the intersection measure ranks F_1 to be preferable to F_2 whereas the union measure ranks F_2 to be preferable to F_1. An interesting question is which restrictions on Γ that guarantee that D_Γ ranks F_1 to be preferable to F_2 or vice versa.

As it will be demonstrated below, the ranking of F_1 and F_2 provided by D_Γ depends on whether Γ is convex or concave, which according to Theorems 3.1A and 3.2B depend on whether the social planner favors second-degree downward or upward count distribution dominance. This judgment can be equivalently expressed in terms of the mean and the dispersion of the deprivation count distributions. The intuition of this result is now presented through the two-dimensional case, then the general r-dimensional case follows.

Let $r=2$, i.e. $X=X_1+X_2$,

$p_{ij} = Pr\left(\left(X_1 = i\right) \cap \left(X_2 = j\right)\right)$, $p_{i+} = \Pr\left(X_1 = i\right)$ and $p_{+j} = \Pr\left(X_2 = j\right)$.
Thus, $q_k = \Pr\left(X = k\right)$ can be expressed by $p_{ij}, i, j = 1,2$ in the following way:

(2.7)
$$\begin{aligned} q_0 &= p_{00} \\ q_1 &= p_{10} + p_{01} \\ q_2 &= p_{11}. \end{aligned}$$

The 2x2 case is illustrated in Table 2.1.

Table 2.1. The distribution of deprivation in two dimensions

		X_2		
		0	1	
X_1	0	p_{00}	p_{01}	p_{0+}
	1	p_{10}	p_{11}	p_{1+}
		p_{+0}	p_{+1}	1

The distribution F of X is given by

(2.8)
$$F(k) = \Pr(X \le k) = \sum_{j=0}^{k} q_j, \ k = 0,1,2,$$

where $F(2) = 1$ and the mean is defined by $\mu = q_1 + 2q_2 = 2 - \sum_{k=0}^{1} F(k)$.

In this case the class of deprivation measures $D_\Gamma(F)$ defined by (2.4) is given by

(2.9)
$$D_\Gamma(F) = 2 - \Gamma(1 - q_2) - \Gamma(q_0).$$

Note that Γ can be interpreted as a preference function of a social planner that assigns lower weights for one than for two deprivation counts.

To supplement the information provided by $D_\Gamma(F)$ and μ, it will be useful to introduce the following measures of dispersion,

(2.10)
$$\Delta_\Gamma(F) = \begin{cases} \sum_{k=0}^{1}\left[F(k) - \Gamma(F(k))\right] \text{ when } \Gamma \text{ is convex} \\ \sum_{k=0}^{1}\left[\Gamma(F(k)) - F(k)\right] \text{ when } \Gamma \text{ is concave,} \end{cases}$$

which by inserting for (2.8) in (2.10) yields

(2.11)
$$\Delta_\Gamma(F) = \begin{cases} q_0 - \Gamma(q_0) + (1 - q_2) - \Gamma(1 - q_2) \text{ when } \Gamma \text{ is convex} \\ \Gamma(q_0) - q_0 + \Gamma(1 - q_2) - (1 - q_2) \text{ when } \Gamma \text{ is concave.} \end{cases}$$

It can easily be observed from (2.11) that $\Delta_\Gamma(F)=0$ if and only if q_0, q_1 or q_2 is equal to 1, which means that every individual suffers from 0, 1 or 2 deprivations. Note that $\Delta_\Gamma(F)$ can be considered as left- or right-spread measures of dispersion (or tail-heaviness), depending on whether Γ is concave or convex[8]. Since $q_0+(1-q_2)=2-q_1-2q_2=2-\mu$, it follows by inserting for μ and (2.10) in (2.9) that the deprivation measure D_Γ admits the following decomposition

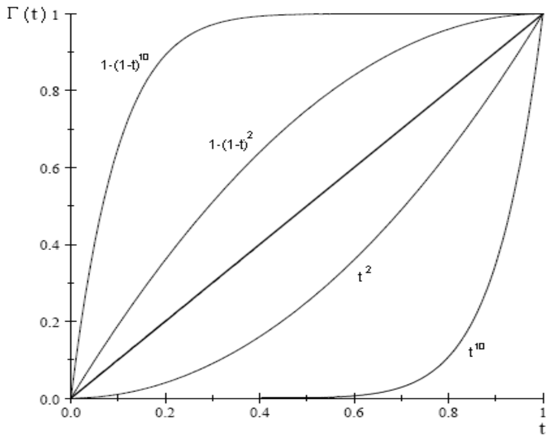
(2.12)
$$D_\Gamma(F)=\begin{cases}\mu+\Delta_\Gamma(F) & \text{when } \Gamma \text{ is convex}\\ \mu-\Delta_\Gamma(F) & \text{when } \Gamma \text{ is concave.}\end{cases}$$

Thus, by using (2.12) we may identify the contribution to D_Γ from the average number of deprivations (μ) as well as from the dispersion of deprivations across the population. Expression (2.12) demonstrates that a social planner with preference function $\Gamma(t)=t$ will only be concerned about reducing the mean number of deprivations, whereas a social planner who is also concerned about reducing the dispersion of deprivations across the population will use a measure D_Γ with a convex Γ. When Γ is convex the social planner pays more attention to people who suffer from many deprivations than to people who suffer from few deprivations. By contrast, when the social planner uses criterion D_Γ with a concave Γ, he/she is more concerned about the number of people who are deprived on at least one dimension (the union approach) than about the number of individuals deprived on all dimensions (the intersection approach). In this case D_Γ can be expressed as the difference between the mean number of deprivations in the population and the dispersion of deprivations across the population. Thus, with Γ concave, D_Γ decreases when Δ_Γ increases.

By employing the criterion $D_\Gamma(F)$ defined by (2.12) to Example 1, it follows that F_1 is preferred if the social planner relies on a convex Γ. By contrast, F_2 is considered to be preferable if a concave Γ represent the preferences of the social planner. Two different concave Γ are represented in Figure 1 below: Γ (t)= 1- $(1\text{-t})^{10}$ and Γ (t)= 1-$(1\text{-t})^2$. Similarly, Γ (t)= t^2 and Γ (t)= t^{10} are two examples of convex distortion functions.

Figure 1. Examples of concave and convex preference functions Γ

[8] See e.g. Fernández-Ponce et al. (1998) and Shaked and Shanthikumar (1998) who provide a discussion on how to compare the right-spread variability of distribution functions.



By inserting for $\Gamma(t) = 2t - t^2$ or $\Gamma(t) = t^2$ in (2.10) and (2.12) we get the following expressions for the Gini measure of deprivation and the associated Gini measure of dispersion (which corresponds to the Gini mean difference $\int F(x)(1-F(x))dx$)[9],

(2.13) $$D_G(F) = D_\Gamma(F) = \begin{cases} \mu + \sum_{k=0}^{1} F(k)\big(1-F(k)\big) & \textit{when } \Gamma(t) = t^2 \\ \mu - \sum_{k=0}^{1} F(k)\big(1-F(k)\big) & \textit{when } \Gamma(t) = 2t - t^2 \end{cases}$$

Note that $\Delta_G \equiv \sum_{k=0}^{1} F(k)\big(1-F(k)\big)$ takes its maximum value 0.5 when $q_0 = q_2 = 0.5$. Thus, when people on average suffer from one deprivation the minimum value of $D_G(F)$ is attained when $\Delta_G(F) = 0$ in the convex case and $\Delta_G(F) = 0.5$ in the concave case; i.e. when each individual of the population suffers from one deprivation in the convex case and when 50 per cent of the population do not suffer from any deprivation and the remaining 50 per cent suffer from two deprivations in the concave case. By contrast, the maximum value of $D_G(F)$ is attained when $\Delta_G(F) = 0.5$ in the convex case and when $\Delta_G(F)$ is 0 in the concave case.

