Multiproduct-Firm Oligopoly: An Aggregative Games Approach[*]

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Abstract

We develop an aggregative games approach to study oligopolistic price competition with multiproduct firms. We introduce a new class of demand systems, derived from discrete/continuous choice, and nesting CES and logit demand systems. The associated pricing game with multiproduct firms is aggregative and a firm's optimal price vector can be summarized by a uni-dimensional sufficient statistic, the ι-markup. We prove existence of equilibrium using a nested fixed-point argument, and provide conditions for equilibrium uniqueness. In equilibrium, firms may choose not to offer some products. We analyze the pricing distortions and provide monotone comparative statics. Under CES and logit demands, another aggregation property obtains: All relevant information for determining a firm's performance and competitive impact is contained in that firm's uni-dimensional type. Finally, we re-visit classic questions in static and dynamic merger analysis, and study the impact of a trade liberalization on the inter- and intra-firm size distributions, productivity and welfare.

1   Introduction

Analyzing the behavior of multiproduct firms in oligopolistic markets appears to be of a first-order importance. Multiproduct firms are endemic and play an important role in the economy. Even when defining products quite broadly at the NAICS 5-digit level, multiproduct firms account for 91% of total output and 41% of the total number of firms (Bernard, Redding,

and Schott, 2010). Similarly, many markets are characterized by oligopolistic competition. Even at the 5-digit industry level, concentration ratios are fairly high: For instance, in U.S. manufacturing, the average NAICS 5-digit industry has a four-firm concentration ratio of 35% (Source: Census of U.S. Manufacturing, 2002).[1] The ubiquitousness of multiproduct firms and oligopoly is reflected in the modern empirical IO literature, where oligopolistic price competition with multiproduct firms abounds (e.g., Berry, 1994; Berry, Levinsohn, and Pakes, 1995; Nevo, 2001).

In contrast to single-product firms, a multiproduct firm must choose not only how aggressive it wants to be in the market place but also how to vary its markups across products within its portfolio. In contrast to monopolistically competitive firms, an oligopolistic multiproduct firm must take self-cannibalization into account, both when setting its markups and when deciding which products to offer. Multiproduct-firm oligopoly therefore gives rise to a number of important questions: What determines the within-firm markup structure, between-firm markup differences, and the industry-wide markup level? What explains firms' scope in oligopoly? Along which dimensions are markups and product offerings distorted by oligopolistic behavior? Due to the technical difficulties discussed below, these questions have been under-researched in the existing literature. In this paper, we develop an aggregative games approach to circumvent the technical difficulties and address these and related questions. We make several contributions.

We introduce a new class of (integrable) quasi-linear demand systems, derived from a discrete/continuous choice model of consumer demand, where each consumer first chooses which product to purchase, and then how much of that product to consume. This class nests standard constant elasticity of substitution (CES) and multinomial logit (MNL) demand systems as special cases. As demand satisfies the Independence of Irrelevant Alternatives (IIA) axiom, consumer surplus depends only on an aggregator that is additively separable in prices.

We use this class of demand systems to analyze oligopolistic price competition between multiproduct firms with arbitrary firm and product heterogeneity. The associated pricing game has two important properties. First, it is aggregative, and the aggregator is the same as the one for consumer surplus. Second, a firm's optimal price vector is such that, for every product in that firm's portfolio, the Lerner index multiplied by a product-level elasticity measure is equal to a firm-level sufficient statistic, called the ι-markup. That ι-markup pins down the price level of the firm.[2] We can therefore think of the firm's maximization problem as one of choosing the right ι-markup.

These two properties allow us to prove existence of a pricing equilibrium under weak con-

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[1]The prevalence of multiproduct-firms and oligopolistic markets would be larger if one were to define products and industries more narrowly, as is usually done in industrial organization. The focus of this paper will be on such narrowly defined industries in which there are substantial demand-side linkages between all products.

[2]With CES (resp. MNL) demands, this implies that the firm charges the same relative (resp. absolute) markup over all its products. Our class of demand systems allows us to generate heterogeneous relative and absolute markups within a firm's set of products.

ditions using a nested fixed point argument. This approach circumvents problems that arise when attempting to apply off-the-shelf equilibrium existence theorems, such as the failure of quasi-concavity, (log-)supermodularity and upper semi-continuity of the profit functions. It also gives rise to an efficient algorithm for computing equilibrium, and allows us to derive sufficient conditions for equilibrium uniqueness.

The aggregative structure of the game and the constant ι-markup property allow us to decompose the welfare distortions from oligopolistic competition between multiproduct firms as follows. First, since firms are setting positive markups, the industry delivers too little consumer surplus. Second, consumer surplus is not delivered efficiently: Although a social planner would like to set the same ι-markup on all the products in the industry, equilibrium ι-markups typically differ across firms. This means that some firms are inefficiently large, while others are inefficiently small. Perhaps surprisingly, there are no within-firm pricing distortions, in the sense that a firm's equilibrium prices maximize social welfare subject to the constraint that that firm's contribution to consumer surplus is held fixed at its equilibrium value.

We also study the determinants of firm scope. Despite the absence of product-specific fixed costs, firms do not necessarily sell all their products in equilibrium. The intuition is that a multiproduct oligopolist has to worry about self-cannibalization effects when setting its prices and when deciding which products to supply. It therefore has incentives to withdraw some of its weaker products so as to channel consumers toward its more profitable products.

Despite our game not being supermodular, we are able to rank equilibria from the consumers' and firms' viewpoints, and to perform comparative statics on the set of equilibria. We explore the impact of entry, trade liberalization, and productivity and quality shocks on industry conduct and performance. Among other results, we find that a shock that makes the industry more competitive (such as a trade liberalization or the entry of new competitors) induces firms to broaden their scope in equilibrium. This is in stark contrast with existing results in the international trade literature on multiproduct firms, which we further discuss in Section 1.1. Intuitively, as the industry becomes more competitive, a firm starts worrying more about consumers purchasing its rivals' products, and less about cannibalizing its own sales. That firm therefore has incentives to introduce "fighting brands" (Johnson and Myatt, 2003) to protect its market share.

In the last part of the paper, we specialize the model to the cases of CES and MNL demands. We show that an additional aggregation property, called type aggregation, obtains: All relevant information for determining a firm's performance and competitive impact is contained in that firm's uni-dimensional type. This property allows us to obtain additional predictions (e.g., about the impact of productivity and quality shocks on social welfare) that are unavailable in the general case. We provide two sets of applications of our framework with CES and MNL demands. First, we revisit classical questions in static and dynamic merger analysis. These issues have been addressed in the literature using the single-product homogeneous-goods Cournot oligopoly model, which is a well-known example of an aggregative game (see Farrell and Shapiro, 1990; Nocke and Whinston, 2010). The aggregative

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structure of our game and the type aggregation property allow us to generalize the insights of that earlier literature to settings with multiproduct firms, price competition and horizontally differentiated products. Second, we study the impact of a unilateral trade liberalization on the inter- and intra-firm size distributions, average industry-level productivity, and welfare.

Road map. In the remainder of this section, we provide a short review of the related literature. In Section 2, we analyze discrete/continuous consumer choice and derive a new class of demand systems. In Section 3, we describe the multiproduct-firm pricing game. This is followed, in Section 4, by the equilibrium analysis. We prove existence of equilibrium under mild conditions and uniqueness of equilibrium under stronger conditions. We characterize the equilibrium pricing structure as well as firms' scope, provide a welfare analysis, and perform monotone comparative statics. In Section 5, we specialize to the cases of CES and MNL demands and show that the type aggregation property obtains. In Section 6, we study applications to static and dynamic merger analysis and to the impact of international trade. Finally, we conclude in Section 7.

1.1 Related Literature

Our paper contributes to the relatively small literature on multiproduct-firm oligopoly pricing with horizontally differentiated products. One strand of that literature focuses on proving equilibrium existence and uniqueness in multiproduct-firm oligopoly pricing games with firm and product heterogeneity and demand systems derived from discrete/continuous choice. Importantly, Caplin and Nalebuff (1991)'s powerful existence theorem for pricing games with single-product firms, the proof of which relies on establishing quasi-concavity of a firm's profit function in own price, does not extend to the case of multiproduct firms. The reason is that, even with standard MNL demand, a multiproduct firm's profit function often fails to be quasi-concave (Spady, 1984; Hanson and Martin, 1996). For this reason, the literature has focused on special cases of discrete/continuous choice demand systems, such as MNL demand (Spady, 1984; Konovalov and Sándor, 2010), CES demand (Konovalov and Sándor, 2010), and nested logit demand where each firm owns a nest of products (Gallego and Wang, 2014). Our aggregative games techniques provide a unified approach to address existence and uniqueness issues in pricing games with discrete/continuous choice demand systems.

   More applied papers make stronger functional forms and/or symmetry assumptions, and study firms' product range decisions in oligopoly. Anderson and de Palma (1992, 2006) analyze the equilibrium (resp. welfare-maximizing) number of firms and number of products per firm in a model with discrete/continuous choice demands, symmetric products and firms, and free entry. Shaked and Sutton (1990) and Dobson and Waterson (1996) develop linear-demand models to analyze product-range decisions. The tools developed in the present paper allow us to do away with the symmetry and linearity assumptions made in these earlier

papers.[3]

More recently, multiproduct firms have received much attention from international trade researchers. A central question in that literature is how multiproduct firms react to trade liberalization by adjusting their product range and product mix. This question is usually addressed in models of monopolistic competition, under either CES demand (Bernard, Redding, and Schott, 2010, 2011; Nocke and Yeaple, 2014) or linear demand (Dhingra, 2013; Mayer, Melitz, and Ottaviano, 2014). An exception is Eckel and Neary (2010) who study (identical) multiproduct firms in a Cournot model with linear demand.[4] A common finding in these papers is that firms react to a trade liberalization by refocusing on their core competencies, i.e., by shrinking their product ranges.[5] In models with CES demand and product-specific fixed costs, this is due to the fact that more intense competition reduces variable profits on all products, and therefore makes it harder to cover fixed costs. In models with linear demand, more intense competition chokes out the demand for products sold at a high price. By contrast, we find that self-cannibalization effects matter relatively less when competition is more intense, which implies that firms respond to trade liberalization by broadening their product ranges.

The concept of aggregative games was introduced by Selten (1970). In such games, a player's payoff depends only on that player's actions and on an aggregate common to all players. In games with additive aggregation, each player has a fitting-in correspondence, and the set of pure-strategy Nash equilibria corresponds to the set of fixed points of the aggregate fitting-in correspondence.[6] McManus (1962, 1964) and Selten (1970) use the aggregate fitting-in correspondence to establish existence of a Nash equilibrium in a homogeneous-goods Cournot model. Szidarovszky and Yakowitz (1977), Novshek (1985) and Kukushkin (1994) refine this approach further.[7] Our proof of equilibrium existence and our characterization of the set of equilibria also rely on the aggregate fitting-in correspondence. Corchon (1994) and Acemoglu and Jensen (2013) show that aggregative games also deliver powerful monotone comparative statics results, in the spirit of Milgrom and Roberts (1994) and Milgrom and Shannon (1994). We perform such monotone comparative statics in Section 4.3. In a recent paper, Anderson, Erkal, and Piccinin (2013) adopt an aggregative games approach to study pricing games similar to ours, but restrict attention to single-product firms. They are mainly interested in long-run comparative statics with free entry and exit.

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[3]Another strand of literature studies price and quantity competition between multiproduct firms selling vertically differentiated products. See, among others, Champsaur and Rochet (1989), and Johnson and Myatt (2003, 2006).

[4]Eckel, Iacovone, Javorcik, and Neary (2015) extend their framework to heterogeneous firms.

[5]In Nocke and Yeaple (2014), the prediction depends on whether the firm sells only domestically or not.

[6]See Section 4.1 for more details. The term "fitting-in correspondence" was coined by Selten (1973). This concept appears under different names in the literature.

[7]The aggregate fitting-in correspondence is not well-defined when aggregation is not additive (Cornes and Hartley, 2012). Dubey, Haimanko, and Zapechelnyuk (2006) and Jensen (2010) show that games with non-additive aggregation and monotone best replies have a pseudo potential or a best-reply potential (depending on the notion of monotonicity employed). This allows them to establish equilibrium existence. In Section 4.4, we show that our multiproduct-firm pricing game has an ordinal potential (Monderer and Shapley, 1996).

The fact that a firm optimally sets the same absolute markup (possibly adjusted by a price-sensitivity parameter) over all its products when demand is of the MNL type, and the same relative markup over all its products when demand is of the CES type, was already pointed out by Anderson, de Palma, and Thisse (1992), Konovalov and Sándor (2010), and Gallego and Wang (2014). The common ι-markup property, discussed above, generalizes these findings to the whole class of demand systems that can be derived from discrete/continuous choice, and allows us to simplify the firm's pricing problem considerably. Similar in spirit, Armstrong and Vickers (2016) reduce the dimensionality of a multiproduct monopolist's quantity-setting problem by confining attention to demand systems that have the property that consumer surplus is homothetic in quantities. They show that the multiproduct monopolist optimally scales down the welfare-maximizing vector of quantities by a common multiplicative factor.[8]

2 Discrete/Continuous Consumer Choice

We consider a demand model in which consumers make discrete/continuous choices: Each consumer first decides which product to purchase, and then, how much of this product to consume. This approach captures Novshek and Sonnenschein (1979)'s idea that price-induced demand changes can be decomposed into two effects: An intensive margin effect (consumers purchase less of the product whose price was raised), and an extensive margin effect (some consumers stop purchasing the commodity whose price increased).[9,10] Discrete/continuous choice models of consumer demand have been used by empirical researchers to estimate demand for electric appliances (Dubin and McFadden, 1984), soft drinks (Chan, 2006), and painkillers (Björnerstedt and Verboven, 2016). In Smith (2004), consumers first choose a supermarket, and then how much to spend at that supermarket based on the price index at that store. In addition to giving rise to tractable multiproduct-firm pricing games, the discrete/continuous approach will also turn out to be useful to interpret some of the comparative statics results derived in Section 4.

We formalize discrete/continuous choice as follows. There is a population of consumers with quasi-linear preferences. Each consumer chooses a single product from a finite and non-empty set of products $\mathcal{N}$ and the quantity of that product; he spends the rest of his income on the outside good (or Hicksian composite commodity), the price of which is normalized to one. Conditional on selecting product i, the consumer receives indirect utility $y + v_i(p_i) + \varepsilon_i$, where p_i is the price of product i, y is the consumer's income, and ε_i is a taste shock. By Roy's identity, the consumer purchases $-v_i'(p_i)$ units of good i. We call $-v_i'(p_i)$ the conditional

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[8]Their class of demand systems does not nest ours (e.g., it does no include CES-like demand with heterogeneous price elasticity parameters) nor does ours nest theirs (e.g., ours does not include linear demand). Armstrong and Vickers (2016) also extend Bergstrom and Varian (1985) to establish equilibrium existence in a Cournot oligopoly model with identical multiproduct firms.

[9]Income effects are absent in our quasi-linear world.

[10]See also Hanemann (1984).

demand for product i. At the product-choice stage, the consumer selects product i only if

$$\forall j \in \mathcal{N}, \quad y + v_k(p_k) + \varepsilon_k \geq y + v_j(p_j) + \varepsilon_j.$$

We assume that the components of vector $(\varepsilon_j)_{j \in \mathcal{N}}$ are identically and independently drawn from a type-1 extreme value distribution. By Holman and Marley's theorem, product i is therefore chosen with probability

$$\mathbb{P}_i(p) = \Pr\left(v_i(p_i) + \varepsilon_i = \max_{j \in \mathcal{N}}\left(v_j(p_j) + \varepsilon_j\right)\right) = \frac{\mathrm{e}^{v_i(p_i)}}{\sum_{j \in \mathcal{N}} \mathrm{e}^{v_j(p_j)}} = \frac{h_i(p_i)}{\sum_{j \in \mathcal{N}} h_j(p_j)},$$

where $h_j \equiv \mathrm{e}^{v_j}$ for every j. It follows that expected demand for product k is given by

$$\mathbb{P}_k(p) q_k(p_k) = \frac{\mathrm{e}^{v_k(p_k)}}{\sum_{j \in \mathcal{N}} \mathrm{e}^{v_j(p_j)}} \left(-v_k'(p_k)\right) = \frac{-h_k'(p_k)}{\sum_{j \in \mathcal{N}} h_j(p_j)}.$$

In the following, we use collection of functions $(h_j)_{j \in \mathcal{N}}$ (rather than $(v_j)_{j \in \mathcal{N}}$) as primitives. We assume that all the h functions are $\mathcal{C}^3$ from $\mathbb{R}_{++}$ to $\mathbb{R}_{++}$, strictly decreasing, and log-convex. The assumption that h_j is non-increasing and log-convex is necessary and sufficient for v_j to be an indirect subutility function (Nocke and Schutz, 2016b). The assumption that h_j is strictly decreasing means that the demand for product j never vanishes.

To sum up, the demand system generated by the discrete/continuous choice model $(h_j)_{j \in \mathcal{N}}$ (when normalizing market size to one) is:

$$D_k\left((p_j)_{j \in \mathcal{N}}\right) = \frac{-h_k'(p_k)}{\sum_{j \in \mathcal{N}} h_j(p_j)}, \quad \forall k \in \mathcal{N}, \quad \forall (p_j)_{j \in \mathcal{N}} \in \mathbb{R}_{++}^{\mathcal{N}}. \tag{1}$$

The conditional demand for good k is $-d \log h_k/dp_k = -h_k'/h_k$. Product k is chosen with probability $h_k/\sum_j h_j$.

Our class of demand systems nests standard MNL (if $h_j(p_j) = \mathrm{e}^{\frac{a_j - p_j}{\lambda}}$ for all $j \in \mathcal{N}$, where $a_j \in \mathbb{R}$ and $\lambda > 0$ are parameters) and CES demands (if $h_j(p_j) = a_j p_j^{1-\sigma}$, where $a_j > 0$ and $\sigma > 1$ are parameters) as special cases. The fact that CES demands can be derived from discrete/continuous choice was already pointed out by Anderson, de Palma, and Thisse (1987) in a slightly different framework without an outside good. As seen in equation (1), the class of demand systems that are derivable from discrete/continuous choice is much wider than CES and MNL demands. Notice also that, by virtue of the i.i.d. type-1 extreme value distribution assumption, all these demand systems have the IIA property. That property will allow us to greatly simplify the multiproduct-firm pricing problem in Section 4.

The consumer's expected utility can be computed using standard formulas (see, e.g.,

Anderson, de Palma, and Thisse, 1992):

$$\mathbb{E}\left(y + \max_{j \in \mathcal{N}} v_j(p_j)\right) = y + \log\left(\sum_{j \in \mathcal{N}} \mathrm{e}^{v_j(p_j)}\right) = y + \log\left(\sum_{j \in \mathcal{N}} h_j(p_j)\right). \tag{2}$$

Therefore, consumer surplus is aggregative, in that it only depends on the value of the aggregator $H \equiv \sum_{j \in \mathcal{N}} h_j(p_j)$.

Representative consumer approach. While much of the empirical industrial organization literature has adopted discrete choice models as a way of deriving consumer demand, other strands of literature, such as the international trade literature, mainly use a representative consumer approach.[11] We show that demand system (1) is quasi-linearly integrable, i.e., it can be obtained from the maximization of the utility function of a representative consumer with quasi-linear preferences:[12]

Proposition 1. *Let D be the demand system generated by discrete/continuous choice model $(h_j)_{j \in \mathcal{N}}$. D is quasi-linearly integrable. Moreover, v is an indirect subutility function for D if and only if there exists a constant $\alpha \in \mathbb{R}$ such that $v((p_j)_{j \in \mathcal{N}}) = \alpha + \log \sum_{j \in \mathcal{N}} h_j(p_j)$.*

Proof. See Online Appendix I. $\square$

Hence, any demand system that can be derived from discrete/continuous choice can also be derived from quasi-linear utility maximization. The second part of the proposition says that the expected utility of a consumer engaging in discrete/continuous choice and the indirect utility of the associated representative consumer coincide (up to an additive constant,). The results we will derive on consumer welfare therefore do not depend on the way the demand system has been generated. Whether we use discrete/continuous choice or a representative consumer approach, all that matters is the value of aggregator H.