The r dimensional case

Next, we consider the *r* dimensional case formed by the multinomial distribution of *r* deprivation indicators $X_1, X_2, ..., X_r$. In this case $\sum_{k=0}^{r} q_k = 1$ and the mean μ is given by (2.2). Similarly as in the 2x2 case we get that $D_\Gamma(F)$ admits the decomposition

(2.14) $$D_\Gamma(F) = \begin{cases} \mu + \Delta_\Gamma(F) & \textit{when } \Gamma \textit{ is convex} \\ \mu - \Delta_\Gamma(F) & \textit{when } \Gamma \textit{ is concave,} \end{cases}$$

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[5] Gini's mean difference was already used by von Andrae (1872) and Helmert (1876) as a measure of spread.

where the dispersion measure $\Delta_\Gamma(F)$ is defined by

(2.15)
$$\Delta_\Gamma(F)=\begin{cases}\sum_{k=0}^{r-1}\left[F(k)-\Gamma(F(k))\right] \text{ when } \Gamma \text{ is convex}\\ \sum_{k=0}^{r-1}\left[\Gamma(\mathrm{F(k)})-F(k)\right] \text{ when } \Gamma \text{ is concave,}\end{cases}$$

Note that $D_\Gamma(F)\geq r-\sum_{k=0}^{r-1}F(k)=\mu$ and $\mu\leq D_\Gamma(F)\leq r$ when Γ *is convex*, and $0\leq D_\Gamma(F)\leq\mu$ when Γ *is concave*.

The decomposition (2.14) suggests that $r-\sum_{k=0}^{r-1}\Gamma(F(k))$ obeys the principle of mean preserving spread when Γ is convex; i.e. $D_\Gamma(F)$ increases when the number of deprivations at the middle of the count distribution is shifted towards the tails, under the condition of fixed total number of deprivations. However, when Γ is concave, the summary measure $D_\Gamma(F)$ decreases as a consequence of a mean preserving spread. This is due to the fact that such an operation will increase the number of people who do not suffer from any deprivation and/or suffer from few dimensions of deprivation.
As for the two-dimensional case, we get by inserting for $\Gamma(t)=t^2$ and $\Gamma(t)=2t-t^2$ in (2.14) and (2.15) the following convenient expressions for the Gini measures of deprivation and dispersion,

(2.16)
$$D_\Gamma(F)=\begin{cases}\mu+\Delta_G(F) & \text{when } \Gamma(t)=t^2\\ \mu-\Delta_G(F) & \text{when } \Gamma(t)=2t-t^2.\end{cases}$$

where

(2.17)
$$\Delta_G(F)=\sum_{k=0}^{r-1}F(k)\left(1-F(k)\right).$$

More generally, by inserting a parametric specification of Γ we can derive alternative parametric subfamilies of Δ and *D*. If the preference function is defined by

(2.18)
$$\Gamma(t)=t^i,$$

then

(2.19)
$$\Delta_\Gamma(F)=\Delta_i(F)=\begin{cases}\sum_{k=0}^{r-1}\left[F(k)\left(1-(F(k))^{i-1}\right)\right], i\geq 1\\ \sum_{k=0}^{r-1}\left[F(k)\left(F(k))^{i-1}-1\right)\right], 0<i\leq 1.\end{cases}$$

Note that Δ_i can be considered as a measure of left-spread when $0 < i < 1$ and a measure of right-spread when $i > 1$. The next sub-section will clarify the relationship between a mean preserving spread, second-degree upward and downward count distribution dominance and association rearrangements. The association rearrangement principles are shown to provide a normative justification of the convexity and concavity of the preference function Γ.

2.3. Measurement of poverty versus deprivation

The above discussion and results concern ranking and measurement of the extent of deprivation exhibited by distributions of deprivation counts, whilst the relationship between deprivation and poverty has been ignored. Whether or not deprivation and poverty should be considered as identical concepts has been subject to discussion in the literature. Bourguignon and Chakravarty (1999, 2003), Tsui (2002) and Bossert et al. (2013) among others do not make a distinction between poverty and deprivation, whereas Alkire and Foster (2011) introduce methods where suffering from poverty can be considered as more serious than suffering from deprivation. To capture this distinction between poverty and deprivation Alkire and Foster (2011) introduce a cut-off z ($1 \le z \le r$) in the deprivation count distribution, where a person is considered as poor if he/she suffers from deprivation in at least z dimensions. Thus, the headcount measure is given by $1 - F(z-1)$. Extending the primal and dual methods for measuring multidimensional deprivation to measuring poverty follows, as was also indicated by Aaberge and Brandolini (2015), from replacing the count distribution F with the conditional distribution F^* defined by

$$\text{(2.20)} \quad F^*(k;\mathrm{z}) = \Pr(X \le k \,|\, X \ge z) = \frac{F(k) - F(\mathrm{z}-1)}{1 - F(\mathrm{z}-1)} = \frac{\sum_{j=z}^{k} q_j}{\sum_{j=z}^{r} q_j},\ k = z, \mathrm{z}+1, \ldots, r,$$

with mean

$$\text{(2.21)} \qquad \mu^*(z) = r - \sum_{k=z}^{r-1} F^*(k) = \frac{\sum_{j=z}^{r} jq_j}{\sum_{j=z}^{r} q_j}.$$

By inserting for F^* in (2.14) and (2.15) we get the following measures of poverty for distributions of deprivation counts,

$$\text{(2.22)} \qquad D_\Gamma(F^*) = r - \sum_{k=1}^{r-1} \Gamma(F^*(k)) = \begin{cases} \mu^* + \Delta_\Gamma(F^*) & \textit{when } \Gamma \textit{ is convex} \\ \mu^* - \Delta_\Gamma(F^*) & \textit{when } \Gamma \textit{ is concave,} \end{cases}$$

and

$$\text{(2.23)} \qquad \Delta_\Gamma(F^*) = \begin{cases} \sum_{k=z}^{r-1} \left[F^*(k) - \Gamma(F^*(k)) \right] & \textit{when } \Gamma \textit{ is convex} \\ \sum_{k=z}^{r-1} \left[\Gamma(F^*(k)) - F^*(k) \right] & \textit{when } \Gamma \textit{ is concave.} \end{cases}$$

Note that the poverty measures defined by (2.22) can be given a similar axiomatic justification based on an order relation defined on the conditional count distributions F^* as was given for the family of deprivation measures $D_\Gamma(F)$ in Section 2.2. Moreover, the poverty measures admit the useful decomposition into the mean and dispersion of deprivation counts for people classified as poor, which means that these poverty measures captures both the mean and the distribution of poverty. By contrast, Alkire and Foster (2011) introduce separate headcount adjusted FGT poverty measures to account for the distribution of deprivations among the poor. Headcount adjusted versions of the distribution-sensitive poverty measures (2.22) is given by

$$(2.24)\quad N_\Gamma(z)=\frac{1-F(z-1)}{r}D_\Gamma(F^*)=\begin{cases}\mu^{**}(z)+\dfrac{1-F(z-1)}{r}\Delta_\Gamma(F^*) & \text{when } \Gamma \text{ is convex}\\ \mu^{**}(z)-\dfrac{1-F(z-1)}{r}\Delta_\Gamma(F^*) & \text{when } \Gamma \text{ is concave,}\end{cases}$$

where $\mu^{**}(z)=(\sum_{j=z}^{r} jq_j)\Big/r$ is the headcount adjusted mean.