Consumer heterogeneity. While the discrete/continuous consumer choice model allows for some type of consumer heterogeneity (different consumers receive different taste shocks and may therefore select different products), it does have the property that all consumers who select the same product choose to purchase the same quantity. However, the model can easily be adapted to accommodate consumer heterogeneity in the quantity purchased of the same

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[11]See however Fajgelbaum, Grossman, and Helpman (2011), Handbury (2013) and Fally and Faber (2016) for recent examples of international trade papers deriving demand from (special cases of) discrete/continuous choice.

[12]Quasi-linear integrability and indirect subutility functions are defined in Nocke and Schutz (2016b), Definitions 3 and 4. In Online Appendix I, we prove a more general result: We derive necessary and sufficient conditions for demand system

$$D_k\left((p_j)_{j \in \mathcal{N}}\right) = \frac{g_k(p_k)}{\sum_{j \in \mathcal{N}} h_j(p_j)}, \quad \forall k \in \mathcal{N}, \quad \forall (p_j)_{j \in \mathcal{N}} \in \mathbb{R}_{++}^{\mathcal{N}}$$

to be quasi-linearly integrable.

product. In particular, suppose that the indirect subutility derived from choosing product j is $v_j(p_j, t_j)$, where $t_j \in \mathbb{R}$ is the consumer's "type" for product j, drawn from probability distribution $G_j(\cdot)$. The realized value of t_j is observed by the consumer only *after* he has chosen product j. Let $v_j(p_j) = \int v_j(p_j, t_j) dG_j(t_j)$ be the expected indirect utility derived from product j. Then, product i is chosen with probability $\exp v_i(p_i)/(\sum_j \exp v_j(p_j))$. Under some technical conditions (which allow us to differentiate under the integral sign), the consumer's expected conditional demand for product j is:

$$\int -\frac{\partial}{\partial p_j} v_j(p_j, t_j) dG_j(t_j) = -\frac{\partial}{\partial p_j} \int v_j(p_j, t_j) dG_j(t_j) = -v_j'(p_j).$$

Therefore, if we define $h_j(p_j) = \exp(v_j(p_j))$ for every j, then the expected (unconditional) demand for product i is still given by equation (1). Differentiating once more under the integral sign, we also see that $v_j(\cdot)$ is decreasing and convex if $v_j(\cdot, t_j)$ is decreasing and convex for every t_j. Therefore, discrete/continuous choice with consumer heterogeneity gives rise to the same class of demand systems as discrete/continuous choice without heterogeneity.[13]

3 The Multiproduct-Firm Pricing Game

In this section, we describe the multiproduct-firm pricing game. The pricing game consists of three elements: $(h_j)_{j \in \mathcal{N}}$ is the demand system defined in the previous section; $\mathcal{F}$, the set of firms, is a partition of $\mathcal{N}$ such that $|\mathcal{F}| \geq 2$;[14] $(c_j)_{j \in \mathcal{N}} \in \mathbb{R}_{++}^{\mathcal{N}}$ is a profile of marginal costs. (In the following, we adopt the discrete/continuous choice interpretation of demand system $(h_j)_{j \in \mathcal{N}}$ but the reader should keep in mind that $(h_j)_{j \in \mathcal{N}}$ may equally well have been generated by utility maximization of a representative consumer.) The profit of firm $f \in \mathcal{F}$ is defined as follows:[15]

$$\Pi^f(p) = \sum_{\substack{k \in f \\ p_k < \infty}} (p_k - c_k) \frac{-h_k'(p_k)}{\displaystyle\sum_{\substack{j \in \mathcal{N} \\ p_j < \infty}} h_j(p_j) + \sum_{\substack{j \in \mathcal{N} \\ p_j = \infty}} \lim_{\infty} h_j}, \quad \forall p \in (0, \infty]^{\mathcal{N}}. \tag{3}$$

Note that we are allowing firms to set infinite prices, which essentially compactifies firms' action sets. We will later see that this compactification ensures that each firm's maximization problem has a solution (provided that rival firms are not pricing all their products at infinity). The assumption we are making here is that, if $p_k = \infty$, then the firm simply does not supply product k, and therefore does not earn any profit on this product. In the following, we write

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[13] If the consumer observes his vector of types *before* choosing a variety, then the implied demand system becomes a mixture of equation (1). We are not able to handle such mixtures of demand systems, because they no longer give rise to an aggregative game. This implies in particular that our approach cannot accommodate random coefficient logit demand systems.

[14] That is, each product in $\mathcal{N}$ is offered by exactly one firm.

[15] Throughout the paper, we adopt the convention that the sum of an empty collection of real numbers is equal to zero. Note that, since h_j is monotone, $\lim_{\infty} h_j$ exists for every j.

$h_j(\infty)$ instead of $\lim_\infty h_j$. We say that product j is active if $p_j < \infty$. We discuss infinite prices in greater detail at the end of this section.

We study the normal-form game in which firms set their prices simultaneously, and payoff functions are given by equation (3). A pure-strategy Nash equilibrium of that normal-form game is called a pricing equilibrium.

Outside option. Since $\bigcup_{f \in \mathcal{F}} f = \mathcal{N}$, our definition of a pricing game does not seem to allow for an exogenously priced outside option.[16] However, such an outside option is easy to incorporate. Let $(h_j)_{j \in \mathcal{N}}$ be a discrete/continuous choice model of consumer demand. Partition $\mathcal{N}$ into two sets: $\widetilde{\mathcal{N}}$, the set of products sold by oligopoly players, and $\mathcal{N}^0$, the set of products sold at exogenous prices $(p_j)_{j \in \mathcal{N}^0}$. Let $H^0 = \sum_{j \in \mathcal{N}^0} h_j(p_j) > 0$ be the value of the outside option. We can now define another discrete/continuous choice model $(\widetilde{h}_j)_{j \in \widetilde{\mathcal{N}}}$ as follows: For every $j \in \widetilde{\mathcal{N}}$, $\widetilde{h}_j = h_j + \frac{H^0}{|\widetilde{\mathcal{N}}|}$.[17] Note that this transformation affects neither consumer surplus nor expected demand. Therefore, the price competition game with discrete/continuous choice model $(h_j)_{j \in \mathcal{N}}$ and exogenous prices $(p_j)_{j \in \mathcal{N}^0}$ is equivalent to the price competition game with discrete/continuous choice model $(\widetilde{h}_j)_{j \in \widetilde{\mathcal{N}}}$ and no outside option.

More on infinite prices. We first argue that the idea that product k is simply not supplied when $p_k = \infty$ is consistent with the discrete/continuous choice interpretation of the demand system. In the discrete/continuous choice model, a consumer receives a type-1 extreme value draw ε_k for product k even when $p_k = \infty$. Three cases can arise when the price is infinite: (i) The conditional demand is positive ($\lim_\infty -h'_k/h_k > 0$), in which case the choice probability must be equal to zero ($\lim_\infty h_k = 0$). (ii) The choice probability is positive ($\lim_\infty h_k > 0$), in which case the conditional demand must be equal to zero ($\lim_\infty -h'_k/h_k = 0$). (iii) Both the conditional demand and the choice probability are equal to zero.[18] The term $\sum_{j:p_j=\infty} \lim_\infty h_j$ appears in the denominator of Π^f to allow for case (ii). In all three cases, the consumer does not consume a positive quantity of the good when the price is infinite, which is consistent with the interpretation that the product is simply not available.

An alternative way of allowing for infinite prices would be to define the profit function for finite prices first, and then extend it by continuity to price vectors that have infinite components. In the proof of Lemma B in the Appendix, we show that, if price vector $p \in (0, \infty]^\mathcal{N}$ has a least one finite component, then $\lim_p \Pi^f$ coincides with the value of $\Pi^f(p)$ defined in equation (3). There is, however, an important exception. If $p_j = \infty$ for every j,

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[16]An exogenously priced outside option should not be confused with the outside good. For example, in the market for automobiles, public transport may represent the outside option whereas the outside good represents the expenditure on other goods.

[17]It is straightforward to check that $\widetilde{h}_j$ is strictly decreasing and log-convex for every j.

[18]To see this, suppose that $\lim_\infty -h'/h = l > 0$ (the limit exists, since h is log-convex), where we have dropped the product subscript to ease notation. There exists $x_0 > 0$ such that $-h'(x)/h(x) > l/2$ for all $x \geq x_0$. Integrating this inequality, we see that $-\log\left(\frac{h(x)}{h(x_0)}\right) > \frac{l}{2}(x - x_0)$ for all $x > x_0$. Taking exponentials on both side, and letting x go to infinity, we obtain that $\lim_\infty h = 0$. Conversely, $\lim_\infty h > 0$ implies that $\lim_\infty -h'/h = 0$.

then $\lim_p \Pi^f$ does not necessarily exist. For instance, with CES or MNL demands, firms' profits do not have a limit when all prices go to infinity.

4 Equilibrium Analysis

In this section, we provide an equilibrium analysis of the multiproduct-firm pricing game. In the first part, we adopt an aggregative games approach to prove existence of equilibrium. In the second part, we investigate how firm behavior is affected by changes in the aggregator. In the third part, we study the equilibrium properties, both from a positive and normative point of view, and comparative statics. In the fourth part, we discuss extensions and provide a cookbook for applied work. Finally, we provide conditions under which the equilibrium is unique.

4.1 An Aggregative Games Approach to Equilibrium Existence

There are three main difficulties associated with the equilibrium existence problem. First, Π^f is not necessarily quasi-concave in $(p_j)_{j \in f}$.[19] Second, Π^f is not necessarily upper semi-continuous in $(p_j)_{j \in f}$.[20,21] Third, if f is a multiproduct firm, then Π^f is neither supermodular nor log-supermodular in $(p_j)_{j \in f}$.[22] The first two difficulties imply that standard existence theorems for compact games (such as Nash or Glicksberg's theorems) based on Kakutani's fixed-point theorem cannot be applied. The last two difficulties imply that existence theorems based on supermodularity theory and Tarski's fixed-point theorem (see Milgrom and Roberts, 1990; Vives, 1990, 2000; Topkis, 1998) have no bite. The second (and, to some extent, the third) difficulty prevents us from using Jensen (2010)'s existence theorem for aggregative games with monotone best replies.

The idea behind our existence proof is to reduce the dimensionality of the problem in two ways. First, we show that a firm's optimal price vector can be fully summarized by a uni-dimensional sufficient statistic, which is pinned down by a single equation in one unknown. Second, the pricing game is aggregative (see Selten, 1970), in that the profit of a firm depends only on its own profile of prices and the uni-dimensional sufficient statistic $H = \sum_{j \in \mathcal{N}} h_j(p_j)$.

In the following, we present a semi-formal sketch of our existence proof. We refer the reader to Appendix A for details. In this sketch, we introduce the key concepts of ι-markup, pricing function, fitting-in function and aggregate fitting-in function, which will prove useful

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[19]Spady (1984) and Hanson and Martin (1996) provide examples of multiproduct-firm pricing games with MNL demand in which quasi-concavity fails.

[20]To see this, suppose that demand is CES, and that firm $f = \{k\}$ is a single-product firm. If firm f's rivals are setting infinite prices for all their products, then $\Pi^f = (\sigma - 1)(p_k - c_k)/p_k$ for every $p_k > 0$. It follows that Π^f goes to $\sigma - 1$ as p_k goes to infinity. This is strictly greater than 0, which is the profit firm f receives when it sets $p_k = \infty$. Therefore, Π^f is not upper semi-continuous in p_k.

[21]If profit functions had been defined over $(0, \infty)^{\mathcal{N}}$ instead of $(0, \infty]^{\mathcal{N}}$, then payoff functions would be upper semi-continuous, but the lack of compactness would become an issue.

[22]See Online Appendix II.2.

to describe the equilibria of our pricing game, and to understand our comparative statics results.

Fix a pricing game $((h_j)_{j \in \mathcal{N}}, \mathcal{F}, (c_j)_{j \in \mathcal{N}})$. For the sake of expositional simplicity, suppose that first-order conditions are sufficient for optimality. Ignoring the possibility of infinite prices for the time being, the first-order conditions for each firm's profit maximization problem hold at price vector $p \in \mathbb{R}_{++}^{\mathcal{N}}$ if and only if for every $f \in \mathcal{F}$ and $k \in f$,

$$\frac{\partial \pi^f}{\partial p_k} = \frac{-h'_k(p_k)}{H} \left(1 - \frac{p_k - c_k}{p_k} p_k \frac{-h''_k(p_k)}{-h'_k(p_k)} + \sum_{j \in f} (p_j - c_j) \frac{-h'_j(p_j)}{H} \right) = 0, \tag{4}$$

where $H = \sum_{j \in \mathcal{N}} h_j(p_j)$ is the aggregator. Let $\iota_k(p_k) = p_k \frac{-h''_k(p_k)}{-h'_k(p_k)}$. Then, the profile of prices p is a pricing equilibrium if and only if

$$\frac{p_k - c_k}{p_k} \iota_k(p_k) = 1 + \sum_{j \in f} (p_j - c_j) \frac{-h'_j(p_j)}{H}, \ \forall f \in \mathcal{F}, \ \forall k \in f, \tag{5}$$

$$\text{and } H = \sum_{j \in \mathcal{N}} h_j(p_j).$$

We learn two facts from equation (5). First, for a given $f \in \mathcal{F}$, the right-hand side of equation (5) is independent of the identity of $k \in f$. It follows that, in any Nash equilibrium, for any $f \in \mathcal{F}$, and for all $k, l \in f$,

$$\frac{p_k - c_k}{p_k} \iota_k(p_k) = \frac{p_l - c_l}{p_l} \iota_l(p_l).$$

Put differently, there exists a scalar μ^f, which we call firm f's ι-markup, such that $\frac{p_k - c_k}{p_k} \iota_k(p_k) = \mu^f$ for every $k \in f$. We say that firm f's profile of prices, $(p_k)_{k \in f}$ satisfies the common ι-markup property. Second, we see from equation (5) that $\mu^f = 1 + \Pi^f(p)$. Put differently, firm f's equilibrium profit is equal to the value of its ι-markup minus one.[23]

The constant ι-markup property can be interpreted as follows. Consider a hypothetical single-product firm selling product k. Suppose that this firm behaves in a monopolistically competitive way, in the sense that it does not internalize the impact of its price on aggregator H. Firm k therefore faces demand $-h'_k(p_k)/H$ and, since it takes H as given, believes that the price elasticity of demand for its product is equal to the elasticity of $-h'_k(p_k)$, which is precisely $\iota_k(p_k)$. Therefore, firm k prices according to the inverse elasticity rule: $\frac{p_k - c_k}{p_k} = \frac{1}{\iota_k(p_k)}$. In our model, firm f internalizes its impact on the aggregator level as well as self-cannibalization effects. It therefore prices in a less aggressive way, according to modified inverse elasticity rule $\frac{p_k - c_k}{p_k} = \frac{\mu^f}{\iota_k(p_k)}$, with $\mu^f > 1$. Put differently, firm f sets the same price that a firm would set under monopolistic competition, if that firm believed that the price

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[23]Recall that market size has been normalized to unity. Without this normalization, firm f's equilibrium profit is $\mu^f - 1$ times market size.

elasticity of demand is equal to $\iota_k(p_k)/\mu^f$, instead of $\iota_k(p_k)$. What is remarkable is that the ι-markup μ^f, which summarizes the impact of firm f's behavior on H, is firm-specific, rather than product-specific.

Next, we use ι-markups to reduce the dimensionality of firms' profit maximization problems. Note first that equation (5) can be rewritten as follows: For every $f \in \mathcal{F}$,

$$\mu^f = 1 + \frac{1}{H}\sum_{j \in f} \underbrace{\frac{p_j - c_j}{p_j} \underbrace{p_j \frac{h_j''(p_j)}{-h_j'(p_j)}}_{=\iota_j(p_j)} \underbrace{\frac{(h_j'(p_j))^2}{h_j''(p_j)}}_{\equiv\gamma_j(p_j)}}_{=\mu^f}.$$

Defining $\gamma_j \equiv (h_j')^2/h_j''$, and rearranging, we obtain:

$$\mu^f \left(1 - \frac{1}{H}\sum_{j \in f}\gamma_j(p_j)\right) = 1. \tag{6}$$

Suppose that function $p_k \mapsto \frac{p_k - c_k}{p_k}\iota_k(p_k)$ is one-to-one for every $k \in \mathcal{N}$, and denote its inverse function by $r_k(\cdot)$. We call r_k the *pricing function* for product k. Then, using equation (6), firm f's pricing strategy can be fully described by a uni-dimensional variable, μ^f, such that

$$\mu^f \left(1 - \frac{1}{H}\sum_{j \in f}\gamma_j(r_j(\mu^f))\right) = 1. \tag{7}$$

Suppose that equation (7) has a unique solution in μ^f, denoted $m^f(H)$. We call m^f firm f's *fitting-in function*. Then, the equilibrium existence problem boils down to finding an H such that

$$H = \underbrace{\sum_{f \in \mathcal{F}}\sum_{j \in f} h_j\left(r_j\left(m^f(H)\right)\right)}_{\equiv\Gamma(H)}.$$

In the parlance of aggregative games, Γ is the *aggregate fitting-in function*. The equilibrium existence problem reduces to finding a fixed point of that function. As we will see later on, the aggregative games approach is also useful to establish equilibrium uniqueness: The pricing game has a unique equilibrium if the following index condition is satisfied: $\Gamma'(H) < 1$ whenever $\Gamma(H) = H$.

This informal exposition leaves a number of questions open. Are first-order conditions sufficient for optimality? Can infinite prices be accommodated? Is function $p_k \mapsto \frac{p_k - c_k}{p_k}\iota_k(p_k)$ one-to-one for every k? Are fitting-in functions well-defined? Does the aggregate fitting-in function have a fixed point? We need one assumption to answer all these questions in the affirmative.

Assumption 1. *For every $j \in \mathcal{N}$ and $p_j > 0$, $\iota_j'(p_j) \geq 0$ whenever $\iota_j(p_j) > 1$.*

Theorem 1. *Suppose that the demand system $(h_j)_{j \in \mathcal{N}}$ satisfies Assumption 1. Then, the pricing game $((h_j)_{j \in \mathcal{N}}, \mathcal{F}, (c_j)_{j \in \mathcal{N}})$ has a pricing equilibrium for every $\mathcal{F}$ and $(c_j)_{j \in \mathcal{N}}$.*

The set of equilibrium aggregator levels coincides with the set of fixed points of the aggregate fitting-in function Γ. If H^* is an equilibrium aggregator level, then, in the associated equilibrium, consumer surplus is given by $\log H^*$, firm $f \in \mathcal{F}$ earns profit $m^f(H^*) - 1$, and product $k \in f$ is priced at $r_k(m^f(H^*))$.

Proof. See Appendix A. □

Broadly speaking, Assumption 1 says that for every product j, ι_j, the price elasticity of the monopolistic competition demand for product j, should be non-decreasing in p_j. This condition is sometimes called Marshall's second law of demand. It clearly holds with CES and MNL demands, where $\iota_j(p_j)$ is equal to σ and p_j/λ, respectively. We discuss how it can be relaxed in Section 4.4. In the following, we denote by $\mathcal{H}^\iota$ the set of $\mathcal{C}^3$, strictly decreasing and log-convex functions from $\mathbb{R}_{++}$ to $\mathbb{R}_{++}$ such that Assumption 1 holds.