Note that

$N_\Gamma(z)\geq\frac{1-F(z-1)}{r}\left(r-\sum_{k=z}^{r-1}F^*(k)\right)=\mu^{**}(z)$ and $\mu^{**}(z)\leq N_\Gamma(z)\leq 1$ when Γ *is convex*, and $0\leq N_\Gamma(z)\leq\mu^{**}$ when Γ *is concave*. For a given $\mu^{**}(z)$ the minimum value of $N_\Gamma(z)$ for convex Γ is attained when $\Delta_\Gamma(F^*)=0$; i.e. when each of the individuals classified as poor suffers from the same number of deprivations.

2.4. Accounting for different weights

A social planner might consider deprivation in certain dimensions to be more detrimental than deprivation in other dimensions. A convenient way to incorporate such preferences is to replace the deprivation count *X* with the weighted counting variable

$$(2.25)\qquad \tilde{X}=\sum_{i=1}^{r}w_iX_i\ .$$

with cumulative distribution $\tilde{F}$. For instance, in the two-dimensional case with $\tilde{X}=\tilde{X}_1+\tilde{X}_2=w_1X_1+w_2X_2$ and $w_1\leq w_2$ the cumulative distribution $\tilde{F}$ of $\tilde{X}$ is given by

$$(2.26)\qquad \tilde{F}(z)=\begin{cases}p_{00} & if\ \ z=0\\ p_{00}+p_{10} & if\ \ z=w_1\\ p_{00}+p_{10}+p_{01} & if\ \ z=w_2\\ 1 & if\ \ z=w_1+w_2.\end{cases}$$

where p_{00}, p_{01}, p_{10} and p_{11} is given by the following table,

Table 2.2. The distribution of weighted deprivation in two dimensions

$\tilde{X}_2$		$\tilde{X}_1$		
		0	w_2	
	0	p_{00}	p_{01}	p_{0+}
	w_1	p_{10}	p_{11}	p_{1+}
		p_{+0}	p_{+1}	1

Let $\tilde{\boldsymbol{F}}$ denote the family of distributions of $\tilde{X}$ for r dimensions and let $\boldsymbol{S}$ be the set of possible outcomes for $\tilde{X}$. Provided that the order relation $\succeq$ defined on $\tilde{\boldsymbol{F}}$ is continuous, transitive and complete and satisfies the dual independence axiom and the following dominance condition,

Axiom *(First-degree Stochastic Dominance FSD). Let* $\tilde{F}_1, \tilde{F}_2 \in \tilde{\boldsymbol{F}}$ *. If* $\tilde{F}_1(z) \geq \tilde{F}_2(z)$ *for all* $z \in \boldsymbol{S}$ *then* $\tilde{F}_1 \geq \tilde{F}_2$,

we get a similar representation for $\succeq$ as was demonstrated for the count distributions in Section 2.2,

$$\tilde{F}_1 \succeq \tilde{F}_2 \quad \Leftrightarrow \quad \tilde{D}_\Gamma(\tilde{F}_1) \leq \tilde{D}_\Gamma(\tilde{F}_2),$$

where $\tilde{D}_\Gamma$ is defined by

$$\tilde{D}_\Gamma(\tilde{F}) = \sum_{z \in S} \left(1 - \Gamma\left(\tilde{F}(z)\right)\right). \tag{2.27}$$

The social planer considers the distribution $\tilde{F}$ that minimizes $\tilde{D}_\Gamma(\tilde{F})$ to be the most favorable among those being compared.

3. Association rearrangements

The axiomatic characterization of the family D_Γ of deprivation measures provides a normative justification of these measures. However, analogous to the role played by the Pigou-Dalton principle of transfers in measurement of income inequality it is useful to introduce a normative principle that justifies employment of the deprivation measures D_Γ and the dominance criteria introduced in Section 2.1. To this end, the previous literature on measurement of multidimensional poverty and inequality in distributions of continuous variables have relied on the principle of correlation increasing transfers defined by Boland and Proschan (1988) and applied by e.g. Tsui, 1999 and Alkire and Foster, 2011, whereas Epstein and Tanny (1980) and Atkinson and Bourguignon (1982) have provided an alternative definition in terms of correlation increasing perturbation which can be employed for discrete distributions[10]. Both definitions, which normally are referred to as a correlation

[10] For further discussion and application of association (correlation) increasing rearrangements under the condition of fixed marginal distributions we refer to Dardanoni (1995) , Tsui (2002), Bourguignon and Chakravarty (2003), Duclos et al. (2006), Weymark (2006) and Kakwani and Silber (2008). See also Tchen (1980) who deals with positive association (or concordance) between bivariate probability measures and Decancq (2012) for a recent generalization of these principles and an analysis of their links with stochastic dominance.

increasing rearrangement, rely on the condition of fixed marginal distributions. To provide a normative justification of upward and downward count distribution dominance as well as for employing the deprivation measures D_{Γ} for concave and convex Γ, we will introduce a generalization of the association intervention principles for multidimensional distributions of dichotomous variables, where the condition of fixed marginal distributions is relaxed and replaced by the less restrictive condition of fixed mean number of deprivations. The general intervention rearrangement principle is illustrated in Table 3.1, where the parameters of the multinomial distribution are affected by small amounts δ and γ in such a way as to leave the mean number of deprivations unchanged. It follows from Table 3.1 and definition (2.7) that the mean of the distribution in Table 3.1 is equal to the mean μ of the distribution of Table 2.1 since $(p_{10}-2\delta+\gamma)+(p_{01}-\gamma)+2(p_{11}+\delta)=p_{10}+p_{01}+2p_{11}=q_1+2q_2=\mu$.

Since the multinomial distribution associated with a 2x2 table has three free parameters the condition of fixed mean implies that there are still two free parameters δ and γ available for the definition of the mean preserving association rearrangement. However, since we only are concerned about rearrangements that affect the counting distribution, i.e. the parameters q_0, q_1 *and* q_2, γ can be considered as a nuisance parameter that only affects the allocation between the two dimensions (X_1 and X_2) of those suffering from one dimension. Note that $\delta>0$ ($\delta<0$) implies that the proportions of people that do not suffer from any deprivation and those suffering from two deprivations increase (decrease).

Table 3.1. Illustration of a mean preserving association rearrangement

			X_2	
		0	1	
X_1	0	$p_{00}+\delta$	$p_{01}-\gamma$	$p_{0+}+\delta-\gamma$
	1	$p_{10}-2\delta+\gamma$	$p_{11}+\delta$	$p_{1+}-\delta+\gamma$
		$p_{+0}-\delta+\gamma$	$p_{+1}+\delta-\gamma$	1

As opposed to the previous economic literature on correlation rearrangements, we will make a distinction between whether an association (or correlation) rearrangement comes from a distribution characterized by positive or negative association between two or several deprivation indicators, which means that application of the association rearrangement principles requires clarification of whether the different dimensions are positively or negatively associated. However, abandoning the condition of fixed marginal distributions requires employment of a measure of association that is invariant with respect to changes in the marginal distributions. Thus, an appropriate measure of association has to be invariant with respect to the transformation

$$p_{ij} \to a_i b_j p_{ij} \qquad (3.1)$$

for any set of positive numbers $\{a_i\}$ and $\{b_j\}$ such that $\sum_{i=0}^{1}\sum_{j=0}^{1} a_i b_j p_{ij} = 1$. Since the correlation coefficient does not satisfy the invariance condition (3.1) it is not fully informative about the association between two variables, and consequently inappropriate as a measure of association for defining *mean preserving* increasing (decreasing) rearrangement principles.