4.2 Properties of Fitting-in and Pricing Functions

In this section, we study the properties of the product-level pricing function r_k and the firm-level fitting-in function m^f, and discuss how these properties shape the behavior of firm f.[24] These functions turn out to be convenient for deriving and interpreting comparative statics in Section 4.3.

For every product $k \in \mathcal{N}$, denote $\bar{\mu}_k = \lim_\infty \iota_k$, and let p_k^{mc} be the unique solution of equation $\frac{p_k - c_k}{p_k} \iota_k(p_k) = 1$. Note that p_k^{mc} is the price at which product k would be sold under monopolistic competition. $\bar{\mu}_k$ is the highest ι-markup that product k can support.

Proposition 2 (Pricing function). *r_k is continuous and strictly increasing on $(1, \bar{\mu}_k)$. Moreover, $\lim_1 r_k = p_k^{mc}$, $\lim_{\bar{\mu}_k} r_k = \infty$, and $r_k(\mu^f) = \infty$ for every $\mu^f \geq \bar{\mu}_k$.*

In words, the price of product k increases when ι-markup μ^f increases. If μ^f approaches unity (the monopolistic competition ι-markup), then p_k approaches p_k^{mc} (the monopolistic competition price). If μ^f is above $\bar{\mu}_k$, then $\iota_k(p_k)/\mu^f$, the adjusted price elasticity of demand under monopolistic competition, is strictly lower than unity for every p_k. Therefore, firm f sets an infinite price for product k, i.e., it does not supply product k.

Next, we turn our attention to firm-level fitting-in function m^f. For every firm f, let $\bar{\mu}^f = \max_{j \in f} \bar{\mu}_j$ denote the highest ι-markup that firm f can sustain.

Proposition 3 (Fitting-in function). *m^f is continuous and strictly decreasing on $(0, \infty)$. Moreover, $\lim_0 m^f = \bar{\mu}^f$, and $\lim_\infty m^f = 1$.*

As competition intensifies (H increases), firm f reacts by lowering its ι-markup. As the industry approaches the monopolistic competition limit ($H \to \infty$), m^f tends to 1, the ι-markup under monopolistic competition. Combining Propositions 2 and 3, we see that, as

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[24]These properties are rigorously established in Appendix A.

competition intensifies, firm f lowers the prices of all its products. An immediate consequence is that the aggregate fitting-in function Γ is strictly increasing.

Propositions 2 and 3 also tell us how firm f's product range varies with the intensity of competition. To fix ideas, let $f = \{1, 2, \ldots, N^f\}$, and assume that products are ranked as follows: $\bar{\mu}_1 > \bar{\mu}_2 > \ldots > \bar{\mu}_{N^f}$. When competition is very soft (H close to 0), $m^f(H)$ is close to $\bar{\mu}_1$, and strictly higher than $\bar{\mu}_2$. Therefore, only product 1 is supplied. As H increases, $m^f(H)$ decreases, and eventually crosses $\bar{\mu}_2$, so that product 2 starts being supplied as well. When H approaches the monopolistic competition limit, $m^f(H)$ is strictly lower than $\bar{\mu}_{N^f}$, and firm f therefore sells all of its products. To summarize, the model predicts that a firm tends to sell more products when it operates in a more competitive environment.

The intuition is easiest to grasp in the case where firm f sells CES products with heterogeneous σ's: For every $j \in f$, $h_j(p_j) = a_j p_j^{1-\sigma_j}$. Then, $\iota_j(p_j) = \sigma_j$ for every j, and $\sigma_1 > \sigma_2 > \ldots > \sigma_{N^f}$. In the discrete/continuous choice interpretation of the demand system, the conditional demand for product j is given by $-d \log h_j(p_j)/dp_j = (\sigma_j - 1)/p_j$. The conditional profit made on product j is therefore $(\sigma_j - 1)\frac{p_j - c_j}{p_j}$. The supremum of that conditional profit is $\sigma_j - 1$. If competition is soft, then firm f has little to worry about consumers substituting away from its products. Firm f cares first and foremost about selling product 1, which is its most profitable product. It therefore sets a high price for product 1, earns a profit close to $\sigma_1 - 1$, and shuts down its other, less profitable, products, to prevent consumers from purchasing them. In other words, when competition is soft, firm f is mostly concerned about self-cannibalization effects, and therefore has an incentive to withdraw relatively unprofitable products.

On the other hand, as competition intensifies, self-cannibalization effects become less important, and firm f worries more about consumers switching to other firms' products. The firm therefore has an incentive to flood the market with its products so as to increase the probability that one of its products ends up being chosen by consumers.

4.3 Properties of Equilibria and Comparative Statics

Markups. Our class of demand systems can generate rich patterns of equilibrium markups within a firm's product portfolio. To see this, let us first consider the special case of CES products with common σ (i.e., $h_j(p_j) = a_j p_j^{1-\sigma}$ for all $j \in \mathcal{N}$). In this case, $\iota_j = \sigma$ for all j, and the common ι-markup property implies that, in equilibrium, $\frac{p_j - c_j}{p_j} = \frac{\mu^f}{\sigma}$ for all $j \in f$. Therefore, firm f sets the same Lerner index for all the products in its portfolio, and thus charges higher absolute markups on products that it produces less efficiently (since $p_j - c_j = \frac{\mu^f}{\sigma - \mu^f} c_j$ is increasing in c_j).

These markup patterns are not robust to changes in the demand system. Suppose for instance that all products are still CES products, but with potentially heterogeneous σ's (i.e., $h_j(p_j) = a_j p_j^{1-\sigma_j}$ for all $j \in \mathcal{N}$). Then, in equilibrium, $\frac{p_j - c_j}{p_j} = \frac{\mu^f}{\sigma_j}$, and firm f no longer sets the same Lerner index over all its products (unless all these products share the same σ). Similarly, it does not necessarily charge higher absolute markups on high marginal cost

products.

The same point could be made about the other special case in which all products are MNL with common λ's ($h_j(p_j) = \exp\left(\frac{a_j - p_j}{\lambda}\right)$ for all $j \in \mathcal{N}$). In this case, in equilibrium, a multiproduct firm charges the same absolute markup over all its products (since $\iota_j(p_j) = p_j/\lambda$ for all j), and sets a lower Lerner index on high marginal cost products. Again, this can be overturned by allowing the λ's to differ across products.[25]

It is straightforward to construct fully parametric demand systems that give rise to a richer pattern of within-firm markups than either CES or MNL.[26] More generally, the pattern of markups within a firm's product portfolio depends on demand-side conditions, as captured by functions $(\iota_j)_{j \in f}$, and on supply-side considerations $((c_j)_{j \in f})$.

Comparing equilibria. If we know that H^* is an equilibrium aggregator level, then we can compute consumer surplus (given by $\log H^*$), the profit of firm $f \in \mathcal{F}$ (given by $m^f(H^*) - 1$) and the price of product $k \in f$ (given by $r_k(m^f(H^*))$). Moreover, Propositions 2 and 3 imply that if there are multiple equilibria, then these equilibria can be Pareto-ranked among firms, with this ranking being the inverse of consumers' ranking of equilibria:

Proposition 4. *Suppose that there are two pricing equilibria with aggregators H_1^* and $H_2^* > H_1^*$, respectively. Then, each firm $f \in \mathcal{F}$ makes a strictly larger profit in the first equilibrium (with aggregator H_1^*), whereas consumers' indirect utility is higher in the second equilibrium (with aggregator H_2^*).*
In addition, the set of equilibrium aggregator levels has a maximal and a minimal element.

Proof. See Online Appendix IX.1. $\qquad\square$

Welfare analysis. Next, we analyze the welfare distortions arising from multiproduct-firm oligopoly pricing.

An immediate observation is that firms' pricing is constrained efficient in the following sense: Firm f's equilibrium prices $(r_k(m^f(H^*)))_{k \in f}$ maximize social welfare subject to the constraint that the firm's contribution to the aggregator, H^f, is held fixed at its equilibrium value. The reason is that consumer surplus and rivals' profits are held fixed by the constraint, but firm f's prices maximize its profit by definition.

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[25]Björnerstedt and Verboven (2016) analyze a merger in the Swedish market for painkillers, and find that a CES demand specification (or, in the authors' own words, a constant expenditure demand specification) with random coefficients gives rise to more plausible markup predictions than an MNL demand specification with random coefficients.

[26]Consider, for example, the following family of h functions: For every $\lambda > 0$, $\phi \in [0, 1]$ and $p > 0$,

$$h^{\phi,\lambda}(p) = \begin{cases} \exp\left(-\lambda \frac{p^\phi - 1 + \phi^2}{\phi}\right) & \text{if } \phi > 0, \\ p^{-\lambda} & \text{if } \phi = 0. \end{cases}$$

It is easy to check that $h^{\phi,\lambda}$ converges pointwise to $h^{0,\lambda}$ (i.e., CES) when ϕ goes to zero, and to MNL when ϕ goes to 1, and that $h^{\phi,\lambda} \in \mathcal{H}^\iota$ for every ϕ, λ.

As there are no within-firm pricing distortions, this leaves us with two types of distortions. The first distortion comes from the fact that, under oligopoly, firms set positive markups. This implies that H^*, the equilibrium aggregator level, is strictly lower than the aggregator level under perfect competition ($\sum_{j \in \mathcal{N}} h_j(c_j)$). The second distortion is due to the fact that, conditional on aggregator level H^*, some firms are contributing too much to H^*, while some others are contributing too little. This is easily seen by maximizing social welfare subject to the constraint that consumer surplus is equal to $\log H^*$:[27]

$$\max_p \sum_{k \in \mathcal{N}} (p_k - c_k) \frac{-h'_k(p_k)}{\sum_{j \in \mathcal{N}} h_j(p_j)} \text{ s.t. } \log \sum_{j \in \mathcal{N}} h_j(p_j) = \log H^*.$$

The first-order condition for product i can be written as follows:[28]

$$\frac{p_i - c_i}{p_i} \iota_i(p_i) = 1 - \Lambda + \sum_{k \in \mathcal{N}} (p_k - c_k) \frac{-h'_k(p_k)}{H^*}, \tag{8}$$

where Λ is the Lagrange multiplier associated with the consumer surplus constraint. Since the right-hand side of equation (8) does not depend on i, it follows that the profile of prices $(p_k)_{k \in \mathcal{N}}$ satisfies the common ι-markup property. Let μ^* be the optimal ι-markup. Then, the profile of $|\mathcal{N}|$ first-order conditions boils down to the following optimality condition:

$$\mu^* \left(1 - \frac{1}{H^*} \sum_{j \in \mathcal{N}} \gamma_j(r_j(\mu^*)) \right) = 1 - \Lambda. \tag{9}$$

This means that pricing equilibrium H^* is constrained efficient if and only if $m^f(H^*) = m^g(H^*)$ for every $f, g \in \mathcal{F}$. This condition is unlikely to hold in general.[29] When it does not hold, some firms set their ι-markups above μ^* and end up producing too little, while some other firms set their ι-markups below μ^* and therefore produce too much.[30] Put differently, social welfare can be increased by raising H^f for some firm f while reducing H^g for some other firm g such that $H^f + H^g$ remains constant. Whether a given firm contributes too much or too little to the aggregator can be assessed by comparing equations (7) and (9).

Comparative statics. Although our pricing game is not supermodular, we can exploit its aggregative structure to perform comparative statics on the set of equilibria. The approach is similar to to the one in Corchon (1994) and Acemoglu and Jensen (2013), and can be

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[27]We drop the consumer surplus term in the social welfare expression, since the constraint implies that that term is a constant.

[28]Infinite prices can be handled by defining generalized first-order conditions, as we do in Appendix A.

[29]For instance, with CES or MNL demands, this condition holds if and only if all firms are of the same type. See Section 5.1.

[30]If all firms set their ι-markups above μ^* (resp. below μ^*), then the consumer surplus constraint is violated.

summarized as follows:[31] Suppose that the value of a parameter changes; study how this change affects firms' pricing and fitting-in functions, and hence, the aggregate fitting-in function; analyze how the associated shift in the aggregate fitting-in function affects the set of equilibrium aggregator levels; finally, use pricing and fitting-in functions to translate these changes in aggregator levels into changes in markups, prices, profits, and sets of active products.

Outside option. We introduce an outside option $H^0 > 0$, and ask how an increase in H^0 affects the set of equilibria:

Proposition 5. *Suppose that H^0 increases. Then, in both the equilibrium with the smallest and largest value of the aggregator H, this induces (i) a decrease in the profit of all firms, (ii) a decrease in the prices of all goods, (iii) an increase in consumer surplus, and (iv) an expansion of the set of active products.*

Proof. See Online Appendix IX.2. □

As the outside option improves, the aggregate fitting-in function shifts upward. Since that function is strictly increasing, it follows that the lowest and highest equilibrium aggregator levels increase. The rest of the proposition follows from the monotonicity properties of the pricing and fitting-in functions (Propositions 2 and 3).

In an international trade context with a competitive (or monopolistically competitive) fringe of importers, parameter H^0 can be interpreted as the consumer surplus derived from foreign varieties. Trade liberalization then corresponds to an increase in H^0. According to Proposition 5, trade liberalization lowers prices and markups, and raises consumer surplus. Perhaps more surprisingly, trade liberalization induces import-competing firms to broaden their product portfolios. As discussed in the introduction, this is in contrast to Eckel and Neary (2010), Bernard, Redding, and Schott (2010, 2011), and Mayer, Melitz, and Ottaviano (2014), who develop models in which firms tend to refocus on their core competencies after trade liberalization.[32] The intuition behind our product portfolio expansion result was already hinted at in Section 4.2. After trade liberalization, the industry is more competitive. Import-competing firms therefore worry more about losing consumers to their rivals than about cannibalizing their own products' sales. This leads them to introduce more products, in order to increase the likelihood that one of these products will be purchased.

Note that by abstracting from product-level fixed costs, we have shut down a countervailing effect:[33] The increase in H induced by a trade liberalization makes it harder to cover the

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[31]Acemoglu and Jensen (2013) make a number of assumptions (compactness, pseudo-concavity and upper semi-continuity) which do not hold in our framework. This prevents us from applying their results off the shelf.

[32]Bernard, Redding, and Schott (2011) provide empirical support for their prediction. However, they define products at a rather aggregate 5-digit SIC level at which the self-cannibalization effects highlighted here are arguably absent.

[33]This effect is emphasized by Bernard, Redding, and Schott (2011) in a model of monopolistic competition with CES demand.

product-level fixed cost, thus providing firms with an incentive to drop products. A downside of introducing such fixed costs is that they would give rise to multiplicity of equilibria and/or discontinuities in firms' fitting-in functions.

Entry. Similar techniques can be used to study the impact of entry. Suppose that firm $f^0 \in \mathcal{F}$ is initially inactive, i.e., $p_j = \infty$ for every $j \in f^0$. Solving pricing game $((h_j)_{j \in \mathcal{N} \setminus f^0}, \mathcal{F} \setminus \{f^0\}$, $(c_j)_{j \in \mathcal{N} \setminus f^0})$ with outside option $H^0 = \sum_{j \in f^0} h_j(\infty)$ gives us the set of pre-entry equilibrium aggregator levels. The set of post-entry equilibrium aggregator levels can be obtained by solving pricing game $((h_j)_{j \in \mathcal{N}}, \mathcal{F}, (c_j)_{j \in \mathcal{N}})$ with outside option 0. We prove the following proposition:

Proposition 6. *Suppose that a new firm enters. Then, in both the equilibrium with the smallest and largest value of the aggregator H, this induces (i) a decrease in the profit of all incumbent firms, (ii) a decrease in the prices of all goods, (iii) an increase in consumer surplus, and (iv) an expansion of the set of active products.*

Proof. The result is proven in the same way as Proposition 5. After entry, function $\Gamma(\cdot)$ shifts upward. □

Using the terminology of Johnson and Myatt (2003), the entry of a competitor induces incumbent firms to introduce "fighting brands" to preserve their market shares.[34]

Productivity and quality. Productivity and quality shocks have more ambiguous effects. Suppose that c_k, the marginal cost of product k owned by firm f, increases. Then, m^f, firm f's fitting-in function shifts downward. Intuitively, product k has now become less profitable, and firm f therefore has less incentives to divert sales toward that product. It therefore lowers the prices of its other products, and, hence, its ι-markup. Despite the fact that μ^f goes down, $p_k = r_k(m^f(H))$ goes up due to the direct impact of the increase in c_k. Therefore, firm f's contribution to the aggregator $(\sum_{j \in f} h_j(r_j(m^f(H))))$ may go up or down, depending on whether the decrease in h_k is offset by the increase in h_j ($j \neq k$). This means that the aggregate fitting-in function may shift upward or downward.[35]

Consider first the perhaps more intuitive case in which Γ shifts downward (as it does under CES and MNL demands; see Proposition 11). Then, by monotonicity of Γ, the highest and lowest equilibrium aggregator levels decrease, which lowers consumer surplus. By Proposition 2, firm f's rivals increase their ι-markups. Therefore, by Proposition 3, they end up earning higher profit, charging higher prices, and supplying fewer products. Whether firm f ends up decreasing its ι-markup (and hence, making lower profit and supplying more products) is unclear, since the direct effect of the increase in c_k may be dominated by the indirect effect of the decrease in H.

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[34]In Johnson and Myatt (2003)'s Cournot model with vertically differentiated products, fighting brands can emerge only if marginal revenue does not decrease everywhere. Under the more standard assumption of decreasing marginal revenue, an incumbent firm always reacts to entry by pruning its product line. In our framework with horizontal product differentiation, fighting brands are the rule rather than the exception.

[35]In principle, it could be that $\Gamma(H)$ shifts upward at H, and downward at $H' \neq H$. To simplify the exposition, we confine attention to the simple cases in which Γ shifts upward (resp. downward) everywhere.

If, instead, Γ shifts upward (see Online Appendix IX.3 for an example where this happens), then consumers end up benefiting from the marginal cost increase, while firm f's rivals make less profit, set lower prices, and supply more products. As for firm f, the direct effect of the increase in c_k is now reinforced by the indirect effect of the increase in H. Therefore, firm f charges a lower ι-markup, makes less profit, and supplies more products.

We can augment our discrete/continuous choice model of demand by introducing quality as follows. The discrete continuous choice model is now $(a_j h_j)_{j \in \mathcal{N}}$, where, for every j, h_j satisfies all the assumptions we have made so far, and a_j is a strictly positive scalar, which we call product j's quality. The idea is that an increase in a_k raises the probability that product k is chosen $(a_k h_k / (\sum_j a_j h_j))$, but does not affect the conditional demand for product k $(d \log(a_j h_j)/dp_j = d \log h_j / dp_j)$. This is consistent with the way in which product quality or vertical product characteristics are usually introduced in CES or MNL demand systems. It is then easy to show that quality shocks also have ambiguous effects on consumer surplus and firms' profits, because an increase in a_k does not necessarily imply an upward shift in the aggregate fitting-in function.

4.4 Discussion

Relaxing Assumption 1. Our aggregative games approach is based on first-order conditions. In Online Appendix III, we show that Assumption 1 is the weakest assumption under which an approach based on first-order conditions is valid, in the following sense. We show that set $\mathcal{H}^\iota$ is the largest set such that for every pricing game $((h_j)_{j \in \mathcal{N}}, \mathcal{F}, (c_j)_{j \in \mathcal{N}})$ such that $h_j \in \mathcal{H}^\iota$ for every j, first-order conditions are sufficient for global optimality for every firm f. We formalize and prove this statement in Online Appendix III.

Assumption 1 can be relaxed if we follow instead a potential games approach (Slade, 1994; Monderer and Shapley, 1996). In Nocke and Schutz (2016a), we show that function

$$P(p) = \frac{\prod_{f \in \mathcal{F}} \sum_{j \in f} (p_j - c_j)(-h'_j(p_j))}{\sum_{j \in \mathcal{N}} h_j(p_j)}$$

is an ordinal potential for our pricing game. The idea is that, starting from a profile of prices, if firm f deviates, then firm f's profit increases if and only if the value of the potential function increases. Without putting any restrictions on the discrete/continuous choice model $(h_j)_{j \in \mathcal{N}}$ (except that the h functions are positive, $\mathcal{C}^1$, strictly decreasing and log-convex), we show that function P has a global maximizer. This implies that the pricing game has an equilibrium.