This limitation of the correlation coefficient motivates the use of the cross-product α as a measure of association. The cross-product α was introduced by Yule (1900) and is defined by

$$\alpha = \frac{p_{00} p_{11}}{p_{01} p_{10}} \tag{3.2}$$

and satisfies the invariance condition (3.1)[11]. Thus, the association measure α and the marginal distributions (p_{0+}, p_{1+}) and (p_{+0}, p_{+1}) provide complete information of Table 2.1. It follows straightforward from the definition of α that $\alpha = 1$ if and only if the indicators X_1 and X_2 are independent, i.e. $p_{ij} = p_{i+} p_{+j} \ for\ i, j = 0,1$. Note that $\alpha \in \left[0,1\right\rangle$ when there is negative association between the two indicators ($p_{ij} < p_{i+} p_{+j} \ for\ i, j = 0,1$), whereas $\alpha \in \left\langle 1,\infty \right\rangle$ when there is positive association between the two indicators ($p_{ij} > p_{i+} p_{+j} \ for\ i, j = 0,1$).

The cross-product for Table 3.1 is given by

$$\alpha(\delta,\gamma) = \frac{(p_{00} + \delta)(p_{11} + \delta)}{(p_{01} - \gamma)(p_{10} - 2\delta + \gamma)}. \tag{3.3}$$

Since $\alpha(\delta,\gamma) > \alpha(0,0)$ ($\alpha(\delta,\gamma) < \alpha(0,0)$) if and only if $\delta > 0$ ($\delta < 0$) the cross-product can be considered as a measure of the effect of the association rearrangement. Although the nuisance parameter γ does not have any effect on the rearrangement of the count distribution it follows from (3.3) that it has an effect on the strength of the rearrangement intervention. The weakest effect is attained when $\gamma = \delta + (p_{01} - p_{10})/2$; i.e. when the proportion suffering from one dimension is equally distributed between the two dimensions ($p_{10} - 2\delta + \gamma = p_{01} - \gamma = (q_1/2) - \delta$). The strongest effect is attained when either $\gamma = 2\delta - p_{10}$ or $\gamma = p_{01}$; i.e. the proportion suffering from one deprivation is either exclusively deprived from indicator X_1 or from indicator X_2.

The discussion above provides a motivation for the following definitions of positive association increasing rearrangements, positive association decreasing rearrangements, negative association increasing rearrangements and negative association decreasing rearrangements.

DEFINITION 3.1A. *Consider a 2x2 table with parameters* $(p_{00}, p_{01}, p_{10}, p_{11})$ *where* $\sum\sum p_{ij} = 1$ *and* $\alpha > 1$. *The following change* $(p_{00} + \delta, p_{01} - \gamma, p_{10} - 2\delta + \gamma, p_{11} + \delta)$ *is said to provide a mean preserving positive association increasing (decreasing) rearrangement if* $\delta > 0$ ($\delta < 0$).

DEFINITION 3.1B. *Consider a 2x2 table with parameters* $(p_{00}, p_{01}, p_{10}, p_{11})$ *where* $\sum\sum p_{ij} = 1$ *and* $\alpha < 1$. *The following change* $(p_{00} + \delta, p_{01} - \gamma, p_{10} - 2\delta + \gamma, p_{11} + \delta)$ *is said to provide a mean preserving negative association increasing (decreasing) rearrangement if* $\delta < 0$ ($\delta > 0$).

[11] Note that the cross-product α is closely associated with the Spearman and Kendall coefficients and the copula measures of association (see Nelsen, 2011).

As illustrated by Table 3.2 the right (left) panel can be obtained from the left (right) panel by a mean preserving negative decreasing (increasing) rearrangement, since the association is negative and the mean is kept fixed equal to 1 under the rearrangement where $\delta = .01$ (and $\gamma = 0$).

Table 3.2. Illustration of a mean preserving negative association decreasing rearrangement

	0	1				0	1	
0	.20	.30	.50		0	.21	.30	.51
1	.30	.20	.50		1	.28	.21	.49
	.50	.50	1			.49	.51	1

As can be observed from Table 3.1 the condition of fixed marginal distributions is satisfied when $\gamma = \delta$. Thus, it follows from Definitions 3.1A and 3.1B that the marginal distribution preserving rearrangement correlation principle can be considered as a special case of the mean preserving rearrangement association principle. In this case the reduction (2δ) in the proportion of those suffering from one deprivation is equally allocated between the two indicators X_1 and X_2. When $\gamma = 0$ or $\gamma = 2\delta$ the proportion suffering from either dimension 1 or from dimension 2 is reduced by 2δ. This case has been considered by Aaberge and Peluso (2011).

Definitions 3.2A and 3.2B can readily be extended to higher dimensions. However, for a large number of dimensions the standard subscript notation becomes cumbersome. Thus, we find it convenient to introduce the following simplified subscript notation p_{ijk}, where *i* and *j* represents the outcomes 0 and 1 of two arbitrary chosen deprivation dimensions and *m* represents the remaining *r-2* dimensions and α_{ijm} is defined by

(3.3)
$$\alpha_{ijm} = \frac{p_{iim} p_{jjm}}{p_{ijm} p_{jim}},$$

where *m* is a r-2 dimensional vector of any combination of zeroes and ones. In this case association is defined by *r(r-1)/2* cross-products.

In order to deal with *r*-dimensional counting data we introduce the following generalization of Definitions 3.1A and 3.1B,

DEFINITION 3.2A. *Consider a 2x2x…x2 table formed by s dichotomous variables with parameters* $(p_{iim}, p_{ijm}, p_{jim}, p_{jjm})$ *where* $\sum\sum\sum p_{ijm} = 1$ *and* $\alpha_{ijm} > 1$. *The following change* $(p_{iim} + \delta, p_{ijm} - \gamma, p_{jim} - 2\delta + \gamma, p_{jjm} + \delta)$ *is said to provide a mean preserving positive association increasing (decreasing) rearrangement if* $\delta > 0$ ($\delta < 0$).

DEFINITION 3.2B. *Consider a 2x2x…x2 table formed by s dichotomous variables with parameters* $(p_{iim}, p_{ijm}, p_{jim}, p_{jjm})$ *where* $\sum\sum\sum p_{ijm} = 1$ *and* $\alpha_{ijm} < 1$. *The following change* $(p_{iim} + \delta, p_{ijm} - \gamma, p_{jim} - 2\delta + \gamma, p_{jjm} + \delta)$ *is said to provide a mean preserving negative association increasing (decreasing) rearrangement if* $\delta < 0$ ($\delta > 0$).

As is demonstrated by Theorems 3.1A below, a social planner who is in favour of second-degree downward dominance will consider a mean preserving positive association increasing rearrangement as well as a mean preserving negative association decreasing rearrangement as a rise in overall deprivation. By contrast, a planner who favours upward second-degree dominance will consider such rearrangement as a reduction in the overall deprivation. Moreover, it is proved that the principles of mean preserving association increasing/decreasing rearrangement are equivalent to the mean preserving spread/contraction defined by

DEFINITION 3.3. *Let F_1 and F_2 be members of the family **F** of count distributions based on s deprivation indicators and where F_1 and F_2 are assumed to have equal means. Then F_2 is said to differ from F_1 by mean preserving spread (contraction) if $\Delta_\Gamma(F_2) > \Delta_\Gamma(F_1)$ for all convex Γ ($\Delta_\Gamma(F_2) < \Delta_\Gamma(F_1)$ for all concave Γ).*

Note that Definition 3.3 is analogous to the mean preserving spread for continuous distributions introduced by Rothschild and Stiglitz (1970).