While this more general existence result is useful, the downside of the potential games approach is that it does not allow us to completely characterize the set of equilibria. This implies in particular that we cannot extend the comparative statics and characterization results derived in Section 4.3

Aggregative games and demand systems. We have shown that a demand system that is derivable from discrete/continuous choice gives rise to aggregative pricing games. A natural question is whether this property extends to a wider class of demand systems. In Online Appendix IV, we show that, if there are at least three products, then the smooth and Slutsky-symmetric demand system D gives rise to an aggregative pricing game (with additive aggregation) for every firm partition $\mathcal{F}$ if and only if there exist smooth functions $(g_i)_{i \in \mathcal{N}}$, $(h_i)_{i \in \mathcal{N}}$ and Ψ such that[36]

$$D_i(p) = -g_i'(p_i) - h_i'(p_i)\Psi'\left(\sum_{j \in \mathcal{N}} h_j(p_j)\right), \quad \forall i \in \mathcal{N}, \ \forall p >> 0. \tag{10}$$

This result generalizes Anderson, Erkal, and Piccinin (2013)'s Propositions 4 and 5. Any pricing game based on demand system (10) has the property that a firm's payoff only depends on its prices and on the value of aggregator $H = \sum_j h_j(p_j)$. Moreover, we show that these pricing games also satisfy a generalized version of the common-ι markup property.

Demand system (1) is a special case of (10), where $g_i' = 0$ for every i, and $\Psi(H) = \log H$. Another special case of demand system (10), which demand specification (1) can *not* accommodate, is linear demand (if h_j, g_j' and Ψ' are all affine functions).

Demand system (10) can be viewed as the sum of a monopoly component $(-g_i'(p_i))$ and an IIA component $(\Psi'(\sum_j h_j(p_j)))$. Integrating equation (10) yields the associated indirect subutility function:

$$V(p) = \sum_{j \in \mathcal{N}} g_j(p_j) + \Psi\left(\sum_{j \in \mathcal{N}} h_j(p_j)\right).$$

Note that aggregator H is a sufficient statistic for consumer surplus if and only if the monopoly component of demand is equal to zero for every product. If the demand system does not have the IIA property, then consumer surplus depends both on aggregator H and on aggregator $G = \sum_j g_j(p_j)$.

One of the appeals of demand system (1) is its discrete/continuous choice foundation. A natural question is whether demand system (10) can be derived in a similar way. We argue that a subset of this class of demand systems can indeed be derived from a generalized version of discrete/continuous choice, which we now describe.

For each product i, there is a mass of captive consumers, whose total demand is given by $-g_i'(p_i)$. Other consumers are not captive, and engage in sequential discrete/continuous choice. Non-captive consumers first decide whether they wish to consume an unpriced option or one of the products in set $\mathcal{N}$. If a non-captive consumer chooses the unpriced option, then he receives indirect utility ε, which is a random variable drawn from cumulative distribution function F. If instead he chooses the product set, then he observes a vector of i.i.d. type-1 extreme value taste shocks $(\varepsilon_j)_{j \in \mathcal{N}}$, chooses the product i that maximizes $\log h_j(p_j) + \varepsilon_j$, and consumes $-h_i'/h_i(p_i)$ units of this product. As shown in Section 2, the inclusive value of

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[36]Recall that Slutsky symmetry is necessary for quasi-linear integrability.

product set $\mathcal{N}$ is equal to $\log \sum_j h_j(p_j)$. Therefore, the mass of non-captive consumers that choose the product set is given by $F(\log \sum_j h_j(p_j))$. Putting everything together, we obtain the expected demand for product i:

$$D_i(p) = -g_i'(p_i) - \frac{h_i'(p_i)}{h_i(p_i)} \frac{h_i(p_i)}{\sum_{j \in \mathcal{N}} h_j(p_j)} F\left(\log \sum_{j \in \mathcal{N}} h_j(p_j)\right),$$

$$= -g_i'(p_i) - h_i'(p_i) \frac{F\left(\log \sum_{j \in \mathcal{N}} h_j(p_j)\right)}{\sum_{j \in \mathcal{N}} h_j(p_j)}.$$

We can conclude: Demand system (10) can be derived from generalized discrete/continuous choice (as described above) if and only if $H \mapsto H\Psi'(H)$ is non-decreasing (so that $F(V) = \exp V \Psi'(\exp V)$ is indeed a cumulative distribution function), and for every $i \in \mathcal{N}$, g_i is non-increasing and convex (so that g_i is indeed an indirect subutility function), and h_i is strictly positive, non-increasing and log-convex (so that $\log h_i$ is indeed an indirect subutility function).

A cookbook for applied work. One way of finding a function h that is consistent with discrete/continuous choice and satisfies Assumption 1 is to start with a function h that is positive, decreasing and log-convex, and to check that the associated ι function is non-decreasing whenever it is strictly greater than 1. This is tedious, because nothing guarantees that ι will have the right monotonicity property. Another possibility is to start with a function ι that is non-decreasing, to integrate a second-order differential equation to obtain a function h, and to adjust constants of integration to ensure that h is positive, decreasing and log-convex. The following proposition states that such constants of integration exist:

Proposition 7. *Let* $\widetilde{\iota} : \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$ *be a* $\mathcal{C}^1$ *function such that* $\widetilde{\iota}$ *is non-decreasing,* $\lim_{0^+} \widetilde{\iota} > 0$, *and* $\widetilde{\iota}(x) > 1$ *for some* $x > 0$. *For every* $(\alpha, \beta) \in \mathbb{R}_{++}^2$, *let*

$$h^{\alpha,\beta}(x) = \alpha \left(\beta - \int_1^x \exp\left(-\int_1^t \frac{\widetilde{\iota}(u)}{u} du\right) dt\right).$$

Then, there exists $\underline{\beta} > 0$ *such that, for every strictly decreasing and log-convex function* h, *the elasticity of* h' *is equal to* $\widetilde{\iota}$ *if and only if* $h = h^{\alpha,\beta}$ *for some* $\alpha > 0$ *and* $\beta \geq \underline{\beta}$.

Proof. See Online Appendix V. $\qquad \square$

The appeal of Proposition 7 is that it allows us to use $(\iota_j)_{j \in \mathcal{N}}$ as a primitive, instead of $(h_j)_{j \in \mathcal{N}}$. This is useful, because markup patterns are governed by the ι functions.

Non-linear pricing. We have assumed throughout that firms compete in linear prices. We now show that our methodology can also be usefully applied to study non-linear pricing. To this end, suppose that firms can charge two-part tariffs: For every $j \in \mathcal{N}$, p_j (resp.

F_j) denotes the variable (resp. fixed) part of the two-part tariff contract for product j. In equilibrium, firms find it optimal to set all variable parts equal to marginal cost, and compete on the fixed parts. Intuitively, if firm f sets $p_j \neq c_j$ for some $j \in f$, then it is profitable for this firm to deviate to $p_j = c_j$, and to adjust the fixed part in such a way that product j is chosen with the same probability as before. Since this deviation raises the joint surplus of the consumer and the firm (the consumer purchases the efficient quantity), and since the consumer receives the same expected surplus as before (otherwise, the choice probabilities would not be the same as before), this deviation is indeed profitable.

When all firms set variable parts equal to marginal costs, the consumer's indirect utility conditional on choosing product j (net of the taste shock and income) is $\log h_j(c_j) - F_j$, and his conditional demand for product j is $-h_j'(c_j)/h_j(c_j)$. Therefore, the profit of firm f is given by:

$$\Pi^f = \sum_{k \in f} \left(F_k - c_k \frac{-h_k'(c_k)}{h_k(c_k)} \right) \frac{h_k(c_k)\, \mathrm{e}^{-F_k}}{\sum_{j \in \mathcal{N}} h_j(c_j)\, \mathrm{e}^{-F_j}}.$$

These are the payoff functions of pricing game $((\widetilde{h}_j)_{j \in \mathcal{N}}, \mathcal{F}, (\widetilde{c}_j)_{j \in \mathcal{N}})$ with linear tariffs, where, for every $j \in \mathcal{N}$, $\widetilde{h}_j(F_j) = h_j(c_j) \exp(-F_j)$, and $\widetilde{c}_j = c_j(-h_j'(c_j))/h_j(c_j)$. Put differently, the pricing game with two-part tariffs is formally equivalent to a pricing game with MNL demands and linear pricing. We know that the MNL pricing game has an equilibrium, and in fact, we will soon show that this equilibrium is unique.

An immediate observation is that all available products always end up being sold under non-linear pricing, whereas, as discussed before, this is not necessarily the case under linear pricing. This comes from the fact that the non-linear pricing game is equivalent to a linear pricing game with MNL demands. Since MNL products are such that $\bar{\mu}_j = \infty$ (recall that $\widetilde{\iota}_j(F_j) = F_j$), such products are always supplied. The intuition is that a firm is better able to extract additive taste shock ε_j under non-linear pricing than it is under linear pricing.

Quantity competition. The techniques developed in this paper can also be used to solve quantity competition games with differentiated products and multiproduct firms. In the following, we briefly describe how our aggregative games approach can be adapted, and refer the reader to Online Appendix VI for more details. Suppose that the inverse demand for product i can be written as $P_i(x_i, H) = h_i'(x_i)/H$, where $H = \sum_{j \in \mathcal{N}} h_j(x_j)$ is the aggregator, and x_j is the output of product j. Taking first-order conditions, it is possible to show that an additive form of the constant ι-markup property must hold in equilibrium: For every firm f, there exists μ^f such that $\frac{P_k - c_k}{P_k} + \iota_k(x_k) = \mu^f$ for all $k \in f$, where ι_k is still the elasticity of h_k'. Inverting the ι-markup equation yields an *output function* $\chi_k(\mu^f, H)$ for every product k.[37] We can then use ι-markup μ^f to reduce firm f's profile of first-order conditions to a

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[37]An important difference with the price competition case is that output function χ_k depends on both μ^f and H, whereas pricing function r_k only depends on μ^f. This makes the analysis somewhat more complicated.

unique equation

$$\mu^f = \frac{1}{H} \sum_{j \in f} \chi_j(\mu^f, H) h'_j(\chi_j(\mu^f, H)),$$

which uniquely pins down firm f's *fitting-in function, $m^f(H)$*. Combining output and fitting-in functions allows us to define the *aggregate fitting-in function*:

$$\Gamma(H) = \sum_{f \in \mathcal{F}} \sum_{j \in f} h_j \left(\chi_j(m^f(H), H) \right).$$

We then derive conditions under which first-order conditions are sufficient for optimality, and Γ has a unique fixed point. Under these conditions, there is a unique equilibrium. These conditions hold, e.g., under CES demand.

4.5   Equilibrium Uniqueness

We now turn our attention to the question of equilibrium uniqueness. The idea is to derive conditions under which $\Gamma'(H) < 1$ whenever $\Gamma(H) = H$,[38] which ensures that there is exactly one value of H such that $\Gamma(H) = H$. To avoid non-differentiability issues (m^f may fail to be differentiable at values of H at which firm f drops a product), we assume that, for every $f \in \mathcal{F}$ and $j \in f$, $\bar{\mu}^f = \bar{\mu}_j$. This ensures that all products are always sold at finite prices and that $\Gamma(\cdot)$ is $\mathcal{C}^1$. We also introduce the following notation: For all $j \in \mathcal{N}$, $\rho_j \equiv h_j/\gamma_j$, $\theta_j \equiv h'_j/\gamma'_j$, and $\underline{p}_j = \inf\{p_j > 0 : \iota_j(p_j) > 1\}$.[39]

We can now state our uniqueness theorem:

Theorem 2. *Assume that Assumption 1 holds and that $\bar{\mu}^f = \bar{\mu}_j$ for every $f \in \mathcal{F}$ and $j \in f$. Suppose that, for every firm $f \in \mathcal{F}$, at least one of the following conditions holds:*

(a) $\min_{j \in f} \inf_{p_j > \underline{p}_j} \rho_j(p_j) \geq \max_{j \in f} \sup_{p_j > \underline{p}_j} \theta_j(p_j)$.

(b) $\bar{\mu}^f \leq \mu^*(\simeq 2.78)$, *and for every $j \in f$, $\lim_\infty h_j = 0$ and ρ_j is non-decreasing on $(\underline{p}_j, \infty)$.*[40]

(c) *There exist a function $h^f \in \mathcal{H}^\iota$ and a marginal cost level $c^f > 0$ such that $h_j = h^f$ and $c_j = c^f$ for all $j \in f$. In addition, ρ^f is non-decreasing on $(\underline{p}, \infty)$.*

Then, the pricing game has a unique equilibrium.

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[38] Another possibility would be to follow an index approach and compute the sign of the determinant of the Jacobian of the first-order conditions map. In Online Appendix VII.8, we show that this approach is equivalent to ours.

[39] By Lemma III in Online Appendix II.1, $\underline{p}_j$ is well-defined.

[40] Condition $\lim_\infty h_j = 0$ can be weakened. See Propositions III and IV, and Corollaries I and II in Online Appendix VII.5.

Proof. The theorem is proven in two steps. We first show that the uniqueness condition – $\Gamma'(H) < 1$ whenever $\Gamma(H) = H$ – can be rewritten as $|\mathcal{F}|$ independent firm-level conditions. In the second step, we show that each of conditions (a), (b) and (c) is sufficient for the firm-level condition. See Online Appendix VII for details. $\qquad\square$

Discussion. We first discuss the condition that ρ_j is non-decreasing on $(\underline{p}_j, \infty)$. Consider a discrete/continuous choice model of demand in which only product j and outside option $H^0 > 0$ are available. Then, the expected demand for product j is given by: $D_j(p_j, H^0) = -h'_j(p_j)/(h_j(p_j) + H^0)$. It is easy to show that function $p_j \in (\underline{p}_j, \infty) \mapsto 1/D_j(p_j, H^0)$ is convex if and only if ρ_j is non-decreasing on $(\underline{p}_j, \infty)$.[41] Caplin and Nalebuff (1991) argue that this convexity condition is "just about as weak as possible" (see the paragraph after their Proposition 3, p. 38). They show that, under this condition, single-product firms' profit functions are quasi-concave in own prices. In their framework, equilibrium existence then follows from Kakutani's fixed-point theorem.

We find that, although this convexity condition is not needed to obtain equilibrium existence, it guarantees equilibrium uniqueness, provided that some additional restrictions, contained in conditions (a), (b) and (c), are satisfied. Note that condition (a) is indeed a stronger version of the assumption that ρ_j is non-decreasing. This is because ρ_j is non-decreasing on $(\underline{p}_j, \infty)$ if and only if $\rho_j \geq \theta_j$ on the same interval.[42] Condition (a) imposes that the highest possible value of θ_j $(j \in f)$ be smaller than the lowest possible value of ρ_j $(j \in f)$, which is indeed stronger.

Finally, note that if $f = \{j\}$ and ρ_j is non-decreasing on $(\underline{p}_j, \infty)$, then condition (c) trivially holds. It is therefore easier to ensure equilibrium uniqueness for single-product firms than for multiproduct firms.

Examples. In the following, we provide examples of demand systems that satisfy (or do not satisfy) our uniqueness conditions. A priori, condition (a) seems tedious to check if the firm under consideration has heterogeneous products. The following proposition shows that a certain type of product heterogeneity can be easily handled, and provides a cookbook for applied work:

Proposition 8. *Let $h \in \mathcal{H}^\iota$ such that $\sup_{p>\underline{p}} \theta(p) \leq \inf_{p>\underline{p}} \rho(x)$. Let f be a finite and non-empty set, and, for every $j \in f$, $(\alpha_j, \beta_j, \delta_j, \epsilon_j) \in \mathbb{R}^2_{++} \times \mathbb{R}^2_+$. For every $j \in f$, define*

$$ h_j(p_j) = \alpha_j h(\beta_j p_j + \delta_j) + \epsilon_j, \quad \forall p_j > 0. $$

Then, for all $j \in f$, $h_j \in \mathcal{H}^\iota$. Moreover, $\max_{j \in f} \sup_{p_j > \underline{p}_j} \theta_j(p_j) \leq \min_{j \in f} \inf_{p_j > \underline{p}_j} \rho_j(p_j)$.

Proof. See Online Appendix VII.6. $\qquad\square$

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[41]See Lemma VII in Online Appendix VII.1.

[42]To see this, note that $(\log \rho_j)' = \frac{-\gamma'_j}{h_j}(\rho_j - \theta_j)$, and that $\gamma'_j < 0$ by Lemma III-(d) in Online Appendix II.1.

Proposition 8 can be applied as follows. Let $h(p) = \mathrm{e}^{-p}$ for all $p > 0$. We already know that $h \in \mathcal{H}^\iota$. In addition, $\rho(p) = \theta(p) = 1$ for all $p > 0$. By Proposition 8, if firm f is such that for all $j \in f$, there exist $\lambda_j > 0$ and $a_j \in \mathbb{R}$ such that $h_j(p_j) = \mathrm{e}^{\frac{a_j - p_j}{\lambda_j}}$ for all $p_j > 0$ (i.e., firm f only has MNL products), then condition (a) in Theorem 2 holds for firm f. This implies in particular that a multiproduct-firm pricing game with MNL demand has a unique equilibrium.

Similarly, let $h(p) = p^{1-\sigma}$ for all $p > 0$ ($\sigma > 1$). Again, we already know that $h \in \mathcal{H}^\iota$. In addition, $\rho(p) = \theta(p) = \sigma/(\sigma - 1)$. Therefore, if firm f is such that for all $j \in f$, there exist $a_j, b_j, d_j > 0$ such that $h_j(p_j) = a_j \left(b_j p_j + d_j\right)^{1-\sigma}$ for all $p_j > 0$, then condition (a) in Theorem 2 holds for firm f. In particular, a pricing game with CES demand has a unique equilibrium.[43]

Some functions satisfy condition (b), but not condition (a). Consider the following function: $h(x) = \frac{1}{\log(1+\mathrm{e}^x)}$. It is easy to show that $h \in \mathcal{H}^\iota$, ρ is non-increasing, and $\bar{\mu} = 2(< 2.78)$. Therefore, condition (b) holds. However, condition $\sup \theta(x) \leq \inf \rho(x)$ is not satisfied.

It is easy to construct a multi-product firm that satisfies none of our uniqueness conditions. For instance, let $f = \{1, 2\}$, $h_1(p_1) = p_1^{1-\sigma_1}$ and $h_2(p_2) = p_2^{1-\sigma_2}$, where $\sigma_1 \neq \sigma_2$. Then, $\bar{\mu}_1 \neq \bar{\mu}_2$, so Theorem 2 does not apply. It is also possible to find single-product firms for which Theorem 2 has no bite.[44]

Equilibrium uniqueness when marginal costs are high or the outside option is attractive enough. As discussed above, Theorem 2 is not powerful enough to guarantee equilibrium uniqueness for every pricing game. In the following, we show that, for a given discrete/continuous choice model of consumer demand $(h_j)_{j \in \mathcal{N}}$ and a given partition of the set of products $\mathcal{F}$, pricing game $((h_j)_{j \in \mathcal{N}}, \mathcal{F}, (c_j)_{j \in \mathcal{N}})$ has a unique equilibrium, provided that firms are sufficiently inefficient and that consumers have access to an outside option. We no longer assume that $\bar{\mu}_j = \bar{\mu}^f$ for every $f \in \mathcal{F}$ and $j \in f$.