Next let Ω_1 be a subset of the D_Γ-family, defined as follows

$$\Omega_1 = \left\{\Gamma : \Gamma'(t) > 0, \Gamma''(t) > 0 \ \textit{for all } t \in (0,1], \textit{and } \Gamma'(0) = 0\right\}$$

and

$$\Omega_2 = \left\{\Gamma : \Gamma'(t) > 0, \Gamma''(t) < 0 \ \textit{for } t \in \langle 0,1 \rangle, \textit{and } \Gamma'(1) = 0\right\}.$$

Note that $\Gamma'(0) = 0$ and $\Gamma'(1) = 0$ can be considered as normalization conditions. The following results provide characterizations of the relationship between second-degree downward and upward count distribution dominance and the general family D_Γ of deprivation measures. Moreover, Theorems 3.1A and 3.1B provide normative justification in terms of principles of spread and association rearrangements for application of the dominance criteria and the deprivation measures, where a distinction has been made between whether an association rearrangement comes from a distribution characterized by positive or negative association.

THEOREM 3.1A. *Let F_1 and F_2 be members of the family **F** of count distributions based on s deprivation indicators and assume that F_1 and F_2 have equal means. Then the following statements are equivalent*

(*i*) *F_1 second-degree downward dominates F_2*

(*ii*) *$D_\Gamma(F_1) < D_\Gamma(F_2)$ for all $\Gamma \in \Omega_1$*

(*iii*) *F_2 can be obtained from F_1 by a sequence of mean preserving positive association increasing rearrangements when $\alpha > 1$ for both F_1 and F_2, a sequence of mean preserving negative association decreasing rearrangements when $\alpha < 1$ for both F_1 and F_2, and a combination of mean preserving positive association increasing and negative association decreasing rearrangements when $\alpha > 1$ for either F_1 and F_2.*

(*iv*) *F_2 can be obtained from F_1 by a mean preserving spread.*

(Proof in Appendix).

THEOREM 3.1B. *Let F_1 and F_2 be members of the family **F** of count distributions based on s deprivation indicators and assume that F_1 and F_2 have equal means. Then the following statements are equivalent*

(*i*) *F_1 second-degree upward dominates F_2*

(*ii*) $D_{\Gamma}(F_1) < D_{\Gamma}(F_2)$ *for all* $\Gamma \in \Omega_2$

(*iii*) *F_2 can be obtained from F_1 by a sequence of mean preserving positive association decreasing rearrangements when $\alpha > 1$ for both F_1 and F_2, and a sequence of mean preserving negative association increasing rearrangements when $\alpha < 1$ for both F_1 and F_2, and a combination of mean preserving positive association decreasing and negative association increasing rearrangements when $\alpha > 1$ for either F_1 and F_2.*

(*iv*) *F_2 can be obtained from F_1 by a mean preserving contraction*

(Proof in Appendix).

It follows straightforward that Theorems 3.1 A and 3.1 B can be generalized to valid for distributions of weighted counts discussed in Section 2.3.

4. Application to EU material deprivation

To give an example of the axiomatic theory at work we illustrate its use on the EU indicators of material deprivation. These ten indicators measure non-pecuniary material deprivation, in particular whether a person or household cannot afford:

1. to pay their mortgage or rent
2. to pay their utility bills
3. to keep their home adequately warm
4. to face unexpected expenses
5. to eat meat or proteins regularly
6. to go on holiday
7. a television set
8. a washing machine
9. a car
10. a telephone.

Data on these variables are collected by the EU Statistics on Income and Living Conditions (EU-SILC) project for an increasing number of European countries. The EU-SILC surveys between 7,000 and 15,000 individuals in each country giving every citizen over the age of 16 a non-zero sampling probability. We use version 2 of the 2011 cross-sectional data which leaves us with 26 countries. Our unit of analysis is the individual, and we link individuals to households in order to attach variables that are available at the household level only. The individual is only considered to be suffering in the particular dimension if she responds that she lacks the particular item because she cannot afford it. Non-response is treated as if the individual do not suffer from the deprivation.

We follow Guio et al. (2009) and use the proportion of respondents in the Special Eurobarometer No 279 on “poverty and exclusion” (see TNS, 2007) that answers that an item is “absolutely necessary” for an acceptable or decent standard of living in their country as weights. We compute the cumulative distributions of the weighted deprivation count given in (4.1) and rank them according to the deprivation measure (4.2) for several choices of Γ. We consider the families

$$\Gamma(t) = t^i,\ i = 1, 2, ..., 10$$

and

$$\Gamma(t) = 1-(1-t)^i,\ i = 1, 2, ..., 10$$

of convex and concave preference functions, respectively. We also look at the ranking according to the union and mean approaches. It does not make sense to rank distributions according to the intersection measure as there exist only 11 observations which suffer in all ten dimensions (all are in Latvia, Bulgaria and Hungary).

Results for a selection of Γ are presented in Figures 1-2. An immediate observation is that the rank order is relatively stable with respect to the choice of Γ, and does not change much even at the cases of extreme concavity and convexity. However, there are some notable changes. For instance Belgium looks worse the more convex is the measure, whereas the opposite is the case for Iceland and Portugal.

To understand the changes in the rank order that occur as we use different Γ it is helpful to look at the case of Austria and Iceland, for which (un-weighted) deprivation distributions are presented in Figure 3. As Austria has 67% suffering in no dimension, against 60% in Iceland, a follower of the union approach or other concave Γ would prefer Austria’s deprivation distribution. However, a social planner with a more convex Γ would put more weight on the fact that Austria has relatively large concentration of individuals suffering in many dimensions, and conclude that the distribution of Iceland is the better.

In Tables 1-3 we present first-degree dominance and second-degree upward and downward dominance among the deprivation distributions. A “>” indicates that the horizontal country dominates the vertical country, a “<” that the vertical country dominates the horizontal, and a “=” that no country dominates. Looking at Table 2 we see that, by using the criteria of second-degree upward dominance, we are able to rank almost any two distributions against each other. Thus, only knowing that a social planner is adhering to a concave Γ is sufficient for us to provide her with an almost complete rank order of material deprivation distributions. However, this optimistic message is turned around in the case of a convex Γ; there are a lot more pairs of distributions which we are not able to distinguish by using the criteria of second-degree downward dominance in Table 3. This is to be expected, however, as second-degree downward dominance depends on comparing the shares of individuals suffering in many dimensions for which the number of observations are few.

Figure 1. Ranking of EU material deprivation distributions for a selection of concave Γ.