Proposition 9. *Let $((h_j)_{j \in \mathcal{N}}, \mathcal{F}, (c_j)_{j \in \mathcal{N}})$ be a pricing game such that $h_j \in \mathcal{H}^\iota$ for every j. Then,*

- *For every $\underline{H}^0 > 0$, there exists $\underline{c} > 0$ such that pricing game $((h_j)_{j \in \mathcal{N}}, \mathcal{F}, (c_j)_{j \in \mathcal{N}})$ with outside option H^0 has a unique equilibrium whenever $(c_j)_{j \in \mathcal{N}} \in [\underline{c}, \infty)^{\mathcal{N}}$ and $H^0 \geq \underline{H}^0$.*

- *For every $\underline{c} > 0$, there exists $\underline{H}^0 \geq 0$ such that pricing game $((h_j)_{j \in \mathcal{N}}, \mathcal{F}, (c_j)_{j \in \mathcal{N}})$ with outside option H^0 has a unique equilibrium whenever $(c_j)_{j \in \mathcal{N}} \in [\underline{c}, \infty)^{\mathcal{N}}$ and $H^0 \geq \underline{H}^0$.*

Proof. See Online Appendix VII.7. □

Intuitively, when the products in $\mathcal{N}$ are relatively unattractive compared to the outside option (either because marginal costs are high, or because the outside option delivers high

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[43]Other candidates for the base h include $h(x) = \exp\left(\mathrm{e}^{-x}\right)$, $h(x) = 1 + \frac{1}{1+\mathrm{e}^{1+x}}$, $h(x) = 1 + \frac{1}{\cosh(2+x)}$, etc.

[44]Consider, for instance, the family of functions $h^{\phi,\lambda} \in \mathcal{H}^\iota$ introduced in footnote 26. It is easy to show that $\rho^{\phi,\lambda}(\cdot)$ is strictly decreasing whenever $\phi \in (0, 1)$. Therefore, none of our uniqueness conditions hold.

consumer surplus), the firms have low market shares, and, hence, little market power. The firms therefore set ι-markups close to those they would set under monopolistic competition, and react relatively little to changes in their rivals' behavior.

5 Type Aggregation: CES and MNL Demands

In this section, we analyze multiproduct-firm pricing when demand is either CES or MNL. In the first part, we show that another aggregation property, called *type aggregation*, obtains: Even though firms may differ in multiple dimensions (number of products, qualities and marginal costs), all the information relevant for determining a firm's fitting-in function can be summarized by its uni-dimensional "type." This type-aggregation property allows us to derive clear-cut comparative statics. In the second part, we provide an efficient computational algorithm for applied work.

5.1 Multiproduct-Firm Oligopoly with CES and MNL Demands

We study a multiproduct-firm pricing game with CES demand (resp. MNL demand) and heterogeneous qualities and productivities. In the CES case, let $h_k(p_k) = a_k p_k^{1-\sigma}$ for every $k \in \mathcal{N}$, where $a_k > 0$ is the quality of product k, and $\sigma > 1$ is the elasticity of substitution. In the MNL case, let $h_k(p_k) = \exp\left(\frac{a_k - p_k}{\lambda}\right)$ for every $k \in \mathcal{N}$, where $a_k \in \mathbb{R}$ is the quality of product k, and $\lambda > 0$ is a price sensitivity parameter. We have already shown in Section 4.5 that any pricing game based on $(h_j)_{j \in \mathcal{N}}$ has a unique equilibrium.

Firm f's fitting-in function is pinned down by equation (7), which involves functions γ_k and r_k. With CES demand, $\gamma_k(p_k) = \frac{\sigma-1}{\sigma} a_k p_k^{1-\sigma}$ and $\iota_k = \sigma$. Therefore, for every $\mu^f \in [1, \sigma)$, $r_k(\mu^f) = c_k/(1 - \mu^f/\sigma)$. Equation (7) can then be rewritten as follows:

$$\mu^f \left(1 - \frac{1}{H} \sum_{k \in f} \frac{\sigma-1}{\sigma} a_k \left(\frac{c_k}{1 - \frac{\mu^f}{\sigma}} \right)^{1-\sigma} \right) = 1. \tag{11}$$

With MNL demand, $\gamma_k(p_k) = h_k(p_k)$, and $\iota_k(p_k) = p_k/\lambda$. Therefore, $r_k(\mu^f) = \lambda \mu^f + c_k$. This allows us to rewrite equation (7) as follows:

$$\mu^f \left(1 - \frac{1}{H} \sum_{k \in f} \exp\left(\frac{a_k - c_k - \lambda \mu^f}{\lambda} \right) \right) = 1. \tag{12}$$

Let $T^f = \sum_{k \in f} a_k c_k^{1-\sigma}$ in the CES case, and $T^f = \sum_{k \in f} \exp\left(\frac{a_k - c_k}{\lambda}\right)$ in the MNL case. Note that T^f would be equal to firm f's contribution to the aggregator H (and thus to consumer surplus) if that firm were to price competitively (i.e., set $p_k = c_k$ for all $k \in f$). We call T^f firm f's type. We will see that all the relevant information about firm f's performance and competitive impact is summarized by its type. In Section 6.3, we will argue that T^f

also provides an appealing measure of firm-level productivity. Using the definition of T^f, equations (11) and (12) can be simplified as follows:

$$\text{(CES)} \quad \mu^f = \cfrac{1}{1 - \frac{\sigma-1}{\sigma}\frac{T^f}{H}\left(1 - \frac{\mu^f}{\sigma}\right)^{\sigma-1}}, \tag{13}$$

$$\text{(MNL)} \quad \mu^f = \cfrac{1}{1 - \frac{T^f}{H}\, e^{-\mu^f}}. \tag{14}$$

Equation (13) (resp. (14)) implicitly defines a function $m(T^f/H)$. Firm f's fitting-in function is simply $H \mapsto m(T^f/H)$. An immediate implication is that firms f and g have the same type $(T^f = T^g)$ if and only if they have the same fitting-in function.

Note that (for fixed T^f) the CES markup equation converges pointwise to the MNL markup equation as σ goes to infinity. A pricing game with MNL demand can therefore be viewed as a limiting case of pricing games with CES demand.

Next, we claim, that if firms f and g have the same type, then their contributions to the aggregator are the same. We introduce the following notation. Under CES demand, for a given aggregator level H, $s_k = a_k p_k^{1-\sigma}/H$ is the market share (in value) of product k. Under MNL demand, market shares are defined in volume: $s_k = e^{\frac{a_k - p_k}{\lambda}}/H$. Firm f's market share is $\sum_{k \in f} s_k$. Note that

$$\text{(CES)} \quad s^f = \sum_{k \in f} \frac{a_k r_k(\mu^f)^{1-\sigma}}{H} = \sum_{k \in f} \frac{a_k c_k^{1-\sigma}\left(1 - \frac{\mu^f}{\sigma}\right)^{\sigma-1}}{H} = \frac{T^f}{H}\left(1 - \frac{m\left(\frac{T^f}{H}\right)}{\sigma}\right)^{\sigma-1} \equiv S\left(\frac{T^f}{H}\right),$$

$$\text{(MNL)} \quad s^f = \sum_{k \in f} \frac{e^{\frac{a_k - r_k(\mu^f)}{\lambda}}}{H} = \sum_{k \in f} \frac{e^{\frac{a_k - c_k}{\lambda} - \mu^f}}{H} = \frac{T^f}{H}\, e^{-m\left(\frac{T^f}{H}\right)} \equiv S\left(\frac{T^f}{H}\right).$$

Firm f's market share function is $H \mapsto S(T^f/H)$. Therefore, firms f and g share the same market share function if and only if $T^f = T^g$. Put differently, firm f and g's contributions to the aggregator are identical if and only if they have the same type. Note also that, as σ tends to infinity, the definition of market shares in the CES case converges pointwise to the one in the MNL case.

Recall that H is an equilibrium aggregator level if and only if $\Gamma(H)/H = 1$. Note that

$$\frac{\Gamma(H)}{H} = \frac{1}{H}\sum_{f \in \mathcal{F}}\sum_{k \in f} h_k(r_k(m^f(H))) = \sum_{f \in \mathcal{F}}\sum_{k \in f} s_k = \sum_{f \in \mathcal{F}} S\left(\frac{T^f}{H}\right). \tag{15}$$

In words, H is an equilibrium aggregator level if and only if market shares add up to 1.

Finally, recall that firm f's equilibrium profit is equal to its equilibrium ι-markup minus one: $\Pi^f = m(T^f/H) - 1 \equiv \pi(T^f/H)$.

We summarize these findings in the following proposition:

Proposition 10. *Consider a pricing game with CES or MNL demands. If firm f is replaced by firm g such that $T^g = T^f$, then the equilibrium value of the aggregator is unaffected, and firm g ends up charging the same markup, earning the same profit, and commanding the same market share as firm f.*

Under CES and MNL demands, firms' types are aggregative as well. Firms f and g may differ widely in terms of product portfolios, productivity and product qualities, but if their types are the same, then they share the same fitting-in functions. This implies that, no matter what the competitive environment is, these two firms will always set the same ι-markup (and therefore make the same profit) and command the same market share. Interestingly, given a multiproduct firm f, there always exists an "equivalent" single-product firm. To see this, fix $T^f > 0$, and define firm $\hat{f}$ as a firm selling only one product with quality $\hat{a} = T^f$ (resp. $\hat{a} = \lambda \log T^f + 1$) in the CES (resp. MNL) case and marginal cost $\hat{c} = 1$. Then, $T^f = T^{\hat{f}}$, and firms f and $\hat{f}$ are therefore equivalent in the sense of Proposition 10.

We also obtain the following comparative statics:

Proposition 11. *In a pricing game with CES or MNL demands,*

(i) m', S', $\pi' > 0$.

(ii) For every firm f, $\frac{dH^*}{dT^f}$, $\frac{d\mu^{f*}}{dT^f}$, $\frac{ds^{f*}}{dT^f}$, $\frac{d\pi^{f*}}{dT^f} > 0$, where the superscript $*$ indicates equilibrium values, and d/dT^f is the total derivative with respect to T^f (taking into account the impact of T^f on the equilibrium aggregator level).

(iii) For every $g \neq f$, $\frac{d\mu^{g*}}{dT^f}$, $\frac{ds^{g*}}{dT^f}$, $\frac{d\pi^{g*}}{dT^f} < 0$.

(iv) For every firm f, $\frac{dW^*}{dT^f} > 0$, where W^* is equilibrium social welfare.

Proof. See Online Appendix VIII.2 . □

Part (i) says that a firm charges a higher markup, commands a larger market share, and makes a larger profit if it has more products, if it is more productive, if it sells higher-quality products (higher T^f), or if it operates in a less competitive environment (lower H). Recall from the discussion in Section 4 that, in general, a reduction in marginal cost c_k or an increase in quality a_k of firm f's product $k \in f$ may not necessarily induce an increase in firm f's contribution to the aggregator, H^f, thus making the effect on rivals' profits ambiguous; moreover, if it does, the competitive response by rivals (which involves a reduction in their prices) makes the overall effect on firm f's profit ambiguous. Parts (ii) and (iii) of Proposition 11 show that when demand is either of the CES or MNL forms, then clear-cut predictions obtain: If firm f's type increases, then consumers benefit whereas firm f's markup, market share and profit increase, to the detriment of its rivals.

The impact of an increase in T^f on social welfare can be decomposed into three effects. First, firm f becomes more efficient (because its marginal costs decrease, or its qualities improve, or its scope expands). Second, the equilibrium aggregator level rises. Third, market

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shares are reallocated toward firm f. The first two effects are clearly positive. As discussed in Section 4.3, the third effect is positive if firm f was initially producing too little, and negative otherwise. Part (iv) of the proposition shows that the market shares reallocation effect is never strong enough to offset the first two effects, so that an increase in T^f always raises social welfare. This is in contrast to standard results in homogeneous-goods Cournot models (Lahiri and Ono, 1988; Zhao, 2001), or in models of price or quantity competition with differentiated products and linear demand (Wang and Zhao, 2007), where a reduction in a firm's marginal cost lowers social welfare if that firm initially has a low market share.[45]

5.2 Computational Algorithm for Applied Work

Numerically solving for the equilibrium of a multiproduct-firm pricing game in an industry with many firms and products can be a daunting task with standard methods, due to the high dimensionality of the problem. Exploiting the aggregative structure of the pricing game allows us to reduce this dimensionality tremendously: Instead of solving a system of $|\mathcal{N}|$ nonlinear equations in $|\mathcal{N}|$ unknowns, we only need to look for an $H > 0$ such that $\Gamma(H) = H$. Of course, there usually will not be a closed-form expression for $\Gamma(\cdot)$, so we still need to compute this function numerically. But $\Gamma(H)$ is simple to compute as well, since all we need to do is solve for $|\mathcal{F}|$ separate equations, each with one unknown. Below, we describe how this general approach can be implemented to solve a multiproduct-firm pricing game with CES or MNL demands.

The algorithm uses two nested loops. The inner loop computes $\Gamma(H)$ for a given H. The outer loop iterates on H. We start by describing the inner loop. Fix some $H > 0$. We first need to compute $\mu^f = m^f(T^f/H)$ for every f. We have shown that μ^f solves equation (13) in the CES case, and equation (14) in the MNL case. These equations can be rewritten as follows:

$$0 = \psi^f(\mu^f) \equiv \begin{cases} \mu^f \left(1 - \frac{\sigma-1}{\sigma} \frac{T^f}{H} \left(1 - \frac{\mu^f}{\sigma}\right)^{\sigma-1}\right) - 1 & \text{(CES)}, \\ \mu^f \left(1 - \frac{T^f}{H} e^{-\mu^f}\right) - 1 & \text{(MNL)}. \end{cases} \tag{16}$$

We solve equation (16) numerically using the Newton-Raphson method with analytical derivatives. The usual problem with the Newton-Raphson method is that it may fail to converge if starting values are not good enough. This is potentially a major issue, because the value of $\Gamma(H)$ used by the outer loop would then be incorrect. The following starting values guarantee convergence:

$$\mu_0^f = \begin{cases} \max\left(1, \sigma\left(1 - \left(\frac{H}{T^f}\right)^{\frac{1}{\sigma-1}}\right)\right) & \text{(CES)}, \\ \max\left(1, \log \frac{T^f}{H}\right) & \text{(MNL)}. \end{cases}$$

In fact, the Newton-Raphson method converges extremely fast (usually less than 5 steps). Notice, in addition, that this method can easily be vectorized by stacking up the μ^fs in a

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[45]Wang and Zhao (2007) claim that, with MNL demand, a reduction in a (single-product) firm's marginal cost can raise social welfare. Proposition 11 shows that this statement is incorrect.

vector. Having computed μ^f for every firm f, we can calculate $\Gamma(H)$ (see equation (15)).

The outer loop iterates on H to solve equation $\Gamma(H)/H - 1 = 0$. This can be done by using standard derivative-based methods. The Jacobian can be computed analytically:
$\frac{d}{dH}\frac{\Gamma(H)}{H} = -\sum_{f\in\mathcal{F}}\frac{T^f}{H^2}S'\left(\frac{T^f}{H}\right)$, where[46]

$$\frac{T^f}{H}S'\left(\frac{T^f}{H}\right) = \begin{cases} \frac{\mu^f-1}{\frac{\sigma-1}{\sigma}\mu^f\left(1+(\sigma-1)(\mu^f-1)\frac{\mu^f}{\sigma-\mu^f}\right)} & \text{(CES)}, \\[2ex] \frac{\mu^f-1}{\mu^f\left(1+\mu^f(\mu^f-1)\right)} & \text{(MNL)}. \end{cases}$$

We use the value of H that would prevail under monopolistic competition as starting value ($H_0 = \sum_{f\in\mathcal{F}}T^f\left(1-\frac{1}{\sigma}\right)^{\sigma-1}$ under CES demand, $H^0 = \sum_{f\in\mathcal{F}}T^f\,\mathrm{e}^{-1}$ under MNL demand), and we always obtain convergence (usually in about 10 steps).[47]

6 Applications to Merger Analysis and International Trade

In this section, we apply the aggregative games approach to analyze static and dynamic merger policy, and to study the effect of trade liberalization on the inter- and intra-firm size distributions, average industry-level productivity, and welfare. Throughout, we assume that demand is either of the CES or MNL forms so that the type aggregation property (see Section 5.1) holds.

Much of the existing literature on merger analysis, including Farrell and Shapiro (1990) and Nocke and Whinston (2010), relies on the homogeneous-goods Cournot model. In the first part of this section, we provide a necessary and sufficient condition for a merger to increase consumer surplus (resp. social welfare), and a sufficient condition for a merger to have a positive external effect, thereby extending the classic results of Farrell and Shapiro (1990) to the case of price competition between multiproduct firms. In the second part, we extend Nocke and Whinston (2010) by showing that a myopic merger approval policy is dynamically optimal if the goal is to maximize discounted consumer surplus. For both our static and dynamic merger analysis, the type aggregation property turns out to be very useful. In principle, merger-specific synergies could take many forms: Some of the marginal costs of the merged firms' may go down (while those of others may go up); some of the products' qualities may improve (while others may degrade); the merged entity may offer new products (while possibly withdrawing others). The type aggregation property implies that we do not need to impose any restrictions as all relevant information can be summarized in the merged firm's post-merger type.

In our trade application, we show that a (unilateral) trade liberalization will magnify

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[46]We derive these formulas in Online Appendix VIII.1.

[47]In Breinlich, Nocke, and Schutz (2015), we use this algorithm to calibrate an international trade model with two countries, 160 manufacturing industries, CES demand and oligopolistic competition.

relative market share differences between domestic firms, and leave the intra-firm distribution of market shares unchanged.[48] Different predictions obtain for both the inter- and intra-firm *sales* distributions in the MNL case. A unilateral trade liberalization shifts relative sales toward firms with high market shares and high average costs, and within each firm toward high-cost products. We also propose a new measure of firm-level productivity that allows us to deal with the fact that firms are heterogeneous in terms of scope, and product-level marginal costs and qualities. We find that a trade liberalization raises average industry-level productivity by reallocating market shares toward more productive firms. Finally, we provide sufficient conditions for a unilateral trade liberalization to raise (or lower) domestic welfare.

6.1 Static Merger Analysis

Partition the set of firms, set $\mathcal{F}$, into $\mathcal{I}$ (the insiders) and $\mathcal{O}$ (the outsiders), and suppose that the insiders merge. Let H^* (resp., $\hat{H}^*$) denote the equilibrium value of the aggregator before (resp., after) the merger. As consumer surplus is increasing in the value of that aggregator, we say that the merger is *CS-increasing* (resp., *CS-decreasing*) if $\hat{H}^* > H^*$ (resp., $\hat{H}^* < H^*$); it is *CS-neutral* if $\hat{H}^* = H^*$. Similarly, we say that a merger is *W-increasing* if it raises social welfare, etc. Let T^M denote the merged firm's post-merger type, which takes into account any possible merger-specific (in-)efficiencies in scope, marginal costs, and qualities.

Consumer surplus and social welfare effects. The following proposition provides necessary and sufficient conditions for a merger to be CS-increasing (resp. W-increasing):

Proposition 12. *There exist two cutoffs $\hat{T}^M, \tilde{T}^M > 0$ such that merger M is*

- *CS-neutral if $T^M = \hat{T}^M$, CS-increasing if $T^M > \hat{T}^M$, and CS-decreasing if $T^M < \hat{T}^M$;*

- *W-neutral if $T^M = \tilde{T}^M$, W-increasing if $T^M > \tilde{T}^M$, and W-decreasing if $T^M < \tilde{T}^M$.*

Moreover, $\hat{T}^M > \max\left(\tilde{T}^M, \sum_{f \in \mathcal{I}} T^f\right)$. If M is CS-nondecreasing, it is profitable in that it raises the joint profit of the merger partners.

Proof. See Online Appendix X.1. □

Inequality $\hat{T}^M > \sum_{f \in \mathcal{I}} T^f$ means that, for a merger to be CS-nondecreasing, the merger has to involve synergies, as in Williamson (1968) and Farrell and Shapiro (1990). To see why a CS-nondecreasing merger is profitable, consider first a CS-neutral merger. As such a merger involves synergies and does not change the equilibrium value of the aggegator, it must be profitable. By part (ii) of Proposition 11, merger M is even more profitable if it is CS-increasing rather than CS-neutral since it involves larger synergies.[49] In contrast to Farrell

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[48]Recall that market shares are defined in value under CES demand and in volume under MNL demand.

[49]The insight that CS-nondecreasing mergers are profitable was first obtained by Nocke and Whinston (2010) in the context of the homogeneous-goods Cournot model.

and Shapiro (1990), we are also able to exploit the monotonicity of social welfare in firms' types to establish the existence of a cutoff type $\tilde{T}^M$ such that the merger is W-increasing if and only if $T^M > \tilde{T}^M$. Inequality $\hat{T}^M > \tilde{T}^M$ follows immediately from the fact that a CS-neutral merger is profitable.[50]

External effects. The aggregative structure of our pricing game allows us to extend Farrell and Shapiro (1990)'s analysis of the external effects of a merger. To the extent that a merger is proposed by the merger partners only if it is in their joint interest to do so, a positive external effect is a sufficient ("safe harbor") condition for the merger to raise social welfare. The idea behind focusing on the external effect is that the profitability of a merger depends on the magnitude of internal cost savings, and that these are hard to assess for an antitrust authority.

The external effect of the merger, defined as the sum of its impact on consumer surplus and outsiders' profits, is given by (recall that $\pi(\cdot) = m(\cdot) - 1$):

$$\log \hat{H}^* - \log H^* + \sum_{f \in \mathcal{O}} \left(m\left(\frac{T^f}{\hat{H}^*}\right) - m\left(\frac{T^f}{H^*}\right) \right) = -\int_{H^*}^{\hat{H}^*} \frac{\eta(H)}{H} dH,$$

where

$$\eta(H) \equiv -1 + \sum_{f \in \mathcal{O}} \frac{T^f}{H} m'\left(\frac{T^f}{H}\right).$$

Hence, as in Farrell and Shapiro (1990), the merger can be thought of as a sequence of infinitesimal mergers dH, where, along the sequence, the value of the aggregator changes progressively from H^* to $\hat{H}^*$. The sign of the external effect of an infinitesimal CS-decreasing (resp. CS-increasing) merger is thus given by $\eta(H)$ (resp. $-\eta(H)$). In the following, we focus on CS-decreasing mergers to fix ideas.

An infinitesimal CS-decreasing merger $dH < 0$ reduces consumer surplus by dH/H, which corresponds to the first term in the definition of η. It also increases the profit of every outsider $f \in \mathcal{O}$ by dH/H times $(T^f/H)m'(T^f/H)$. Defining $\phi_\alpha(S(T^f/H)) \equiv (T^f/H)m'(T^f/H)$ with $\alpha = (\sigma - 1)/\sigma$ in the CES case and $\alpha = 1$ in the MNL case, we show in Online Appendix X.2 that this profit change can be rewritten as:

$$\phi_\alpha(s) = \frac{\alpha s(1-s)}{(1-\alpha s)(1-s+\alpha s^2)}, \quad \forall s \in (0,1).$$

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[50]Whether or not $\tilde{T}^M > \sum_{f \in \mathcal{I}} T^f$ is unclear. On the one hand, a merger that does not involve synergies lowers the equilibrium aggregator level. On the other hand, it reallocates market shares toward the outsiders, which can raise social welfare if the outsiders are initially producing too little.

Hence, the external effect of an infinitesimal CS-decreasing merger is given by

$$\eta(H)\frac{dH}{H} = \left(-1 + \sum_{f \in \mathcal{O}} \phi_\alpha(s^f)\right)\frac{dH}{H}.$$

Computing the external effect of an infinitesimal merger therefore only requires knowledge of the outsiders' market shares and of the value of the demand-side parameter α.

To go further, we study the behavior of function $\phi_\alpha(\cdot)$:

Proposition 13. *There exists $\bar{\alpha}(\simeq 0.82)$ such that:*

(i) If $\alpha \leq \bar{\alpha}$, then the external effect of any CS-decreasing merger is negative.

(ii) If $\alpha > \bar{\alpha}$, then there exist CS-decreasing mergers that have positive external effects, and CS-increasing mergers that have negative external effects.

Moreover, if $\alpha > \bar{\alpha}$, then there exist thresholds $s^*(\alpha) \in (0, 1]$ and $\hat{s}(\alpha) \in (0, 1)$ such that:

(iii) $s \mapsto \phi_\alpha(s)$ is strictly increasing on $(0, s^*(\alpha))$ and strictly decreasing on $(s^*(\alpha), 1)$.[51]

(iv) $s \mapsto \phi_\alpha(s)$ is strictly convex on $(0, \hat{s}(\alpha))$ and strictly concave on $(\hat{s}(\alpha), 1)$.[52]

Proof. See Online Appendix X.2. □

Part (i) of the proposition implies that, if $\alpha < \bar{\alpha}$ (i.e., demand is of the CES form and $\sigma < \bar{\sigma} \simeq 5.53$), then the positive effect of the CS-decreasing merger on outsiders' profits is always outweighed by the negative effect on consumer surplus, implying a negative external effect.

Next, suppose that $\alpha > \bar{\alpha}$ (i.e., demand is either of the MNL form or of the CES form with $\sigma > \bar{\sigma}$). Then, by part (ii) of the proposition, there exist CS-decreasing mergers that have positive (resp. negative) external effects. Parts (iii) and (iv) provide conditions under which the external effect of an infinitesimal CS-decreasing merger is more likely to be positive. To understand these conditions, note that the decrease in H has two effects on an outsider's profit. First, holding fixed outsiders' markups, it increases the profit of each outsider f by $\Pi^f \times |dH/H|$. Hence, the "direct" effect on outsiders' joint profit is proportional to their joint profit. Second, outsiders respond by increasing their markups.

To grasp the intuition for parts (iii) and (iv), it is useful to focus on the first, direct effect. By part (iii), the merger is more likely to have a positive external effect if the outsiders have high market shares (provided no outsider has a market share above $s^*(\alpha)$). The intuition is that if outsiders command larger market shares, they make larger profits, and therefore benefit more from the direct effect of the reduction in H. By part (iv), the merger is more likely to have a positive external effect if the outsiders' market shares are more concentrated

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[51]In the MNL case, $s^*(\alpha) = 1$; in the CES case, numerically, we find that $s^*(\alpha) \geq 0.68$ for every α.
[52]Numerically, we find that $\hat{s}(\alpha) \geq 0.28$ for every α.

(provided no outsider has a market share above $\hat{s}(\alpha)$).[53] The intuition is that if outsiders' market shares are more concentrated, their joint profit is larger, implying that they jointly benefit more from the direct effect of the reduction in H.[54]

Part (iv) implies that relying on the pre-merger Herfindahl-Hirschman index (HHI) to evaluate the social desirability of a merger can be misguided. To see this, consider two industries, and suppose that the vector of insiders' market shares is the same in both industries. Suppose also that outsiders' market shares are more concentrated in the first industry than in the second. Then, the first industry's HHI is higher than the second's. However, the merger in the first industry is more likely to have a positive external effect than the one in the second industry.

The external effect of the non-infinitesimal CS-decreasing merger M is the integral of the external effects of the infinitesimal mergers along the path from H^* to $\hat{H}^* < H^*$. If products are relatively poor substitutes ($\alpha < \bar{\alpha}$), then, by part (i) of Proposition 13, merger M has a negative external effect. Suppose instead that products are relatively good substitutes ($\alpha > \bar{\alpha}$). As the merger is CS-decreasing by assumption, outsiders' market shares increase along the sequence of infinitesimal mergers from H^* to $\hat{H}^*$. Hence, if $\eta(H^*) > 0$ (i.e., at the pre-merger aggregator level, an infinitesimal CS-decreasing merger has a positive external effect), then, by part (iii) of Proposition 13, $\eta(H)$ remains positive along the sequence (provided no outsider reaches a market share larger than s^*), and so the external effect of the merger is positive. But to check whether $\eta(H^*) > 0$ involves using only the outsiders' *pre*-merger market shares.

6.2   Dynamic Merger Analysis

In many industries, mergers are not isolated events. Evaluating a proposed merger on the basis of current market conditions therefore appears inappropriate. However, future market conditions are affected by today's decision on the proposed merger, for two reasons. First, today's decision changes the welfare effects of potential future mergers and therefore the set of mergers that will be approved in the future. Second, today's decision changes the profitability of potential future mergers and therefore the set of mergers that will be proposed in the future.

For the case of Cournot competition with homogeneous goods, Nocke and Whinston (2010) show that this problem has a surprisingly simple solution when the authority's objective is to maximize discounted consumer surplus: A myopic merger approval policy that approves a proposed merger today if and only if it does not lower current consumer sur-

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[53]We say that the profile of market shares s' is *more concentrated* than the profile s if the cumulative distribution function of s' second-order stochastically dominates that of s. See Online Appendix X.2.

[54]The reason why this intuition may not hold if some of the outsiders are too large is the result of the second, indirect effect. Holding H fixed, the induced increase in an outsider's markup decreases its profit. This holds since oligopolistic markups are always above those of monopolistically competitive firms that perceive H as fixed, so any further increase must reduce profit for a fixed H. The qualifiers in parts (iii) and (iv) arise because the effect of a given increase in the markup is heterogeneous across outsiders, as is the extent of the induced markup increase.

plus (completely ignoring the possibility of future mergers) is dynamically optimal.[55] They consider a T-period model in which merger opportunities arise stochastically over time and, in each period, firms with a feasible but not-yet-approved merger have to decide whether to propose it and the antitrust authority has to decide which of the proposed mergers to approve (if any). In addition to Cournot competition, the two key assumptions are: First, the set of potential mergers is disjoint (i.e., no firm can be party to more than one potential merger). Second, rejected mergers can be proposed again in the future (i.e., merger opportunities do not disappear).

As we show in Online Appendix X.4, the optimality result carries over to the case of price competition between multiproduct firms when demand takes either the CES or MNL forms. The optimality result comes in two parts. First, assuming that all feasible and not-yet-approved mergers are proposed in every period, a myopic merger approval policy is dynamically optimal in that it maximizes the discounted sum of consumer surplus. Second, if the antitrust authority adopts a myopic merger approval policy, then in any subgame-perfect equilibrium, every merger that the authority would want to approve in the dynamically optimal solution will be proposed.

The key observation for the first part of the optimality result is the following:

Proposition 14. *For any merger M, the post-merger cutoff type $\hat{T}^M$ is decreasing in the pre-merger value H^* of the aggregator.*

Proof. See Online Appendix X.3. □

The intuition is straightforward: An increase in the aggregator reduces the market share of the merger partners and thereby the market power effect of the merger. The proposition implies a certain sign-preserving complementarity in the consumer surplus effects of mergers. Suppose mergers M_1 and M_2 are both CS-nondecreasing in isolation (i.e., given the pre-merger aggregator H^*). Then, each merger M_i remains CS-nondecreasing after the other merger M_{-i} has been implemented. Conversely, suppose both mergers are CS-decreasing in isolation. Then, each remains CS-decreasing after the other one has been implemented. The proposition also implies that a CS-decreasing merger M_2 may become CS-nondecreasing once a CS-increasing merger M_1 has been implemented; if so, M_1 remains CS-increasing conditional on M_2 taking place.[56]

The complementarity result implies that if the antitrust authority approves only mergers that are CS-nondecreasing at the time of approval, then it will never have ex-post regret about previously approved mergers as these will always remain CS-nondecreasing given the set of approved mergers. In conjunction with the assumption that merger opportunities do

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[55]Dynamic optimality obtains in a strong sense: The antitrust authority could not improve upon this outcome even if it had perfect foresight about future merger possibilities (which, by assumption, it does not) nor if it could undo previously approved mergers (which, by assumption, it cannot).

[56]To see this, note that if M_1 is implemented before M_2, then consumer surplus increases at each step. Hence, conditional on implementing M_2 (which, by assumption, decreases consumer surplus), M_1 must increase consumer surplus.

not disappear, the complementarity also implies that the authority will never have ex-post regret about having blocked mergers that were CS-decreasing at the time of the decision since these mergers can (and will) be proposed and approved once they become CS-nondecreasing.

Turning to the second part of the optimality result, Proposition 12 implies that any CS-nondecreasing merger M is profitable at the time of its approval. Moreover, if the authority adopts a CS-based merger approval policy, the merger remains CS-nondecreasing and therefore profitable given the set of other mergers that will be approved along the equilibrium path. Surprisingly, a CS-nondecreasing merger M remains profitable even if it induces (directly or indirectly) the implementation of additional mergers, all of which are CS-nondecreasing at the time of their approval (and therefore reduce the profit of the merged entity M).[57] As a result, a CS-based myopic merger approval policy solves the moral hazard problem arising from the fact that the authority can approve only mergers that are proposed in the first place: In any equilibrium, any merger that the authority would like to approve will be proposed.

6.3   Trade Analysis

We now apply our framework to study classic questions in international trade. As discussed in Section 1.1, existing papers with multiproduct firms have studied the impact of trade mostly in the context of monopolistic competition with either CES or linear demands, and specific assumptions on how marginal costs are allowed to vary within firms. We have already seen in Section 4.3 that our model's predictions on firm scope differ markedly from those in the existing literature. As we will show below, our model also provides novel predictions on the effects of a unilateral trade liberalization on the inter- and intra-firm size distributions, average productivity, and welfare.

As argued in Section 4, a trade liberalization that improves the access of foreign firms to the domestic market can be thought of as an increase in the value of the outside option, H^0. By Proposition 5, the increase in the value of H^0 in turn leads to an increase in the equilibrium value of the aggregator, H, thereby benefiting consumers but hurting domestic firms.

Inter-firm size distribution. As we show in Online Appendix X.5, the ratio of market shares between domestic firm f and a smaller domestic firm g of type $T^g < T^f$, $S(T^f/H)/S(T^g/H)$, is increasing in H. That is, a trade liberalization leads to a smaller fractional decrease in the market share of a larger than a smaller domestic firm, and thereby magnifies relative size differences. To put this result into perspective, consider the case of monopolistic competition, where firms take H as given. As demand is inversely proportional to H, a trade liberalization would, in that case, not affect prices and would thus have no impact on the market share ratio of two domestic firms. In contrast, under oligopolistic competition, firms reduce their markups in response to the increased competition from abroad. But a small oligopolistic firm (small T) will not reduce its markup much as it cannot affect H much; as a result, its

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[57]A merger proposal can only increase, not decrease the set of other mergers that will be implemented.

fractional reduction in market share is almost as large as the fractional increase in H, and therefore larger than that of a large firm (large T).

Recall from Section 5 that firm f's market share $S(T^f/H)$ is measured in value (i.e., sales) in the case of CES demand, and in volume (i.e., output) in the case of MNL demand. As sales data are often more readily available than output data, in the following we also derive the effect of a trade liberalization on the domestic sales distribution in the MNL case. In that case, firm f's sales share depends not only on f's type T^f (and the aggregator H) but also on its (output-weighted) average cost $\bar{c}^f$:

$$\text{Sales}^f = \frac{1}{H} \left(\sum_{j \in f} p_j \exp \frac{a_j - p_j}{\lambda} \right) = \frac{T^f}{H} \, \mathrm{e}^{-\mu^f} \left(\lambda \mu^f + \bar{c}^f \right),$$

where

$$\bar{c}^f \equiv \sum_{k \in f} \left(\frac{e^{\frac{a_k - c_k}{\lambda} - \mu^f}}{\sum_{j \in f} e^{\frac{a_j - c_j}{\lambda} - \mu^f}} \right) c_k = \sum_{k \in f} \left(\frac{e^{\frac{a_k - c_k}{\lambda}}}{\sum_{j \in f} e^{\frac{a_j - c_j}{\lambda}}} \right) c_k.$$

As a result, the effect of a trade liberalization on the domestic inter-firm sales distribution is more subtle than in the CES case. As shown in Online Appendix X.5, there exists a function ϕ, decreasing in both arguments, such that a trade liberalization shifts sales from firm g toward firm f if and only if $\phi(T^f, \bar{c}^f) < \phi(T^g, \bar{c}^g)$. In particular, if (i) $T^f > T^g$ and $\bar{c}^f = \bar{c}^g$ or (ii) $T^f = T^g$ and $\bar{c}^f > \bar{c}^g$, firm f has larger sales than firm g, and a trade liberalization will magnify this size difference. But note that in case (ii) more sales are shifted toward the firm with the higher average cost.

Intra-firm size distribution. In Online Appendix X.5, we show that the ratio of market shares between any two products $j \in f$ and $k \in f$ offered by the same domestic firm f, s_j/s_k, is independent of the equilibrium value of the aggregator H. That is, the intra-firm distribution of market shares is invariant to the degree of international trade integration. This follows from two observations. First, the common ι-markup property implies a common relative markup under CES demand and a common absolute markup under MNL demand. While different firms reduce their markups to different degrees in response to the trade liberalization, each firm reduces all its prices by the same fraction (CES) or the same absolute amount (MNL). Second, all products have the same (constant) price elasticity of revenue (CES) or the same (constant) price semi-elasticity of demand (MNL).

Under MNL demand, a different picture emerges for the intra-firm sales distribution. As each firm reduces all its prices by the same absolute amount, a trade liberalization induces a shift in relative sales toward products with higher marginal costs: If $c_j > c_k$, then $(p_j s_j)/(p_k s_k)$ increases with an increase in H.

Firm-level productivity. The effects of a trade liberalization on industry-level productivity is a key question in the field of international trade. To address this question, we first

define productivity at the firm level. As marginal costs and qualities are allowed to vary across products within the same firm and as firms may differ in the number of products they offer, such a firm-level definition is not obvious. Below, we provide two arguments suggesting that a monotonic transform of the firm's type T^f, $\varphi^f \equiv \varphi(T^f)$ for some strictly increasing function φ, is the theoretically correct measure of productivity.

The composite commodity approach. Suppose the representative consumer has CES preferences. Define the firm-level composite commodity $Q^f \equiv \left(\sum_{j \in f} a_j^{\frac{1}{\sigma}} q_j^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}$ and the associated price index $P^f \equiv \left(\sum_{j \in f} a_j p_j^{1-\sigma} \right)^{\frac{1}{1-\sigma}}$ for every firm f. It is well known that the demand system can then equivalently be derived through two-stage budgeting, where the consumer first decides on $(Q^f)_{f \in \mathcal{F}}$, and then on $(q_j)_{j \in f}$ for every f. For this composite commodity approach, firm f's productivity is equal to the inverse of the firm's (constant) unit cost of producing the composite commodity Q^f, which can be computed as $(T^f)^{\frac{1}{\sigma-1}}$. Equivalently, it is equal to the ratio of firm f's sales over costs, with sales being deflated by the firm-level price index P^f. Unfortunately, such composite commodities are unavailable for the MNL case.

The indirect utility approach. Suppose the consumer makes choices in two steps. He first chooses a firm, and then a product in that firm's "nest." After having chosen firm f, the consumer observes a vector of i.i.d. Gumbel taste shocks and chooses the product that delivers the largest indirect utility. The inclusive value of nest f is then given by $V^f \equiv \log \sum_{j \in f} h_j(p_j)$. Suppose firm f is tasked to deliver the inclusive value in a profit-maximizing way (i.e., conditional on the consumer choosing its products' nest). It is straightforward to show that the resulting prices satisfy the common ι-markup property, and that a firm with a higher T^f delivers any inclusive value V^f in a more efficient way.