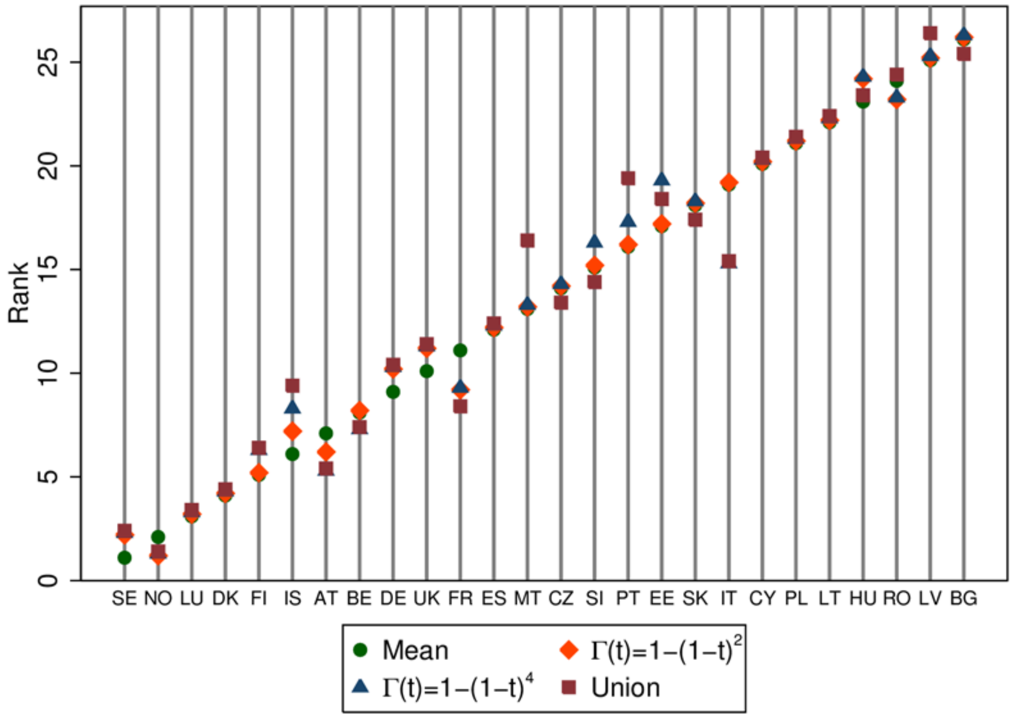

Figure 2. Ranking of EU material deprivation distributions for a selection of convex Γ.

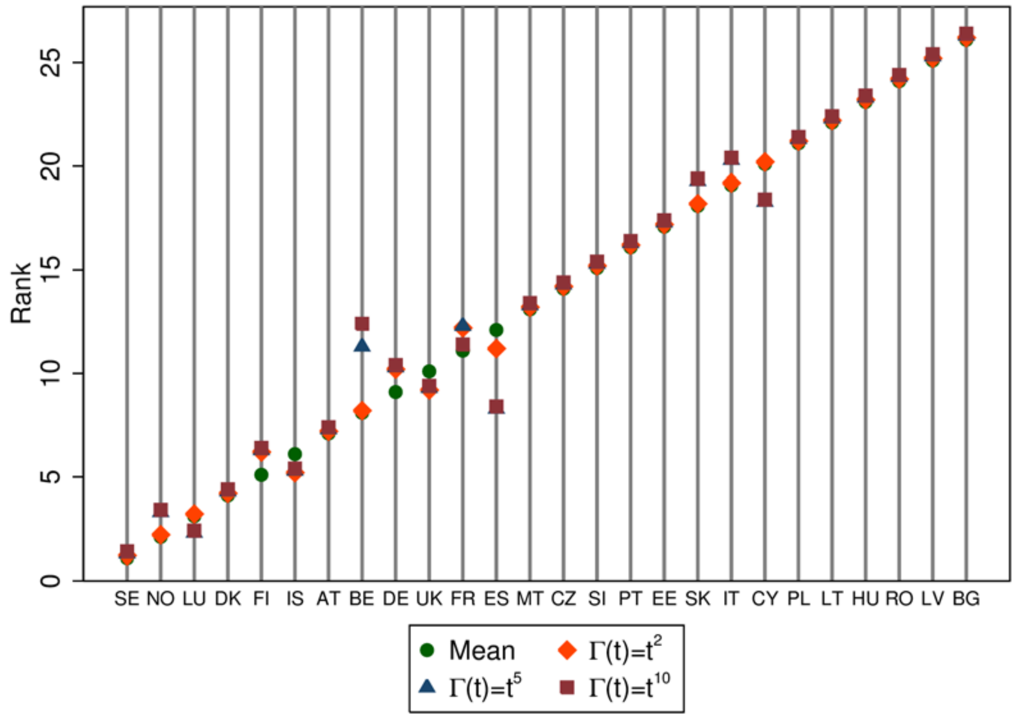

Figure 3. Un-weighted EU material deprivation distribution for Iceland and Austria.

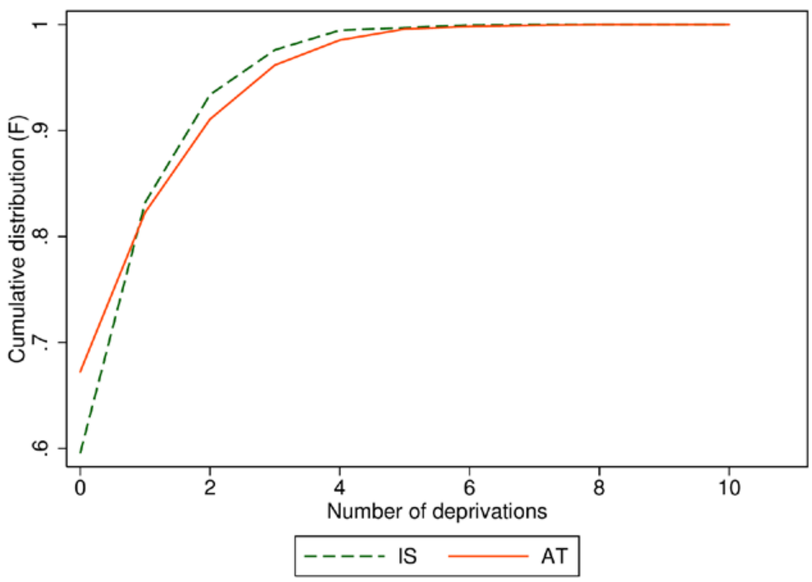

Table 4. First-degree dominance among EU material deprivation distributions

	BG	HU	RO	LT	EE	SK	LV	PL	PT	BE	IT	CZ	SI	FR	FI	CY	MT	UK	SE	DK	NO	AT	DE	ES	LU	IS
IS	>	>	>	>	>	>	>	>	>		>	>	>													
LU	>	>	>	>	>	>		>	>	>				>	>											
ES	>	>	>	>	>	>	>	>			>	>	>													
DE	>	>	>	>	>	>	>	>	>		>															
AT	>	>	>	>	>		>	>	>	>																
NO	>	>	>	>	>	>				>																
SE	>	>		>	>	>																				
DK	>	>		>	>	>																				
UK	>	>		>	>	>																				
MT	>	>	>				>																			
CY	>	>		>																						
FI	>	>	>		>				>																	
FR	>	>	>	>																						
SI	>	>	>	>			>	>																		
CZ	>	>	>				>																			
IT	>	>	>				>																			
BE	>	>	>	>																						
PT	>		>																							
PL	>	>	>																							
LV																										
SK																										
EE	>																									
LT	>																									
RO	>																									
HU																										
BG																										

Table 5. Second-degree upward dominance among EU material deprivation distributions.

	LV	BG	HU	RO	LT	PL	CY	EE	PT	SK	IT	SI	MT	CZ	ES	FR	UK	DE	BE	IS	AT	FI	DK	LU	NO	SE
NO	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>		
SE	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>		
LU	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>			
DK	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>				
FI	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>						
AT	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>							
IS	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>											
BE	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>								
FR	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>											
DE	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>											
UK	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>											
ES	>	>	>	>	>	>	>	>	>	>	>	>	>	>												
CZ	>	>	>	>	>	>	>	>	>	>	>	>														
MT	>	>	>	>	>	>	>	>	>	>																
SI	>	>	>	>	>	>	>	>																		
IT	>	>	>	>	>	>	>																			
SK	>	>	>	>	>	>	>																			
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Table 6. Second-degree downward dominance among EU material deprivation distributions

	BG	HU	RO	LT	EE	SK	PT	LV	PL	IT	CZ	BE	CY	FR	MT	SI	UK	FI	AT	DK	SE	NO	LU	ES	DE	IS
IS	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>		>							
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5. Summary and discussion

The conventional approach in official statistics as well as in most empirical studies of multidimensional deprivation is to focus on distributions of the number of dimensions in which people suffer from deprivation. The conventional “expected utility” type approaches for analysing count distributions have previously been discussed by Chakravarty and D’Ambrosio (2006), Bossert, D’Ambrosio and Peragine (2007) and Aaberge and Brandolini (2014, 2015). This paper relies on the counterpart of the dual (rank-dependent) approach for measuring social welfare (Yaari, 1988) to rank and quantify deprivation and poverty in a multidimensional setting, where individual deprivation on the different dimensions is represented by dichotomized variables. The proposed family of deprivation and poverty measures are shown to be decomposable into extent of and dispersion of deprivation and poverty. The main theoretical result establishes a link between two well-defined subclasses of deprivation measures and two families of dominance criteria easily testable starting from the CDF of the deprivation count distribution. To strengthen the normative justification of the proposed deprivation measures two intervention principles affecting the association between the different deprivation indicators and the spread of the deprivation counts are adopted. The framework provided in this paper is extended to allow for different weighting profiles across the multidimensional distribution of deprivations. Several issues remain open for further research: even if counting the number of deprivations gives a trivial cardinal representation of individual achievements, the empirical implementation of deprivation measurement can easily reveal strict connections with the problem of assessing ordinal inequality (see Allison and Foster 2004 and Gravel et al 2015, among others). Both the rule eventually used to fix a threshold for each attribute and the comparison of populations where the number of dimensions with available data differ are two further difficulties related to this issue that are not considered in this paper and represent interesting directions for further research.

Appendix - *Proofs*

Proof of Theorems 3.1A and 3.1B.

To make the proof more transparent the two-dimensional case will be considered below. However, since intersections between distributions formed by *r* dimensions can be described by 2x2 tables formed by the affected dimensions, the generalization to the *r*-dimensional case is straightforward. More precisely, since interventions affecting two specific dimensions are described by a two-dimensional table, when several dimensions are affected then the procedure demonstrated below for the two-dimensional case is carried out stepwise for the involved two-dimensional tables.

We begin by proving the equivalence between statements (i) and (iii).

Let $F_1(k)=\sum_{j=0}^{k} q_{1j} \quad and \quad F_2(k)=\sum_{j=0}^{k} q_{2j}$, $k=0,1,2.$

By inserting for F_1 and F_2 in Definition 2.1A we get that F_1 second-degree downward dominates F_2 if and only if

(A1) $$\sum_{k=i}^{1}\sum_{j=0}^{k} q_{1j} \geq \sum_{k=i}^{1}\sum_{j=0}^{k} q_{2j} \quad for\ i=0,1.$$

Note that the distance between F_2 and F_1 can be described by two parameters, which will be denoted θ_0 and θ_1, i.e.

(A2) $$F_2(k)-F_1(k)=\sum_{j=0}^{k} q_{2j}-\sum_{j=0}^{k} q_{1j}=\begin{cases}\theta_0, & k=0\\ \theta_0+\theta_1, & k=1\\ 0, & \mathrm{k}=2.\end{cases}$$

The condition of fixed mean assumes that

$$0=q_{21}+2q_{22}-q_{11}-2q_{12}=q_{11}+\theta_1+2\left(1-q_{11}-\theta_1-q_{10}-\theta_0\right)-q_{11}-2\left(1-q_{11}-q_{10}\right)=$$
$$-2\theta_0-\theta_1,$$

which implies that $\theta_1 = -2\theta_0$ and that

(A3) $$F_2(k) - F_1(k) = \sum_{j=0}^{k} q_{2j} - \sum_{j=0}^{k} q_{1j} = \begin{cases} \theta_0, & k = 0 \\ -\theta_0, & k = 1 \\ 0, & \text{k} = 2. \end{cases}$$

and

(A4) $$\sum_{k=i}^{1} F_1(k) - \sum_{k=i}^{1} F_2(k) = \begin{cases} 0, k = 0 \\ \theta_0, k = 1. \end{cases}$$

Since θ_0 is the ratio of two integers we have that $\theta_0 = 2s\delta$, where δ is a small proportion and s an integer.

Next, assume that the two dimensions are positively associated, i.e. $\alpha > 1$, and that F_1 is affected by a mean preserving positive increasing rearrangement. The distance between the resulting distribution F^* and F_1 is given by

(A5) $$F^*(k) - F_1(k) = \begin{cases} \delta, & k = 0 \\ -2\delta, & k = 1 \\ 0, & \text{k} = 2, \end{cases}$$

which means that the distance of the aggregated distributions is given by

(A6) $$\sum_{k=i}^{1} F_1(k) - \sum_{k=i}^{1} F^*(k) = \begin{cases} 0, k = 0 \\ 2\delta, k = 1 \\ 0, k = 2. \end{cases}$$

When $\delta > 0$ it follows from (A6) and (A4) by choosing s=1 and $F_2(k) = F^*(k)$ that (iii) implies (i). To prove the converse statement let $F_1^*, F_2^*, ..., F_s^*$ be a sequence of discrete distribution functions such that $F_1 = F_1^*$, $F_2 = F_s^*$ and F_{i+1}^* differs from F_i^* by a mean preserving positive association increasing rearrangement, i.e. $F_{i+1}^* - F_i^*$ is given by (A4)

Next, we use (A5) to construct F_1^* from F_1, F_2^* from F_1^* and finally F_2 from F_s^*. The required number of iterations (s) depends on the number of steps exhibited by the difference θ_0 .

Next, we will prove the equivalence between (i) and (iv). As was demonstrated above the distance between two distributions F_2 and F_1 with equal mean can be described by equation (A3). Inserting for (A3) in (2.10) for the convex case yields

(A7) $$\Delta_\Gamma(F_2) - \Delta_\Gamma(F_1) = \big(\Gamma(q_{10} + q_{11}) - \Gamma(q_{10} + q_{11} - \theta_0)\big) - \big(\Gamma(q_{10} + \theta_0) - \Gamma(q_{10})\big).$$

It follows from (A7) and the definition of convexity that $\Delta_\Gamma(F_2) - \Delta_\Gamma(F_1) > 0$ for a (non-decreasing) convex function $\Gamma(t)$ if and only if $\theta_0 > 0$, which according to equation (A4) means that F_1 second-degree downward dominates F_2.

What remains to be proved is the equivalence between (ii) and (iv), which follows directly from the decomposition (2.12).

The proof for the concave case has been omitted since it is analogous to the prrof for the convex case.

The proof of Theorem 3.1B is analogous to the proof of Theorem 3.1A. Thus, by using arguments like those in the proof of Theorem 3.1A the results of Theorem 3.1B are obtained.

References

Aaberge, R. (2009): “Ranking Intersecting Lorenz Curves”, *Social Choice and Welfare,* **33**, 235 – 259.