Industry-level productivity. We define domestic industry-level productivity Φ as the market-share-weighted average firm-level productivity:

$$\Phi \equiv \sum_{f \in \mathcal{F}} \left(\frac{s^f}{\sum_{g \in \mathcal{F}} s^g} \right) \varphi^f.$$

The reason for using market shares as weights is that, under both CES and MNL demands, these market shares are equal to firm-level choice probabilities in the discrete/continuous choice model.

We now consider the effect of a (unilateral) trade liberalization on Φ. An immediate observation is that firm-level productivities remain unchanged.[58] However, as a trade liberalization increases the relative market shares of high-productivity firms, it induces an increase

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[58]Recall that, in the CES case, productivity can be measured by the ratio of firm-level sales over costs, with sales being deflated by the firm-level CES price index. Without access to the ideal firm-level price index, it would appear that a trade-liberalization does affect firm-level productivity. The same applies to the MNL case, for which ideal firm-level price indices do not even exist theoretically.

in the average industry-level productivity Φ. This is shown formally in Online Appendix X.5.

Welfare effects. By increasing the equilibrium value of the aggregator H, a unilateral trade liberalization benefits domestic consumers. As shown above, it also raises the average industry-level productivity of domestic firms. The underlying shift in market shares from less productive to more productive domestic firms reduces the relative markup distortions discussed in Section 4.3. The downside of a unilateral trade liberalization is that it increases the market share of foreign firms (the outside option).

The overall effect on domestic welfare, defined as the sum of consumer surplus and the profit of domestic firms, is identical to the external effect of a CS-increasing merger. From our analysis in Section 6.1, we obtain the following. First, if $\sigma < \bar{\sigma}$ under CES demand, then a unilateral trade liberalization always increases domestic welfare. Second, under MNL demand or if $\sigma > \bar{\sigma}$ under CES demand, a small unilateral trade liberalization is more likely to have a positive domestic welfare effect if (i) the joint market share of domestic firms is smaller, and if (ii) the domestic industry is less concentrated. In contrast, under monopolistic competition with CES or MNL demands (see Online Appendix X.5) or linear demand (see Mayer, Melitz, and Ottaviano, 2014), a unilateral trade liberalization would have an unambiguously positive effect on domestic welfare (for a fixed set of firms).

7 Conclusion

The main contribution of this paper consists in developing a tractable approach to multiproduct-firm oligopoly. The aggregative structure of the game and the common ι-markup property deliver simple, yet powerful existence, uniqueness and characterization results. Our approach gives rise to a computationally efficient algorithm, and to a simple decomposition of the welfare distortions in multiproduct-firm oligopoly. Monotone comparative statics results allow us to make predictions on how markups and firm scope vary with the competitive environment.

Under CES and MNL demands, for which type aggregation obtains, we provide applications to classic questions in static and dynamic merger analysis, and to the effects of a trade liberalization. Specifically, we derive necessary and sufficient conditions for a merger to increase consumer surplus and social welfare; we analyze the external effect of a merger; we show the dynamic optimality of a myopic merger approval policy; and we study the effects of a trade liberalization on the inter- and intra-firm size distributions, industry-level average productivity, and domestic welfare.

Another contribution of the paper consists in providing a complete characterization of the class of demand systems that can be derived from discrete/continuous choice with i.i.d. Gumbel taste shocks. In recent empirical work, Björnerstedt and Verboven (2016) use (random coefficient) CES demand rather than the ubiquitous (random coefficient) MNL demand because of its different implications for markup behavior. Our paper demonstrates that richer patterns of markups can be obtained by using conditional demand specifications that

go beyond the special cases of CES and MNL.

Our paper suggests a number of exciting avenues for future research. In Section 4.4, we have provided a complete characterization of the class of demand systems that give rise to an aggregative game with additive aggregation. The next step in this direction involves obtaining a similar characterization for the case of non-additive aggregation. This is equivalent to characterizing the set of demand systems that satisfy generalized separability in the sense of Pollak (1972) with quasi-linear preferences. In Section 4.4, we have also shown that our multiproduct-firm pricing game has an ordinal potential. Conversely, it would be interesting to analyze which properties of the demand system give rise to a potential game.[59] In Section 2, we have characterized the class of demand systems that can be derived from discrete/continuous choice with i.i.d. Gumbel taste shocks. An open question is what can be said when the distributional assumption on taste shocks is relaxed.[60]

A Proof of Theorem 1

Let $((h_j)_{j \in \mathcal{N}}, \mathcal{F}, (c_j)_{j \in \mathcal{N}})$ be a pricing game satisfying Assumption 1. The following lemma ensures that each firm sets at least one finite price in any equilibrium.

Lemma A. *In any Nash equilibrium $(p_j^*)_{j \in \mathcal{N}}$, for every firm $f \in \mathcal{F}$, there exists $k \in f$ such that $p_k^* < \infty$.*

Proof. This follows from profit function (3): If firm f sets $p_j^* = \infty$ for all $j \in f$, then it makes zero profit; if instead it sets $p_k \in (c_k, \infty)$ for some $k \in f$, then its profit is strictly positive. $\square$

Next, we show that first-order conditions (appropriately generalized to handle infinite prices) are necessary and sufficient for global optimality. To this end, fix a firm $f \in \mathcal{F}$ and a price vector for firm f's rivals $(p_j)_{j \in \mathcal{N} \setminus \{f\}}$, and denote $H^0 = \sum_{j \in \mathcal{N} \setminus \{f\}} h_j(p_j)$. Suppose that at least one of the prices set by firm f's rivals is finite, and define

$$\Pi^f((p_j)_{j \in f}, H^0) = \sum_{\substack{k \in f \\ p_k < \infty}} (p_k - c_k) \frac{-h_k'(p_k)}{\sum_{j \in f} h_j(p_j) + H^0}. \tag{17}$$

Note that $\Pi^f((p_j)_{j \in f}, H^0)$ is the profit of firm f when it sets price vector $(p_j)_{j \in f}$ and its rivals set price vector $(p_j)_{j \in \mathcal{N} \setminus \{f\}}$. We study the following maximization problem:

$$\max_{(p_j)_{j \in f} \in (0, \infty]^f} \Pi^f\left((p_j)_{j \in f}, H^0\right). \tag{18}$$

We prove the following lemma:

---

[59]In Nocke and Schutz (2016a), we make some headway in this direction.
[60]In the same spirit, Anderson, de Palma, and Thisse (1989) and Armstrong and Vickers (2015) analyze the class of demand systems that can be derived from pure discrete choice.

Lemma B. *Maximization problem* (18) *has a solution. Moreover, if* $(p_j)_{j\in f}$ *solves maximization problem* (18)*, then* $p_j \geq c_j$ *for all* $j \in f$*, and* $p_k < \infty$ *for some* $k \in f$*.*

Proof. The observation that firm f sets a least one finite price follows from the argument in the proof of Lemma A. Next, we show that firm f does not set any price below marginal costs. Let $(p_j)_{j\in f} >> 0$. Suppose that $p_k < c_k$ for some k in f, and let $\tilde{p}_j \equiv \max(c_j, p_j)$ for every j in f. When firm f deviates from $(p_j)_{j\in f}$ to $(\tilde{p}_j)_{j\in f}$, it stops making losses on products j such that $p_j < c_j$, and it raises the demand for products j such that $p_j \geq c_j$. Therefore, $\Pi^f((p_j)_{j\in f}, H^0) < \Pi^f((\tilde{p}_j)_{j\in f}, H^0)$, and $(p_j)_{j\in f}$ is not a solution of maximization problem (18).

Next, we show that maximization problem (18) has a solution. Assume without loss of generality that $f = \{1, \ldots, n\}$. For every $k \in f$ and $x_k \in [0, 1]$, define

$$
\phi_k(x_k) = \begin{cases} c_k + \frac{x_k}{1-x_k} & \text{if } x_k < 1, \\ \infty & \text{if } x_k = 1. \end{cases}
$$

Note that $\lim_1 \phi = \phi(1) = \infty$. For every $x \in [0, 1]^n$, define $\phi(x) = (\phi_1(x_1), \ldots, \phi_n(x_n))$. ϕ is a bijection from $[0, 1]^n$ to $\prod_{k=1}^n [c_k, \infty]$. Finally, let

$$
\Psi(x) = \Pi^f\left(\phi(x), H^0\right), \quad \forall x \in [0, 1]^n.
$$

Since ϕ is a bijection, maximization problem $\max_{x\in[0,1]^n} \Psi(x)$ has a solution if and only if maximization problem (18) has a solution. All we need to do now is show that Ψ is continuous on $[0, 1]^n$.

Clearly, Ψ is continuous at every point x such that $x_k < 1$ for every $1 \leq k \leq n$. Next, let x such that $x_k = 1$ for some $1 \leq k \leq n$. To fix ideas, suppose that $x_k = 1$ for all $1 \leq k \leq K$, and that $x_k < 1$ for all $K + 1 \leq k \leq n$, where $K \geq 1$. Then,

$$
\begin{aligned}
\lim_{\tilde{x}\to x} \Psi(\tilde{x}) &= \lim_{\tilde{x}\to x} \sum_{k=1}^n (\phi_k(\tilde{x}_k) - c_k) \frac{-h'_k(\phi_k(\tilde{x}_k))}{\sum_{j=1}^n h_j(\phi_j(\tilde{x}_j)) + H^0}, \\
&= \frac{\sum_{k=1}^n \lim_{\tilde{x}_k\to x_k} (\phi_k(\tilde{x}_k) - c_k)(-h'_k(\phi_k(\tilde{x}_k)))}{\sum_{j=1}^n \lim_{\tilde{x}_j\to x_j} h_j(\phi_j(\tilde{x}_j)) + H^0}, \\
&= \frac{0 + \sum_{k=K+1}^n (\phi_k(x_k) - c_k)(-h'_k(\phi_k(x_k)))}{\sum_{j=1}^K h_j(\infty) + \sum_{j=K+1}^n h_j(\phi_j(p_j)) + H^0}, \\
&= \Psi(x),
\end{aligned}
$$

where the third line follows by Lemma III-(a) in Online Appendix II.1. Therefore, Ψ is continuous. Combining this with the fact that $[0, 1]^n$ is compact implies that maximization problem $\max_{x\in[0,1]^n} \Psi(x)$ has a solution. $\square$

The next step is to solve the firm's maximization problem using first-order conditions. The problem is that the profit function is not necessarily differentiable at infinite prices, so

we will need to generalize the definition of first-order conditions to account for that. Note first that, if all products in $f' \subsetneq f$ are priced at infinity, then profit function $\Pi^f(., H^0)$ is still $\mathcal{C}^2$ in $(p_j)_{j \in f \setminus f'} \in \mathbb{R}_{++}^{f \setminus f'}$, as can be seen by inspecting profit function (17). Next, we slightly abuse notation, by denoting $\left(p_k, (p_j)_{j \in f \setminus \{k\}}\right)$ the price vector with k-th component p_k, and with other components given by $(p_j)_{j \in f \setminus \{k\}}$. We generalize first-order conditions as follows:

Definition A. *We say that the generalized first-order conditions of maximization problem (18) hold at price vector $(\tilde{p}_j)_{j \in f} \in (0, \infty]^f$ if for every $k \in \mathcal{N}$,*

(a) $\frac{\partial \Pi^f}{\partial p_k}((\tilde{p}_j)_{j \in f}, H^0) = 0$ *whenever* $\tilde{p}_k < \infty$, *and*

(b) $\Pi^f((\tilde{p}_j)_{j \in f}, H^0) \geq \Pi^f\left(\left(p_k, (\tilde{p}_j)_{j \in f \setminus \{k\}}\right), H^0\right)$ *for every* $p_k \in \mathbb{R}_{++}$ *whenever* $\tilde{p}_k = \infty$.

It is obvious that generalized first-order conditions are necessary for optimality:

Lemma C. *If $(p_j)_{j \in f} \in (0, \infty]^f$ solves maximization problem (18), then the generalized first-order conditions are satisfied at price profile $(p_j)_{j \in f}$.*

Next, we want to show that, if the generalized first-order conditions hold at a price vector, then this price vector satisfies a generalized version of the common ι-markup property introduced in Section 4.1. To define this generalized common ι-markup property, we first need to establish a few facts about functions $\nu_k : p_k \mapsto \frac{p_k - c_k}{p_k} \iota_k(p_k)$. Let p_k^{mc} be the unique solution of equation $\nu_k(p_k) = 1$. Product k would be priced at p_k^{mc} under monopolistic competition. We prove the following lemma:

Lemma D. *For every $k \in f$, function ν_k is a strictly increasing $\mathcal{C}^1$-diffeomorphism from (p_k^{mc}, ∞) to $(1, \bar{\mu}_k)$. Denote its inverse function by r_k. Then, for all $\mu^f \in (0, \bar{\mu}_k)$,*

$$r_k'(\mu^f) = \frac{\gamma_k\left(r_k(\mu^f)\right)}{\mu^f\left(-\gamma_k'\left(r_k(\mu^f)\right)\right) - (\mu^f - 1)\left(-h_k'\left(r_k(\mu^f)\right)\right)} > 0. \tag{19}$$

Moreover, for every $\mu^f \in (1, \bar{\mu}_k)$, $r_k(\mu^f)$ is the unique solution of equation $\nu_k(p_k) = \mu^f$ on interval $(0, \infty)$.

Proof. We first argue that p_k^{mc} is well-defined. To see this, note that ν_k is continuous and, by Lemma III in Online Appendix II.1, $\nu_k(p_k) < 1$ for every $p_k < \max(\underline{p}_k, c_k)$ (where $\underline{p}_k$ is defined in Lemma III), $\lim_\infty \nu_k = \bar{\mu}_k > 1$, and ν_k is strictly increasing on $(\max(\underline{p}_k, c_k), \infty)$. Therefore, p_k^{mc} is well-defined, and $p_k^{mc} > \max(\underline{p}_k, c_k)$.

Since $p_k^{mc} > \max(\underline{p}_k, c_k)$, it follows from by Lemma III-(b) in Online Appendix II.1 that function ι_k is non-decreasing on (p_k^{mc}, ∞), and that $\nu_k'(p_k) > 0$ for every $p_k > p_k^{mc}$. By the inverse function theorem, ν_k is a $\mathcal{C}^1$-diffeomorphism from (p_k^{mc}, ∞) to $\nu_k((p_k^{mc}, \infty))$, and $r_k'(\mu^f) = 1/\nu_k'(r_k(\mu^f))$. Note that

$$\nu_k'(p_k) = \left(\frac{-(p_k - c_k)h_k'(p_k)}{\gamma_k(p_k)}\right)' = \frac{-h_k' - (p_k - c_k)h_k' + \gamma_k'(p_k - c_k)\frac{h_k'}{\gamma_k}}{\gamma_k} = \frac{(\nu_k - 1)h_k' - \nu_k\gamma_k'}{\gamma_k}.$$

This proves equation (19). Since ν_k is strictly increasing,

$$\nu_k\left((p_k^{mc}, \infty)\right) = \left(\lim_{p_k^{mc}} \nu_k, \lim_\infty \nu_k\right) = (1, \bar{\mu}_k).$$

Finally, let $\mu^f \in (1, \bar{\mu}_k)$ and $p_k > 0$ such that $\nu_k(p_k) = \mu^f$. Then, $\iota_k(p_k) > 1$, and, by Lemma III-(b) in Online Appendix II.1, $p_k > \underline{p}_k$. Since ν_k is strictly increasing on interval $(\underline{p}_k, \infty)$, we can conclude that p_k is unique, and that $p_k = r_k(\mu^f)$. □

We extend functions ν_k and r_k by continuity as follows: $\nu_k(\infty) = \bar{\mu}_k$, $r_k(1) = p_k^{mc}$, and $r_k(\mu^f) = \infty$ for every $\mu^f \geq \bar{\mu}_k$. We can now generalize the common ι-markup property to price vectors with infinite components:

Definition B. *We say that price vector* $(p_j)_{j \in f} \in (0, \infty]^f$ *satisfies the common ι-markup property if there exists a scalar $\mu^f \geq 1$, called the ι-markup, such that $p_k = r_k(\mu^f)$ for every $k \in f$.*

For every $k \in \mathcal{N}$, extend γ_k by continuity at infinity: $\gamma_k(\infty) = 0$ (see Lemma III-(e) in Online Appendix II.1). The following lemma allows us to simplify first-order conditions considerably:

Lemma E. *Suppose that the generalized first-order conditions for maximization problem (18) hold at price vector $(p_j)_{j \in f} \in (0, \infty]^f$. Then, $(p_j)_{j \in f}$ satisfies the common ι-markup property. The corresponding ι-markup, μ^f, solves the following equation on interval $(1, \infty)$:*

$$\mu^f = 1 + \mu^f \frac{\sum_{j \in f} \gamma_j\left(r_j(\mu^f)\right)}{\sum_{j \in f} h_j\left(r_j(\mu^f)\right) + H^0}. \tag{20}$$

In addition, $\Pi^f\left((p_j)_{j \in f}, H^0\right) = \mu^f - 1$.

Proof. Assume without loss of generality that $f = \{1, \ldots, n\}$, and let $f' = \{k \in f : p_k < \infty\}$. Clearly, $f' \neq \emptyset$, because if this set were empty, then the firm could obtain a strictly positive profit by pricing, say, product 1, at some finite price $p_1 > c_1$, which would violate condition (b) in Definition A. Assume without loss of generality that $f' = \{1, \ldots, K\}$, where $1 \leq K \leq n$. Then, following the same reasoning as in Section 4.1, we obtain the existence of a $\mu^f > 1$ such that

$$\mu^f = 1 + \mu^f \frac{\sum_{k=1}^K \gamma_k\left(r_k(\mu^f)\right)}{\sum_{j=1}^K h_j\left(r_j(\mu^f)\right) + \sum_{j=K+1}^n h_j(\infty) + H^0}, \tag{21}$$

and, for every $1 \leq k \leq K$, $p_k = r_k(\mu^f)$. Moreover, $\Pi^f((p_j)_{j \in f}, H^0) = \mu^f - 1$.

Next, we claim that, for every $j \geq K+1$, $r_j(\mu^f) = \infty$, or, equivalently, $\bar{\mu}_j \leq \mu^f$. Assume for a contradiction, that, for some $j \geq K+1$, $\bar{\mu}_j > \mu^f$. To fix ideas, assume that this j is equal to $K+1$. Let $\tilde{\Pi}^f(x)$ and $\tilde{D}_{K+1}(x)$ be the profit of firm f and the demand for product $K+1$ at price vector $(p_1, \ldots, p_K, x, \infty, \ldots, \infty)$. Note that $\tilde{\Pi}^f(x)$ tends to $\Pi^f((p_j)_{j \in f}, H^0) = \mu^f - 1$

as x goes to infinity (see the proof of Lemma B). Using the expression of marginal profit given in equation (4), we see that, for every $x \in \mathbb{R}_{++}$,

$$\tilde{\Pi}^{f\prime}(x) = \tilde{D}_{K+1}(x) \left(1 - \nu_{K+1}(x) + \frac{\sum_{j=1}^{K}(p_j - c_j)\left(-h_j'(p_j)\right) + (x - c_{K+1})h_{K+1}'(x)}{\sum_{j=1}^{K} h_j(p_j) + h_{K+1}(x) + \sum_{j=K+2}^{n} h_j(\infty) + H^0} \right),$$

$$= \tilde{D}_{K+1}(x) \left(1 - \underbrace{\nu_{K+1}(x)}_{\underset{x\to\infty}{\longrightarrow} \bar{\mu}_{K+1}} + \underbrace{\tilde{\Pi}^f(x)}_{\underset{x\to\infty}{\longrightarrow} \mu^f - 1} \right).$$

Since, by assumption, $\bar{\mu}_{K+1} > \mu^f$, this implies that

$$\frac{\partial \Pi^f}{\partial p_{K+1}} \left((p_1, \ldots, p_K, x, \infty, \ldots, \infty), H^0 \right)$$

is strictly negative when x is high enough. Therefore, there exists $\tilde{p}_{K+1} \in \mathbb{R}_{++}$ such that

$$\Pi^f \left((p_1, \ldots, p_K, \tilde{p}_{K+1}, \infty, \ldots, \infty), H^0 \right) > \Pi^f \left((p_1, \ldots, p_K, \infty, \ldots, \infty), H^0 \right),$$

which contradicts condition (b) in Definition A.