Aaberge, R. and A. Brandolini (2014): “Social Evaluation of Deprivation Count Distributions”, ECINEQ WP 2014 - 342.

Aaberge, R. and A. Brandolini (2015): “Multidimensional Poverty and Inequality”. Chapter 3 in Anthony B. Atkinson and Francois Bourguignon (Eds.): *Handbook of Income Distribution - Volume 2A.* North-Holland, Elsevier, Oxford OX5 1GB, UK.

Alkire, S. and J. Foster (2011). "Counting and Multidimensional Poverty Measurement", *Journal of Public Economics*, **95**, 476-487.

Allison R A, Foster J E. 2004. “Measuring health inequality using qualitative data”, *Journal of Health Economics* 23(3): 505-524.

Anand, S. and A. K. Sen (1997): “Concepts of Human Development and Poverty: A Multidimensional Perspective”. Human Development Papers, United Nations Development Programme (UNDP), New York

Atkinson, A.B. (1970): On the measurement of inequality, *Journal of Economic Theory*, **2**, 244-263.

Atkinson, T., B. Cantillon, E. Marlier and B. Nolan (2002): *Social Indicators: The EU and SocialInclusion*, Oxford University Press, Oxford.

Atkinson, A. B. (2003): “Multidimensional Deprivation: Contrasting Social Welfare and Counting Approaches”, *Journal of Economic Inequality,* **1,** 51-65.

Atkinson, A. B. and F. Bourguignon (1982): “The Comparison of Multidimensional Distribution of Economic Status,” *The Review of Economic Studies,* **49**, 183 – 201.

Boland, P. J. and F. Prochan (1988): “Multivariate Arrangement Increasing Functions with Applications in Probability and Statistics,” *Journal of Multivariate Analysis*, **25**, 286 – 298.

Bossert, W., C. D’Ambrosio and V. Peragine (2007): “Deprivation and Social Exclusion,” *Economica*, **74**, 777-803.

Bossert, W., Chakravarty, S. R. and D’Ambrosio C. (2013): "Multidimensional Poverty and Material Deprivation with Discrete Data," *Review of Income and Wealth*, **59**(1), 29-43.

Bourguignon, F. and S. Chakravarty (2003): “The Measurement of Multidimensional Poverty”, *Journal of Economic Inequality*, **1,** 25-49.

Chakravarty, S. and C. D'Ambrosio (2006). "The Measurement Of Social Exclusion," *Review of Income and Wealth*, **52**, 377-398.

Dardanoni, V. (1995): “On Multidimensional Inequality Measurement.” In C. Dagum and A. Lemmi (eds): *Income Distribution, Social Welfare, Inequality and Poverty*. Research of Economic Inequality, Stamford, CT: JAI Press.

Debreu G. (1964) “Continuity properties of Pareto Utility” *International Economic Review,* 5, 285-293.

Decancq, K. (2012) “Elementary Multivariate Rearrangements and Stochastic Dominance on a Fréchet class”, *Journal of Economic Theory* 147(4), 1450-1459.

Duclos, J.Y., D. E. Sahn and S. D. Younger (2006): “Robust Multidimensional Poverty Comparisons,” *Economic Journal*, **116,** 943-968.

Epstein, L. and S. M. Tanny (1980): “Increasing Generalized Correlation: A Definition and some Economic Consequences, “ *Canadian Journal of Economics*, **13**, 16 – 34.

Fernández-Ponce, J.M., S.C. Kochar and J. Muñoz-Pérez (1998): Partial Orderings of Distributions based on Right Spread Functions, Journal of Applied Probability, 35, 221-228.

Foster, J.E., J. Greer and E. Thorbecke, (1984), “A Class of Decomposable Poverty Measures”, *Econometrica*, **52**, 761-766.

Gravel N., Magdalou, B. and Moyes, P. (2014). "Ranking distributions of an ordinal attribute," Working Papers 14-13, LAMETA, Universtiy of Montpellier.

Guio, A C., Fusco A. and E. Marlier (2009) “A European Union Approach to Material Deprivation using EU-SILC and Eurobarometer data” IRISS Working Paper Series 2009-19, IRISS at CEPS/INSTEAD.

Helmert, F.R. (1876): Die Berechnung des wahrscheinlichen Beobachtungsfehlers aus den ersten Potenzen der Differenzen gleichgenauer directer Beobachtungen. *Astron. Nach.*, **88**, 127-132.

Kakwani, N. and J. Silber (2008): *Quantitative Approaches to Multidimensional Poverty Measurement*. Basingstoke: Palgrave Macmillan.

Lasso de La Vega, C. and A. Urrutia (2011) “Characterizing how to Aggregate the Individuals’ Deprivations in a Multidimensional Framework”, *Journal of Economic Inequality*, **9**, 173-184.

Le Breton, M. and Peluso, E. (2010). “Smooth Inequality Measurement: Approximation Theorems”, Journal of Mathematical Economics, 46(4), 405-415.

Nelsen, Roger B. (1999), *An Introduction to Copulas*, New York: Springer

Shaked, M. and J.G. Shanthikumar (1998): Two Variability Orders, Probability in the Engineering and Informational Sciences, 12, 1-23.

Rothschild, M. and J.E. Stiglitz (1970): “Increasing Risk: A Definition”, *Journal of Economic Theory*, **2**, 225-243.

Sen, A.K., (1976), “Poverty: An Ordinal Approach to Measurement”, *Econometrica*, **44**, 219-231.

Tchen, A.H., (1980): “Inequality for Distributions with Given Marginals”, *Annals of Probability*, **8**, 814-827.

TNS (2007). “Poverty and Social Exclusion”, *Report on the Special Eurobarometer 279/Wave 67.1*.

Tsui, K. Y. (1995): “Multidimensional Generalizations of the Relative and Absolute Inequality Indices: The Atkinson-Kolm-Sen Approach,” *Journal of Economic Theory*, **67,** 251-265.

Tsui, K. Y. (1999): “Multidimensional Inequality and Multidimensional Generalised Entropy Measures: An Axiomatic Approach,” *Social Choice and Welfare*, **16**, 145 – 158.

Tsui, K.Y. (2002). “Multidimensional Poverty Indices”. *Social Choice and Welfare* 19, 69-93.

von Andrae (1872): Ueber die Bestimmung des wahrscheinlichen Fehlers durch die gebedenen Differenzen von *m* gleich genauen Beobachtungen einer Unbekannten. *Astron. Nach.*, **79**, 257-272.

Weymark, J. (1981): “Generalized Gini inequality indices”, *Mathematical Social Sciences*, **1**, 409-430.

Weymark, J. (2006): “The Normative Approach to the Measurement of Multidimensional Inequality,” in F. Farina and E. Savaglio, eds., *Inequality and Economic Integration*, Routledge: London, 303–328.

Yaari, M.E. (1986): “Univariate and multivariate comparisons of risk aversion: a new approach,” *Essays in Honor of Kenneth Arrow* (W. Heller, R. Starr and D. Starrett, Eds.), Cambridge University Press, pp. 173–187.

Yaari, M.E. (1987): “The Dual Theory of Choice under Risk”, *Econometrica*, **55**, 95-115.

Yaari, M.E. (1988). “A controversial proposal concerning inequality measurement”. Journal of Economic Theory **44**, 381-397.

Yule, G. U. (1900): “On the Association of Attributes in Statistics: With Illustrations from the Material of the Childhood Society”, *&c, Philosophical Transactions of the Royal Society of London, Series A*, **194**, 257 – 319.