Therefore, $r_j\left(\mu^f\right) = \infty$ for all $j \geq K+1$, and equations (20) and (21) are equivalent. $\quad\square$

All we need to do now is study equation (20):

Lemma F. *Equation* (20) *has a unique solution on interval* $(1, \infty)$.

Proof. Assume without loss of generality that $f = \{1, \ldots, n\}$ and $\bar{\mu}_1 \leq \bar{\mu}_2 \leq \ldots \bar{\mu}_n$. Let $S = \{\bar{\mu}_j\}_{1 \leq j \leq n}$. Set S contains $K \leq n$ distinct elements: $\hat{\mu}_1 < \hat{\mu}_2 < \ldots < \hat{\mu}_K$. We define the following function:

$$\phi : \mu^f \in (1, \infty) \mapsto (\mu^f - 1) \left(\sum_{j=1}^{n} h_j\left(r_j(\mu^f)\right) + H^0 \right) - \mu^f \sum_{j=1}^{n} \gamma_j\left(r_j(\mu^f)\right).$$

Note that μ^f solves equation (20) if and only if $\phi(\mu^f) = 0$. ϕ is continuous, and

$$\lim_{1^+} \phi = -\sum_{j=1}^{n} \gamma_j\left(p_j^{mc}\right) < 0.$$

Next, we show that $\phi(\mu^f)$ is positive for μ^f high enough. Assume first that $\hat{\mu}_K < \infty$. Then, for all $\mu^f \geq \hat{\mu}_K$, $\phi(\mu^f) = (\mu^f - 1)H^0 > 0$. If instead $\hat{\mu}_K = \infty$, then, for every μ^f,

$$\phi(\mu^f) \geq (\mu^f - 1)H^0 - \sum_{j\in f} \gamma_j(r_j(\mu^f)) \xrightarrow[\mu^f\to\infty]{} \infty,$$

where the inequality follows from log-convexity ($h_j \geq \gamma_j$),[61] and the limit follows from the fact that $\gamma_j(\infty) = 0$ (see Lemma III-(e) in Online Appendix II.1). It follows from the intermediate value theorem that ϕ has a zero. Moreover, all the zeros of ϕ are contained in interval $(1, \hat{\mu}_K)$.

Next, we claim that ϕ is strictly increasing. To see this, we partition interval $(1, \hat{\mu}_K)$ into sub-intervals $(1, \hat{\mu}_1)$, $(\hat{\mu}_1, \hat{\mu}_2)$, ..., $(\hat{\mu}_{K-1}, \hat{\mu}_K)$. Pick one of these sub-intervals, call it (a, b), and let i be the smallest $j \in \{1, \ldots, n\}$ such that $\bar{\mu}_j = b$. Then, for all $\mu^f \in (a, b)$,

$$\phi(\mu^f) = (\mu^f - 1) \left(\sum_{j=1}^{i-1} h_j(\infty) + \sum_{j=i}^{n} h_j \left(r_j(\mu^f) \right) + H^0 \right) - \mu^f \sum_{j=i}^{n} \gamma_j \left(r_j(\mu^f) \right),$$

and, by Lemma D, ϕ is $\mathcal{C}^1$ on (a, b). $\phi'(\mu^f)$ is given by (we omit the arguments of functions to save space):

$$\phi'(\mu^f) = H^0 + \sum_{j=1}^{i-1} h_j(\infty) + \sum_{j=i}^{n} (h_j - \gamma_j) + (\mu^f - 1) \left(\sum_{j=i}^{n} r_j' h_j' \right) - \mu^f \left(\sum_{j=i}^{n} r_j' \gamma_j' \right),$$

$$= H^0 + \sum_{j=1}^{i-1} h_j(\infty) + \sum_{j=i}^{n} (h_j - \gamma_j) + \sum_{j=i}^{n} \underbrace{r_j' \left(\mu^f(-\gamma_j') - (\mu^f - 1)(-h_j') \right)}_{=\gamma_j \text{ by Lemma D}},$$

$$= H^0 + \sum_{j=1}^{n} h_j > 0.$$

Therefore, ϕ is strictly increasing on intervals $(1, \hat{\mu}_1)$, $(\hat{\mu}_1, \hat{\mu}_2)$, ..., $(\hat{\mu}_{K-1}, \hat{\mu}_K)$. By continuity, it follows that ϕ is strictly increasing on $(1, \hat{\mu}_K)$. Therefore, equation (20) has a unique solution. $\square$

Combining Lemmas B–F allows us to conclude the analysis of maximization problem (18):

Lemma G. *Maximization problem* (18) *has a unique solution. The generalized first-order conditions associated with this maximization problem are necessary and sufficient for global optimality. The optimal price vector (which contains at least one finite component) satisfies the common ι-markup property, and the corresponding ι-markup, μ^{f*}, is the unique solution of equation* (20). *The maximized value of the objective function is $\mu^{f*} - 1$.*

Proof. Let $(p_j^*)_{j \in f}$ be a solution of maximization problem (18). By Lemma B, such a $(p_j^*)_{j \in f}$ exists and $p_k^* < \infty$ for some $k \in f$. By Lemmas C, $(p_j^*)_{j \in f}$ satisfies the generalized first-order conditions. Therefore, by Lemma E, $(p_j^*)_{j \in f}$ satisfies the common ι-markup property, and the corresponding μ^f solves equation (20). By Lemma F, this equation has a unique solution, which we denote μ^{f*}. Therefore, $(p_j^*)_{j \in f} = \left(r_j(\mu^{f*}) \right)_{j \in f}$, and maximization problem (18) has

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[61] $(\log h_j)'' = \frac{h_j'' h_j - (h_j')^2}{h_j^2} = \frac{h_j''}{h_j^2} \left(h_j - \frac{(h_j')^2}{h_j''} \right) = \frac{h_j''}{h_j^2} (h_j - \gamma_j) \geq 0.$

a unique solution. The fact that the maximized value of the objective function is $\mu^{f*} - 1$ follows immediately from Lemma E.

Conversely, assume that the generalized first-order conditions hold at price vector $(\tilde{p}_j)_{j \in f}$. Then, by Lemmas E and F, $(\tilde{p}_j)_{j \in f} = \left(r_j(\mu^{f*}) \right)_{j \in f} = (p_j^*)_{j \in f}$. It follows that generalized first-order conditions are sufficient for global optimality. $\qquad \square$

We now turn our attention to the equilibrium existence problem. Price vector $p \in (0, \infty]^{\mathcal{N}}$ is a Nash equilibrium if and only if, for every $f \in \mathcal{F}$, $(p_j)_{j \in f}$ maximizes $\Pi^f \left(\cdot, \sum_{j \in \mathcal{N} \setminus \{f\}} h_j(p_j) \right)$. From Lemma A, each firm sets at least one finite price in any Nash equilibrium. Therefore, $\sum_{j \in \mathcal{N} \setminus \{f\}} h_j(p_j) > 0$ for every f, and we can apply Lemma G: There exists a pricing equilibrium if and only if there exists a profile of ι-markups $\left(\mu^f \right)_{f \in \mathcal{F}} \in (1, \infty)^{\mathcal{F}}$ such that

$$\mu^f = 1 + \mu^f \frac{\sum_{j \in f} \gamma_j \left(r_j(\mu^f) \right)}{\sum_{g \in \mathcal{F}} \sum_{j \in g} h_j \left(r_j(\mu^g) \right)}, \quad \forall f \in \mathcal{F}.$$

This is, in turn, equivalent to finding an aggregator level $H > 0$ and a profile of ι-markups $\left(\mu^f \right)_{f \in \mathcal{F}} \in (1, \infty)^{\mathcal{F}}$ such that $H = \sum_{g \in \mathcal{F}} \sum_{j \in g} h_j \left(r_j(\mu^g) \right)$ and for all $f \in \mathcal{F}$,

$$\mu^f = 1 + \mu^f \frac{\sum_{j \in f} \gamma_j \left(r_j(\mu^f) \right)}{H}. \tag{22}$$

Our approach to equilibrium existence consists in showing that this nested fixed point problem has a solution. We start by studying the inner fixed point problem:

Lemma H. *For every $f \in \mathcal{F}$, for every $H > 0$, equation (22) has a unique solution in μ^f on interval $(1, \infty)$. Denote this solution by $m^f(H)$.*
Function $m^f(.)$ is continuous, strictly decreasing, and satisfies $\lim_\infty m^f = 1$ and $\lim_{0^+} m^f = \bar{\mu}^f$.

Proof. μ^f solves equation (22) if and only if

$$\psi(\mu^f, H) \equiv \mu^f \left(1 - \frac{\sum_{j \in f} \gamma_j \left(r_j(\mu^f) \right)}{H} \right) = 1.$$

Note that ψ is continuous on $(1, \infty) \times \mathbb{R}_{++}$. In addition, if $\bar{\mu}^f < \infty$, then $\psi(\bar{\mu}^f, H) = \mu^f > 1$ for all $\mu^f \geq \bar{\mu}^f$. Therefore, equation (22) does not have a solution on $[\bar{\mu}^f, \infty)$.

Next, note that $\psi(1, H) = 1 - \frac{\sum_{j \in f} \gamma_j \left(p_j^{mc} \right)}{H} < 1$. Suppose there exists $\mu^f > 1$ such that $\sum_{j \in f} \gamma_j \left(r_j(\mu^f) \right) \geq H$. Then, since γ_j is decreasing and r_j is increasing (see Lemmas D and III-(d)), $\sum_{j \in f} \gamma_j \left(r_j(\mu^{f\prime}) \right) \geq H$ for all $\mu^{f\prime} \leq \mu^f$. Define

$$\underline{\mu}^f(H) = \sup \left\{ \mu^f \in (1, \infty) : \quad \sum_{j \in f} \gamma_j \left(r_j(\mu^f) \right) \geq H \right\},$$

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and note that $\underline{\mu}^f(H) < \bar{\mu}^f$, since $\sum_{j \in f} \gamma_j \left(r_j(\mu^f) \right)$ goes to zero as μ^f goes to $\bar{\mu}^f$. For all $\mu^f \leq \underline{\mu}^f(H)$, $\psi(\mu^f, H) < 1$.

Therefore, equation (22) does not have solutions outside interval $(\underline{\mu}^f(H), \bar{\mu}^f)$ (define $\underline{\mu}^f(H) = 1$ if $\sum_{j \in f} \gamma_j \left(r_j(\mu^f) \right) < H$ for all $\mu^f > 1$). In addition, $\psi \left(\underline{\mu}^f(H), H \right) < 1$ and $\lim_{\bar{\mu}^f} \psi > 1$. By the intermediate value theorem, equation (22) has a solution on interval $(\underline{\mu}^f(H), \bar{\mu}^f)$. Since ψ is strictly increasing in μ^f on that interval (see Lemmas III-(d) and D), the solution is unique. This establishes the existence and uniqueness of $m^f(H)$. In addition, since ψ is strictly increasing in μ^f and H, m^f is strictly decreasing in H.

Let $H > 0$. Assume for a contradiction that m^f is not continuous at H. There exists $\varepsilon_0 > 0$ and $(H^n)_{n \in \mathbb{N}} \in \mathbb{R}_{++}^{\mathbb{N}}$ such that $H^n \xrightarrow[n \to \infty]{} H$ and $\left| m^f(H^n) - m^f(H) \right| > \varepsilon_0$ for every $n \geq 0$. Let $0 < \eta < H$. Then, for high enough n, $H^n \in [H - \eta, H + \eta]$. By monotonicity of m^f, it follows that $m^f(H^n) \in \left[m^f(H + \eta), m^f(H - \eta) \right]$. Therefore, $\left(m^f(H^n) \right)_{n \geq 0}$ is bounded, and we can extract a convergent subsequence: There exists $\xi : \mathbb{N} \longrightarrow \mathbb{N}$ strictly increasing and $\mu^f \in (1, \bar{\mu}^f)$ such that $m^f \left(H^{\xi(n)} \right) \xrightarrow[n \to \infty]{} \mu^f$. Since $m^f(H^n)$ is bounded away from $m^f(H)$, it follows that $\mu^f \neq m^f(H)$. By definition of m^f, for every $n \geq 0$, $\psi \left(m^f \left(H^{\xi(n)} \right), H^{\xi(n)} \right) = 1$. Since ψ is continuous, we can take limits, and obtain that $\psi(\mu^f, H) = 1$. By uniqueness, it follows that $\mu^f = m^f(H)$, which is a contradiction. Therefore, m^f is continuous. $\qquad \square$

We can now take care of the outer fixed-point problem, which, by Lemma H consists in finding an $H > 0$ such that $\Omega(H) = 1$, where $\Omega(H) \equiv \sum_{f \in \mathcal{F}} \sum_{k \in f} h_k(r_k(m^f(H)))/H$. The following lemma guarantees that the outer fixed-point problem has a solution:

Lemma I. *There exists $H^* > 0$ such that $\Omega(H^*) = 1$.*

Proof. By Lemmas D and H, Ω is continuous on $\mathbb{R}_{++}$. In addition, when H goes to ∞, the numerator of Ω goes to $\sum_{f \in \mathcal{F}} \sum_{k \in f} h_k(p_k^{mc})$, which is finite. Therefore, $\lim_{\infty} \Omega = 0$. If we show that Ω is strictly greater than 1 in the neighborhood of 0^+, then we can apply the intermediate value theorem to obtain the existence of H^*.

Assume first that there exists $j \in \mathcal{N}$ such that $\lim_{\infty} h_j = l > 0$. Since h_j is decreasing, $h_j(x) \geq l$ for all $x > 0$. Let $f \in \mathcal{F}$ such that $j \in f$. Then, for all $H > 0$,

$$\Omega(H) \geq \frac{h_j \left(r_j \left(m^f(H) \right) \right)}{H} \geq \frac{l}{H} \xrightarrow[H \to 0^+]{} \infty.$$

Therefore, $\lim_{0^+} \Omega = \infty > 1$.

Next, assume that $\lim_{\infty} h_k = 0$ for all $k \in \mathcal{N}$. For every $f \in \mathcal{F}$, we define threshold $\underline{H}^{f\prime} > 0$ as follows. If $\bar{\mu}_k = \bar{\mu}^f$ for all $k \in f$, then let $\underline{H}^{f\prime} = 1$. If $\bar{\mu}_k < \bar{\mu}^f$ for some $k \in f$, then, since $\lim_{0^+} m^f = \bar{\mu}^f$, there exists $\hat{H}^f > 0$ such that $m^f(H) > \max(\{\bar{\mu}_k\}_{k \in f} \setminus \{\bar{\mu}^f\})$ whenever $H < \hat{H}^f$. In this case, let $\underline{H}^{f\prime} \equiv \hat{H}^f$. Having done that for every $f \in \mathcal{F}$, let $\underline{H}' = \min_{f \in \mathcal{F}} \underline{H}^{f\prime}$. Then, for every $H \in (0, \underline{H}')$,

$$\Omega(H) = \frac{1}{H} \sum_{f \in \mathcal{F}} \sum_{j \in f: \ \bar{\mu}_j = \bar{\mu}^f} h_j \left(r_j \left(m^f(H) \right) \right).$$

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We partition the set of firms into two sets: $\mathcal{F}'$ and $\mathcal{F}''$, where $\mathcal{F}' = \left\{ f \in \mathcal{F} : \bar{\mu}^f = \infty \right\}$, and $\mathcal{F}'' = \mathcal{F} \backslash \mathcal{F}'$.

Let $f \in \mathcal{F}''$. Then, by Lemma III-(f) in Online Appendix II.1, $\lim_\infty \rho_k = \frac{\bar{\mu}^f}{\bar{\mu}^f - 1}$ for every $k \in f$ such that $\bar{\mu}_k = \bar{\mu}^f$. In addition, by Lemmas D and H, for every $k \in f$, $r_k \left(m^f(H) \right) \xrightarrow[H \to 0^+]{} \infty$. Therefore, there exists $\underline{H}^{f''} > 0$ such that

$$\rho_k \left(r_k \left(m^f(H) \right) \right) \geq \frac{\bar{\mu}^f}{\bar{\mu}^f - 1} \left(1 - \frac{1}{2|\mathcal{F}|} \right), \quad \forall H < \underline{H}^{f''}, \ \forall k \in f \ \text{s.t.} \ \bar{\mu}_k = \bar{\mu}^f.$$

Let $\underline{H}'' = \min_{f \in \mathcal{F}''} \underline{H}^{f''}$ (or any strictly positive real number if $\mathcal{F}''$ is empty), and $\underline{H} = \min \left(\underline{H}', \underline{H}'' \right)$. For every $H < \underline{H}$,

$$\Omega(H) = \frac{1}{H} \left(\sum_{f \in \mathcal{F}'} \sum_{\substack{k \in f \\ \bar{\mu}_k = \bar{\mu}^f}} h_k \left(r_k \left(m^f(H) \right) \right) + \sum_{f \in \mathcal{F}''} \sum_{\substack{k \in f \\ \bar{\mu}_k = \bar{\mu}^f}} h_k \left(r_k \left(m^f(H) \right) \right) \right),$$

$$\geq \frac{1}{H} \left(\sum_{f \in \mathcal{F}'} \sum_{\substack{k \in f \\ \bar{\mu}_k = \bar{\mu}^f}} \gamma_k \left(r_k \left(m^f(H) \right) \right) + \sum_{f \in \mathcal{F}''} \sum_{\substack{k \in f \\ \bar{\mu}_k = \bar{\mu}^f}} \gamma_k \left(r_k \left(m^f(H) \right) \right) \rho_k \left(r_k \left(m^f(H) \right) \right) \right),$$

$$\geq \sum_{f \in \mathcal{F}'} \frac{1}{H} \sum_{\substack{k \in f \\ \bar{\mu}_k = \bar{\mu}^f}} \gamma_k \left(r_k \left(m^f(H) \right) \right) + \sum_{f \in \mathcal{F}''} \frac{\bar{\mu}^f}{\bar{\mu}^f - 1} \left(1 - \frac{1}{2|\mathcal{F}|} \right) \frac{1}{H} \sum_{\substack{k \in f \\ \bar{\mu}_k = \bar{\mu}^f}} \gamma_k \left(r_k \left(m^f(H) \right) \right),$$

$$= \sum_{f \in \mathcal{F}'} \frac{\sum_{k \in f} \gamma_k \left(r_k \left(m^f(H) \right) \right)}{H} + \sum_{f \in \mathcal{F}''} \frac{\bar{\mu}^f}{\bar{\mu}^f - 1} \left(1 - \frac{1}{2|\mathcal{F}|} \right) \frac{\sum_{k \in f} \gamma_k \left(r_k \left(m^f(H) \right) \right)}{H},$$

$$= \sum_{f \in \mathcal{F}'} \frac{m^f(H) - 1}{m^f(H)} + \sum_{f \in \mathcal{F}''} \frac{m^f(H) - 1}{m^f(H)} \frac{\bar{\mu}^f}{\bar{\mu}^f - 1} \left(1 - \frac{1}{2|\mathcal{F}|} \right), \quad \text{using equation (22)},$$

where the second line follows by log-convexity ($h_k \geq \gamma_k$ for all k). When H goes to 0^+, the right-hand side term on the last line goes to

$$|\mathcal{F}'| + |\mathcal{F}''| \left(1 - \frac{1}{2|\mathcal{F}|} \right) \geq |\mathcal{F}| - \frac{1}{2},$$

which is strictly greater than 1. Therefore, $\Omega(H) > 1$ when H is small enough. $\quad \square$

This concludes the proof of Theorem 1.

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